



HAL
open science

Global existence for small data of the viscous Green-Naghdi type equations

Dena Kazerani

► **To cite this version:**

Dena Kazerani. Global existence for small data of the viscous Green-Naghdi type equations. *Journal of Differential Equations*, 2016, 261 (1), pp.762-796. 10.1016/j.jde.2016.03.022 . hal-01111941v1

HAL Id: hal-01111941

<https://hal.science/hal-01111941v1>

Submitted on 1 Feb 2015 (v1), last revised 31 Aug 2016 (v2)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Global existence for small data of the viscous Green–Naghdi type equations

Dena Kazerani ^{*†‡}

February 1, 2015

Abstract

We consider the Cauchy problem for the Green–Naghdi equations with viscosity, for small initial data. It is well-known that adding a second order diffusion term to a hyperbolic system leads to the existence of global smooth solutions, as soon as the hyperbolic system is symmetrizable and the so-called Kawashima–Shizuta condition is satisfied. In a previous work, we have proved that the Green–Naghdi equations can be written in a symmetric form, using the associated Hamiltonian. This system being dispersive, in the sense that it involves third order derivatives, the symmetric form is based on symmetric differential operators. In this paper, we use this structure for an appropriate change of variable to prove that adding viscosity effects through a second order term leads to global existence of smooth solutions, for small data. We also deduce that constant solutions are asymptotically stable.

Keywords : Green–Naghdi equations, viscosity, small solutions, symmetric structure, energy equality, global existence

1 Introduction

The Green–Naghdi system is a shallow water approximation of the water wave problem which models incompressible flows. The vertical and horizontal speeds are averaged vertically. Moreover, vertical acceleration is supposed too small to be considered [9]. In other words, Green–Naghdi equations is one order higher in approximation compared to the Saint–Venant (called also isentropic Euler) system [3]. To obtain the latter system, not only the vertical acceleration but also the vertical speed are neglected. This leads to a hyperbolic system of equations contrary to the Green–Naghdi equation which is dispersive due to the term $\alpha h^2 \ddot{h}$

*Sorbonne Universités, UPMC Univ Paris 06, UMR 7598, Laboratoire Jacques-Louis Lions, F-75005, Paris, France

†CNRS, UMR 7598, Laboratoire Jacques-Louis Lions, F-75005, Paris, France

‡INRIA-Paris-Rocquencourt, EPC ANGE, Domaine de Voluceau, BP105, 78153 Le Chesnay Cedex

defined below. In this work, we focus on the Green–Naghdi type equation with a second order viscosity:

$$\begin{cases} \partial_t h + \partial_x h u = 0, \\ \partial_t h u + \partial_x h u^2 + \partial_x (g h^2 / 2 + \alpha h^2 \ddot{h}) = \mu \partial_x (h \partial_x u) \end{cases} \quad (1)$$

We assume that $h(x, t) > 0$, α and μ are strictly positive and g is the gravity constant. The unknown h represents the fluid height and u its average horizontal speed. Moreover, the material derivative $(\dot{})$ is defined by $(\dot{}) = \partial_t() + u \partial_x()$.

Remark 1.1 *Let us note that the $\alpha = 0$ case gives us the Saint-Venant system. We can also learn more about the derivation of the system in [17, 1, 11] for $(\mu, \alpha) = (0, \frac{1}{3})$, and in [6] for $(\mu, \alpha) = (0, \frac{1}{4})$.*

It is worth remarking that (1) admits the following energy equality [8, 6],

$$\partial_t E + \partial_x (u(E + p)) = \mu u \partial_x (h \partial_x u), \quad (2)$$

where

$$E(h, u) = g h^2 / 2 + h u^2 / 2 + \alpha h^3 (\partial_x u)^2 / 2,$$

and

$$p(h, u) = g h^2 / 2 + \alpha h^2 \ddot{h}.$$

We can also check that (1) admits a family of energy conversation equality

$$\partial_t E_{h_e, u_e} + \partial_x P_{h_e, u_e} = \mu (u - u_e) \partial_x (h \partial_x u). \quad (3)$$

where

$$E_{h_e, u_e}(h, u) = g (h - h_e)^2 / 2 + h (u - u_e)^2 / 2 + \alpha h^3 (\partial_x u)^2 / 2, \quad (4a)$$

and

$$P_{h_e, u_e}(h, u) = u E_{h_e, u_e}(h, u) + (u - u_e) p(h, u) - \frac{g h_e^2}{2} u. \quad (4b)$$

for all $h_e > 0$ and $u_e \in \mathbb{R}$.

Remark 1.2 *Let us assume that $\alpha = 0$. Then, $E(h, u)$ and $E_{h_e, u_e}(h, u)$ are convex entropies for Saint-Venant system.*

The aim of this paper is to study the stability of the equilibrium based on the symmetric structure of the system. The intuition comes from the Kawashima works on hyperbolic-parabolic systems ([19] and [13]) and Hanouzet–Natalini and Yong ([10] and [20]) on entropy dissipative symmetric hyperbolic systems. These studies can be physically sketched by the stability of constant solutions of symmetric hyperbolic systems considering friction ([10] and [20]) or viscosity ([19] and [13]) right hand side terms. All these results have been proved using the symmetric structure of hyperbolic systems. A generalization of the symmetric structure for dispersive equations admitting an additional energy conversation law with a locally convex energy integral is done in [14]. The symmetric structure used in this work to extend

the symmetric hyperbolic equations' technique to Green-Naghdi equation is recalled in Sub-section 1.1.

Note that the higher order of this equation compared to hyperbolic systems makes us consider right hand side terms of order 2. In fact, this result can not be generalized to the system with friction in the absence of viscosity. This is due to the incoherence between the order of a priori estimates and the estimation of the time integral of the norm of the solution. However, the coherence is guaranteed while we consider a second order viscosity. Likewise, if we consider viscosities of order 4 or more, with or without any second order viscosity, we are not able to generalize this method to estimate the time integral of the norm of the solution. A viscous version of the system has been derived with a fourth order viscosity in [6]. Therefore, other techniques must be used to study this last system.

In all this work, partial derivatives with respect to x of any continuous function f are presented by either f_x or $\partial_x f$. The differential of the application F at U is symbolized by $D_U F(U)$. The adjoint of the operator \mathcal{A} is denoted by \mathcal{A}^* .

1.1 Symmetric structure

Following Li's notations in [17], we use the unknown $U = (h, m)$ defined by a Sturm-Liouville operator called \mathcal{L} :

$$m = hu - \alpha \partial_x (h^3 u_x) = \mathcal{L}_h(u).$$

Then, the system (1) is of the form

$$\partial_t U + \partial_x F(U) = Q(U),$$

where

$$F(U) = \begin{pmatrix} h \mathcal{L}_h^{-1}(m) \\ m \mathcal{L}_h^{-1}(m) - 2\alpha h^3 (\partial_x \mathcal{L}_h^{-1}(m))^2 + \frac{g}{2} h^2 - \frac{g}{2} h_e^2 \end{pmatrix}, \quad (5)$$

and

$$Q(U) = \begin{pmatrix} 0 \\ \mu \partial_x (hu_x) \end{pmatrix} \quad (6)$$

Based on the structure presented in [17], it is easy to check that the unknown U enables us to write (1) under a Hamiltonian structure with the Hamiltonian \mathcal{H}_{h_e, u_e} defined by the energy integral, $\mathcal{H}_{h_e, u_e} = \int_{\mathbb{R}} E_{h_e, u_e}$. This unknown presents also another advantage. In fact, we can recover the physical variable $V = (h, u)$ from U using the interesting change of variable $V = (h, \delta_m \mathcal{H}_{h_e, u_e}(U))$, where δ_m denotes the variational derivative with respect to m ¹. This

¹We have,

$$\delta_m \mathcal{H}_{h_e, u_e}(U) = g(h - h_e) - \frac{u^2 - u_e^2}{2} - \frac{3}{2} \alpha h^2 (u_x)^2,$$

and

$$\delta_h \mathcal{H}_{h_e, u_e}(U) = u - u_e.$$

consideration, as suggested in the following theorem, enables us to symmetrize the system in the physical variable with a diagonal locally definite positive operator (See Appendix A for more details).

Theorem 1.3 [14]. *Let $V_e = (h_e, u_e)$ be a constant solution of (1) with $h_e > 0$. Then, there exists $\delta > 0$ such that, while the solution is in $B_{\bar{s}}(V_e, \delta)$, the system is equivalent to the following symmetric form:*

$$\mathcal{A}_0(V)\partial_t V + \mathcal{A}_1(V)\partial_x V = \begin{pmatrix} 0 \\ \mu\partial_x(hu_x) \end{pmatrix} \quad (7)$$

where

$$\begin{aligned} \mathcal{A}_0(V) &= D_V U^*(V) (\delta_U^2 \mathcal{H}) D_V U(V) \\ &= \begin{pmatrix} g - 3\alpha h(u_x)^2 & 0 \\ 0 & \mathcal{L}_h \end{pmatrix} \end{aligned} \quad (8)$$

is a positive definite operator and

$$\begin{aligned} \mathcal{A}_1(V) &= D_V U^*(V) (\delta_U^2 \mathcal{H}) (D_U F(U)) D_V U(V) \\ &= \begin{pmatrix} gu - 3\alpha hu(u_x)^2 & gh - 3\alpha h^2(u_x)^2 \\ gh - 3\alpha h^2(u_x)^2 & hu + 2\alpha(h^3 u_x)_x - \alpha(h^3 u)_x \partial_x - \alpha h^3 u \partial_x^2 \end{pmatrix} \end{aligned} \quad (9)$$

is a symmetric one.

Proof Let us consider the writing

$$\partial_t U + \partial_x F(U) = Q(U).$$

Obviously, we have

$$D_V U(V)\partial_t V + D_U F(U)D_V U(V)\partial_x V = Q(U).$$

The, acting $D_V U^*(V) (\delta_U^2 \mathcal{H}_{h_e, u_e})$ on the system and considering the fact that $Q(U)$ is an invariant vector of $D_V U^*(V) (\delta_U^2 \mathcal{H}_{h_e, u_e})$, we get the result (See Appendix A for more details). ■

Let us note that $\mathcal{A}_0(V)$ and $\mathcal{A}_1(V)$ are linear second order operators. Therefore, they can be decomposed as

$$\mathcal{A}_0(V) = \mathcal{A}_0^0(V) + \mathcal{A}_0^1(V)\partial_x + \mathcal{A}_0^2(V)\partial_x^2 \quad (10)$$

$$\mathcal{A}_1(V) = \mathcal{A}_1^0(V) + \mathcal{A}_1^1(V)\partial_x + \mathcal{A}_1^2(V)\partial_x^2 \quad (11)$$

where the expressions of symmetric matrix $\mathcal{A}_i^j(V)$ for $i, j \in \{0, 1, 2\}$ are given by

$$\mathcal{A}_0^0(V) = \begin{pmatrix} g - 3\alpha h(u_x)^2 & 0 \\ 0 & h \end{pmatrix}, \quad \mathcal{A}_0^1(V) = \begin{pmatrix} 0 & 0 \\ 0 & -3\alpha h^2 h_x \end{pmatrix}, \quad \mathcal{A}_0^2(V) = \begin{pmatrix} 0 & 0 \\ 0 & -\alpha h^3 \end{pmatrix},$$

$$\mathcal{A}_1^0(V) = \begin{pmatrix} gu - 3\alpha hu(u_x)^2 & gh - 3\alpha h^2(u_x)^2 \\ gh - 3\alpha h^2(u_x)^2 & hu + 2\alpha(h^3 u_x)_x \end{pmatrix}, \quad \mathcal{A}_1^1(V) = \begin{pmatrix} 0 & 0 \\ 0 & -\alpha(h^3 u)_x \end{pmatrix},$$

$$\mathcal{A}_1^2(V) = \begin{pmatrix} 0 & 0 \\ 0 & -\alpha h^3 u \end{pmatrix}.$$

Remark 1.4 *The definite positivity of a real matrix is equivalent to its coercivity. However, this fact does not necessary hold true for definite positive operators i.e. some definite positive operators are not coercive. It is important to point out that, as we will see in Section 3, what lets us generalize the hyperbolic methods to our symmetric system is actually the coercivity of $\mathcal{A}_0(V)$ for the convenient norm. This is to say that we would not be able to generalize the method if $\mathcal{A}_0(V)$ was definite positive but not coercive.*

1.2 Outline

We are going to see the global existence of solutions of Green–Naghdi equations for initial data close enough to the equilibrium. A local well-posedness is proved in Appendix B. To prove the global existence of the local solution, we will use the dissipative character of the viscosity as well as the symmetric structure of the system. In fact, the first step of the proof is initial estimates obtained taking the scalar product of the s^{th} derivative of the equation with the s^{th} derivative of the solution. As it is exposed in Subsections 3.1 and 3.2, these estimates are obtained by almost the same approach as in the hyperbolic case ([10] and [20]). Then, the second step is to estimate the time integral of the norm of the solution. For hyperbolic systems, this estimate is found using the Kawashima–Shizuta condition. This condition has been introduced in [19] for computational necessities. It is based on the existence of a skew-symmetric matrix such that its product with a definite positive matrix (equivalent to \mathcal{A}_0) is skew-symmetric at equilibrium and the symmetric part of its product with the symmetric matrix (equivalent to \mathcal{A}) may become a definite positive matrix by modifying the components of the down diagonal bloc. However, in the case of Green–Naghdi system, we have not been able to find any operator generalization of the Kawashima–Shizuta condition. Hence, we must use a slightly different approach to find a convenient estimate. Indeed, we can find a null diagonal real matrix K such that $K\mathcal{A}_1(V_e)$ is a symmetric definite positive matrix for all equilibrium V_e such that $u_e = 0$. However, $K\mathcal{A}_0(V_e)$ is not a skew-symmetric operator. Therefore, we try to extract a part from its associated terms which we know the time primitive of. We can see that this extraction is suitable because we are able to control the remaining terms in a convenient way. This is what is done in Subsection 3.3. Then, using a symmetry group of the system, we can generalize the result to equilibriums V_e such that $u_e \neq 0$.

