Stability in the energy space of the sum of N peakons for the Degasperis–Procesi equation

André Kabakouala

Stability in the Energy Space of the Sum of N Peakons for the Degasperis-Procesi Equation

André Kabakouala
L.M.P.T., U.F.R Sciences et Techniques, Université de Tours, Parc Grandmont, 37200 Tours, France.

Andre.Kabakouala@lmpt.univ-tours.fr

Abstract

The Degasperis-Procesi equation possesses well-known peaked solitary waves that are called peakons. Their stability has been established by Lin and Liu in [5]. In this paper, we localize the proof (in some suitable sense detailed in Section 3) of the stability of a single peakon. Thanks to this, we extend the result of stability to the sum of N peakons traveling to the right with respective speeds $c_1, \dots, c_N$, such that the difference between consecutive locations of peakons is large enough.

1 Introduction

The Degasperis-Procesi (DP) equation

$$u_t - u_{txx} + 4uu_x = 3u_x u_{xx} + uu_{xxx}, \quad (t,x) \in \mathbb{R}_+^* \times \mathbb{R} \tag{1.1}$$

is completely integrable (see [1]) and possesses among others the following invariants

$$E(u) = \int_{\mathbb{R}} yv \quad \text{and} \quad F(u) = \int_{\mathbb{R}} u^3, \tag{1.2}$$

where $y = (1-\partial_x^2)u$ and $v = (4-\partial_x^2)^{-1}u$. Substituting u by $4v - v_{xx}$ in (1.2) and using integration by parts, the conservation laws can be rewritten as

$$E(u) = \int_{\mathbb{R}} \left(4v^2 + 5v_x^2 + v_{xx}^2\right) \quad \text{and} \quad F(u) = \int_{\mathbb{R}} \left(-v_{xx}^3 + 12vv_{xx}^2 - 48v^2 v_{xx} + 64v^3\right). \tag{1.3}$$

One can see that the conservation law $E(\cdot)$ is equivalent to $\|\cdot\|_{L^2(\mathbb{R})}^2$. Indeed, using integration by parts

$$\|u\|_{L^2(\mathbb{R})}^2 = \int_{\mathbb{R}} u^2 = \int_{\mathbb{R}} (4v - v_{xx})^2 = \int_{\mathbb{R}} \left(16v^2 + 8v_x^2 + v_{xx}^2\right) \le 4E(u), \tag{1.4}$$

and applying Plancherel-Parseval identity

$$E(u) = \int_{\mathbb{R}} yv = \int_{\mathbb{R}} \frac{1+\omega^2}{4+\omega^2} |\widehat{u}(\omega)|^2 \le \int_{\mathbb{R}} |\widehat{u}(\omega)|^2 = \|u\|_{L^2(\mathbb{R})}^2, \tag{1.5}$$

where $\widehat{u}$ denotes the Fourier transform of u. In the sequel we will denote

$$\|u\|_{\mathcal{H}} = \sqrt{E(u)}. \tag{1.6}$$

Note that, by reversing the operator $(1-\partial_x^2)$ in (1.1), the DP equation can be rewritten in conservation form as

$$u_t + \frac{1}{2}\partial_x u^2 + \frac{3}{2}(1-\partial_x^2)^{-1}\partial_x u^2 = 0, \quad (t,x) \in \mathbb{R}_+^* \times \mathbb{R}. \tag{1.7}$$

The DP equation possesses solitary waves called *peakons* (see Figure 1a) and defined by

$$u(t,x) = \varphi_c(x-ct) = c\varphi(x-ct) = ce^{-|x-ct|}, \quad c \in \mathbb{R}, \tag{1.8}$$

but they are not smooth since $\varphi_c \notin C^1(\mathbb{R})$ (see Figure 1b). The peakons are only global weak solutions of (1.7). It means, for any smooth test function $\phi \in C^\infty(\mathbb{R}_+ \times \mathbb{R})$, it holds

$$\int_0^{+\infty}\int_{\mathbb{R}} \varphi_c(x-ct)\phi_t(t,x)dtdx + \frac{1}{2}\int_0^{+\infty}\int_{\mathbb{R}} \varphi_c^2(x-ct)\phi_x(t,x)dtdx$$
$$+ \frac{3}{2}\int_0^{+\infty}\int_{\mathbb{R}} (1-\partial_x^2)^{-1}\varphi_c^2(x-ct)\phi_x(t,x)dtdx + \int_{\mathbb{R}} \varphi_c(x)\phi(0,x)dx = 0.$$

The goal of our work is to prove that ordered trains of peakons are stable under small perturbations in the energy space $\mathcal{H}$ (equivalent to L^2).

Definition 1.1 (Stability)**.** *Let $c>0$ be given. The peakon φ_c is said stable in $\mathcal{H}$, if for all $\delta>0$, there exists $\varepsilon>0$ such that if*

$$\|u_0 - \varphi_c\|_{\mathcal{H}} \le \varepsilon, \tag{1.9}$$

then for all $t \ge 0$, there exists $\xi(t)$ such that

$$\|u(t,\cdot) - \varphi_c(\cdot - \xi(t))\|_{\mathcal{H}} \le \delta, \tag{1.10}$$

where u is the solution to (1.1) *emanating from u_0.*

Lin and Liu proved in [5] the stability of a single peakon under the additional condition that $(1-\partial_x^2)u_0 \in \mathcal{M}^+(\mathbb{R})$. Using this result and the general strategy introduced by Martel, Merle and Tsai in [7] for the generalized Korteweg-de Vries (gKdV) equation and adapted by Dika and Molinet in [3] and [2] for the Camassa-Holm (CH) equation, we prove here the stability of the sum of N peakons for the DP equation.

Before stating the main result we introduce the function space where will live our class of solutions to the equation. For I a finite or infinite time interval of $\mathbb{R}_+$, we denote by $\mathcal{X}(I)$ the function space [1]

$$\mathcal{X}(I) = \left\{u \in C\left(I; H^1(\mathbb{R})\right) \cap L^\infty\left(I; W^{1,1}(\mathbb{R})\right),\ u_x \in L^\infty\left(I; BV(\mathbb{R})\right)\right\}. \tag{1.11}$$

The main result of the present paper is the following theorem.

Theorem 1.1 (Stability of the sum of N peakons)**.** *Let be given N velocities $c_1,\dots,c_N$ such that $0 < c_1 < \dots < c_N$. Let $u \in \mathcal{X}([0,T[)$, with $0 < T \le +\infty$, be a solution of the DP equation. There exist $C>0$, $L_0>0$ and $\varepsilon_0>0$ only depending on the speeds $(c_i)_{i=1}^N$, such that if*

$$y_0 = (1-\partial_x^2)u_0 \in \mathcal{M}^+(\mathbb{R}) \tag{1.12}$$

and

$$\left\| u_0 - \sum_{i=1}^N \varphi_{c_i}(\cdot - z_i^0)\right\|_{\mathcal{H}} \le \varepsilon^2, \quad \textit{with} \quad 0<\varepsilon<\varepsilon_0, \tag{1.13}$$

for some $z_1^0,\dots,z_N^0$ satisfying

$$z_1^0 < \dots < z_N^0 \quad \textit{and} \quad z_i^0 - z_{i-1}^0 \ge L, \quad \textit{with} \quad L > L_0 > 0, \quad i = 2,\dots,N, \tag{1.14}$$

[1] $W^{1,1}(\mathbb{R})$ is the space of $L^1(\mathbb{R})$ functions with derivatives in $L^1(\mathbb{R})$ and $BV(\mathbb{R})$ is the space of function with bounded variation.

then there exist $\xi_1^1(t),\dots,\xi_1^N(t)$ *such that*

$$\left\| u(t) - \sum_{i=1}^{N} \varphi_{c_i}(\cdot - \xi_1^i(t)) \right\|_{\mathcal{H}} \leq C(\sqrt{\varepsilon} + L^{-1/8}), \quad \forall t \in [0,T[\tag{1.15}$$

and

$$\xi_1^i(t) - \xi_1^{i-1}(t) > \frac{L}{2}, \quad \forall t \in [0,T[, \quad i = 2,\dots,N, \tag{1.16}$$

where $\xi_1^1(t),\dots,\xi_1^N(t)$ *are defined in Subsection 4.1.*

2 Preliminaries

In this section we briefly recall the global well-posedness results for the DP equation and its consequences (see [4] and [6] for details).

Theorem 2.1 (Global weak solution; see [4])**.** *Assume that* $u_0 \in L^2(\mathbb{R})$ *with* $y_0 = (1-\partial_x^2)u_0 \in \mathcal{M}^+(\mathbb{R})$. *Then the DP equation has a unique global weak solution* $u \in \mathcal{X}(\mathbb{R}_+)$ *such that*

$$y(t,\cdot) = (1-\partial_x^2)u(t,\cdot) \in \mathcal{M}^+(\mathbb{R}), \quad \forall t \in \mathbb{R}_+. \tag{2.1}$$

Moreover $E(\cdot)$ *and* $F(\cdot)$ *are conserved by the flow.*

Remark 2.1 (Control of L^∞ norm by L^2 norm)**.** From (2.1), it holds

$$u(x) = \frac{e^{-x}}{2}\int_{-\infty}^{x} e^{x'}y(x')dx' + \frac{e^{x}}{2}\int_{x}^{+\infty} e^{-x'}y(x')dx'$$

and

$$u_x(x) = -\frac{e^{-x}}{2}\int_{-\infty}^{x} e^{x'}y(x')dx' + \frac{e^{x}}{2}\int_{x}^{+\infty} e^{-x'}y(x')dx',$$

which lead to

$$|u_x(x)| \leq u(x), \quad \forall x \in \mathbb{R}. \tag{2.2}$$

Then, using the Sobolev embedding of $H^1(\mathbb{R})$ into $L^\infty(\mathbb{R})$ and (2.2), we infer that there exists a constant $C_S > 0$ such that

$$\|u\|_{L^\infty(\mathbb{R})} \leq C_S\|u\|_{H^1(\mathbb{R})} \leq 2C_S\|u\|_{L^2(\mathbb{R})}. \tag{2.3}$$

