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Imaging of anomalous components in unknown background

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Abstract: We introduce a qualitative method capable of imaging defects in an unknown complex environment using differential measurements. The main difficulty is that the background medium is unknown and too complex to obtain a reliable estimate of the associated Green function. To overcome this difficulty our approach exploits two measurements of the farfield operators, one without defects and one with defects. The analysis of our method relies on the recently introduced Generalized Linear Sampling Methods (GLSM) and the link to the solutions of the interior transmission problems. We give numerical examples related to non destructive testing in concrete-like materials, illustrating the performance of our method.

Keywords: Inverse scattering, Sampling methods, differential measurement

1 Introduction

We are interested in the imaging of defects inside unknown heterogeneous medium from multi-static measurements of waves at a fixed frequency. The main difficulty of our setting is that the background medium is unknown and complex, which means that one cannot obtain a good estimate for the background Green function. This discards the possibility of using classical (qualitative) imaging methods [2]. We assume that two sets of measurements, one for the defect free and one for the defect containing medium, are at our disposal and we design a numerical inversion algorithm that exploits those supplementary data to visualize the defect location. The analysis of our method relies on the GLSM and the link it has with the solution to the interior transmission problems (see [4] and [5]). One ends up with an indicator function that combines the indicator functions from the GLSM and a filtered difference term computed without additional significant numerical cost. We shall briefly introduce the scattering problem by a heterogeneous medium and the needed notations in Section 2. In Section 3 we review the main theoretical results of [5]. We give in Section 4 some numerical experiments on concrete like materials, illustrating the performance of our method.

2 Scattering by an inhomogeneous medium

We restrict ourselves to the case of scalar time harmonic waves and we focus on full aperture farfield measurements associated to incident plane waves. For a wave number $k > 0$, the total field solves the Helmholtz equation:

$$\Delta u + k^2 n u = 0 \text{ in } \mathbb{R}^d$$

for $d = 2$ or 3 and n the refractive index, where $\Im(n) \geq 0$. We denote by \bar{D} the support of $n - 1$ and assume that D is a bounded domain with Lipschitz boundary and connected complement. We are interested by the case where u is generated by an incident plane wave, $u^i(\theta, x) := e^{ikx \cdot \theta}$ for $x \in \mathbb{R}^d$ and $\theta \in \mathbb{S}^{d-1}$. We also introduce the scattered field u^s defined by:

$$\begin{cases} u^s(\theta, \cdot) := u - u^i(\theta, \cdot) \text{ in } \mathbb{R}^d, \\ \lim_{r \rightarrow \infty} \int_{|x|=r} \left| \frac{\partial u^s}{\partial r} - ik u^s \right|^2 ds = 0. \end{cases} \quad (1)$$

We introduce the farfield $u^\infty(\theta, \hat{x})$ defined through the following expansion: $u^s(\theta, x) = \frac{e^{ik|x|}}{|x|^{(d-1)/2}} (u^\infty(\theta, \hat{x}) + O(1/|x|))$ for $|x| \rightarrow \infty$ and for all $(\theta, \hat{x}) \in \mathbb{S}^{d-1} \times \mathbb{S}^{d-1}$.

Leading to the farfield operator:

$$Fg(\hat{x}) := \int_{\mathbb{S}^{d-1}} u^\infty(\theta, \hat{x}) g(\theta) ds(\theta).$$

It is well known that the farfield operator admits two factorisations $F = GH = H^*TH$. The compact operator $H : L^2(\mathbb{S}^{d-1}) \rightarrow L^2(D)$ is defined by :

$$Hg := \int_{\mathbb{S}^{d-1}} e^{ikx \cdot \theta} g(\theta) ds(\theta), \quad g \in L^2(\mathbb{S}^{d-1}), \quad x \in D, \quad (2)$$

and is dense in $\{v \in L^2(D) \text{ s.t. } \Delta v + k^2 v = 0 \text{ in } D\}$. Its adjoint $H^* : L^2(D) \rightarrow L^2(\mathbb{S}^{d-1})$ is defined by :

$$H^* \varphi(\hat{x}) := \int_D e^{-iky \cdot \hat{x}} \varphi(y) dy, \quad \varphi \in L^2(D), \quad \hat{x} \in \mathbb{S}^{d-1}.$$