This paper is organized on 5 sections. The global existence theorem and its corollaries are presented in Section 2. Section 3 contains the steps of the proof. Then, in Section 4 we investigate the relevance of our approach when $\alpha = 0$ or $\mu = 0$. Some perspectives are suggested in Section 5. The advantages of the symmetric structure used in this study are explained in Appendix A. So we can see why this symmetric structure is more interesting than others. Appendix B contains the proof of the local well-posedness theorem 2.1. Appendix C highlights one of the other utilities of the symmetric structure. In fact, linear stability of equilibrium of non viscous Green–Naghdi can be proved using this structure.

2 Main results

The local well-posedness of (1) has been studied in [12] and [17] for the case $\mu = 0$. We see here that we can prove the local well-posedness of (1), around constant solutions, based on the idea used for symmetric hyperbolic systems. To do so, we first note that the set of constant solutions of (1) is

$$\{V_e = (h_e, u_e); h_e > 0, u_e \in \mathbb{R}\}.$$

We may also call these solutions the equilibriums of the system.

We denote the norm associated with the affine space $\mathbb{X}^{\bar{s}}(\mathbb{R}) = (\mathbb{H}^{\bar{s}}(\mathbb{R}) + h_e) \times (\mathbb{H}^{\bar{s}+1}(\mathbb{R}) + u_e)$ by

$$\|(f, g)\|_{\mathbb{X}^{\bar{s}}}^2 = \|f\|_{\mathbb{H}^{\bar{s}}}^2 + \|g\|_{\mathbb{H}^{\bar{s}+1}}^2.$$

Moreover, the \bar{s} -neighborhood of radius δ and center $V_e \in \mathbb{X}^{\bar{s}}(\mathbb{R})$ is presented by $B_{\bar{s}}(V_e, \delta) = \{V \in \mathbb{X}^{\bar{s}}(\mathbb{R}), \|V - V_e\|_{\mathbb{X}^{\bar{s}}}^2 \leq \delta\}$ for all integer $\bar{s} \in \mathbb{R}$.

We are now able to announce the local well-posedness theorem,

Theorem 2.1 *Let $\bar{s} \geq 2$ be an integer and consider the constant solution V_e of system (1). Then, there exists $\delta > 0$ such that for all initial data $V_0 \in B_{\bar{s}}(V_e, \delta)$, the system is locally well-posed in $\mathbb{X}^{\bar{s}}(\mathbb{R})$.*

The proof of the theorem is given in Appendix B. The steps of the proof are the same as for hyperbolic systems (see [7, 5] for instance). However, the necessary estimate to reach the final result of each step, is obtained by the same technique used in Section 3.2. In fact, we can see again in this part, how the generalized symmetric structure (7) of the system enables us to generalize the techniques used for symmetric hyperbolic systems.

The main result of this study is the following theorem on the asymptotic stability of equilibriums.

Theorem 2.2 *Let us consider the equilibrium $V_e = (h_e, u_e)$ of (1) and $\bar{s} \geq 2$ an integer. Then, there exists $\delta > 0$ such that for all initial data $V_0 = (h_0, u_0) \in B_{\bar{s}}(V_e, \delta)$, the solution V exists for all time and converges asymptotically to V_e .*

In other words, every constant solution $V_e = (h_e, u_e)$ of (1) is asymptotically stable.

Let us remark that we can prove Theorem 2.2 by considering $u_e = 0$. This is due to the fact that $v = t\partial_x + \partial_u$ is a infinitesimal generator of a symmetry group of (1). That is to say that

$$V_\beta = (h(x - \beta t, t), u(x - \beta t, t) + \beta)$$

is also a solution of (1) for all solution $V = (h, u)$ and all $\beta \in \mathbb{R}$. This fact has been mentioned in [16, 2] for the case $\mu = 0$. It is easy to check that the second order viscosity right hand side does not change this symmetry group. Hence, from now on, all the equilibriums considered in this work are of the form

$$V_e = (h_e, 0).$$

The key of this study is the following proposition which is a consequence of the primitive estimates in \mathbb{X}^s and the estimation of the time integral of the \mathbb{H}^{s-1} norm of h_x obtained in Section 3. In order to understand this study, let us mention that symbol $C_S(\delta)$ stands for a function of δ , defined by the elements of the set S , which converges to a limit strictly different from zero while δ goes to 0. On the other hand, $\Theta_S(\delta)$ stands for a function, defined by the elements of the set S , which converges to zero while δ goes to 0. Let us also mention that the estimate suggested in Proposition 2.3 has a similar structure to the estimate given in Theorem 3.1 of [20].

Proposition 2.3 *Let $\bar{s} \geq 2$ be an integer and $\delta > 0$ small enough such that (1) is locally well-posed for the initial data $V_0 \in B_{\bar{s}}(V_e, \delta)$ with $V_e = (h_e, 0)$. Assume also that there exists $\bar{T} > 0$ such that the unique local solution V satisfies $V(T) \in B_{\bar{s}}(V_e, \delta)$ for all $0 \leq T < \bar{T}$. Then, the following estimate holds true for all $T \in [0, \bar{T})$,*

$$(1 - \Theta_{\{h_e, \alpha\}}(\delta)) \|V(T) - V_e\|_{\mathbb{X}^{\bar{s}}}^2 + C_{\{h_e, \mu\}}(\delta) \int_0^T \|u_x\|_{\mathbb{H}^{\bar{s}}}^2 \leq C_{\{h_e, \alpha\}}(\delta) \|V(0) - V_e\|_{\mathbb{X}^{\bar{s}}}^2 + \Theta_{\{h_e, \mu, \alpha\}}(\delta) \int_0^T \|u_x\|_{\mathbb{H}^{\bar{s}}}^2$$

Now, we get the global existence theorem as a result. In fact, we have

Theorem 2.4 *Let us consider the equilibrium $V_e = (h_e, 0)$ of (1) and an integer $\bar{s} \geq 2$. Then, there exists $\nu > 0$ such that for all initial data $V_0 = (h_0, u_0) \in B_{\bar{s}}(V_e, \nu)$, the solution V exists for all time.*

In other words, the equilibrium solutions $V_e = (h_e, 0)$ of (1) are stable.

Proof *Let us first remark that if $\delta > 0$ is small enough, we have*

$$1 - \Theta_{\{h_e, \alpha\}}(\delta) > \frac{1}{2} \quad \text{and} \quad \frac{C_{\{h_e, \mu\}}(\delta) - \Theta_{\{h_e, \mu, \alpha\}}(\delta)}{1 - \Theta_{\{h_e, \alpha\}}(\delta)} > 0.$$

Let us also assume that δ satisfies the assumptions of Proposition 2.3. Then, as long as $V \in B_{\bar{s}}(V_e, \delta)$, it satisfies

$$\|V(T) - V_e\|_{\mathbb{X}^{\bar{s}}}^2 + \frac{C_{\{h_e, \mu\}}(\delta) - \Theta_{\{h_e, \mu, \alpha\}}(\delta)}{1 - \Theta_{\{h_e, \alpha\}}(\delta)} \int_0^T \|u_x\|_{\mathbb{H}^{\bar{s}}}^2 \leq C_{\{g, h_e, \alpha\}}(\delta) \|V_0 - V_e\|_{\mathbb{X}^{\bar{s}}}^2$$

Therefore, while $V \in B_{\bar{s}}(V_e, \delta)$,

$$\|V(T) - V_e\|_{\mathbb{X}^{\bar{s}}}^2 \leq L(\delta) \|V_0 - V_e\|_{\mathbb{X}^{\bar{s}}}^2$$

where L is a function of δ such that $\lim_{\delta \rightarrow 0} L(\delta) = l > 0$. Setting $\nu \leq \delta$ such that $L(\delta)\nu \leq \delta/2$, we have

$$\|V(T) - V_e\|_{\mathbb{X}^{\bar{s}}}^2 \leq \delta/2, \quad \text{while} \quad V(T) \in B_{\bar{s}}(V_e, \delta).$$

This is to say that for $V(0) \in B_{\bar{s}}(V_e, \nu)$, the local solution can not go out from $B_{\bar{s}}(V_e, \delta/2)$ for any time. Hence, the norm of the local solution does not blow up. Moreover, h stays away from 0. Therefore, the unique local solution exists for all time. ■

Corollary 2.5 (Asymptotic stability of equilibrium solutions) *Let us $\bar{s} \geq 2$ be an integer and consider the equilibrium $V_e = (h_e, 0)$ of (1). Then, there exists $\delta > 0$ such that for all initial data $V_0 = (h_0, u_0)$ in $B_{\bar{s}}(V_e, \delta)$, the global solution $V(x, t)$ in $\mathbb{X}^{\bar{s}}(\mathbb{R})$ of (1) converges asymptotically to V_e . In other words, $\lim_{t \rightarrow \infty} V(x, t) = V_e$ for all $x \in \mathbb{R}$.*

Proof *We use a similar logic to the one used in [20] for symmetric entropy dissipative hyperbolic systems satisfying the stability condition. In fact, we consider the system under the form*

$$\partial_t U + \partial_x F(U) = Q(U), \quad (12)$$

where F and Q are defined by (5) and (6). We then take the x derivative of (12), the time integral on $[t_1, t_2]$ and consider the $\mathbb{H}^1 \times \mathbb{L}^2$ norm. This leads us to

$$\| U_x(t_2) - U_x(t_1) \|_{\mathbb{H}^1 \times \mathbb{L}^2} = \left\| \int_{t_1}^{t_2} \partial_{xx} F(U) + \begin{pmatrix} 0 \\ \mu \partial_x^2 (hu_x) \end{pmatrix} dt \right\|_{\mathbb{H}^1 \times \mathbb{L}^2}. \quad (13)$$

Hence,

$$\| U_x(t_2) - U_x(t_1) \|_{\mathbb{H}^1 \times \mathbb{L}^2} \leq |t_2 - t_1| \left(\sup_{t_1 \leq t \leq t_2} \| \partial_{xx} F(U) \|_{\mathbb{H}^1 \times \mathbb{L}^2} + \mu \sup_{t_1 \leq t \leq t_2} \| \partial_x^2 (hu_x) \|_{\mathbb{H}^1 \times \mathbb{L}^2} \right).$$

On the other hand, we have, by Proposition 2.3, that there exists a positive constant C such that the $\| U_x(t) \|_{\mathbb{H}^1 \times \mathbb{L}^2} \leq C$ for all time. This together with the continuity of F gives us a $\tilde{C} > 0$ such that we have for all t_1, t_2 positive,

$$| \| U_x(t_1) \|_{\mathbb{H}^1 \times \mathbb{L}^2} - \| U_x(t_2) \|_{\mathbb{H}^1 \times \mathbb{L}^2} | \leq \| U_x(t_2) - U_x(t_1) \|_{\mathbb{H}^1 \times \mathbb{L}^2} \leq \tilde{C} |t_2 - t_1|.$$

Hence, $t \mapsto \| U_x(t) \|_{\mathbb{H}^1 \times \mathbb{L}^2}$ is Lipschitz continuous. It is also $\mathbb{L}^2([0, \infty))$ by the estimate of the same proposition. Therefore, $\| U_x(t) \|_{\mathbb{H}^1 \times \mathbb{L}^2}$ converges to 0 at the limit $t \rightarrow \infty$, so does $\| V_x(t) \|_{\mathbb{X}^1}$. Then, the continuous embedding of $\mathbb{H}^s(\mathbb{R}) \rightarrow \mathbb{L}^\infty(\mathbb{R})$ for $s \geq 1$ (the Gagliardo–Nirenberg inequality) leads us to

$$\lim_{t \rightarrow \infty} \| V_x(t) \|_{\mathbb{L}^\infty \times \mathbb{L}^\infty} = 0.$$

In other words, $V_x(x, t)$ converges uniformly to 0:

$$\lim_{t \rightarrow \infty} \sup_{x \in \mathbb{R}} |V_x(x, t)| = 0.$$

Therefore, $\lim_{t \rightarrow \infty} V(x, t) = \beta \in \mathbb{R}$ for all $x \in \mathbb{R}$. That is to say that

$$\lim_{t \rightarrow \infty} V(x, t) = \lim_{t \rightarrow \infty} \lim_{x' \rightarrow \infty} V(x', t) = V_e \quad \forall x \in \mathbb{R}.$$

■

3 A priori estimates

The main of this part is to obtain some a priori estimates of (1) similar to the estimate obtained in [10, 20, 19, 13] for hyperbolic systems. To do so, we use the Hamiltonian dissipation to find a 0th order estimate. We then take the s^{th} order derivative of the symmetric equation and consider the scalar product with the s^{th} order spatial derivative of the solution for all $1 \leq s \leq \bar{s}$. Then, using the properties of the operators $\mathcal{A}_0(V)$ and $\mathcal{A}_1(V)$ especially the coercivity of $\mathcal{A}_0(V)$ and their symmetry, we get a s^{th} order primary estimate for the solution $V \in B_{\bar{s}}(V_e, \delta)$. Then, in Subsection 3.3, we get an estimation of $\int_0^T \|\partial_x^s h\|_{\mathbb{L}^2}^2$ for all $1 \leq s \leq \bar{s}$ which together with the primary estimates leads us to Proposition (2.3). These estimates are obtained by acting a hollow real matrix on the system. The equilibrium V_e we consider in all this section is of the form $V_e = (h_e, 0)$ and \bar{s} is an integer equal or greater than 2.