Lemma 2.1 (Positivity; see [6])**.** *Let* $u \in H^1(\mathbb{R})$ *with* $y = (1-\partial_x^2)u \in \mathcal{M}^+(\mathbb{R})$. *If* $k_1 \geq 1$, *then we have*

$$(k_1 \pm \partial_x)u(x) \geq 0, \quad \forall x \in \mathbb{R}. \tag{2.4}$$

Lemma 2.2 (Positivity; see [6])**.** *Let* $w(x) = (k_1 \pm \partial_x)u(x)$. *Assume that* $u \in H^1(\mathbb{R})$ *with* $y = (1-\partial_x^2)u \in \mathcal{M}^+(\mathbb{R})$. *If* $k_1 \geq 1$ *and* $k_2 \geq 2$, *then we have*

$$(k_2 \pm \partial_x)(4-\partial_x^2)^{-1}w(x) \geq 0, \quad \forall x \in \mathbb{R}. \tag{2.5}$$

3 Stability of a single peakon

The proof of Lin and Liu in [5] is not entirely suitable for our work, because it involves all local extrema of the function $v = (4-\partial_x^2)^{-1}u$ on $\mathbb{R}$, and thus is not local. For our work, we have to localize the estimates. Therefore, we need to modify a little the proof of Lin and Liu. We do this first for a single peakon.

Theorem 3.1 (Stability of peakons). *Let $u \in \mathcal{X}([0,T[)$, with $0 < T \leq +\infty$, be a solution of the DP equation and φ_c be the peakon defined in (1.8), traveling to the right at the speed $c > 0$. There exit $C > 0$ and $\varepsilon_0 > 0$ only depending on the speed $c > 0$, such that if*

$$y_0 = (1 - \partial_x^2)u_0 \in \mathcal{M}^+(\mathbb{R}) \tag{3.1}$$

and

$$\|u_0 - \varphi_c\|_{\mathcal{H}} \leq \varepsilon^2, \quad \text{with } \ 0 < \varepsilon < \varepsilon_0, \tag{3.2}$$

then

$$\|u(t,\cdot) - \varphi_c(\cdot - \xi_1(t))\|_{\mathcal{H}} \leq C\sqrt{\varepsilon}, \quad \forall t \in [0,T[, \tag{3.3}$$

where $\xi_1(t) \in \mathbb{R}$ is any point where the function $v(t,\cdot) = (4 - \partial_x^2)^{-1}u(t,\cdot)$ attains its maximum.

To prove this theorem we first need the following lemma that enables to control the distance of $E(u)$ and $F(u)$ to respectively $E(\varphi_c)$ and $F(\varphi_c)$.

Lemma 3.1 (Control of distances between energies). *Let $u \in H^1(\mathbb{R})$ with $y = (1 - \partial_x^2)u \in \mathcal{M}^+(\mathbb{R})$. If $\|u - \varphi_c\|_{\mathcal{H}} \leq \varepsilon^2$, then*

$$|E(u) - E(\varphi_c)| \leq O(\varepsilon^2) \tag{3.4}$$

and

$$|F(u) - F(\varphi_c)| \leq O(\varepsilon^2), \tag{3.5}$$

where $O(\cdot)$ only depends on c.

Proof. For the first estimate, applying triangular inequality, and using that $\|u - \varphi_c\|_{\mathcal{H}} \leq \varepsilon^2$ and $\|\varphi_c\|_{\mathcal{H}} = c/\sqrt{3}$, we have

$$\begin{aligned}
|E(u) - E(\varphi_c)| &= |\,\|u\|_{\mathcal{H}} - \|\varphi_c\|_{\mathcal{H}}|\,(\|u\|_{\mathcal{H}} + \|\varphi_c\|_{\mathcal{H}}) \\
&\leq \|u - \varphi_c\|_{\mathcal{H}}\,(\|u - \varphi_c\|_{\mathcal{H}} + 2\|\varphi_c\|_{\mathcal{H}}) \\
&\leq \varepsilon^2 \left(\varepsilon^2 + \frac{2c}{\sqrt{3}}\right) \\
&\leq O(\varepsilon^2).
\end{aligned}$$

For the second estimate, applying the Hölder inegality, and using that $\|u - \varphi_c\|_{\mathcal{H}} \leq \varepsilon^2$ and (2.3), we have

$$\begin{aligned}
|F(u) - F(\varphi_c)| &\leq \int_{\mathbb{R}} |u^3 - \varphi_c^3| \\
&\leq \int_{\mathbb{R}} |u - \varphi_c|(u^2 + u\varphi_c + \varphi_c^2) \\
&\leq \|u - \varphi_c\|_{L^2(\mathbb{R})} \left(\int_{\mathbb{R}} (u^2 + u\varphi_c + \varphi_c^2)^2\right)^{1/2} \\
&= \|u - \varphi_c\|_{L^2(\mathbb{R})} \left(\int_{\mathbb{R}} (u^4 + 2u^3\varphi_c + 3u^2\varphi_c^2 + 2u\varphi_c^3 + \varphi_c^4)\right)^{1/2} \\
&\leq \|u - \varphi_c\|_{L^2(\mathbb{R})} \\
&\qquad \cdot \left(4C_S^2\|u\|_{L^2(\mathbb{R})}^4 + 4cC_S\|u\|_{L^2(\mathbb{R})}^2 + 3c^2\|u\|_{L^2(\mathbb{R})}^2 + \frac{8}{3}c^3C_S\|u\|_{L^2(\mathbb{R})} + \frac{1}{2}c^4\right)^{1/2} \\
&\leq O(\varepsilon^2),
\end{aligned}$$

where we also use that the L^2 norm of u is bounded, and the following measures of peakon:

$$\|\varphi_c\|_{L^\infty(\mathbb{R})} = c, \quad \|\varphi_c\|_{L^3(\mathbb{R})}^3 = \frac{2}{3}c^3 \ \text{ and } \ \|\varphi_c\|_{L^4(\mathbb{R})}^4 = \frac{1}{2}c^4.$$

This proves the lemma.

Now, to prove Theorem 3.1, by the conservation of $E(\cdot)$, $F(\cdot)$ and the continuity of the map $t \mapsto u(t)$ from $[0,T[$ to $H^1(\mathbb{R}) \hookrightarrow \mathcal{H}$ (since $\mathcal{H} \simeq L^2$ and $\|u\|_{L^2(\mathbb{R})} \leq \|u\|_{H^1(\mathbb{R})}$), it suffices to prove that for any function $u \in H^1(\mathbb{R})$ satisfying (3.1), (3.2), (3.4) and (3.5), if

$$\inf_{\xi\in\mathbb{R}} \|u - \varphi_c(\cdot - \xi)\|_{\mathcal{H}} \leq \varepsilon^{1/4}, \tag{3.6}$$

then

$$\|u(t,\cdot) - \varphi_c(\cdot - \xi_1)\|_{\mathcal{H}} \leq C\sqrt{\varepsilon}, \quad \forall t \in [0,T[, \tag{3.7}$$

where $\xi_1 \in \mathbb{R}$ is any point where the function $v = (4-\partial_x^2)^{-1}u$ attains its maximum.

We divide the proof of Theorem 3.1 into a sequence of lemmas. In the sequel we will need to introduce the following *smooth-peakons* defined for all $x \in \mathbb{R}$ by

$$\rho_c(x) = c\rho(x) = (4-\partial_x^2)^{-1}\varphi_c(x) = \frac{c}{3}e^{-|x|} - \frac{c}{6}e^{-2|x|}. \tag{3.8}$$

One can check that $\rho_c \in H^3(\mathbb{R}) \hookrightarrow C^2(\mathbb{R})$ (by the Sobolev embedding) since $\varphi_c \in H^1(\mathbb{R})$. Indeed, we have

$$\|\rho_c\|^2_{H^3(\mathbb{R})} = \int_{\mathbb{R}} \frac{(1+\omega^2)^3}{(4+\omega^2)^2}|\widehat{\varphi}_c(\omega)|^2 \leq \int_{\mathbb{R}}(4+\omega^2)|\widehat{\varphi}_c(\omega)|^2 \leq 4\|\varphi_c\|^2_{H^1(\mathbb{R})} = 8c^2. \tag{3.9}$$

Also, it is a positive even function which decays at infinity, and admits a single maximum $c/6$ at point 0 (see Figures 1a-1c).

Lemma 3.2 (Uniform estimates)**.** *Let $u \in H^1(\mathbb{R})$ with $y = (1-\partial_x^2)u \in \mathcal{M}^+(\mathbb{R})$, and $\xi \in \mathbb{R}$. If*

$$\|u - \varphi_c(\cdot - \xi)\|_{\mathcal{H}} \leq \varepsilon^{1/4}, \tag{3.10}$$

then

$$\|u - \varphi_c(\cdot - \xi)\|_{L^\infty(\mathbb{R})} \leq O(\varepsilon^{1/8}) \tag{3.11}$$

and

$$\|v - \rho_c(\cdot - \xi)\|_{L^\infty(\mathbb{R})} \leq O(\varepsilon^{1/4}), \tag{3.12}$$

where $v = (4-\partial_x^2)^{-1}u$ and ρ_c is defined in (3.8).