We define the compact operator $G : \overline{\mathcal{R}(H)} \subset L^2(D) \rightarrow L^2(\mathbb{S}^{d-1})$ defined by: $Gv := w^\infty$ where w^∞ is the farfield of $w \in H_{loc}^1(\mathbb{R}^d)$ associated to the incident wave v :

$$\begin{cases} \Delta w + nk^2 w = k^2(1-n)v \text{ in } \mathbb{R}^d, \\ \lim_{r \rightarrow \infty} \int_{|x|=r} \left| \frac{\partial w}{\partial r} - ikw \right|^2 ds = 0. \end{cases} \quad (3)$$

Finally we define $T : L^2(D) \rightarrow L^2(D)$ by:

$$Tv := -k^2(1-n)(v+w). \quad (4)$$

In the following we will use the operator $F_\# = |\Re(F)| + |\Im(F)|$, which can be factorised as $F_\# = H^*T_\#H$, where the operator $T_\#$ is a real selfadjoint operator [1] that satisfies (under hypothesis 1) :

$$|(T_\#h, h)| = \left\| (T_\#)^{\frac{1}{2}} h \right\|^2 \geq \mu \|h\|^2 \quad \forall h \in \mathcal{R}(H), \quad (5)$$

where $\mu > 0$ is a constant independent of h .

Hypothesis 1. *The index of refraction n and the domain D satisfy $n \in L^\infty(\mathbb{R}^d)$, $\text{supp}(n-1) = \bar{D}$, $\Im(n) \geq 0$ and there exist a constant $n_* > 0$ such that $\Re(n(x)-1) \geq n_*$ for a.e. $x \in D$.*

3 Theoretical Results

The GLSM relies on the solvability of the so-called interior transmission problem defined by $(u, v) \in L^2(D) \times L^2(D)$ such that $u - v \in H^2(D)$ and

$$\text{ITP}(D, f, g, n, n') = \begin{cases} \Delta u + k^2 n u = 0 & \text{in } D, \\ \Delta v + k^2 n' v = 0 & \text{in } D, \\ (u - v) = f & \text{on } \partial D, \\ \frac{\partial}{\partial \nu} (u - v) = g & \text{on } \partial D, \end{cases} \quad (6)$$

We denote by $\sigma(D, n, n')$ the set of wave number $k \in \mathbb{R}$ for which the $\text{ITP}(D, f, g, n, n')$ is not well posed for all $f \in H^{\frac{1}{2}}(\partial D)$ and $g \in H^{-\frac{1}{2}}(\partial D)$.

Hypothesis 2. We assumed that $k^2 \in \mathbb{R}_+$ and n are such that for all $f \in H^{\frac{1}{2}}(\partial D)$ and $g \in H^{-\frac{1}{2}}(\partial D)$ the ITP(D, f, g, n) has a unique solution $(u, v) \in L^2(D) \times L^2(D)$ such that $u - v \in H^2(D)$.

This hypothesis is verified [6] for all $k^2 \in \mathbb{R}_+$ except a countable set without finite point of accumulation if n verifies $1/(n-1) \in L^\infty(D)$ and $\Re(n-1)$ is either positive or negative definite in the neighbourhood of ∂D . We introduce the farfield pattern of the Green function:

$$\phi_z(\hat{x}) := e^{-ik\hat{x} \cdot z}$$

and the key ingredient of the GLSM:

Theorem 1. Assume that $k \notin \sigma(D, n, 1)$. Then G is compact, injective with dense range and $\phi_z \in \mathcal{R}(G)$ if and only if $z \in D$. Moreover, if $z \in D$ then $G(v) = \phi_z$ if and only if there exists $u \in L^2(D)$ such that (u, v) is a solution of ITP($D, \Phi_z, \frac{\partial \Phi_z}{\partial \nu}, n, 1$).

We outline the main results of the GLSM in the case of noisy data (see [5]). The noisy operators, F^δ and $F_{\#}^\delta$ are such that $\|F^\delta - F\| \leq c\delta$ and $\|F_{\#}^\delta - F_{\#}\| \leq \delta$ where c is a real constant. Let $g_z^{\alpha, \delta} \in L^2(\mathbb{S}^{d-1})$ be the minimizer of

$$J_\alpha^\delta(\phi_z; g) := \alpha(|(F_{\#}^\delta g, g)| + \delta \alpha^{-\eta} \|g\|^2) + \|F^\delta g - \phi_z\|^2, \quad (7)$$

for $\alpha > 0$, $\delta > 0$, $\eta \in]0, 1[$ and $\phi_z \in L^2(\mathbb{S}^{d-1})$. The functional

$$\mathcal{A}^{\alpha, \delta}(g) := |(F_{\#}^\delta g, g)| + \alpha^{-\eta} \delta \|g\|^2 \quad (8)$$

gives a characterisation of D through the following result.