3.1 Estimate in \mathbb{X}^0

System (1) admits a global \mathbb{X}^0 estimation considering the dissipation of the integral $\mathcal{H}_{h_e,0}$ of the energy $E_{h_e,0}$ defined in Section 1. In fact, the following proposition holds true,

Proposition 3.1 . *The local solution of (1) satisfies during its time existence:*

$$\|u(t)\|_{\mathbb{H}^1}^2 \leq \frac{\mathcal{H}_{h_e,0}(h_0, u_0)}{\min\{\inf_{x \in \mathbb{R}} h(t)/2, \alpha \inf_{x \in \mathbb{R}} h^3(t)/2\}}, \quad (14)$$

and

$$\|h(t) - h_e\|_{\mathbb{L}^2}^2 \leq \frac{2}{g} \mathcal{H}_{h_e,0}(h_0, u_0) \quad (15)$$

Proof We integrate the energy equality (3), for $u_e = 0$, with respect to the spatial variable x on \mathbb{R} . Then, considering the fact that $(h, u) \in (\mathbb{H}^{\bar{s}}(\mathbb{R}) + h_e) \times \mathbb{H}^{\bar{s}+1}(\mathbb{R})$, we have the dissipation of the Hamiltonian $\mathcal{H}_{h_e,0}$ by a simple integration by part:

$$\frac{d}{dt} \mathcal{H}_{h_e,0}(h, u) = -\mu \int_{\mathbb{R}} h(u_x)^2 \leq 0.$$

In other words,

$$\mathcal{H}_{h_e}(h(t), u(t)) - \mathcal{H}_{h_e,0}(h(0), u(0)) = -\mu \int_0^t \int_{\mathbb{R}} h(u_x)^2 \leq 0.$$

Now, using the definition of $\mathcal{H}_{h_e,0} = \int_{\mathbb{R}} g(h - h_e)^2/2 + hu^2/2 + \alpha h^3(u_x)^2$, we get the result. ■

Let us also remark that the Hamiltonian $\mathcal{H}_{h_e,0}$ is locally \mathbb{X}^0 -quadratic on V_e , in the sense that we have for all $\bar{s} \geq 2$, $\delta > 0$, $V \in B_{\bar{s}}(V_e, \delta)$,

$$C_{\{h_e\}}(\delta) \|V - V_e\|_{\mathbb{X}^0}^2 \leq \mathcal{H}_{h_e,0}(h, u) \leq C_{\{h_e\}}(\delta) \|V - V_e\|_{\mathbb{X}^0}^2$$

This together with the dissipation of $\mathcal{H}_{h_e,0}$ gives us the following estimate of order 0 for solutions close to equilibrium. In fact, we have,

Proposition 3.2 *Let $\bar{s} \geq 2$ be an integer and consider the equilibrium V_e and the initial data $V_0 \in \mathbb{X}^{\bar{s}}(\mathbb{R})$. Then, there exists $\delta, \bar{T} > 0$ such that the solution V of (1) satisfies $V(T) \in B_{\bar{s}}(V_e, \delta)$ for all $0 \leq T < \bar{T}$. Then, it satisfies for such T*

$$\|V(T) - V_e\|_{\mathbb{X}^0}^2 + C_{\{h_e\}}(\delta) \int_0^T \|u_x\|_{\mathbb{L}^2}^2 \leq C_{\{h_e\}}(\delta) \|V(0) - V_e\|_{\mathbb{X}^0}^2. \quad (16)$$

3.2 Estimate in \mathbb{X}^s

The main of this part is to obtain a convenient a priori estimate of s^{th} order for all integer $s \in [1, \bar{s}]$. To do so, we take the s^{th} derivate of the system (7) with respect to the spatial variable, take the scalar product with $\partial_x^s V$ and integrate on $[0, T) \times \mathbb{R}$:

$$\int_0^T \int_{\mathbb{R}} \partial_x^s (\mathcal{A}_0(V) \partial_t V) \cdot \partial_x^s V + \int_0^T \int_{\mathbb{R}} \partial_x^s (\mathcal{A}_1(V) \partial_x V) \cdot \partial_x^s V = \mu \int_0^T \int_{\mathbb{R}} \partial_x^{s+1} (hu_x) \partial_x^s u \quad (17)$$

Then, using basic computations and the Leibniz formula, we remark that²

$$\begin{aligned} \int_{\mathbb{R}} \partial_x^s (\mathcal{A}_0(V) \partial_t V) \cdot \partial_x^s V &= \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} \mathcal{A}_0(V) \partial_x^s V \cdot \partial_x^s V - \frac{1}{2} \int_{\mathbb{R}} (\mathcal{A}_{0t}^0 + \mathcal{A}_{0t}^1 \partial_x + \mathcal{A}_{0t}^2 \partial_x^2) \partial_x^s V \cdot \partial_x^s V \\ &\quad + \sum_{i=1}^s \binom{s}{i} \int_{\mathbb{R}} (\mathcal{A}_{0i}^0 + \mathcal{A}_{0i}^1 \partial_x + \mathcal{A}_{0i}^2 \partial_x^2) \partial_t \partial_x^{s-i} V \cdot \partial_x^s V, \end{aligned}$$

where, \mathcal{A}_{0i}^j is another notation for $\partial_x^i (\mathcal{A}_0^j(V))$, the i th spatial derivative of $\mathcal{A}_0^j(V)$, for all $j \in \{0, 1, 2\}$ and for any $i \in \mathbb{N}$. $\binom{s}{i}$ represents the number of i -combinations of s elements.

In other words, $\binom{s}{i} = \frac{s!}{i!(s-i)!}$. On the other hand, the integration by part and the symmetry of \mathcal{A}_1 imply that

$$\begin{aligned} \int_{\mathbb{R}} \partial_x^s (\mathcal{A}_1(V) \partial_x V) \cdot \partial_x^s V &= \left(s - \frac{1}{2}\right) \int_{\mathbb{R}} (\mathcal{A}_{1x}^0 + \mathcal{A}_{1x}^1 \partial_x + \mathcal{A}_{1x}^2 \partial_x^2) \partial_x^s V \cdot \partial_x^s V \\ &\quad + \sum_{i=2}^s \binom{s}{i} \int_{\mathbb{R}} (\mathcal{A}_{1i}^0 + \mathcal{A}_{1i}^1 \partial_x + \mathcal{A}_{1i}^2 \partial_x^2) \partial_x^{s-i+1} V \cdot \partial_x^s V. \end{aligned}$$

We have also

$$\int_{\mathbb{R}} \partial_x^{s+1} (hu_x) \partial_x^s u = - \int_{\mathbb{R}} h (\partial_x^{s+1} u)^2 - \sum_{i=1}^s \binom{s}{i} \int_{\mathbb{R}} (\partial_x^i h) (\partial_x^{s-i+1} u) (\partial_x^{s+1} u).$$

Hence, (17) becomes

$$\int_{\mathbb{R}} \mathcal{A}_0(V) \partial_x^s V(T) \cdot \partial_x^s V(T) + 2\mu \int_0^T \int_{\mathbb{R}} h (\partial_x^{s+1} u)^2 = \int_{\mathbb{R}} \mathcal{A}_0(V) \partial_x^s V(0) \cdot \partial_x^s V(0)$$

²For sake of simplicity, we use sometimes \mathcal{A}_1 or \mathcal{A}_0 instead of $\mathcal{A}_1(V)$ or $\mathcal{A}_0(V)$.

$$\begin{aligned}
& -2 \sum_{i=1}^s \binom{s}{i} \int_0^T \int_{\mathbb{R}} (\mathcal{A}_{0i}^0 + \mathcal{A}_{0i}^1 \partial_x + \mathcal{A}_{0i}^2 \partial_x^2) \partial_t \partial_x^{s-i} V \cdot \partial_x^s V \\
& + \int_0^T \int_{\mathbb{R}} (\mathcal{A}_{0t}^0 + \mathcal{A}_{0t}^1 \partial_x + \mathcal{A}_{0t}^2 \partial_x^2) \partial_x^s V \cdot \partial_x^s V \\
& + (1-2s) \int_0^T \int_{\mathbb{R}} (\mathcal{A}_{1x}^0 + \mathcal{A}_{1x}^1 \partial_x + \mathcal{A}_{1x}^2 \partial_x^2) \partial_x^s V \cdot \partial_x^s V \\
& - 2 \sum_{i=2}^s \binom{s}{i} \int_0^T \int_{\mathbb{R}} (\mathcal{A}_{1i}^0 + \mathcal{A}_{1i}^1 \partial_x + \mathcal{A}_{1i}^2 \partial_x^2) \partial_x^{s-i+1} V \cdot \partial_x^s V \\
& - 2\mu \sum_{i=1}^s \binom{s}{i} \int_{\mathbb{R}} (\partial_x^i h) (\partial_x^{s-i+1} u) (\partial_x^{s+1} u). \tag{18}
\end{aligned}$$

The two following lemmas present two results that we will use several times in the rest of this Section. The first one is on the \mathbb{X}^0 -quadraticity of $\mathcal{A}_0(V)$:

Lemma 3.3 *There exists $\delta > 0$ such that $\mathcal{A}_0(V)$ is quadratic on $B_{\bar{s}}(V_e, \delta)$. That is to say that we have for all $V \in B_{\bar{s}}(V_e, \delta)$ and all $f = (f_1, f_2) \in \mathbb{X}^0(\mathbb{R})$,*

$$C_{\{h_e\}}(\delta) \|f\|_{\mathbb{X}^0} \leq \int_{\mathbb{R}} \mathcal{A}_0(V) f \cdot f \leq C_{\{h_e\}}(\delta) \|f\|_{\mathbb{X}^0}.$$

Proof *Obviously we have*

$$\int_{\mathbb{R}} \mathcal{A}_0(V) f \cdot f = \int_{\mathbb{R}} (g - 3\alpha h(u_x)^2) f_1^2 + f_2 \mathcal{L}_h f_2.$$

Therefore, we have for δ small enough and $V \in B_{\bar{s}}(V_e, \delta)$,

$$\begin{aligned}
& \frac{g \|f_1\|_{\mathbb{L}^2}^2}{2} + \min\{h_e - \delta, \alpha(h_e - \delta)^3\} \|f_2\|_{\mathbb{H}^1}^2 \leq \int_{\mathbb{R}} \mathcal{A}_0(V) f \cdot f \\
& \leq g \|f_1\|_{\mathbb{L}^2}^2 + \max\{h_e + \delta, \alpha(h_e + \delta)^3\} \|f_2\|_{\mathbb{H}^1}^2.
\end{aligned}$$

■

The second lemma is on the smallness of the \mathbb{L}^∞ norm (in time and space) of h_t and u_t and some of their spatial derivatives as long as $V \in B_{\bar{s}}(V_e, \delta)$. Actually, the following lemma holds true,

Lemma 3.4 *Let us assume that the solution $V(t)$ of (1) belongs to $B_{\bar{s}}(V_e, \delta)$ for all $t \in [0, T)$. Then, we have for all $0 \leq j \leq \bar{s} - 1$ and all $0 \leq l \leq \bar{s}$,*

$$\lim_{\substack{\delta \rightarrow 0 \\ V \in B(V_e, \delta)}} \|\partial_x^j \partial_t h\|_{\mathbb{L}^\infty} = 0, \quad \lim_{\substack{\delta \rightarrow 0 \\ V \in B(V_e, \delta)}} \|\partial_x^l \partial_t u\|_{\mathbb{L}^\infty} = 0, \tag{19}$$