Proof. For the second estimate, applying the Hölder inegality and using (3.10), we get for all $x \in \mathbb{R}$,

$$\begin{aligned}
|v(x) - \rho_c(x-\xi)| &\leq \frac{1}{4}\int_{\mathbb{R}} e^{-2|x'|}\,|u(x-x') - \varphi_c\left[(x-x')-\xi\right]|\,dx' \\
&\leq \frac{1}{4}\left(\int_{\mathbb{R}} e^{-4|x'|}dx'\right)^{1/2}\left(\int_{\mathbb{R}} |u(x') - \varphi_c(x'-\xi)|^2\,dx'\right)^{1/2} \\
&\leq \frac{1}{2\sqrt{2}}\|u - \varphi_c(\cdot-\xi)\|_{\mathcal{H}} \\
&\leq O(\varepsilon^{1/4}).
\end{aligned}$$

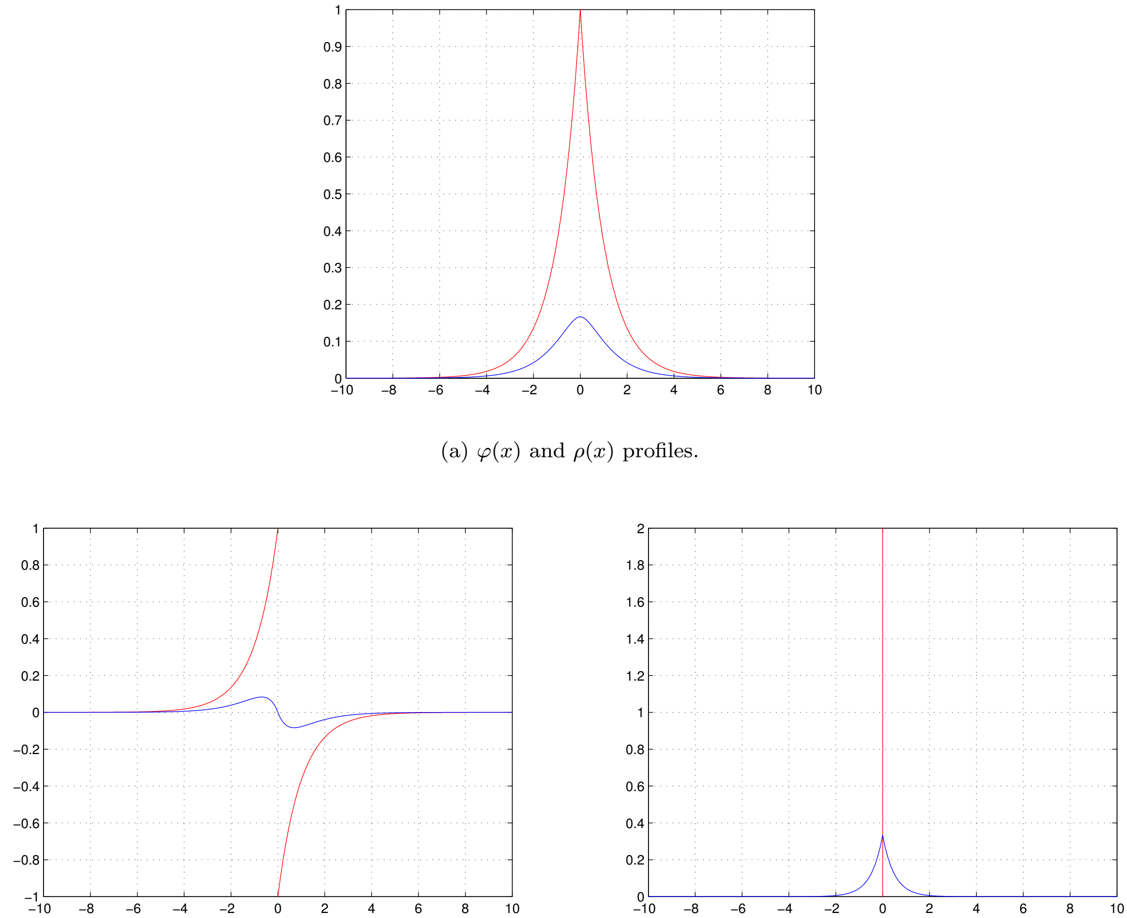
For the first estimate, note that the assumption $y = (1-\partial_x^2)u \geq 0$ implies that $u = (1-\partial_x^2)^{-1}y \geq 0$ and satisfies (2.2). Then, applying triangular inegality, and using that $|\varphi_c'| = \varphi_c$ on $\mathbb{R}$ and (2.3), we have

$$\begin{aligned}
\|u - \varphi_c(\cdot-\xi)\|_{H^1(\mathbb{R})} &\leq \|u\|_{H^1(\mathbb{R})} + \|\varphi_c\|_{H^1(\mathbb{R})} \\
&\leq 2\|u\|_{L^2(\mathbb{R})} + 2\|\varphi_c\|_{L^2(\mathbb{R})} \\
&\leq 2\|u - \varphi_c(\cdot-\xi)\|_{L^2(\mathbb{R})} + 4\|\varphi_c\|_{L^2(\mathbb{R})} \\
&\leq O(\varepsilon^{1/4}) + O(1).
\end{aligned}$$

Now, applying the Gagliardo-Nirenberg inegality and using (3.10), we obtain

$$\begin{aligned}\|u-\varphi_c(\cdot-\xi)\|_{L^\infty(\mathbb{R})} &\le C_G\|u-\varphi_c(\cdot-\xi)\|^{1/2}_{L^2(\mathbb{R})}\|u-\varphi_c(\cdot-\xi)\|^{1/2}_{H^1(\mathbb{R})}\\ &\le O(\varepsilon^{1/8})\left(O(\varepsilon^{1/8})+O(1)\right)\\ &\le O(\varepsilon^{1/8}).\end{aligned}$$

This proves the lemma.



(a) $\varphi(x)$ and $\rho(x)$ profiles.

(b) $\varphi'(x)$ and $\rho'(x)$ profiles.

(c) $(1-\partial_x^2)\varphi(x) = 2\delta_0$ and $(1-\partial_x^2)\rho(x) = (1/3)e^{-2|x|}$ profiles

Figure 1: Peakon and smooth-peakon at initial time with the speed $c = 1$.

Lemma 3.3 (Quadratic identity; see [5])**.** *For any $u \in L^2(\mathbb{R})$ and $\xi \in \mathbb{R}$, it holds*

$$E(u) - E(\varphi_c) = \|u-\varphi_c(\cdot-\xi)\|^2_{\mathcal{H}} + 4c\left(v(\xi) - \frac{c}{6}\right), \tag{3.13}$$

where $v = (4-\partial_x^2)^{-1}u$ and $c/6 = \rho_c(0) = \max_{\xi\in\mathbb{R}}\rho_c(\cdot-\xi)$.

Sketch of proof. The proof follows by direct computation, with the aid of two integration by parts, and using that $(1-\partial_x^2)\varphi_c(\cdot-\xi) = 2c\delta_\xi$, where δ_ξ denotes the Dirac mass applied at point ξ.

Let $u \in H^1(\mathbb{R})$ with $y = (1-\partial_x^2)u \in \mathcal{M}^+(\mathbb{R})$, and assume that there exists $\xi \in \mathbb{R}$ such that (3.10) holds. We consider now the interval in which the peakon (respectively smooth-peakon) is concentrated, and we will decompose this interval according to the variation of $v = (4-\partial_x^2)^{-1}u$ in the following way: we set

$$\alpha = \sup\left\{x < \xi,\ v(x) = \frac{c}{2400}\right\} \quad \text{and} \quad \beta = \inf\left\{x > \xi,\ v(x) = \frac{c}{2400}\right\}. \tag{3.14}$$

According to Lemma 3.2, we know that v is close to $\rho_c(\cdot - \xi)$ in L^∞ norm with $\rho_c(0) = c/6$. Therefore v must have at least one local maximum on $[\alpha, \beta]$. Assume that on $[\alpha, \beta]$ the function v admits $k+1$ points $(\xi_j)_{j=1}^{k+1}$ with local maximal values for some integer $k \geq 0$, where ξ_1 is the first local maximum point and ξ_{k+1} the last local maximum point[2]. Then between ξ_1 and ξ_{k+1}, the function v admits k points $(\eta_j)_{j=1}^k$ with local minimal values. We rename $\alpha = \eta_0$ and $\beta = \eta_{k+1}$ so that it holds

$$\eta_0 < \xi_1 < \eta_1 < \ldots < \xi_j < \eta_j < \xi_{j+1} < \eta_{j+1} < \ldots < \eta_k < \xi_{k+1} < \eta_{k+1}. \tag{3.15}$$

Let

$$M_j = v(\xi_j), \quad j = 1, \ldots, k+1, \quad \text{and} \quad m_j = v(\eta_j), \quad j = 1, \ldots, k. \tag{3.16}$$

By construction

$$v_x(x) \geq 0, \quad \forall x \in [\eta_{j-1}, \xi_j], \quad j = 1, \ldots, k \tag{3.17}$$

and

$$v_x(x) \leq 0, \quad \forall x \in [\xi_j, \eta_j], \quad j = 1, \ldots, k+1. \tag{3.18}$$

We claim that

$$v(x) \leq \frac{c}{300}, \quad \forall x \in \mathbb{R} \setminus [\eta_0, \eta_{k+1}], \tag{3.19}$$

$$u(x) \leq \frac{c}{300}, \quad \forall x \in \mathbb{R} \setminus [\eta_0, \eta_{k+1}], \tag{3.20}$$

and there exits $C_0 > 0$ such that

$$[\eta_0, \eta_{k+1}] \subset [\xi - C_0, \xi + C_0]. \tag{3.21}$$

Indeed, from (3.12) we have

$$\rho_c(\eta_0 - \xi) = v(\eta_0) + O(\varepsilon^{1/4}) = \frac{c}{2400} + O(\varepsilon^{1/4}) \leq \frac{c}{350},$$

please note that we abuse notation by writing that the difference between v and $\rho_c(\cdot - \xi)$ is equal to $O(\varepsilon^{1/4})$. Therefore, using that $\rho_c(\cdot - \xi)$ is increasing on $]-\infty, \xi]$, it holds for all $x \in]-\infty, \eta_0[$,

$$v(x) = \rho_c(x - \xi) + O(\varepsilon^{1/4}) \leq \frac{c}{350} + O(\varepsilon^{1/4}) \leq \frac{c}{300}.$$

Proceeding in the same way for $x \in]\eta_{k+1}, +\infty[$, we obtain (3.19).