Theorem 2. Under hypothesis 2 and 1 we have:

- $z \in D$ implies $\limsup_{\alpha \rightarrow 0} \limsup_{\delta \rightarrow 0} \mathcal{A}^{\alpha, \delta}(g_z^{\alpha, \delta}) < \infty$,
- $z \notin D$ implies $\liminf_{\alpha \rightarrow 0} \liminf_{\delta \rightarrow 0} \mathcal{A}^{\alpha, \delta}(g_z^{\alpha, \delta}) = \infty$.

When we have two measurements campaigns, the same results applies to $\bar{D}_0 = \text{supp}(n_0 - 1) \subset \bar{D}$ where $\mathcal{A}_0^{\alpha, \delta}$ is defined as above using F_0^δ (the farfield associated with n_0 and D_0) and

$$g_{0,z}^{\alpha, \delta} = \arg \min_{g \in L^2(\mathbb{S}^{d-1})} \alpha(|(F_{0, \#}^\delta g, g)| + \delta \alpha^{-\eta} \|g\|^2) + \|F_0^\delta g - \phi_z\|^2.$$

However we are interested in $\text{supp}(n - n_0)$. The filtered difference term defined by:

$$\mathcal{D}^\delta(g, g_0) := |(F_{0, \#}^\delta (g - g_0), g - g_0)| + \delta \|g - g_0\|^2,$$

will be used to image the simply connected part of D_0 that have been modified between the two measurements. We denote this domain by \tilde{D}_0 . Let Ω be the part of $\text{supp}(n - n_0)$ such that $\bar{\Omega} \cap \bar{D}_0 = \emptyset$. We introduce the following indicator function:

$$\mathcal{I}_T^{\alpha, \delta}(g_z^{\alpha, \delta}, g_{0,z}^{\alpha, \delta}) = \frac{1}{\sqrt{\mathcal{A}^{\alpha, \delta}(g_z^{\alpha, \delta}) \left(1 + \mathcal{A}^{\alpha, \delta}(g_z^{\alpha, \delta}) \mathcal{D}^\delta(g_z^{\alpha, \delta}, g_{0,z}^{\alpha, \delta}) - 1\right)}}, \quad (9)$$

which will image a domain larger than the defect as follows:

Theorem 3. If we assume that $k \notin \sigma(D, n, 1) \cup \sigma(D_0, n_0, 1) \cup \sigma(D, n, n_0)$ and, n and n_0 verify hypothesis 1, then $z \in \tilde{D}_0 \cup \Omega$ if and only if $\lim_{\alpha \rightarrow 0} \liminf_{\delta \rightarrow 0} \mathcal{I}_T^{\alpha, \delta}(g_z^{\alpha, \delta}, g_{0,z}^{\alpha, \delta}) > 0$.

4 Numerical Simulations

In order to fix the ideas, we shall limit ourselves to the two dimensional case and will introduce the algorithms for the discrete setting. We identify \mathbb{S}^1 with the interval $[0, 2\pi[$. In order to collect the data of the inverse problem we solve numerically (3) for N incident fields $u^i(\frac{2\pi j}{N}, \cdot)$, $j \in \{0 \dots N-1\}$ using a finite element method implemented with Freefem++ [8]. The discrete version of F is then the matrix $F := (u^\infty(\frac{2\pi j}{N}, \frac{2\pi k}{N}))_{0 \leq j, k \leq N-1}$. We add some noise to the data to build a noisy far field matrix F^δ where $(F^\delta)_{j,k} = (F)_{j,k}(1 + \sigma N_{ij})$ for $\sigma > 0$ and N_{ij} an uniform complex random variable in $[-1, 1]^2$. We similarly generate F_0^δ . We denote by $\phi_z \in \mathbb{C}^N$, the vector defined by $\phi_z(j) = \exp(-ik(z_1 \cos(\frac{2\pi j}{N}) + z_2 \sin(\frac{2\pi j}{N})))$ for $0 \leq j \leq N-1$. The analysis of previous sections suggests to consider

$$g_z^{\text{GLSM}} := \operatorname{argmin}_{g \in L^2(\mathbb{S}^1)} \left(\alpha \left\| (F_\#^\delta)^{\frac{1}{2}} g \right\|_{L^2(\mathbb{S}^1)}^2 + \alpha^{1-\eta} \delta \|g\|_{L^2(\mathbb{S}^1)}^2 + \|F^\delta g - \phi_z\|_{L^2(\mathbb{S}^1)}^2 \right).$$