Proof The first equation of the system (1) gives us $h_t = -hu_x - uh_x$. Therefore,

$$\| \partial_x^j \partial_t h \|_{\mathbb{L}^\infty} \leq \| \sum_{k=0}^j \partial_x^k h \partial_x^{j-k+1} u \|_{\mathbb{L}^\infty} + \| \sum_{k=0}^j \partial_x^k u \partial_x^{j-k+1} h \|_{\mathbb{L}^\infty} \leq \Theta_{\{h_e\}}(\delta)$$

Likewise, the second equation of the system can be written under the following form (see [12, 15]),

$$u_t = -uu_x + \mathcal{L}_h^{-1} \partial_x (gh^2/2 + 2\alpha h^3(u_x)^2 + \mu hu_x). \quad (20)$$

Then, the continuous embedding $\mathbb{H}^1(\mathbb{R})$ in $\mathbb{L}^\infty(\mathbb{R})$ implies that there exists $C > 0$ such that

$$\begin{aligned} \| \partial_x^l \partial_t u \|_{\mathbb{L}^\infty} &\leq \| \partial_x^l (uu_x) \|_{\mathbb{L}^\infty} + \| \partial_x^l \mathcal{L}_h^{-1} \partial_x (gh^2/2 + 2\alpha h^3(u_x)^2 + \mu hu_x) \|_{\mathbb{L}^\infty} \\ &\leq \Theta(\delta) + C \| \partial_x^l \mathcal{L}_h^{-1} \partial_x (gh^2/2 + 2\alpha h^3(u_x)^2 + \mu hu_x) \|_{\mathbb{H}^1} \\ &\leq \Theta(\delta) + C \| \mathcal{L}_h^{-1} \partial_x (gh^2/2 + 2\alpha h^3(u_x)^2 + \mu hu_x) \|_{\mathbb{H}^{l+1}} \\ &\leq \Theta(\delta) + C_{\{h_e, \alpha\}}(\delta) \| \partial_x (gh^2/2 + 2\alpha h^3(u_x)^2 + \mu hu_x) \|_{\mathbb{H}^{l-1}} \\ &\leq \Theta(\delta) + C_{\{h_e, \alpha\}}(\delta) \| \partial_x (gh^2/2 + 2\alpha h^3(u_x)^2 + \mu hu_x) \|_{\mathbb{H}^{s-1}} \\ &\leq \Theta(\delta) + C_{\{h_e, \alpha\}}(\delta) \Theta_{\{h_e, \alpha, \mu\}}(\delta) \leq \Theta_{\{h_e, \alpha, \mu\}}(\delta). \end{aligned}$$

■

All of these tools give us facilities to prove the following lemma which is the key step to achieve the appropriate s^{th} order estimate.

Lemma 3.5 *Let us consider the solution V of (7) and assume that it belongs to $B_{\bar{s}}(V_e, \delta)$ for some $\delta > 0$. Then, the following estimates hold true,*

$$\left| \sum_{i=1}^s \binom{s}{i} \int_{\mathbb{R}} \mathcal{A}_{0i}^0 \partial_x^{s-i} \partial_t V \cdot \partial_x^s V \right| \leq \Theta_{\{h_e, \alpha, \mu\}}(\delta) \left(\sum_{j=1}^{s+1} \| \partial_x^j u \|_{\mathbb{L}^2}^2 + \sum_{j=1}^s \| \partial_x^j h \|_{\mathbb{L}^2}^2 \right). \quad (21)$$

$$\left| \int_{\mathbb{R}} \mathcal{A}_{0t}^0 \partial_x^s V \cdot \partial_x^s V \right| \leq \Theta_{\{h_e, \alpha, \mu\}}(\delta) (\| \partial_x^s u \|_{\mathbb{L}^2}^2 + \| \partial_x^s h \|_{\mathbb{L}^2}^2). \quad (22)$$

$$\left| \sum_{i=1}^s \binom{s}{i} \int_{\mathbb{R}} (\mathcal{A}_{0i}^1 \partial_x + \mathcal{A}_{0i}^2 \partial_x^2) \partial_x^{s-i} \partial_t V \cdot \partial_x^s V \right| \leq \Theta_{\{h_e, \alpha, \mu\}}(\delta) \left(\sum_{i=1}^s \| \partial_x^i h \|_{\mathbb{L}^2}^2 + \| \partial_x^{s+1} u \|_{\mathbb{L}^2}^2 \right) \quad (23)$$

$$\left| \int_{\mathbb{R}} (\mathcal{A}_{0t}^1 \partial_x + \mathcal{A}_{0t}^2 \partial_x^2) \partial_x^s V \cdot \partial_x^s V \right| \leq \Theta_{\{h_e, \alpha, \mu\}}(\delta) (\| \partial_x^{s+1} u \|_{\mathbb{L}^2}^2). \quad (24)$$

$$\left| \int_{\mathbb{R}} (\mathcal{A}_{1x}^0 + \mathcal{A}_{1x}^1 \partial_x + \mathcal{A}_{1x}^2 \partial_x^2) \partial_x^s V \cdot \partial_x^s V \right| \leq \Theta_{\{h_e, \alpha\}}(\delta) (\| \partial_x^s h \|_{\mathbb{L}^2}^2 + \| \partial_x^s u \|_{\mathbb{H}^1}^2). \quad (25)$$

$$\left| \sum_{i=2}^s \binom{s}{i} \int_{\mathbb{R}} (\mathcal{A}_{1i}^0 + \mathcal{A}_{1i}^1 \partial_x + \mathcal{A}_{1i}^2 \partial_x^2) \partial_x^{s-i+1} V \cdot \partial_x^s V \right| \leq \Theta_{\{h_e, \alpha\}}(\delta) (\| \partial_x h \|_{\mathbb{H}^{s-1}}^2 + \| \partial_x u \|_{\mathbb{H}^s}^2) \quad (26)$$

Proof Let us first prove (21). We have, by definition of \mathcal{A}_{0i}^0 , for all $1 \leq i \leq s$,

$$\int_{\mathbb{R}} \mathcal{A}_{0i}^0 \partial_x^{s-i} \partial_t V \cdot \partial_x^s V = -3\alpha \partial_x^i (h(u_x)^2) \partial_x^{s-i} \partial_t h \partial_x^s h + \partial_x^i h \partial_x^{s-i} \partial_t u \partial_x^s u.$$

Therefore,

$$\begin{aligned}
& \left| \int_{\mathbb{R}} \mathcal{A}_{0t}^0 \partial_x^{s-i} \partial_t V \cdot \partial_x^s V \right| \leq \frac{\| \partial_x^{s-i} \partial_t u \|_{\mathbb{L}^\infty}}{2} (\| \partial_x^i h \|_{\mathbb{L}^2}^2 + \| \partial_x^s u \|_{\mathbb{L}^2}^2) \\
& + \| \partial_x^{s-i} \partial_t h \|_{\mathbb{L}^\infty} \left(\| \partial_x^s h \|_{\mathbb{L}^2}^2 + \| 3\alpha \partial_x^i h (u_x)^2 \|_{\mathbb{L}^2}^2 + \| 6\alpha \sum_{j=0}^{i-1} \partial_x^j h \partial_x u \partial_x^{i-j+1} u \|_{\mathbb{L}^2}^2 \right) \\
& \leq \max \left\{ \frac{\| \partial_x^{s-i} \partial_t u \|_{\mathbb{L}^\infty}}{2}, C_{h_e, \alpha}(\delta) \| \partial_x^{s-i} \partial_t h \|_{\mathbb{L}^\infty} \right\} (\| h_x \|_{\mathbb{H}^{s-1}}^2 + \| u_x \|_{\mathbb{H}^s}^2).
\end{aligned}$$

Then, considering (19), the proof of (21) is complete.

We are now going to prove (22). To do so, we should first remark that

$$\int_{\mathbb{R}} \mathcal{A}_{0t}^0 \partial_x^s V \cdot \partial_x^s V = -3\alpha \partial_t (h(u_x)^2) (\partial_x^s h)^2 + h_t (\partial_x^s u)^2.$$

Then,

$$\left| \int_{\mathbb{R}} \mathcal{A}_{0t}^0 \partial_x^s V \cdot \partial_x^s V \right| \leq \| 3\alpha \partial_t (h(u_x)^2) \|_{\mathbb{L}^\infty} \| \partial_x^s h \|_{\mathbb{L}^2}^2 + \| h_t \|_{\mathbb{L}^\infty} \| \partial_x^s u \|_{\mathbb{L}^2}^2.$$

Now, we use (19) to get the result.

The first step to prove (23) is to notice that we have for all $1 \leq i \leq s$

$$\int_{\mathbb{R}} (\mathcal{A}_{0t}^1 \partial_x + \mathcal{A}_{0t}^2 \partial_x^2) \partial_x^{s-i} \partial_t V \cdot \partial_x^s V = \alpha \int_{\mathbb{R}} (\partial_x^i h^3) \partial_x^{s+1} u (\partial_x^{s-i+1} \partial_t u).$$

Hence,

$$\begin{aligned}
& \left| \int_{\mathbb{R}} (\mathcal{A}_{0t}^1 \partial_x + \mathcal{A}_{0t}^2 \partial_x^2) \partial_x^{s-i} \partial_t V \cdot \partial_x^s V \right| \leq \frac{\| \alpha \partial_x^{s-i+1} \partial_t u \|_{\mathbb{L}^\infty}}{2} (\| \partial_x^i h^3 \|_{\mathbb{L}^2}^2 + \| \partial_x^{s+1} u \|_{\mathbb{L}^2}^2) \\
& \leq \frac{\| \alpha \partial_x^{s-i+1} \partial_t u \|_{\mathbb{L}^\infty}}{2} \left(C_{\{h_e\}}(\delta) \sum_{j=1}^i \| \partial_x^j h \|_{\mathbb{L}^2}^2 + \| \partial_x^{s+1} u \|_{\mathbb{L}^2}^2 \right).
\end{aligned}$$

Considering 19, we obtain the estimate.

In order to prove (24), we first remark that

$$\int_{\mathbb{R}} (\mathcal{A}_{0t}^1 \partial_x + \mathcal{A}_{0t}^2 \partial_x^2) \partial_x^s V \cdot \partial_x^s V = \int_{\mathbb{R}} 3\alpha h^2 h_t (\partial_x^{s+1} u)^2.$$

Again, using (19), we find

$$\left| \int_{\mathbb{R}} (\mathcal{A}_{0t}^1 \partial_x + \mathcal{A}_{0t}^2 \partial_x^2) \partial_x^s V \cdot \partial_x^s V \right| \leq \Theta_{\{h_e, \alpha\}}(\delta) \| \partial_x^{s+1} u \|_{\mathbb{L}^2}^2.$$

To prove (25), we use an integration by part,

$$\begin{aligned} \int_{\mathbb{R}} (\mathcal{A}_{1x}^1 \partial_x + \mathcal{A}_{1x}^2 \partial_x^2) \partial_x^s V \cdot \partial_x^s V &= \int_{\mathbb{R}} -\alpha (h^3 u)_{xx} \partial_x^{s+1} u \partial_x^s u - \alpha (h^3 u)_x \partial_x^{s+2} u \partial_x^s u \\ &= \int_{\mathbb{R}} -\alpha \partial_x ((h^3 u)_x \partial_x^{s+1} u) \partial_x^{s+1} u = \int_{\mathbb{R}} \alpha (h^3 u)_x (\partial_x^{s+1} u)^2. \end{aligned}$$

Hence,

$$\left| \int_{\mathbb{R}} (\mathcal{A}_{1x}^1 \partial_x + \mathcal{A}_{1x}^2 \partial_x^2) \partial_x^s V \cdot \partial_x^s V \right| \leq \Theta_{\{h_e, \alpha\}}(\delta) \|\partial_x^{s+1} u\|_{\mathbb{L}^2}^2.$$

We have also

$$\int_{\mathbb{R}} \mathcal{A}_{1x}^0 \partial_x^s V \cdot \partial_x^s V = \|\mathcal{A}_{1x}^0\|_{\mathbb{L}^\infty} \int_{\mathbb{R}} (\partial_x^s h)^2 + (\partial_x^s u)^2 + 2\partial_x^s h \partial_x^s u,$$

where

$$\|\mathcal{A}_{1x}^0\|_{\mathbb{L}^\infty} = \max_{\substack{i \in \{1,2\} \\ j \in \{1,2\}}} \{ \|\mathcal{A}_{1x}^0\|_{ij} \}_{\mathbb{L}^\infty},$$

where $(\mathcal{A}_{1x}^0)_{ij}$ represents the element of row i and column j of the matrix $(\mathcal{A}_1^0(V))_x$. We now just need to remark that

$$\lim_{\substack{\delta \rightarrow 0 \\ V \in B(V_e, \delta)}} \|\mathcal{A}_{1x}^0\|_{\mathbb{L}^\infty} = 0. \quad (27)$$

Indeed, this fact is obvious using the structure of \mathcal{A}_1^0 ,

$$\|\mathcal{A}_{1x}^0\|_{\mathbb{L}^\infty} \leq \|gu_x\|_{\mathbb{L}^\infty} + \|gh_x\|_{\mathbb{L}^\infty} + \|\partial_x(hu) + 2\alpha \partial_x^2(h^3 u_x)\|_{\mathbb{L}^\infty} + \|6\alpha \partial_x(hu(u_x)^2)\|_{\mathbb{L}^\infty}.$$

Therefore,

$$\|\mathcal{A}_{1x}^0\|_{\mathbb{L}^\infty} \leq \Theta_{\{h_e, \alpha\}}(\delta).$$

and we get the result.