One can remark that for all $x \in \mathbb{R}$,

$$\varphi_c(x) - 6\rho_c(x) = ce^{-|x|} - 6\left(\frac{c}{3}e^{-|x|} - \frac{c}{6}e^{-2|x|}\right) = -ce^{-|x|} + ce^{-2|x|} \leq 0. \tag{3.22}$$

Thus, combining (3.11), (3.22) and proceeding as for the estimate (3.19), we infer (3.20).

Finally, from (3.12) we have

$$\rho_c(\eta_0 - \xi) = v(\eta_0) + O(\varepsilon^{1/4}) = \frac{c}{2400} + O(\varepsilon^{1/4}) \geq \frac{c}{3000}.$$

Therefore, since $\rho_c = (c/3)e^{-|x|} - (c/6)e^{-2|x|}$ and that $x \mapsto (1/3)e^{-|x|} - (1/6)e^{-2|x|}$ is a positive even function decreasing to 0 on $\mathbb{R}_+$ (see Figure 1a), there exists a universal constant $C_0 > 0$ such that (3.21) holds.

We now are ready to establish the connection between the conservation laws. Please note that, we will change the order of the extrema of $v = (4-\partial_x^2)^{-1}u$ while keeping the same notations as in (3.16).

[2]In the case of an infinite countable number of local maximal values, the proof is exactly the same.

Lemma 3.4 (Connection between $E(\cdot)$ and the local extrema of v)**.** *Let $u \in H^1(\mathbb{R})$ with $u \geq 0$. Define the function g by*

$$g(x) = \begin{cases} 2v + v_{xx} - 3v_x, & x < \xi_1, \\ 2v + v_{xx} + 3v_x, & \xi_j < x < \eta_j, \\ 2v + v_{xx} - 3v_x, & \eta_j < x < \xi_{j+1}, \\ 2v + v_{xx} + 3v_x, & x > \xi_{k+1}, \end{cases} \quad j = 1, \ldots, k. \tag{3.23}$$

Then it holds

$$\int_{\mathbb{R}} g^2(x)dx = E(u) - 12 \left(\sum_{j=0}^{k} M_{j+1}^2 - \sum_{j=1}^{k} m_j^2 \right). \tag{3.24}$$

Proof. We have

$$\int_{\mathbb{R}} g^2(x)dx = \int_{-\infty}^{\xi_1} g^2(x)dx + \sum_{j=1}^{k} \int_{\xi_j}^{\xi_{j+1}} g^2(x)dx + \int_{\xi_{k+1}}^{+\infty} g^2(x)dx. \tag{3.25}$$

For $j = 1, \ldots, k$,

$$\begin{aligned} \int_{\xi_j}^{\xi_{j+1}} g^2(x)(x)dx &= \int_{\xi_j}^{\eta_j} (2v + v_{xx} + 3v_x)^2 + \int_{\eta_j}^{\xi_{j+1}} (2v + v_{xx} - 3v_x)^2 \\ &= J + I. \end{aligned}$$

Let us compute I,

$$\begin{aligned} I &= \int_{\eta_j}^{\xi_{j+1}} \left(4v^2 + v_{xx}^2 + 9v_x^2 + 4vv_{xx} - 12vv_x - 6v_xv_{xx}\right) \\ &= \int_{\eta_j}^{\xi_{j+1}} \left(4v^2 + v_{xx}^2 + 9v_x^2\right) + 4\int_{\eta_j}^{\xi_{j+1}} vv_{xx} - 12\int_{\eta_j}^{\xi_{j+1}} vv_x - 6\int_{\eta_j}^{\xi_{j+1}} v_xv_{xx} \\ &= \int_{\eta_j}^{\xi_{j+1}} \left(4v^2 + v_{xx}^2 + 9v_x^2\right) + I_1 + I_2 + I_3. \end{aligned}$$

Applying integration by parts and using that $v_x(\xi_j) = v_x(\eta_j) = 0$, we get

$$I_1 = -4\int_{\eta_j}^{\xi_{j+1}} v_x^2, \quad I_2 = -6\int_{\eta_j}^{\xi_{j+1}} \partial_x(v^2) = -6v^2(\xi_{j+1}) + 6v^2(\eta_j) \;\text{ and }\; I_3 = -3\int_{\eta_j}^{\xi_{j+1}} \partial_x(v_x^2) = 0.$$

Therefore

$$I = \int_{\eta_j}^{\xi_{j+1}} \left(4v^2 + 5v_x^2 + v_{xx}^2\right) - 6v^2(\xi_{j+1}) + 6v^2(\eta_j). \tag{3.26}$$

Similar computations lead to

$$J = \int_{\xi_j}^{\eta_j} \left(4v^2 + 5v_x^2 + v_{xx}^2\right) - 6v^2(\xi_j) + 6v^2(\eta_j), \quad \int_{-\infty}^{\xi_1} g^2(x)\,dx = \int_{-\infty}^{\xi_1} \left(4v^2 + 5v_x^2 + v_{xx}^2\right) - 6v^2(\xi_1) \tag{3.27}$$

and

$$\int_{\xi_{k+1}}^{+\infty} g^2(x)\,dx = \int_{\xi_{k+1}}^{+\infty} \left(4v^2 + 5v_x^2 + v_{xx}^2\right) - 6v^2(\xi_{k+1}). \tag{3.28}$$

Adding I and J, and summing over $j \in \{1, \ldots, k\}$, we obtain

$$\int_{\xi_1}^{\xi_{k+1}} g^2(x)dx = \int_{\xi_1}^{\xi_{k+1}} \left(4v^2 + 5v_x^2 + v_{xx}^2\right) - 6\sum_{j=1}^{k} v^2(\xi_{j+1}) - 6\sum_{j=1}^{k} v^2(\xi_j) + 12\sum_{j=1}^{k} v^2(\eta_j). \tag{3.29}$$

The lemma follows by combining (3.25) and (3.27)-(3.29).

Lemma 3.5 (Connection between $F(\cdot)$ and the local extrema of v)**.** *Let $u \in H^1(\mathbb{R})$ with $u \geq 0$. Define the function h by*

$$h(x) = \begin{cases} -v_{xx} - 6v_x + 16v, & x < \xi_1, \\ -v_{xx} + 6v_x + 16v, & \xi_j < x < \eta_j, \\ -v_{xx} - 6v_x + 16v, & \eta_j < x < \xi_{j+1}, \\ -v_{xx} + 6v_x + 16v, & x > \xi_{k+1}, \end{cases} \quad j = 1, \dots, k. \tag{3.30}$$

Then it holds

$$\int_{\mathbb{R}} h(x) g^2(x) dx = F(u) - 144 \left(\sum_{j=0}^{k} M_{j+1}^3 - \sum_{j=1}^{k} m_j^3 \right). \tag{3.31}$$

Proof. We have

$$\int_{\mathbb{R}} h(x) g^2(x) dx = \int_{-\infty}^{\xi_1} h(x) g^2(x) dx + \sum_{j=1}^{k} \int_{\xi_j}^{\xi_{j+1}} h(x) g^2(x) dx + \int_{\xi_{k+1}}^{+\infty} h(x) g^2(x) dx. \tag{3.32}$$

For $j = 1, \dots, k$,

$$\begin{aligned} \int_{\xi_j}^{\xi_{j+1}} h(x) g^2(x) dx &= \int_{\xi_j}^{\eta_j} (-v_{xx} - 6v_x + 16v) (2v + v_{xx} - 3v_x)^2 \\ &\quad + \int_{\eta_j}^{\xi_{j+1}} (-v_{xx} + 6v_x + 16v) (2v + v_{xx} + 3v_x)^2 \\ &= J + I. \end{aligned}$$

Let us compute I,

$$\begin{aligned} I &= \int_{\eta_j}^{\xi_{j+1}} \left(-v_{xx}^3 + 12 v v_{xx}^2 + 64 v^3 + 60 v^2 v_{xx}\right) - 54 \int_{\eta_j}^{\xi_{j+1}} v_x^3 + 27 \int_{\eta_j}^{\xi_{j+1}} v_x^2 v_{xx} \\ &\quad - 108 \int_{\eta_j}^{\xi_{j+1}} v v_x v_{xx} - 216 \int_{\eta_j}^{\xi_{j+1}} v^2 v_x + 216 \int_{\eta_j}^{\xi_{j+1}} v v_x^2 \\ &= \int_{\eta_j}^{\xi_{j+1}} \left(-v_{xx}^3 + 12 v v_{xx}^2 + 64 v^3 + 60 v^2 v_{xx}\right) \\ &\quad - 54 \int_{\eta_j}^{\xi_{j+1}} v_x^3 + I_1 + I_2 + I_3 + I_4. \end{aligned}$$

Applying integration by parts and using that $v_x(\xi_j) = v_x(\eta_j) = 0$, we get

$$I_1 = 9 \int_{\eta_j}^{\xi_{j+1}} \partial_x(v_x^3) = 0, \quad I_2 = 54 \int_{\eta_j}^{\xi_{j+1}} v_x^3, \quad I_3 = -72 \int_{\eta_j}^{\xi_{j+1}} \partial_x(v^3) = -72 v^3(\xi_{j+1}) + 72 v^3(\eta_j)$$

and

$$I_4 = 108 \int_{\eta_j}^{\xi_{j+1}} \partial_x(v^2) v_x = -108 \int_{\eta_j}^{\xi_{j+1}} v^2 v_{xx}.$$

Therefore

$$I = \int_{\eta_j}^{\xi_{j+1}} \left(-v_{xx}^3 + 12 v v_{xx}^2 + 64 v^3 + 60 v^2 v_{xx}\right) - 72 v^3(\xi_{j+1}) + 72 v^3(\eta_j). \tag{3.33}$$