The minimizer is explicitly given by

$$g_z^{\text{GLSM}} = (\alpha F_\#^\delta + \alpha^{1-\eta} \delta Id + F^{\delta*} F^\delta)^{-1} F^{\delta*} \phi_z.$$

We similarly construct $g_{0,z}^{\text{GLSM}}$ using F_0^δ . In our numerical simulations we choose $\eta = 0$ (which corresponds to the one used in [4]) and set α with the same heuristic as in [4]. We then look at $z \rightarrow \mathcal{I}_T^{\alpha, \delta}(g_z^{\alpha, \delta}, g_{0,z}^{\alpha, \delta})$ as an indicator function.

All our experiments are conducted for the background medium n_0 shown in Figure 1. This background medium is a simplified numerical description of a concrete like material. The wave frequency is $150kHz$, the celerity of the medium is $4300m.s^{-1}$ (which means a wavelength of $2.87cm$) and the celerity inside the inclusion is $5700m.s^{-1}$.

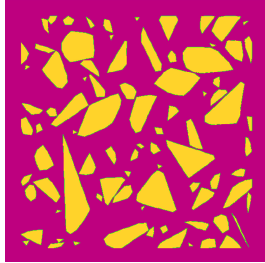


Figure 1. The background medium, $n = 0.57$ inside the yellow inclusions

Our theoretical analysis is only valid for inhomogeneous perturbations n and n_0 . One example of this setting is shown in Figure 2 where we modified the celerity (3 times higher) in two of the inclusions between the two measurements.

The main concern with concrete non destructive testing is the case of cracks. Our analysis do not include this case and its extension to it is the subject of an ongoing work. However the results shown in Figure 3 for a crack being either inside or outside the inhomogeneity n_0 gives promising results. To obtain this results we do not test with ϕ_z but with its normal derivatives as explained in [7].

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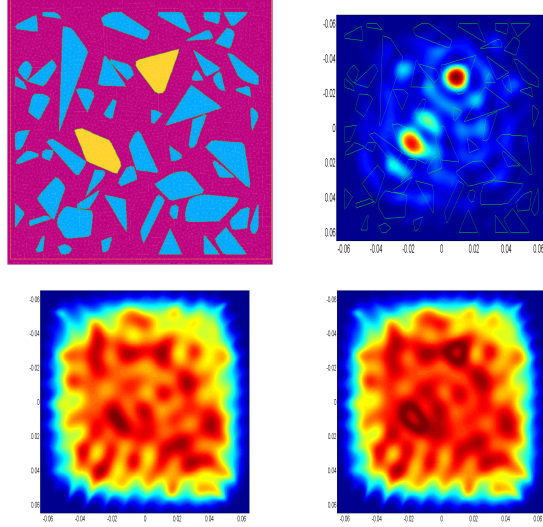


Figure 2. From left to right and up to down : The index n , $\mathcal{I}_T^{\alpha, \delta}(g_z^{\alpha, \delta}, g_{0,z}^{\alpha, \delta})$, $\mathcal{A}_0^{\alpha, \delta}(g_{0,z}^{\alpha, \delta})^{-1}$ and $\mathcal{A}^{\alpha, \delta}(g_z^{\alpha, \delta})^{-1}$

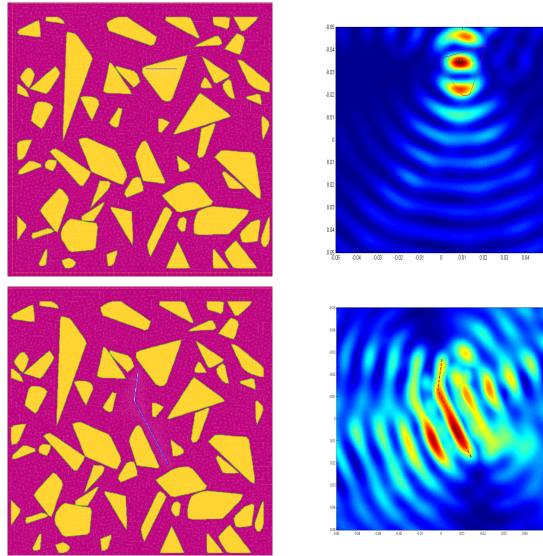


Figure 3. Cracks: On the left the medium with the cracks and on the right the corresponding $\mathcal{I}_T^{\alpha, \delta}(g_z^{\alpha, \delta}, g_{0,z}^{\alpha, \delta})$

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