The last estimate (26) is just a consequence of the following fact which holds true for all $2 \leq i \leq s$. It is due to the structure of $\mathcal{A}_1^1(V)$ and $\mathcal{A}_1^2(V)$ together with an integration by part:

$$\int_{\mathbb{R}} (\mathcal{A}_{1i}^1 \partial_x + \mathcal{A}_{1i}^2 \partial_x^2) \partial_x^{s-i+1} V \cdot \partial_x^s V = \alpha \int_{\mathbb{R}} (\partial_x^i(h^3 u)) \partial_x^{s-i+2} u (\partial_x^{s+1} u).$$

Hence, as long as $V \in B_{\bar{s}}(V_e, \delta)$,

$$\begin{aligned} \left| \int_{\mathbb{R}} (\mathcal{A}_{1i}^1 \partial_x + \mathcal{A}_{1i}^2 \partial_x^2) \partial_x^{s-i+1} V \cdot \partial_x^s V \right| &\leq \frac{\|\alpha \partial_x^i(h^3 u)\|_{\mathbb{L}^\infty}}{2} (\|\partial_x^{s-i+2} u\|_{\mathbb{L}^2}^2 + \|\partial_x^{s+1} u\|_{\mathbb{L}^2}^2) \\ &\leq \Theta_{\{h_e, \alpha\}}(\delta) (\|\partial_x^{s-i+2} u\|_{\mathbb{L}^2}^2 + \|\partial_x^{s+1} u\|_{\mathbb{L}^2}^2). \end{aligned}$$

On the other hand, we have for all $2 \leq i \leq s - 1$,

$$\begin{aligned} \left| \int_{\mathbb{R}} \mathcal{A}_{1i}^0 \partial_x^{s-i+1} V \cdot \partial_x^s V \right| &\leq 2 \|\mathcal{A}_{1i}^0\|_{\mathbb{L}^\infty} (\|\partial_x^{s-i+1} u\|_{\mathbb{L}^2}^2 + \|\partial_x^s u\|_{\mathbb{L}^2}^2 + \|\partial_x^{s-i+1} h\|_{\mathbb{L}^2}^2 + \|\partial_x^s h\|_{\mathbb{L}^2}^2) \\ &\leq \Theta_{\{h_e, \alpha\}}(\delta) (\|\partial_x^{s-i+1} u\|_{\mathbb{L}^2}^2 + \|\partial_x^s u\|_{\mathbb{L}^2}^2 + \|\partial_x^{s-i+1} h\|_{\mathbb{L}^2}^2 + \|\partial_x^s h\|_{\mathbb{L}^2}^2), \end{aligned}$$

since the structure of \mathcal{A}_1^0 gives us for all integer $i \in [2, s - 1]$,

$$\lim_{\substack{\delta \rightarrow 0 \\ V \in B_{\bar{s}}(V_e, \delta)}} \|\mathcal{A}_{1i}^0\|_{\mathbb{L}^\infty} = 0.$$

We now need to give an estimation on the norm of the last term of the sum, $\left| \int_{\mathbb{R}} \mathcal{A}_{1s}^0 \partial_x V \cdot \partial_x^s V \right|$. This term can not be treated by the same way as the previous terms because

$$\lim_{\substack{\delta \rightarrow 0 \\ V \in B(V_e, \delta)}} \|\mathcal{A}_{1s}^0\|_{\mathbb{L}^\infty} \neq 0.$$

However, we can use the structure of $\mathcal{A}_1(V)$, a basic Young's inequality and an integration by part to get

$$\begin{aligned} \left| \int_{\mathbb{R}} \mathcal{A}_{1s}^0 \partial_x V \cdot \partial_x^s V \right| &\leq \frac{\|\partial_x V\|_{\mathbb{L}^\infty}}{2} (\|\partial_x^s (gu - 3\alpha hu(u_x)^2)\|_{\mathbb{L}^2}^2 + \|\partial_x^s (gh - 3\alpha h^2(u_x)^2)\|_{\mathbb{L}^2}^2) \\ &+ \|\partial_x V\|_{\mathbb{L}^\infty} \left(\frac{\|\partial_x^s (hu)\|_{\mathbb{L}^2}^2}{2} + \|\partial_x^s h\|_{\mathbb{L}^2}^2 + \|\partial_x^s u\|_{\mathbb{L}^2}^2 \right) + \left| \int_{\mathbb{R}} 2\alpha \partial_x^{s+1} (h^3 u_x) \partial_x u \partial_x^s u \right| \\ &\leq \Theta_{\{h_e\}}(\delta) \left(\|\partial_x h\|_{\mathbb{H}^{s-1}}^2 + \|\partial_x u\|_{\mathbb{H}^s}^2 + \left| \int_{\mathbb{R}} 2\alpha \partial_x^s (h^3 u_x) \partial_x (\partial_x u \partial_x^s u) \right| \right) \\ &\leq \Theta_{\{h_e, \alpha\}}(\delta) (\|\partial_x h\|_{\mathbb{H}^{s-1}}^2 + \|\partial_x u\|_{\mathbb{H}^s}^2). \end{aligned}$$

Hence, estimate (26) is totally proved. \blacksquare

This lemma together with the coercivity of \mathcal{A}_0 and relation (18) leads us to the following propositions

Proposition 3.6 *Let us assume that there exists $\delta > 0$, $\bar{T} > 0$ such that the solution V of (1) satisfies $V(T) \in B_{\bar{s}}(V_e, \delta)$ for all $T \in [0, \bar{T}]$. Then, we have for all $1 \leq s \leq \bar{s}$,*

$$\begin{aligned} \|\partial_x^s (V(T) - V_e)\|_{\mathbb{X}^0}^2 + C_{\{h_e, \mu\}}(\delta) \int_0^T \|\partial_x^{s+1} u\|_{\mathbb{L}^2}^2 &\leq C_{\{h_e, \alpha\}}(\delta) \|\partial_x^s (V(0) - V_e)\|_{\mathbb{X}^0}^2 \\ &+ \Theta_{\{h_e, \alpha, \mu\}}(\delta) \int_0^T \|V_x\|_{\mathbb{X}^{s-1}}^2. \end{aligned}$$

Then, considering this proposition together with the 0th order estimate of Subsection 3.1, we reach the final primary estimate which is given in the following proposition. This estimate together with the result of the next part enables us to prove the main theorem.

Proposition 3.7 *Let us assume that there exists $\delta > 0$, $\bar{T} > 0$ such that the solution V of (1) satisfies $V(T) \in B_{\bar{s}}(V_e, \delta)$ for all $T \in [0, \bar{T})$. Then, we have for such T ,*

$$\begin{aligned} \|V(T) - V_e\|_{\mathbb{X}^{\bar{s}}}^2 + C_{\{h_e, \mu\}}(\delta) \int_0^T \|u_x\|_{\mathbb{H}^{\bar{s}}}^2 &\leq C_{\{h_e, \alpha\}}(\delta) \|V(0) - V_e\|_{\mathbb{X}^{\bar{s}}}^2 \\ &+ \Theta_{\{h_e, \mu\}}(\delta) \int_0^T \|V_x\|_{\mathbb{X}^{\bar{s}-1}}^2. \end{aligned} \quad (28)$$

3.3 Estimate on $\int_0^t \|\partial_x^s h\|_{\mathbb{L}^2}^2$

This part is the final step to prove Proposition 2.3. In fact, considering (28), we need to find a convenient estimate on $\int_0^t \|\partial_x V\|_{\mathbb{X}^{\bar{s}-1}}^2$. This idea has been used in [20], [10] and [19]. Actually, estimate (28) has a similar appearance than the estimate found in these references for symmetric hyperbolic systems with dissipative terms. Then, they use the Kawashima stability condition to control the norm of spatial derivatives of first components of the solution. Let us note that, as in the case of hyperbolic system, we do not need to control the norm of second components. This is due to the presence of the second term of the left hand side of inequality (28). What we need to control in the case of Green–Naghi equation, is the time integral of norm of the spatial derivative of h . Nevertheless, the main difficulty is the generalization of the Kawashima–Shizuta condition. Actually, we have not been able to find any operator version of the Shizuta–Kawashima condition for Green–Naghdi equation. However, we are going to see that it is possible to find an appropriate upper bound for $\int_0^t \|\partial_x^s h\|_{\mathbb{L}^2}^2$ by using a slightly different technique as in the hyperbolic case. To do so, we consider the 2×2 hollow real matrix $K(V_e)$ defined by

$$K(V_e) = \begin{pmatrix} 0 & 1 \\ -\frac{h_e}{g} & 0 \end{pmatrix}. \quad (29)$$

As we will see further, the reason why we consider this matrix, is the fact that $K(V_e)\mathcal{A}_1(V_e)$ is a diagonal real matrix with a strictly positive first component. In other words, there exists a matrix of the form $B = \begin{pmatrix} 0 & 0 \\ 0 & L \end{pmatrix}$ with $L \geq 0$ such that $K(V_e)\mathcal{A}_1(V_e) + B$ is definite positive.

This enables us, as in [20], [10], to get an upper bound for $\int_0^t \|\partial_x^s h\|_{\mathbb{L}^2}^2$. This upper bound is convenient even though, unlike the case of hyperbolic systems, $K(V_e)\mathcal{A}_0(V_e)$ is not a skew-symmetric operator. This is due to the fact that we can extract a part from $K(V_e)\mathcal{A}_0(V)$, which plays a quite similar role to a skew-symmetric operator such that the norm of the remaining part is controllable in a suitable manner. So, let us write (7) under the form

$$\mathcal{A}_0(V)\partial_t V + \mathcal{A}_1(V_e)\partial_x V = H(V), \quad (30)$$

where $H(V)$ is defined by

$$H(V) = [A_1(V_e) - A_1(V)]\partial_x V + \begin{pmatrix} 0 \\ \mu\partial_x(hu_x) \end{pmatrix}. \quad (31)$$

We then take the action of the operator $K(V_e)\partial_x^{s-1}$ on (30) and take the scalar product with $\partial_x^s V$. This leads us to

$$\begin{aligned} \int_0^T \int_{\mathbb{R}} K(V_e)\partial_x^{s-1}\mathcal{A}_0(V)\partial_t V \cdot \partial_x^s V + \int_0^T \int_{\mathbb{R}} K(V_e)\mathcal{A}_1(V_e)\partial_x^s V \cdot \partial_x^s V \\ = \int_0^T \int_{\mathbb{R}} K(V_e)\partial_x^{s-1}H(V) \cdot \partial_x^s V, \end{aligned}$$

or equivalently to

$$\begin{aligned} \int_0^T \int_{\mathbb{R}} K(V_e)\mathcal{A}_1(V_e)\partial_x^s V \cdot \partial_x^s V = \int_0^T \int_{\mathbb{R}} K(V_e)\partial_x^{s-1}H(V) \cdot \partial_x^s V \\ - \int_0^T \int_{\mathbb{R}} K(V_e)\partial_x^{s-1}\mathcal{A}_0(V)\partial_t V \cdot \partial_x^s V. \end{aligned} \quad (32)$$

Let us note that

$$K(V_e)\mathcal{A}_1(V_e) = \begin{pmatrix} gh_e & 0 \\ 0 & -h_e^2 \end{pmatrix}. \quad (33)$$

Hence,

$$\begin{aligned} \int_0^T \int_{\mathbb{R}} K(V_e)\mathcal{A}_1(V_e)\partial_x^s V \cdot \partial_x^s V = \int_0^T \int_{\mathbb{R}} (gh_e(\partial_x^s h)^2 - h_e^2(\partial_x^s u)^2) \\ = gh_e \int_0^T \|\partial_x^s h\|_{\mathbb{L}^2}^2 - h_e^2 \int_0^T \|\partial_x^s u\|_{\mathbb{L}^2}^2. \end{aligned} \quad (34)$$

Gathering (32) and (34), we get

$$g \int_0^T \|\partial_x^s h\|_{\mathbb{L}^2}^2 = h_e \int_0^T \|\partial_x^s u\|_{\mathbb{L}^2}^2 + \frac{1}{h_e} \int_0^T \int_{\mathbb{R}} \partial_x^{s-1} (K(V_e)H(V) - K(V_e)\mathcal{A}_0(V)\partial_t V) \cdot \partial_x^s V. \quad (35)$$

It is now sufficient to give a convenient estimate on the last term of (35). To do so, we first remark that

$$A_1(V_e) - A_1(V) = \begin{pmatrix} -gu + 3\alpha hu(u_x)^2 & g(h_e - h) + 3\alpha h^2(u_x)^2 \\ g(h_e - h) + 3\alpha h^2(u_x)^2 & -hu - 2\alpha(h^3 u_x)_x + \alpha \partial_x(h^3 u \partial_x(\cdot)) \end{pmatrix}. \quad (36)$$

Then we have

$$\begin{aligned} K(V_e)\partial_x^{s-1}H(V) \cdot \partial_x^s V = \mu \partial_x^s(hu_x)\partial_x^s h + \partial_x^{s-1} (g(h_e - h)(h_x) + 3\alpha h^2(u_x)^2 h_x) \partial_x^s h \\ + \partial_x^{s-1} \left(h_e u h_x - \frac{3\alpha h_e}{g} h u h_x (u_x)^2 \right) \partial_x^s u - \partial_x^{s-1} \left(h_e (h_e - h)(u_x) + \frac{3\alpha h_e}{g} h^2 (u_x)^3 \right) \partial_x^s u \\ - \partial_x^{s-1} (h u u_x + 2\alpha \partial_x(h^3 u_x) u_x) \partial_x^s h + \alpha \partial_x^s (h^3 u u_{xx}) \partial_x^s h. \end{aligned} \quad (37)$$

On the other hand,

$$K(V_e)A_0(V) = \begin{pmatrix} 0 & \mathcal{L}_h \\ -h_e + \frac{3\alpha}{g} h_e h (u_x)^2 & 0 \end{pmatrix}, \quad (38)$$

and,

$$K(V_e)\partial_x^{s-1}(\mathcal{A}_0(V)\partial_t V)\cdot\partial_x^s V = \partial_x^{s-1}\mathcal{L}_h(u_t)\cdot\partial_x^s h + \partial_x^{s-1}\left(\frac{3\alpha h_e}{g}h(u_x)^2 h_t - h_e h_t\right)\partial_x^s u. \quad (39)$$

Now, we use the following lemma to visualize the part of $K(V_e)\partial_x^{s-1}(\mathcal{A}_0(V)\partial_t V)\cdot\partial_x^s V$ that we would extract.