Similar computations lead to

$$J=\int_{\xi_j}^{\eta_j}\left(-v_x^3+12vv_{xx}^2+64v^3+60v^2v_{xx}\right)-72v^3(\xi_j)+72v^3(\eta_j), \tag{3.34}$$

$$\int_{-\infty}^{\xi_1}h(x)g^2(x)dx=\int_{-\infty}^{\xi_1}\left(-v_x^3+12vv_{xx}^2+64v^3+60v^2v_{xx}\right)-72v^3(\xi_1) \tag{3.35}$$

and

$$\int_{\xi_{k+1}}^{+\infty}h(x)g^2(x)\,dx=\int_{\xi_{k+1}}^{+\infty}\left(-v_x^3+12vv_{xx}^2+64v^3+60v^2v_{xx}\right)-72v^3(\xi_{k+1}). \tag{3.36}$$

Adding (3.26) and (3.34), and summing over $j\in\{1,\ldots,k\}$, we obtain

$$\begin{aligned}\int_{\xi_1}^{\xi_{k+1}}h(x)g^2(x)dx=\int_{\xi_1}^{\xi_{k+1}}&\left(-v_x^3+12vv_{xx}^2+64v^3+60v^2v_{xx}\right)-72\sum_{j=1}^{k}v^3(\xi_{j+1})\\&-72\sum_{j=1}^{k}v^3(\xi_j)+72\sum_{j=1}^{k}v^3(\eta_j).\end{aligned} \tag{3.37}$$

The lemma follows by combining (3.32) and (3.35)-(3.37).

Lemma 3.6 (Connection between $E(\cdot)$ and $F(\cdot)$)**.** *Let $u\in H^1(\mathbb{R})$, with $y=(1-\partial_x^2)u\in\mathcal{M}^+(\mathbb{R})$, that satisfies* (3.10) *for some $\xi\in\mathbb{R}$. Assume that $v=(4-\partial_x^2)^{-1}u$ satisfies* (3.14)-(3.21)*, with local extrema on $[\eta_0,\eta_{k+1}]$ arranged in decreasing order in the following way:*

$$M_1\geq M_2\geq\ldots\geq M_{k+1}\geq 0,\quad m_1\geq m_2\geq\ldots\geq m_k\geq 0,\quad M_{j+1}\geq m_j,\ j=1,\ldots,k. \tag{3.38}$$

Then it holds,

$$M_1^3-\frac{1}{4}E(u)M_1+\frac{1}{72}F(u)\leq 0. \tag{3.39}$$

Proof. The key is to show that $h\leq 18M_1$ on $\mathbb{R}$. Note that by (3.10) we know that $18M_1\geq c/4$. We rewrite the function h as

$$h(x)=\begin{cases}-v_{xx}-6v_x+16v, & x<\eta_0,\\ -\left(\partial_x^2+3\partial_x+2\right)v-3v_x+18v, & \eta_0<x<\xi_1,\\ -\left(\partial_x^2-3\partial_x+2\right)v+3v_x+18v, & \xi_j<x<\eta_j,\\ -\left(\partial_x^2+3\partial_x+2\right)v-3v_x+18v, & \eta_j<x<\xi_{j+1},\\ -\left(\partial_x^2-3\partial_x+2\right)v+3v_x+18v, & \xi_{k+1}<x<\eta_{k+1},\\ -v_{xx}+6v_x+16v, & x>\eta_{k+1},\end{cases}\qquad j=1,\ldots,k.$$

First, one can remark that for all $x\in\mathbb{R}$,

$$v(x)=\frac{e^{-2x}}{4}\int_{-\infty}^{x}e^{2x'}u(x')dx'+\frac{e^{2x}}{4}\int_{x}^{+\infty}e^{-2x'}u(x')dx'$$

and

$$v_x(x)=-\frac{e^{-2x}}{2}\int_{-\infty}^{x}e^{2x'}u(x')dx'+\frac{e^{2x}}{2}\int_{x}^{+\infty}e^{-2x'}u(x')dx'.$$

Then using that $u=(1-\partial_x^2)^{-1}y\geq 0$ on $\mathbb{R}$, we get

$$|v_x(x)|\leq 2v(x),\quad\forall x\in\mathbb{R}. \tag{3.40}$$

Next, if $x \in \mathbb{R} \setminus [\eta_0, \eta_{k+1}]$, using that $v_{xx} = 4v - u$, (3.19), (3.20) and (3.40), it holds

$$h \leq |v_{xx}| + 6|v_x| + 16v \leq u + 32v \leq \frac{c}{9}.$$

If $\eta_0 < x < \xi_1$, then $v_x \geq 0$, and using that $y = (1 - \partial_x^2)u \geq 0$, it follows from Lemma 2.2 that

$$\begin{aligned} h &= -(\partial_x^2 + 3\partial_x + 2)v - 3v_x + 18v \\ &= -(2 + \partial_x)(4 - \partial_x^2)^{-1}(1 + \partial_x)u - 3v_x + 18v \\ &\leq 18v. \end{aligned}$$

If $\xi_j < x < \eta_j$, then $v_x \leq 0$, and similarly using that $y = (1 - \partial_x^2)u \geq 0$, it follows from Lemma 2.2 that

$$\begin{aligned} h &= -(\partial_x^2 - 3\partial_x + 2)v + 3v_x + 18v \\ &= -(2 - \partial_x)(4 - \partial_x^2)^{-1}(1 - \partial_x)u + 3v_x + 18v \\ &\leq 18v. \end{aligned}$$

Therefore, it holds

$$h(x) \leq 18 \max_{x \in \mathbb{R}} v(x) = 18M_1, \quad \forall x \in \mathbb{R}. \tag{3.41}$$

Now, combining (3.24), (3.31) and (3.41), we get

$$\begin{aligned} &F(u) - 144 \left(\sum_{j=0}^{k} M_{j+1}^3 - \sum_{j=1}^{k} m_j^3 \right) \\ &\quad = \int_{\mathbb{R}} h(x) g^2(x) dx \\ &\quad \leq \|h\|_{L^\infty(\mathbb{R})} \int_{\mathbb{R}} g^2(x) dx \\ &\quad \leq 18M_1 \left[E(u) - 12 \left(\sum_{j=0}^{k} M_{j+1}^2 - \sum_{j=1}^{k} m_j^2 \right) \right]. \end{aligned}$$

For $j = 1, \dots, k$, we set

$$A_j = M_{j+1}^3 - m_j^3 \quad \text{and} \quad B_j = M_{j+1}^2 - m_j^2,$$

and our inegality becomes

$$M_1^3 - \frac{1}{4} E(u) M_1 + \frac{1}{72} F(u) \leq 2 \sum_{j=1}^{k} \left(A_j - \frac{3}{2} M_1 B_j \right). \tag{3.42}$$

On the other hand, using that $M_{j+1} \geq m_j$, we have

$$A_j - \frac{3}{2} M_1 B_j = -\frac{1}{2} \left(M_{j+1} - m_j \right) \left(3M_1 m_j + 3M_1 M_{j+1} - 2M_{j+1} m_j - 2M_{j+1}^2 - 2m_j^2 \right) \leq 0. \tag{3.43}$$

Finally, combining (3.42) and (3.43), we obtain the lemma.

Proof of Theorem 3.1. As noticed after the statement of the theorem, it suffices to prove (3.7) assuming that $u \in H^1(\mathbb{R})$ satisfies (3.1), (3.2) and (3.4)-(3.6). We set $M_1 = v(\xi_1) = \max_{x \in \mathbb{R}} v(x)$ and $\delta = c/6 - M_1$. We first remark that if $\delta \leq 0$, combining (3.4) and (3.13), it holds

$$\|u - \varphi_c(\cdot - \xi_1)\|_{\mathcal{H}} \leq |E(u_0) - E(\varphi_c)|^{1/2} \leq O(\varepsilon),$$

that yields the desired result. Now suppose that $\delta > 0$, that is the maximum of the function v is less than the maximum of ρ_c. Combining (3.4), (3.5) and (3.39), we get

$$M_1^3 - \frac{1}{4}E(\varphi_c)M_1 + \frac{1}{72}F(\varphi_c) \leq O(\varepsilon^2).$$

Using that $E(\varphi_c) = c^2/3$ and $F(\varphi_c) = 2c^3/3$, our inequality becomes

$$\left(M_1 - \frac{c}{6}\right)^2 \left(M_1 + \frac{c}{3}\right) \leq O(\varepsilon^2).$$

Substituting M_1 by $c/6 - \delta$ and using that $(M_1 + c/3)^{-1} < 3/c$, it holds

$$\delta^2 \leq O(\varepsilon^2) \Rightarrow \delta \leq O(\varepsilon). \tag{3.44}$$

Finally, combining (3.4), (3.13) and (3.44), we obtain

$$\|u - \varphi_c(\cdot - \xi_1)\|_{\mathcal{H}} \leq C\sqrt{\varepsilon},$$

where $C > 0$ only depends on the speed c. This completes the proof of the stability of a single peakon.

4 Stability of the train of peakons

For $\gamma > 0$ and $L > 0$, we define the following neighborhood of all the sums of N peakons of speed $c_1, ..., c_N$ with spatial shifts z_i that satisfied $z_i - z_{i-1} \geq L$,

$$U(\gamma, L) = \left\{ u \in H^1(\mathbb{R}), \inf_{z_i - z_{i-1} > L} \left\| u - \sum_{i=1}^{N} \varphi_{c_i}(\cdot - z_i) \right\|_{\mathcal{H}} < \gamma \right\}. \tag{4.1}$$

By the continuity of the map $t \mapsto u(t)$ from $[0, T[$ into $H^1(\mathbb{R}) \hookrightarrow \mathcal{H}$, to prove Theorem 1.1 it suffices to prove that there exist $A > 0$, $\varepsilon_0 > 0$ and $L_0 > 0$ such that for all $L > L_0$ and $0 < \varepsilon < \varepsilon_0$, if u_0 satisfies (1.12)-(1.14), and if for some $0 < t_0 < T$,

$$u(t) \in U\left(A(\sqrt{\varepsilon} + L^{-1/8}), \frac{L}{2}\right), \quad \forall t \in [0, t_0], \tag{4.2}$$

then

$$u(t_0) \in U\left(\frac{A}{2}(\sqrt{\varepsilon} + L^{-1/8}), \frac{2L}{3}\right). \tag{4.3}$$

Therefore, in the sequel of this section we will assume (4.2) for some $0 < \varepsilon < \varepsilon_0$ and $L > L_0$, with A, ε_0 and L_0 to be specified later, and we will prove (4.3).

Remark 4.1 (Distance between v and the sum of N smooth-peakons)**.** From the definition of $E(\cdot)$ and $\mathcal{H}$ (see respectively (1.3) and (1.6)), one can clearly see that $\|u\|_{\mathcal{H}}$ is equivalent to $\|v\|_{H^2(\mathbb{R})}$, where $v = (4 - \partial_x^2)^{-1}u$. Therefore if $u(t) \in U(\gamma, L/2)$ on $[0, t_0]$, then $v(t) \in H^3(\mathbb{R}) \hookrightarrow C^2(\mathbb{R})$ stays close to $\sum_{i=1}^{N} \rho_{c_i}(\cdot - z_i)$ in the H^2 norm for all $t \in [0, t_0]$, where ρ_{c_i} is defined in (3.8).

4.1 Control of the distance between the peakons

In this subsection we want to prove that the different bumps of u that are individually close to a peakon get away from each others as time is increasing. This is crucial in our analysis since we do not know how to manage strong interactions.

Lemma 4.1 (Decomposition of the solution)**.** *Let u_0 satisfying (1.12)-(1.14). There exist $\gamma_0 > 0$, $L_0 > 0$ and $C_0 > 0$ such that for all $0 < \gamma < \gamma_0$ and $0 < L_0 < L$, if $u(t) \in U(\gamma, L/2)$ on $[0, t_0]$ for some $0 < t_0 < T$, then there exist N C^1 functions $\tilde{x}_1, \ldots, \tilde{x}_N$ defined on $[0, t_0]$ such that*

$$\left\| u(t) - \sum_{i=1}^{N} \varphi_{c_i}(\cdot - \tilde{x}_i(t)) \right\|_{\mathcal{H}} \leq O(\gamma), \tag{4.4}$$

$$\left\| v(t) - \sum_{i=1}^{N} \rho_{c_i}(\cdot - \tilde{x}_i(t)) \right\|_{C^1(\mathbb{R})} \leq O(\gamma), \tag{4.5}$$

$$\left| \dot{\tilde{x}}_i(t) - c_i \right| \leq c_1^{-2} \left(O(\gamma) + O(e^{-L/4}) \right), \quad i = 1, \ldots, N, \tag{4.6}$$

and

$$\tilde{x}_i(t) - \tilde{x}_{i-1}(t) \geq \frac{3L}{4} + \frac{(c_i - c_{i-1})t}{2}, \quad i = 2, \ldots, N. \tag{4.7}$$

Moreover, for $i = 1, \ldots, N$, setting $J_i = [y_i(t), y_{i+1}(t)]$, with

$$\begin{cases} y_1 = -\infty, \\ y_i(t) = \dfrac{\tilde{x}_{i-1}(t) + \tilde{x}_i(t)}{2}, \quad i = 2, \ldots, N, \\ y_{N+1} = +\infty, \end{cases} \tag{4.8}$$

it holds

$$\left| \xi_1^i(t) - \tilde{x}_i(t) \right| \leq \frac{L}{12}, \quad i = 1, \ldots, N, \tag{4.9}$$

where $\xi_1^1(t), \ldots, \xi_1^N(t)$ are any point such that

$$v\left(t, \xi_1^i(t)\right) = \max_{x \in J_i} v(t, x), \quad i = 1, \ldots, N, \tag{4.10}$$

and where $v = (4 - \partial_x^2)^{-1} u$ and $O(\cdot)$ only depends on $(c_i)_{i=1}^N$.

Proof. We will slightly modify the construction done in [3]. One can remark that the functions $\varphi_{c_i}(\cdot - c_i t)$ and $\rho_{c_i}(\cdot - c_i t)$ travel at the same speed c_i, thanks to this, we will do our construction with $v = (4 - \partial_x^2)^{-1} u$ instead of u. We do that because the $\mathcal{H}$ (equivalent to L^2) approximation (4.2) does not permit us to construct a C^1 function, which is crucial for application of the implicit function theorem. We note that the same approach can also be used for the CH equation.

For $Z = (z_1, \ldots, z_N) \in \mathbb{R}^N$ fixed such that $|z_i - z_{i-1}| > L/2$, we set

$$R_Z(\cdot) = \sum_{i=1}^{N} \rho_{c_i}(\cdot - z_i) \quad \text{and} \quad S_Z(\cdot) = \sum_{i=1}^{N} \varphi_{c_i}(\cdot - z_i). \tag{4.11}$$

For $0 < \gamma < \gamma_0$, we define the function

$$\begin{aligned} \mathcal{Y} : (-\gamma, \gamma)^N \times B_{H^2}(R_Z, \gamma) &\to \mathbb{R}^N, \\ (y_1, \ldots, y_N, v) &\mapsto \left(\mathcal{Y}^1(y_1, \ldots, y_N, v), \ldots, \mathcal{Y}^N(y_1, \ldots, y_N, v) \right) \end{aligned}$$

with

$$\mathcal{Y}^i(y_1, \ldots, y_N, v) = \int_{\mathbb{R}} \left(v - \sum_{j=1}^{N} \rho_{c_j}(\cdot - z_j - y_j) \right) \partial_x \rho_{c_i}(\cdot - z_i - y_i).$$

Y is clearly of class C^1. For $i = 1, \dots, N$,

$$\frac{\partial \mathcal{Y}^i}{\partial y_i}(y_1, \dots, y_N, v) = -\int_{\mathbb{R}} \left(v - \sum_{\substack{1 \le j \le N \\ j \ne i}} \rho_{c_j}(\cdot - z_j - y_j) \right) \partial_x^2 \rho_{c_i}(\cdot - z_i - y_i)$$

and for $j \neq i$,

$$\frac{\partial \mathcal{Y}^i}{\partial y_j}(y_1, \dots, y_N, v) = \int_{\mathbb{R}} \partial_x \rho_{c_j}(\cdot - z_j - y_j) \partial_x \rho_{c_i}(\cdot - z_i - y_i).$$

Hence

$$\frac{\partial \mathcal{Y}^i}{\partial y_i}(0, \dots, 0, R_Z) = \|\partial_x \rho_{c_i}\|_{L^2(\mathbb{R})}^2 = \frac{c_i^2}{54} \ge \frac{c_1^2}{54}$$

and for $j \neq i$, using the exponential decay of φ_{c_i} and that $|z_i - z_{i-1}| > L/2$, for $L > L_0 > 0$ with $L_0 \gg 1$, it holds

$$\begin{aligned}
\left| \frac{\partial \mathcal{Y}^i}{\partial y_j}(0, \dots, 0, R_Z) \right| &= \left| \int_{\mathbb{R}} \partial_x \rho_{c_j}(\cdot - z_j) \partial_x \rho_{c_i}(\cdot - z_i) \right| \\
&= \left| \int_{\mathbb{R}} \rho_{c_j}(\cdot - z_j) \partial_x^2 \rho_{c_i}(\cdot - z_i) \right| \\
&\le \frac{1}{9} c_i c_j \left\{ \int_{\mathbb{R}} e^{-|x-z_j|-|x-z_i|} dx + 2 \int_{\mathbb{R}} e^{-|x-z_j|-2|x-z_i|} dx \right. \\
&\qquad \left. + 2 \int_{\mathbb{R}} e^{-2|x-z_j|-|x-z_i|} dx + 4 \int_{\mathbb{R}} e^{-2|x-z_j|-2|x-z_i|} dx \right\} \\
&\le O(e^{-L/4}).
\end{aligned}$$

We deduce that, for $L > 0$ large enough, $D_{(y_1, \dots, y_N)} \mathcal{Y}(0, \dots, 0, R_Z) = D + P$ where D is an invertible diagonal matrix with $\|D^{-1}\| \le (c_1/3\sqrt{6})^{-2}$ and $\|P\| \le O(e^{-L/4})$. Hence there exists $L_0 > 0$ such that for $L > L_0$, $D_{(y_1, \dots, y_N)} \mathcal{Y}(0, \dots, 0, R_Z)$ is invertible with an inverse matrix of norm smaller than $2(c_1/3\sqrt{6})^{-2}$. From the implicit function theorem we deduce that there exists $\beta_0 > 0$ and C^1 functions $(y_1, \dots, y_N)$ from $B_{H^2}(R_Z, \beta_0)$ to a neighborhood of $(0, \dots, 0)$ which are uniquely determined such that

$$\mathcal{Y}(y_1(v), \dots, y_N(v), v) = 0, \quad \forall v \in B_{H^2}(R_Z, \beta_0).$$

In particular, there exists $C_0 > 0$ such that if $v \in B_{C^0}(R_Z, \beta)$, with $0 < \beta \le \beta_0$, then

$$\sum_{i=1}^N |y_i(v)| \le C_0 \beta. \tag{4.12}$$

Note that β_0 and C_0 only depend on c_1 and L_0 and not on the point $(z_1, \dots, z_N)$. For $v \in B_{H^2}(R_Z, \beta_0)$ we set $\tilde{x}_i(v) = z_i + y_i(v)$. Assuming that $\beta_0 \le L_0/8C_0$, $(\tilde{x}_1(v), \dots, \tilde{x}_N(v))$ are thus C^1 functions on $B_{H^2}(R_Z, \beta)$ satisfying

$$\tilde{x}_i(v) - \tilde{x}_{i-1}(v) = z_i - z_{i-1} + y_i(v) - y_{i-1}(v) > \frac{L}{2} - 2C_0\beta > \frac{L}{4}. \tag{4.13}$$