Lemma 3.8 *Assume that $f \in C^0([0, T], \mathbb{X}^s(\mathbb{R}))$ for some $T > 0$. Then, we have*

$$\mathcal{L}_h f_t = \partial_t \mathcal{L}_h f - f h_t + 3\alpha \partial_x (h^2 h_t f_x). \quad (40)$$

In fact, using the lemma, (39) becomes

$$\begin{aligned} K(V_e)\partial_x^{s-1}(\mathcal{A}_0(V)\partial_t V)\cdot\partial_x^s V &= \partial_t(\partial_x^{s-1}\mathcal{L}_h u \cdot \partial_x^s h) - \partial_x^{s-1}\mathcal{L}_h u \cdot \partial_t \partial_x^s h \\ &\quad - \partial_x^{s-1}(u h_t) \cdot \partial_x^s h + 3\alpha \partial_x^s (h^2 h_t u_x) \partial_x^s h + \partial_x^{s-1}\left(\frac{3\alpha h_e}{g}h(u_x)^2 h_t - h_e h_t\right)\partial_x^s u. \end{aligned}$$

Therefore, $\partial_t(\partial_x^{s-1}\mathcal{L}_h u \cdot \partial_x^s h)$ is the term we are going to find a lower bound for. Then, we find an upper bound for all the other terms to get to the final estimate. All of the operations and simplifications we use in this part are done in order to lead us to the desired upper-bound estimate. First, we use the mass conservation $h_t = -(hu)_x$, and we find

$$\begin{aligned} K(V_e)\partial_x^{s-1}(\mathcal{A}_0(V)\partial_t V)\cdot\partial_x^s V &= \partial_t(\partial_x^{s-1}\mathcal{L}_h u \cdot \partial_x^s h) + \partial_x^{s-1}\mathcal{L}_h u \cdot \partial_x^{s+1}(hu) \\ &\quad + \partial_x^{s-1}(u(hu)_x) \cdot \partial_x^s h - 3\alpha \partial_x^s (h^2(hu)_x u_x) \partial_x^s h - \partial_x^{s-1}\left(\frac{3\alpha h_e}{g}h(u_x)^2(hu)_x - h_e(hu)_x\right)\partial_x^s u. \end{aligned}$$

Then, a simple development gives us

$$\begin{aligned} K(V_e)\partial_x^{s-1}(\mathcal{A}_0(V)\partial_t V)\cdot\partial_x^s V &= \partial_t(\partial_x^{s-1}\mathcal{L}_h u \cdot \partial_x^s h) - 3\alpha \partial_x^s (h^2(hu)_x u_x) \partial_x^s h \\ &\quad + \partial_x^{s-1}(u(hu)_x) \cdot \partial_x^s h - \partial_x^{s-1}\left(\frac{3\alpha h_e}{g}h(u_x)^2(hu)_x\right)\partial_x^s u + h_e \partial_x^s (hu) \partial_x^s u + \partial_x^{s-1}\mathcal{L}_h u \cdot \partial_x^{s+1}(hu). \end{aligned}$$

We now need the following lemmas to be able to deal with terms the order of which is too high to be controlled as desired. These lemmas are consequences of basic computations. We are going to see that $\int_{\mathbb{R}} \alpha h^3 u \partial_x^{s+2} u \partial_x^s h$ is the only term the order of which is too high. It appears in the expressions of both $\int_{\mathbb{R}} K(V_e)\partial_x^{s-1}(\mathcal{A}_0(V)\partial_t V)\cdot\partial_x^s V$ and $\int_{\mathbb{R}} K(V_e)\partial_x^{s-1}(\mathcal{A}_0(V)\partial_t V)\cdot\partial_x^s V$. This is to say that higher order terms compensate each other. This is what leads us to the convenient estimate.

Lemma 3.9 *Assume that $V \in \mathbb{X}^s(\mathbb{R})$. Then, we have*

$$\begin{aligned} &\int_{\mathbb{R}} h_e \partial_x^s (hu) \partial_x^s u + \partial_x^{s-1}\mathcal{L}_h u \cdot \partial_x^{s+1}(hu) = \int_{\mathbb{R}} h(h_e - h)(\partial_x^s u)^2 + \alpha h^3 u \partial_x^{s+2} u \partial_x^s h \\ &+ \int_{\mathbb{R}} h^2 h_e \sum_{i=1}^s \partial_x^i h \partial_x^{s-i} u \partial_x^s u \left(\sum_{i=1}^s \partial_x^i h \partial_x^{s-i} u\right) \left(\sum_{j=1}^s \partial_x^j h \partial_x^{s-j} u\right) - 2h \partial_x^s u \left(\sum_{j=1}^s \partial_x^j h \partial_x^{s-j} u\right) \\ &+ \alpha \int_{\mathbb{R}} \partial_x^s (hu) \left(\sum_{i=1}^{s+1} \partial_x^i (h^3) \partial_x^{s+1-i} u\right). \end{aligned}$$

Lemma 3.10 Assume that $V \in X^{\bar{s}}(\mathbb{R})$. Then, we have

$$\alpha \int_{\mathbb{R}} \partial_x^s (h^3 u u_{xx}) \partial_x^s h = \int_{\mathbb{R}} \alpha h^3 u \partial_x^{s+2} u \partial_x^s h + \alpha \sum_{i=1}^s \int_{\mathbb{R}} \partial_x^i (h^3 u) \partial_x^{s+2-i} u \partial_x^s h. \quad (41)$$

These lemmas are consequences of the definition of \mathcal{L} , integration by part and Leibniz formula. Now using (37), (41) together with previous lemmas, we get

$$\begin{aligned} \int_{\mathbb{R}} K(V_e) \partial_x^{s-1} (H(V) - \mathcal{A}_0(V) \partial_t V) \cdot \partial_x^s V &= \int_{\mathbb{R}} \partial_t (\partial_x^{s-1} \mathcal{L}_h u \cdot \partial_x^s h) + \mu \partial_x^s (h u_x) \partial_x^s h \\ &+ R[h, u], \end{aligned} \quad (42)$$

where

$$\left| \int_{\mathbb{R}} R[h, u] \right| \leq \Theta_{\{h_e, \alpha\}}(\delta) \|h_x\|_{\mathbb{H}^{s-1}}^2 + C_{\{h_e, \alpha\}}(\delta) \|u_x\|_{\mathbb{H}^s}^2.$$

Let us also note that the first two terms of the right hand side of (42) satisfy the relations given in the following lemma,

Lemma 3.11 Let $V = (h, u)$ be in $\mathbb{X}^{\bar{s}}(\mathbb{R})$. Then,

$$|\mu \partial_x^s (h u_x) \partial_x^s h| \leq \Theta_{\{\mu\}}(\delta) \|\partial_x V\|_{\mathbb{X}^{s-1}}^2 + C_{\{\mu, h_e\}}(\delta) \|\partial_x^{s+1} u\|_{\mathbb{L}^2}^2 + \frac{g}{2} \|\partial_x^s h\|_{\mathbb{L}^2}^2, \quad (43)$$

and

$$\begin{aligned} \int_0^T \int_{\mathbb{R}} \partial_t (\partial_x^{s-1} \mathcal{L}_h u \cdot \partial_x^s h) &\leq C_{\{h_e, \alpha\}}(\delta) (\|u(T)\|_{\mathbb{H}^{s+1}}^2 + \|\partial_x^s h(T)\|_{\mathbb{L}^2}^2) \\ &+ C_{\{h_e, \alpha\}}(\delta) (\|u(0)\|_{\mathbb{H}^{s+1}}^2 + \|\partial_x^s h(0)\|_{\mathbb{L}^2}^2). \end{aligned} \quad (44)$$

Proof The first estimate (43) is a consequence of Leibniz formula and the fact that

$$|\partial_x^{s+1} u \partial_x^s h| \leq \frac{2\mu(h_e + \delta)}{g} (\partial_x^{s+1} u)^2 + \frac{g}{2\mu(h_e + \delta)} (\partial_x^s h)^2.$$

The second one (44) is a consequence of the facts that

$$\partial_x^{s-1} \mathcal{L}_h u \cdot \partial_x^s h \leq \frac{1}{2} (\partial_x^{s-1} \mathcal{L}_h u)^2 + \frac{1}{2} (\partial_x^s h)^2,$$

and of the continuity of \mathcal{L}_h expressed by

$$\|\mathcal{L}_h u\|_{\mathbb{L}^2}^2 \leq C_{\{h_e, \alpha\}}(\delta) \|u\|_{\mathbb{H}^2}^2.$$

■

We now sum (35) for $1 \leq s \leq \bar{s}$. This together with (42) and Lemma 3.11 enables us to give an estimation on $\int_0^T \|h_x\|_{\mathbb{H}^{\bar{s}-1}}^2$:

Proposition 3.12 Let us assume that there exists $\bar{T} > 0$ such that the local $\mathbb{X}^{\bar{s}}$ solution of (1) exists and satisfies $V(T) \in B_{\bar{s}}(V_e, \delta)$ for all $T \in [0, \bar{T})$. Then, we have for such T ,

$$\begin{aligned} \int_0^T \|h_x\|_{\mathbb{H}^{\bar{s}-1}}^2 &\leq C_{\{h_e, \mu\}}(\delta) \int_0^T \|u_x\|_{\mathbb{H}^{\bar{s}}}^2 + C_{\{h_e, \alpha\}}(\delta) (\|u(T)\|_{\mathbb{H}^{\bar{s}+1}}^2 + \|\partial_x h(T)\|_{\mathbb{H}^{\bar{s}-1}}^2) \\ &+ C_{\{h_e, \alpha\}}(\delta) (\|u(0)\|_{\mathbb{H}^{\bar{s}+1}}^2 + \|\partial_x h(0)\|_{\mathbb{H}^{\bar{s}-1}}^2). \end{aligned} \quad (45)$$

This proposition together with Proposition 3.7 gives the a priori estimate of Proposition 2.3.

4 Further remarks

4.1 What happens if $\alpha = 0$?

During this study we have assumed that $\alpha, \mu > 0$. However, we may be interested for instance in $\alpha = 0$ case (Saint-Venant system). Actually we can easily check that the Hamiltonian $\mathcal{H}_{h_e,0}$ as well as the operator $\mathcal{A}_0(V)$ are quadratic for $\mathbb{H}^{\bar{s}} \times \mathbb{H}^{\bar{s}}$ norm for $\alpha = 0$. This fact is completely coherent with the well-posedness space of the system because the Saint-Venant system is well-posed in $\mathbb{H}^{\bar{s}}(\mathbb{R}) + h_e \times \mathbb{H}^{\bar{s}}(\mathbb{R})$ and not any more in $\mathbb{X}^{\bar{s}}(\mathbb{R})$. We can also easily remark that, in this case, the operators $\mathcal{A}_0(V)$ and $\mathcal{A}_1(V)$ are real symmetric matrix and the symmetric structure (7) of the system is equivalent to the normal symmetric structure of Saint-Venant system seen as a hyperbolic system. Then, all the computations of the previous section can be reconsidered for this case to lead us to the following final proposition

Proposition 4.1 *Let $\bar{s} \geq 2$ be an integer, $\alpha = 0$ and $\delta > 0$ and consider the initial data $V_0 \in B_{\bar{s}}(V_e, \delta)$ with $V_e = (h_e, 0)$. Assume also that there exists $\bar{T} > 0$ such that the unique local solution $V \in C([0, \bar{T}); (\mathbb{H}^{\bar{s}} + h_e) \times \mathbb{H}^{\bar{s}})$ of (1) satisfies $V(T) \in B_{\bar{s}}(V_e, \delta)$ for all $0 \leq T < \bar{T}$. Then, the following estimate holds true for such T ,*

$$(1 - \Theta_{\{h_e\}}(\delta)) \| V(T) - V_e \|_{\mathbb{H}^{\bar{s}} \times \mathbb{H}^{\bar{s}}}^2 + C_{\{h_e, \mu\}}(\delta) \int_0^T \| u_x \|_{\mathbb{H}^{\bar{s}}}^2 \leq C_{\{h_e\}}(\delta) \| V(0) - V_e \|_{\mathbb{H}^{\bar{s}} \times \mathbb{H}^{\bar{s}}}^2 + \Theta_{\{h_e, \mu\}}(\delta) \int_0^T \| u_x \|_{\mathbb{H}^{\bar{s}}}^2$$

This gives us by the same logic as in 2 the global existence of this solution. In other words, our approach in the case $\alpha = 0$ is quite similar to the approach presented in [13] for hyperbolic-parabolic systems and gives the same result for Saint-Venant system.