For $L > L_0$ and $0 < \gamma < \gamma_0 < \beta_0/2$ to be chosen later, we define the modulation of v in the following way: we cover the trajectory of v by a finite number of open balls in the following way:

$$\{v(t), t \in [0, t_0]\} \subset \bigcup_{k=1, \dots, M} B_{H^2}(R_{Z^k}, 2\gamma).$$

This is possible thanks to Remark 4.1. It is worth noticing that, since $0 < \gamma < \gamma_0 < \beta_0/2$, the functions $\tilde{x}_i(v)$ are uniquely determined for $v \in B_{H^2}(R_{Z^k}, 2\gamma) \cap B_{H^2}(R_{Z^{k'}}, 2\gamma)$. We can thus define the functions $t \mapsto \tilde{x}_i(t)$ on $[0, t_0]$ by setting $\tilde{x}_i(t) = \tilde{x}_i(v(t))$. By construction

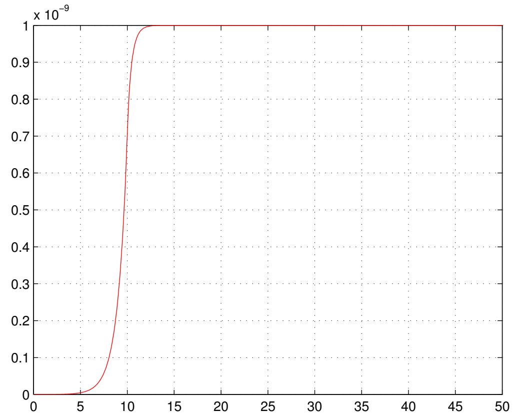
$$\int_{\mathbb{R}} \left(v(t, \cdot) - \sum_{j=1}^{N} \rho_{c_j}(\cdot - \tilde{x}_j(t)) \right) \partial_x \rho_{c_i}(\cdot - \tilde{x}_i(t)) = 0. \tag{4.14}$$

For $0 < \gamma < \gamma_0$, with $\gamma_0 \ll 1$, using that $u \in U(\gamma, L/2)$ and (4.12), we have

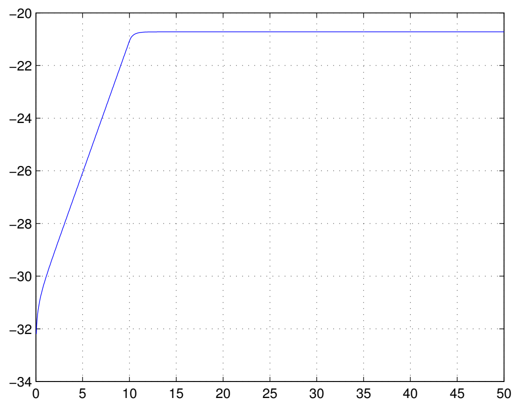
$$\begin{aligned} &\|u(t) - S_{\tilde{X}(t)}\|_{\mathcal{H}} \\ &\quad \leq \|u(t) - S_Z(t)\|_{\mathcal{H}} + \sum_{i=1}^{N} \|\varphi_{c_i}(\cdot - z_i) - \varphi_{c_i}(\cdot - z_i - y_i(v(t)))\|_{L^2(\mathbb{R})} \\ &\quad \leq \gamma + \sqrt{2} \sum_{i=1}^{N} \left(\int_{\mathbb{R}} \varphi_{c_i}^2(x) dx - \int_{\mathbb{R}} \varphi_{c_i}(x - z_i) \varphi_{c_i}(x - z_i - y_i(v(t))) dx \right)^{1/2} \\ &\quad = \gamma + \sqrt{2} \sum_{i=1}^{N} c_i \left(1 - e^{-|y_i(v(t))|} - |y_i(v(t))| e^{-|y_i(v(t))|} \right)^{1/2} \\ &\quad \leq \gamma + \sum_{i=1}^{N} O(|y_i(v(t))|) \\ &\quad \leq O(\gamma), \end{aligned}$$

where we apply two time the mean value theorem with the function φ on $[0, |y_i(v(t))|]$ for substituting $(1 - e^{-|y_i(v(t))|})$ by $|y_i(v(t))| e^{-\theta}$, with $\theta \in]0, |y_i(v(t))|[$, and this proves (4.4) (see Figures 2a-2b).

The estimate (4.5) follows directly by using (4.4), Remark 4.1 and the Sobolev embedding of $H^2(\mathbb{R})$ into $C^1(\mathbb{R})$.



(a) $\left(\int_0^x |\varphi(x-10) - \varphi(x-10-10^{-9})|^2\right)^{1/2} \approx 10^{-9}$



(b) $\log \left(\int_0^x |\varphi(x-10) - \varphi(x-10-10^{-9})|^2\right)^{1/2} \approx -20.7233$

Figure 2: Distance between two very close peakons (at time $t = 10$ with respective speeds 1 and $1+10^{-10}$).

To prove that the speed of $\tilde{x}_i(\cdot)$ stays close to c_i, we set

$$S_j(t) = \varphi_{c_j}(\cdot - \tilde{x}_j(t)), \quad \varepsilon_1(t) = u(t) - \sum_{j=1}^{N} S_j(t)$$

and

$$R_j(t) = \rho_{c_j}(\cdot - \tilde{x}_j(t)), \quad \varepsilon_2(t) = v(t) - \sum_{j=1}^{N} R_j(t).$$

One can notice that

$$\partial_x^2 R_i = 4R_i - S_i, \tag{4.15}$$

and using the Fourier transformation

$$\begin{aligned}(1-\partial_x^2)^{-1}(4-\partial_x^2)^{-1}(\cdot) &= \mathcal{F}^{-1}\left[\frac{1}{(1+\omega^2)(4+\omega^2)}\right](\cdot) \\ &= \mathcal{F}^{-1}\left[\frac{1}{3(1+\omega^2)} - \frac{1}{3(4+\omega^2)}\right](\cdot) \\ &= \frac{1}{3}(1-\partial_x^2)^{-1}(\cdot) - \frac{1}{3}(4-\partial_x^2)^{-1}(\cdot).\end{aligned} \tag{4.16}$$

Differentiating (4.14) with respect to time and using (4.15), we get

$$\int_{\mathbb{R}} \partial_t \varepsilon_2 \partial_x R_i = \dot{\tilde{x}}_i(t)\left(4\int_{\mathbb{R}} \varepsilon_2 R_i - \int_{\mathbb{R}} \varepsilon_2 S_i\right)$$

and thus

$$\begin{aligned}\left|\int_{\mathbb{R}} \partial_t \varepsilon_2 \partial_x R_i\right| &\le |\dot{\tilde{x}}_i(t)| \left(4\|\varepsilon_2\|_{L^\infty(\mathbb{R})}\|R_i\|_{L^1(\mathbb{R})} + \|\varepsilon_2\|_{L^\infty(\mathbb{R})}\|S_i\|_{L^1(\mathbb{R})}\right) \\ &\le |\dot{\tilde{x}}_i(t) - c_i| O(\gamma) + O(\gamma).\end{aligned} \tag{4.17}$$

Substituting u by $\varepsilon_1 + \sum_{j=1}^N S_j$ in (1.7) and using that S_j satisfies

$$\partial_t S_j = -(\dot{\tilde{x}}_j(t) - c_j)\partial_x S_j - \frac{1}{2}\partial_x S_j^2 - \frac{3}{2}(1-\partial_x^2)^{-1}\partial_x S_j^2,$$

we infer that ε_1 satisfies on $[0, t_0]$,

$$\begin{aligned}&\partial_t \varepsilon_1 - \sum_{j=1}^{N}(\dot{\tilde{x}}_j(t) - c_j)\partial_x S_j \\ &\quad = -\frac{1}{2}\partial_x\left[\left(\varepsilon_1 + \sum_{j=1}^{N} S_j\right)^2 - \sum_{j=1}^{N} S_j^2\right] \\ &\qquad - \frac{3}{2}\partial_x(1-\partial_x^2)^{-1}\left[\left(\varepsilon_1 + \sum_{j=1}^{N} S_j\right)^2 - \sum_{j=1}^{N} S_j^2\right].\end{aligned}$$

Multiplying by $(4-\partial_x^2)^{-1}$ and using (4.16), we get

$$\partial_t \varepsilon_2 - \sum_{j=1}^{N}(\dot{\tilde{x}}_j(t) - c_j)\partial_x R_j = -\frac{1}{2}\partial_x(1-\partial_x^2)^{-1}\left[\left(\varepsilon_1 + \sum_{j=1}^{N} S_j\right)^2 - \sum_{j=1}^{N} S_j^2\right].$$

Taking the L^2 scalar product with $\partial_x R_i$, integrating by parts, we find

$$\begin{aligned}
&-(\dot{\tilde{x}}_i(t)-c_i)\int_{\mathbb{R}}(\partial_x R_i)^2 \\
&\qquad = -\int_{\mathbb{R}}\partial_t\varepsilon_2\partial_x R_i + \sum_{\substack{1\le j\le N\\ j\neq i}}(\dot{\tilde{x}}_j(t)-c_j)\int_{\mathbb{R}}(\partial_x R_i)(\partial_x R_j) \\
&\qquad\qquad + \frac{1}{2}\int_{\mathbb{R}}(1-\partial_x^2)^{-1}\left[\left(\varepsilon_1+\sum_{j=1}^N S_j\right)^2-\sum_{j=1}^N S_j^2\right]\partial_x^2 R_i. \qquad (4.18)
\end{aligned}$$