4.2 What happens if $\mu = 0$?

In order to see the importance of the right hand side dissipative term $\mu \partial_x (h \partial_x u)$, we can reconsider our approach assuming $\mu = 0$. Then, we will reach the following proposition

Proposition 4.2 *Let $\bar{s} \geq 2$ be an integer, $\mu = 0$ and $\delta > 0$ small enough such that (1) is locally well-posed for the initial data $V_0 \in B_{\bar{s}}(V_e, \delta)$ with $V_e = (h_e, 0)$. Assume also that there exists $\bar{T} > 0$ such that the unique local solution V satisfies $V(T) \in B_{\bar{s}}(V_e, \delta)$ for all $0 \leq T < \bar{T}$. Then, the following estimate holds true,*

$$(1 - \Theta_{\{h_e, \alpha\}}(\delta)) \| V(T) - V_e \|_{\mathbb{X}^{\bar{s}}}^2 \leq C_{\{h_e, \alpha\}}(\delta) \| V(0) - V_e \|_{\mathbb{X}^{\bar{s}}}^2 + \Theta_{\{h_e, \alpha\}}(\delta) \int_0^T \| u_x \|_{\mathbb{H}^{\bar{s}}}^2. \quad (46)$$

Hence, the global existence theorem can not be generalized in this case because we miss the very important term $\int_0^T \| u_x \|_{\mathbb{H}^{\bar{s}}}^2$ on the right hand side of estimate (46). In other words the factor $C_{\{h_e, \mu, \alpha\}}(\delta)$ associated with this term in the estimate of Proposition 2.3 is linearly

dependent to μ and it is strictly equal to 0 while $\mu = 0$. However, the factor $\Theta_{\{h_e, \mu, \alpha\}}(\delta)$ in front of $\int_0^T \|u_x\|_{\mathbb{H}^{\bar{s}}}^2$ in the left hand side of the estimate of Proposition 2.3 is the addition of a continuous function equal to 0 for $\mu = 0$ with other terms independent of μ , dependent only on α .

5 Conclusion and Perspectives

During this study, we proved the global existence and the asymptotic stability of the equilibrium solutions of Green–Naghdi system with a second order viscosity. This result is obtained by generalizing the technique used for symmetric entropy dissipative hyperbolic equations thanks to the generalized symmetric structure of the system. The study of the rate of the convergence to equilibrium could be one of the perspectives of this work. In fact, this point has been studied for hyperbolic parabolic systems in [13]. We believe that the result may be generalized to the Green–Naghdi system with a second order viscosity.

Let us however recall that the result found in this study can not be generalized by this method to the Green–Naghdi system with friction $-\kappa u$ (with $\kappa > 0$), without the viscosity $\mu u \partial_x(hu_x)$. In fact, in absence of this term, the first estimations are not coherent with the estimation of $\int_0^T \|h_x\|_{\mathbb{H}^{\bar{s}-1}}^2$, in the sense that there are of one order less than the estimation of $\int_0^T \|h_x\|_{\mathbb{H}^{\bar{s}-1}}^2$. Furthermore, if we add higher order viscous terms (order 4 or more), we are not able any more to generalize the technique used in this work. In fact, in this last case, the order of the first estimations are always less than the order of $\int_0^T \|h_x\|_{\mathbb{H}^{\bar{s}-1}}^2$ with or without $-\kappa u^2 + \mu u \partial_x(hu_x)$. This is to say that the order 2 seems to be the only order of viscosity, the main result can be obtained by a similar way as for hyperbolic systems for.

One of the other perspectives of this work is to study, in a general frame, the stability of equilibriums of locally-wellposed symmetrizable systems with a convenient friction or viscous term. In fact the main difficulty of this generalization is to find the condition which lets us get a convenient estimates on the time integral of the spatial derivative of the solution. Let us note that in the case of hyperbolic systems, there are other equivalent formulations of the Kawashima–Shizuta condition [19, 13] which may be more convenient for the generalization. One of these formulations for hyperbolic systems is the emptiness of the intersection of the eigenspaces of the symmetric positive definite matrix (equivalent to \mathcal{A}_0) and of the symmetric matrix (equivalent to \mathcal{A}_1) with the kernel of the viscosity matrix at equilibriums. It is also interesting to mention that the Kawashima–Shizuta condition is not sharp for hyperbolic systems (see [18] or [4] for instance). A generalization of less sharper conditions may be another way to follow. The answer to this question can lead us for instance, to the stability of equilibriums of 2D Green–Naghdi system. In fact, it would be possible to repeat the computations of this work for the two dimensional system to get a similar result. Let us remark that $\mathcal{A}_0(V)$ in

2-dimensional case is given by

$$\mathcal{A}_0(V) = \begin{pmatrix} g - 3\alpha h(\operatorname{div}(u, v))^2 & 0 & 0 \\ 0 & h - \alpha \partial_x(h^3 \partial_x) & -\alpha \partial_x(h^3 \partial_y) \\ 0 & -\alpha \partial_y(h^3 \partial_x) & h - \alpha \partial_y(h^3 \partial_y) \end{pmatrix}$$

where v represents the vertically averaged vertical speed. In this case, $\mathcal{A}_0(V)$ is quadratic, near equilibriums, for the norm $\| \cdot \|_{\mathbb{X}^s}$ defined by

$$\| f \|_{\mathbb{X}^s}^2 = \| f \|_{\mathbb{L}^2}^2 + \| \operatorname{div}(f) \|_{\mathbb{L}^2}^2.$$

This is also the norm of the local well-posedness of 2-dimensional system ([1]). Hence, the symmetric structure is coherent with the well-posedness space.

Acknowledgments

I would like to thank gratefully Nicolas Seguin. He has been a big source of intuition for the creation of this work and played a very important role in its improvement.

A Special Symmetric Structure

In this section, we consider the following conservative system

$$\partial_t W + \partial_x F(W) = 0, \tag{47}$$

where F is a differential application of the unknown W . We also assume that the system admits an additional conservation law of the form

$$\partial_t E(W) + \partial_x P(W) = 0$$

where E and P are differential applications of W such that the second variation of the Hamiltonian $\mathcal{H}(W) = \int_{\mathbb{R}} E(W)$ is definite positive. This is to say, see [14] for more details, that the system is symmetrizable. Moreover, we have the following proposition

Proposition A.1 *Assume that there exists a decomposition $W = (U, V)$ such that the application $\delta_V \mathcal{H}(U, \cdot)$ is invertible. Then, (47) is symmetrizable of the form*

$$\mathcal{A}_0(w) \partial_t w + \mathcal{A}_1(w) \partial_x w = 0.$$

where

$$w = (U, \delta_V \mathcal{H}(W)).$$

Moreover, $\mathcal{A}_0(w)$ is a symmetric definite positive bloc diagonal operator and $\mathcal{A}_1(w)$ is a symmetric one.

Proof Let us first note $w = (u, v)$ where $u = U$ and $v = \delta_V \mathcal{H}(W)$. Then,

$$D_w W = \begin{pmatrix} 1 & 0 \\ D_u V & D_v V \end{pmatrix},$$

and

$$\delta_W^2 \mathcal{H}(W) = \begin{pmatrix} \delta_U^2 \mathcal{H}(W) & \delta_{UV}^2 \mathcal{H}(W) \\ \delta_{UV}^2 \mathcal{H}(W) & \delta_V^2 \mathcal{H}(W) \end{pmatrix}.$$

On the other hand, it is shown in [14] that the expressions of the symmetric operators $\mathcal{A}_0(V)$ and $\mathcal{A}_1(V)$ are given by

$$\mathcal{A}_0(V) = D_w W^*(w) \delta_W^2 \mathcal{H}(W) D_w W(w),$$

and

$$\mathcal{A}_1(V) = D_w W^*(w) \delta_W^2 \mathcal{H}(W) D_W F(W) D_w W(w).$$

Hence,

$$\mathcal{A}_0(w) = \begin{pmatrix} \delta_U^2 \mathcal{H}(W) + \delta_{UV}^2 \mathcal{H}(W) D_u V + (D_u V)^T \delta_{UV}^2 \mathcal{H}(W) + (D_u V)^T \delta_V^2 \mathcal{H}(W) D_u V & \delta_{UV}^2 \mathcal{H}(W) D_v V + (D_u V)^T \delta_V^2 \mathcal{H}(W) D_v V \\ (D_v V)^T \delta_{UV}^2 \mathcal{H}(W) + (D_v V)^T \delta_V^2 \mathcal{H}(W) D_u V & (D_v V)^T \delta_V^2 \mathcal{H}(W) D_v V \end{pmatrix}.$$

Then, $\mathcal{A}_0(w)$ is bloc diagonal considering the fact that

$$(D_v V)^T \delta_{UV}^2 \mathcal{H}(W) + (D_v V)^T \delta_V^2 \mathcal{H}(W) D_u V = 0.$$

Indeed, $v = \delta_V \mathcal{H}(W)$ and $u = U$ give us

$$\begin{aligned} & (D_v V)^T \delta_{UV}^2 \mathcal{H}(W) + (D_v V)^T \delta_V^2 \mathcal{H}(W) D_u V = (D_v V)^T D_U v + (D_v V)^T D_V v D_u V \\ & = (D_v V)^T D_U v D_u U + (D_v V)^T D_V v D_u V = (D_v V)^T (D_U v D_u U + D_V v D_u V) \\ & = (D_v V)^T D_u v = 0. \end{aligned}$$

■

Let us now add a right hand side term of the following form to (47)

$$\begin{cases} \partial_t U + \partial_x F_1(U) = 0, \\ \partial_t V + \partial_x F_2(V) = q(W), \end{cases} \quad (48)$$

where q is a differential application of W and (U, V) is a decomposition of W satisfying the assumptions of Proposition (A.1). Hence, symmetrizing the system as suggested in the proposition, we find a system of the type

$$\mathcal{A}_0(w) \partial_t w + \mathcal{A}_1(w) \partial_x w = G(w),$$

with

$$G(w) = (D_w W)^T \delta_W^2 \mathcal{H}(W) Q(W).$$

We are now going to see that $Q(W) = (0, q(W))$ is an eigenvector for the eigenvalue 1 of $(D_w W)^T \delta_W^2 \mathcal{H}(W)$. In fact, the following proposition holds true.

Proposition A.2 *The right hand side term $G(w)$ is equal to $Q(W)$.*

Proof *We have by assumptions*

$$G(W) = (D_w W)^T \delta_W^2 \mathcal{H}(W) Q(W) = \begin{pmatrix} \delta_{VU}^2 \mathcal{H} q(W) + (D_u V)^T \delta_V^2 \mathcal{H} q(W) \\ (D_v V)^T \delta_V^2 \mathcal{H} q(W) \end{pmatrix}.$$

Considering the fact that the first components (associated to U) of $G(W)$ are the same as the up non diagonal bloc of the operator $\mathcal{A}_0(w)$ considered in the proof of Proposition A.1 acting on $q(W)$, these components vanish. On the other hand,

$$\begin{aligned} (D_v V)^T \delta_V^2 \mathcal{H} q(W) &= (D_v V)^T (\delta_V^2 \mathcal{H})^T q(W) = (D_v V)^T (D_V v)^T q(W) \\ &= (D_v V D_V v)^T q(W) = (D_v V (D_v V)^{-1})^T q(W) = q(W). \end{aligned}$$

■

B Local well-posedness

Let us first note that there exists $\delta > 0$ such that $\mathcal{A}_0(V)$ is invertible for all $V \in B_{\bar{s}}(V_e, \delta)$. Then, consider the associated linear problem

$$\begin{cases} \partial_t V + \mathcal{A}_0^{-1}(V) \mathcal{A}_1(V) \partial_x V = \begin{pmatrix} 0 \\ \mu \mathcal{L}_h^{-1}(\partial_x(hu_x)) \end{pmatrix} \\ V(0, x) = g_0(x) \end{cases} \quad (49)$$

with $V \in C([0, T]; \mathbb{X}^{\bar{s}}(\mathbb{R}))$ for some $\bar{s} \geq 2$ and $g_0 \in \mathbb{X}^{\bar{s}}(\mathbb{R})$. It is proved in [12] that the problem admits a unique solution V in $C([0, T]; \mathbb{X}^{\bar{s}}(\mathbb{R}))$. We now consider the following iteration scheme

$$\begin{cases} \mathcal{A}_0(V^k) \partial_t V^{k+1} + \mathcal{A}_1(V^k) \partial_x V^{k+1} = \begin{pmatrix} 0 \\ \mu \partial_x(h^k u_x^k) \end{pmatrix} \\ V^{k+1}(0, x) = g^{k+1}(x) \end{cases} \quad (50)$$

where $g^{k+1} = \epsilon^k V_0 \star \rho(\frac{\cdot}{\epsilon^k})$ for some mollifier ρ^3 with the positive real set $\epsilon^k = \frac{\beta}{2^k}$, with $\beta > 0$. We initialize the iteration by $g^0 = V_0$. We know that (50) admits a unique solution for all entire k . Let us now assume that $V^l(t) \in B_{\bar{s}}(V_e, \delta)$ for all $l \leq k$ and all $t \in [0, T]$. This implies by triangle inequality that

$$\|V^l - g^0\|_{C([0, T]; \mathbb{X}^{\bar{s}})} \leq 2\delta \quad (51)$$

for all $l \leq k$. We can show that there exists a suitable $T > 0$ such that the estimate (51) holds also true for $l = k + 1$. In fact, we consider the s^{th} derivative of (50), take the scalar product

³ $\rho \in \mathcal{D}(\mathbb{R}; \mathbb{R}^+)$ is supported in the unit ball such that $\int_{\mathbb{R}} \rho = 1$.

with $\partial_x^{s+1}(V^{k+1} - g^0)$ and we sum over $s \in \{0, \dots, \bar{s}\}$. Then, using very similar logics as in 3.2, we find for all $0 \leq t \leq T$,

$$\begin{aligned} \| V^{k+1}(t) - g^0 \|_{\mathbb{X}^{\bar{s}}}^2 &\leq C_{\{\|g^0\|_{\mathbb{L}^\infty}\}}(\delta) \| g^{k+1} - g^0 \|_{\mathbb{X}^{\bar{s}}}^2 + C_{\{\|g^0\|_{\mathbb{L}^\infty}, \mu\}}(\delta) \int_0^t \| V^{k+1}(t') - g^0 \|_{\mathbb{X}^{\bar{s}}}^2 dt' \\ &\quad + C_{\{\|g^0\|_{\mathbb{L}^\infty}, \mu\}}(\delta)t. \end{aligned}$$

Then, Gronwall lemma leads us, for δ small enough, to

$$\| V^{k+1} - g^0 \|_{C([0, T]; \mathbb{X}^{\bar{s}})}^2 \leq C e^{\lambda T} (\| g^{k+1} - g^0 \|_{\mathbb{X}^{\bar{s}}}^2 + T).$$

where C and λ are strictly positive reals independent of k . On the other hand, there exists by assumption, $\epsilon_0 > 0$ such that

$$\| g^{k+1} - g^0 \|_{\mathbb{X}^{\bar{s}}} \leq \epsilon_0 \quad \text{for all } k \in \mathbb{N}.$$

Then, choosing β small enough (therefore ϵ_0 small enough), there exists $T > 0$ such that the condition (51) is satisfied for all $l \in \mathbb{N}$. We assume from now that T and β are small enough to give us (51) for all entire. Then, we consider the s^{th} derivative of (50) for iterations k and $k - 1$, take the scalar product with $\partial_x^{s+1}(V^{k+1} - V^k)$, subtract the two equations and sum over $s \in \{0, \dots, \bar{s}\}$. Likewise, we get

$$\begin{aligned} \| V^{k+1}(t) - V^k(t) \|_{\mathbb{X}^{\bar{s}}}^2 &\leq \gamma \| g^{k+1} - g^k \|_{\mathbb{X}^{\bar{s}}}^2 + \theta \int_0^t \| V^k(t') - V^{k-1}(t') \|_{\mathbb{X}^{\bar{s}}}^2 dt' \\ &\quad + \theta \int_0^t \| V^{k+1}(t') - V^k(t') \|_{\mathbb{X}^{\bar{s}}}^2 dt' \end{aligned}$$

for some convenient positive γ, θ . Applying the Gronwall lemma, we have for all $k \in \mathbb{N}$

$$\| V^{k+1} - V^k \|_{C([0, T]; \mathbb{X}^{\bar{s}})}^2 \leq e^{\lambda T} \left(\| g^{k+1} - g^k \|_{\mathbb{X}^{\bar{s}}}^2 + \theta \int_0^T \| V^k(t') - V^{k-1}(t') \|_{\mathbb{X}^{\bar{s}}}^2 dt' \right). \quad (52)$$

Now, we sum (52) on $k \in \mathbb{N}$. This leads us to

$$(1 - \theta T e^{\lambda T}) \sum_{k \in \mathbb{N}} \| V^{k+1} - V^k \|_{C([0, T]; \mathbb{X}^{\bar{s}})}^2 \leq e^{\lambda T} \sum_{k \in \mathbb{N}} \| g^{k+1} - g^k \|_{\mathbb{X}^{\bar{s}}}^2.$$

Then, considering the fact the T is small enough and the fact that the sum $\sum_{k \in \mathbb{N}} \| g^{k+1} - g^k \|_{\mathbb{X}^{\bar{s}}}^2$ is convergent, we find that the set V^k is convergent in $C([0, T]; \mathbb{X}^{\bar{s}})$. The unicity can be proved by the same way. In fact, we obtain a very similar approximation to (52) for $\| V^1 - V^2 \|_{\mathbb{X}^{\bar{s}}}$ considering two solutions $V^1(x, t)$ and $V^2(x, t)$ for the initial conditions $V_1(x)$ and $V_2(x)$. Hence, the local well-posedness is proved.

C Linear stability of equilibriums of the Green-Naghdi equation

In this part we are going to see another use of the symmetric structure of the Green-Naghdi equation. In fact, this structure enables us to prove the linear stability of the equilibrium $V_e = (h_e, u_e)$, with $h_e > 0$, of the system without any right hand side term. To see this, let us consider the solution $V \in \mathbb{X}^{\bar{s}}(\mathbb{R})$ of the linearized system

$$\mathcal{A}_0(V_e)\partial_t V + \mathcal{A}_1(V_e)\partial_x V = 0, \quad (53)$$

act ∂_x^s on (53) for $0 \leq s \leq \bar{s}$, and take the scalar product by $\partial_x^s(V - V_e)$:

$$\int_0^T \int_{\mathbb{R}} \mathcal{A}_0(V_e)\partial_t \partial_x^s V \cdot \partial_x^s(V - V_e) + \int_0^T \int_{\mathbb{R}} \mathcal{A}_1(V_e)\partial_x^{s+1} V \cdot \partial_x^s(V - V_e) = 0. \quad (54)$$

Now, considering the facts that

$$\int_{\mathbb{R}} \mathcal{A}_0(V_e)\partial_t \partial_x^s V \cdot \partial_x^s(V - V_e) = \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} \mathcal{A}_0(V_e)\partial_x^s(V - V_e) \cdot \partial_x^s(V - V_e),$$

and

$$\int_0^T \int_{\mathbb{R}} \mathcal{A}_1(V_e)\partial_x^{s+1} V \cdot \partial_x^s(V - V_e) = 0,$$

together with the \mathbb{X}^0 -quadraticity of $\mathcal{A}_0(V_e)$, we get the following estimate,

$$\| \partial_x^s(V(T) - V_e) \|_{\mathbb{X}^0}^2 \leq C \| \partial_x^s(V(0) - V_e) \|_{\mathbb{X}^0}^2. \quad (55)$$

where C is a strictly positive constant depending only on h_e, α and g . Hence, we have the following proposition,

Proposition C.1 *Let $\bar{s} \geq 2$ be an integer and consider the initial data $V_0 \in \mathbb{X}^{\bar{s}}(\mathbb{R})$. Then, there exists $C > 0$ such that the solution V of (53) satisfies for all time,*

$$\| V(T) - V_e \|_{\mathbb{X}^{\bar{s}}}^2 \leq C \| V_0 - V_e \|_{\mathbb{X}^{\bar{s}}}^2. \quad (56)$$

This gives us the linear stability of the equilibrium of (1).

Theorem C.2 *Let $\bar{s} \geq 2$ be an integer and consider the Green-Naghdi system,*

$$\begin{cases} \partial_t h + \partial_x h u = 0, \\ \partial_t h u + \partial_x h u^2 + \partial_x (g h^2 / 2 + \alpha h^2 \ddot{h}) = 0. \end{cases} \quad (57)$$

Then, the equilibrium solutions $V_e = (h_e, u_e)$, with $h_e > 0$, are linearly stable for the $\mathbb{X}^{\bar{s}}$ norm.

Let us note that this theorem can be generalized to all locally well-posed symmetrizable system of the form (53) such that $\mathcal{A}_0(V_e)$ is quadratic.

References

- [1] Borys Alvarez-Samaniego and David Lannes. A Nash-Moser theorem for singular evolution equations. Application to the Serre and Green-Naghdi equations. *Indiana Univ. Math. J.*, 57(1):97–131, 2008.
- [2] Yulia Yu. Bagderina and Alexander P. Chupakhin. Invariant and partially invariant solutions of the Green-Naghdi equations. *Prikl. Mekh. Tekhn. Fiz.*, 46(6):26–35, 2005.
- [3] Adhémar-Jean-Claude Barré de Saint-Venant. Théorie du mouvement non permanent des eaux, avec application aux crues des rivières et à l'introduction des marées dans leur lit. *C. R. Acad. Sci. Paris*, 73:147–154, 1871.
- [4] Karine Beauchard and Enrique Zuazua. Large time asymptotics for partially dissipative hyperbolic systems. *Arch. Ration. Mech. Anal.*, 199(1):177–227, 2011.
- [5] Sylvie Benzoni-Gavage and Denis Serre. *Multidimensional hyperbolic partial differential equations*. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, Oxford, 2007. First-order systems and applications.
- [6] Marie-Odile Bristeau, Anne Mangeney, Jacques Sainte-Marie, and Nicolas Seguin. A non-hydrostatic model for shallow water flows. derivation and validation. *Int. J. Finite Volumes*, 2015. In press.
- [7] Constantine M. Dafermos. *Hyperbolic conservation laws in continuum physics*, volume 325 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, troisième édition, 2010.
- [8] Sergey L. Gavriluk and Vladimir M. Teshukov. Generalized vorticity for bubbly liquid and dispersive shallow water equations. *Contin. Mech. Thermodyn.*, 13(6):365–382, 2001.
- [9] Albert E. Green and Paul M. Naghdi. A derivation of equations for wave propagation in water of variable depth. *Journal of Fluid Mechanics*, 78:237–246, 11 1976.
- [10] Bernard Hanouzet and Roberto Natalini. Global existence of smooth solutions for partially dissipative hyperbolic systems with a convex entropy. *Arch. Ration. Mech. Anal.*, 169(2):89–117, 2003.
- [11] Delia Ionescu-Kruse. Variational derivation of the Green-Naghdi shallow-water equations. *J. Nonlinear Math. Phys.*, 19(suppl. 1):1240001, 12, 2012.
- [12] Samer Israwi. Large time existence for 1D Green-Naghdi equations. *Nonlinear Anal.*, 74(1):81–93, 2011.
- [13] Shuichi Kawashima. Large-time behavior of solutions for hyperbolic-parabolic systems of conservation laws. *Proc. Japan Acad. Ser. A Math. Sci.*, 62(8):285–287, 1986.

- [14] Dena Kazerani. The symmetric structure of the Green-Naghdi type equations. October 2014.<hal-01074488v2>.
- [15] David Lannes. *The water waves problem*, volume 188 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2013. Mathematical analysis and asymptotics.
- [16] Yi A. Li. Hamiltonian structure and linear stability of solitary waves of the Green-Naghdi equations. *J. Nonlinear Math. Phys.*, 9(suppl. 1):99–105, 2002. Recent advances in integrable systems (Kowloon, 2000).
- [17] Yi A. Li. A shallow-water approximation to the full water wave problem. *Comm. Pure Appl. Math.*, 59(9):1225–1285, 2006.
- [18] Corrado Mascia and Roberto Natalini. On relaxation hyperbolic systems violating the Shizuta-Kawashima condition. *Arch. Ration. Mech. Anal.*, 195(3):729–762, 2010.
- [19] Yasushi Shizuta and Shuichi Kawashima. Systems of equations of hyperbolic-parabolic type with applications to the discrete Boltzmann equation. *Hokkaido Math. J.*, 14(2):249–275, 1985.
- [20] Wen-An Yong. Entropy and global existence for hyperbolic balance laws. *Arch. Ration. Mech. Anal.*, 172(2):247–266, 2004.