We set

$$Q=\left(\varepsilon_1+\sum_{j=1}^N S_j\right)^2-\sum_{j=1}^N S_j^2=\varepsilon_1^2+2\varepsilon_1\left(\sum_{j=1}^N S_j\right)+\sum_{\substack{1\le i,j\le N\\ j\neq i}} S_iS_j,$$

then

$$\begin{aligned}
(1-\partial_x^2)^{-1}Q &= (1-\partial_x^2)^{-1}\varepsilon_1^2+2\sum_{j=1}^N(1-\partial_x^2)^{-1}(\varepsilon_1 S_j)+\sum_{\substack{1\le i,j\le N\\ j\neq i}}(1-\partial_x^2)^{-1}(S_iS_j) \\
&= I+J+K.
\end{aligned}$$

We have the following estimates

$$I=\frac{1}{2}\int_{\mathbb{R}}e^{-|x-x'|}\varepsilon_1^2(x')\le\frac{1}{2}\|e^{-|\cdot|}\|_{L^\infty(\mathbb{R})}\|\varepsilon_1\|_{L^2(\mathbb{R})}^2\le\frac{1}{2}\|\varepsilon_1\|_{L^2(\mathbb{R})}^2,$$

$$J=\sum_{j=1}^N\int_{\mathbb{R}}e^{-|x-x'|}\varepsilon_1(x')S_j(x')\le\|e^{-|\cdot|}\|_{L^\infty(\mathbb{R})}\|\varepsilon_1\|_{L^2(\mathbb{R})}\sum_{j=1}^N\|S_j\|_{L^2(\mathbb{R})}\le\left(\sum_{j=1}^N c_j\right)\|\varepsilon_1\|_{L^2(\mathbb{R})}$$

and

$$K=\frac{1}{2}\sum_{\substack{1\le i,j\le N\\ j\neq i}}\int_{\mathbb{R}}e^{-|x-x'|}S_j(x')S_i(x')\le\frac{1}{2}\sum_{\substack{1\le i,j\le N\\ j\neq i}}\int_{\mathbb{R}}S_j(x')S_i(x').$$

Thus, using (4.4) and the exponential decay of S_j, it holds

$$\|(1-\partial_x^2)^{-1}Q\|_{L^\infty(\mathbb{R})}\le O(\gamma)+O(e^{-L/4})$$

and then

$$\begin{aligned}
\left|\frac{1}{2}\int_{\mathbb{R}}[(1-\partial_x^2)^{-1}Q]\partial_x^2 R_i\right| &\le \frac{1}{2}\|(1-\partial_x^2)^{-1}Q\|_{L^\infty(\mathbb{R})}\|\partial_x^2 R_i\|_{L^1(\mathbb{R})} \\
&\le O(\gamma)+O(e^{-L/4}). \qquad (4.19)
\end{aligned}$$

Now, combining (4.17), (4.19), and using the exponential decay of $\partial_x R_i$, it holds

$$\left|\dot{\tilde{x}}_i(t)-c_i\right|\|\partial_x R_i\|_{L^2(\mathbb{R})}^2\le\left|\dot{\tilde{x}}_i(t)-c_i\right|O(\gamma)+O(\gamma)+O(e^{-L/4}),$$

then

$$\left|\dot{\tilde{x}}_i(t)-c_i\right|\left(\frac{c_i^2}{54}-O(\gamma)\right)\le O(\gamma)+O(e^{-L/4}),$$

which yields (4.6).

Taking $0 < \gamma < \gamma_0$ and $L > L_0 > 0$ with $\gamma_0 \ll 1$ and $L_0 \gg 1$, combining (1.12)-(1.14), (4.6) and (4.13), we deduce that

$$\begin{aligned}\tilde{x}_i(t) - \tilde{x}_{i-1}(t) &= \tilde{x}_i(0) - \tilde{x}_{i-1}(0) + (c_i - c_{i-1})t \\ &\geq L - 2C_0\gamma_0 + \frac{(c_i - c_{i-1})t}{2} \\ &\geq \frac{3L}{4} + \frac{(c_i - c_{i-1})t}{2},\end{aligned}$$

this proves (4.7).

From (4.5), we infer that

$$v(x) = \sum_{j=1}^{N} \rho_{c_j}(x - \tilde{x}_j) + O(\gamma), \quad \forall x \in \mathbb{R},$$

please note that we abuse notation by writing $\varepsilon_2(x) = O(\gamma)$. Applying this formula with $x = \xi_1^i$ and $v(\xi_1^i) = \max_{x \in J_i} v(x)$, and using (4.7), it holds

$$\begin{aligned}v(\xi_1^i) &= \max_{x \in J_i} \left\{ \sum_{j=1}^{N} \rho_{c_j}(x - \tilde{x}_j) \right\} + O(\gamma) \\ &= \frac{c_i}{6} + O(e^{-L/4}) + O(\gamma) \\ &\geq \frac{c_i}{7}.\end{aligned}$$

On the other hand, for $x \in J_i \setminus [\tilde{x}_i(t) - L/12,\ \tilde{x}_i(t) + L/12]$, we get

$$v(x) \leq \frac{c_i}{3} e^{-L/12} + O(e^{-L/4}) + O(\gamma) \leq \frac{c_i}{8}.$$

This ensures that $\xi_1^i \in [\tilde{x}_i(t) - L/12,\ \tilde{x}_i(t) + L/12]$, and this concluded the proof of the lemma.

4.2 Monotonicity property

Thanks to the preceding lemma, for $\varepsilon_0 > 0$ small enough and $L_0 > 0$ large enough, one can construct N C^1 functions $\tilde{x}_1, \ldots, \tilde{x}_N$ defined on $[0, t_0]$ such that (4.4)-(4.8) are satisfied. In this subsection we state the almost monotonicity of functionals that are very close to the energy at the right of ith bump, $i = 1, \ldots, N-1$ of u. Let ψ be a C^∞ test-function (see Figure 3) such that

$$\begin{cases} 0 < \psi(x) < 1,\ \psi'(x) > 0, & x \in \mathbb{R}, \\ |\psi^{(q)}(x)| \leq 10\psi'(x),\ q = 2,3,4,5, & x \in [-10, 10], \end{cases} \tag{4.20}$$

and

$$\psi(x) = \begin{cases} e^{-|x|}, & x < -10, \\ 1 - e^{-|x|}, & x > 10. \end{cases} \tag{4.21}$$

Setting $\psi_K = \psi(\cdot/K)$, we introduce for $i = 2, \ldots, N$,

$$\mathcal{J}_{i,K}(t) = \int_{\mathbb{R}} \left(4v^2 + 5v_x^2 + v_{xx}^2\right) \psi_{i,K}(t), \tag{4.22}$$

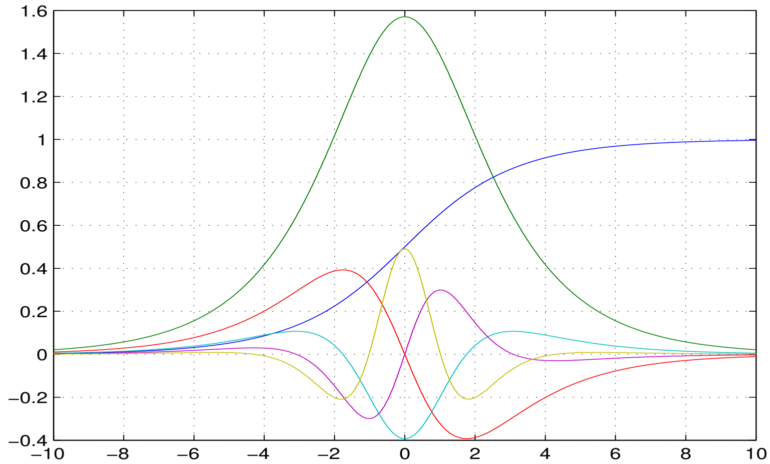


Figure 3: $\psi^{(q)}(x)$, $q = 0, 1, 2, 3, 4, 5$, profiles. Note that $\psi^{(5)}$ will not be used.

where $\psi_{i,K}(t,x) = \psi_K(x - y_i(t))$ with y_i's as in (4.8). Note that $\mathcal{J}_{i,K}(t)$ is close to $\|u(t)\|^2_{\mathcal{H}(x>y_i(t))}$ and thus measures the energy at the right of the $(i-1)$th bump of u. Finally, we set

$$\sigma_0 = \frac{1}{4}\min\{c_1, c_2 - c_1, \ldots, c_N - c_{N-1}\}. \tag{4.23}$$

We have the following monotonicity result.

Proposition 4.1 (Exponential decay of the functional $\mathcal{J}_{i,K}(t)$)**.** *Let $u \in \mathcal{X}([0,T[)$, with $0 < T \leq +\infty$, be a solution of equation* (1.1) *that satisfies* (1.12)*-*(1.14) *and* (4.4)*-*(4.5)*. There exist $\gamma_0 > 0$ and $L_0 > 0$ only depending on c_1 such that if $0 < \gamma < \gamma_0$ and $L > L_0 > 0$, then for any $4 \leq K \lesssim \sqrt{L}$,*

$$\mathcal{J}_{i,K}(t) - \mathcal{J}_{i,K}(0) \leq O(e^{-\frac{L}{8K}}), \quad \forall t \in [0, t_0], \quad i = 2, \ldots, N. \tag{4.24}$$

The proof of Proposition 4.1 relies on the following Virial type identity.

Lemma 4.2 (Virial type identity)**.** *Let $u \in \mathcal{X}([0,T[)$, with $0 < T \leq +\infty$, be a solution of equation* (1.1) *that satisfies* (1.12)*-*(1.14)*. For any smooth space function $g : \mathbb{R} \mapsto \mathbb{R}$, it holds*

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}} \left(4v^2 + 5v_x^2 + v_{xx}^2\right) g \\
&\quad = \frac{2}{3}\int_{\mathbb{R}} u^3 g' - 4\int_{\mathbb{R}} u^2 v g' - \frac{1}{2}\int_{\mathbb{R}} u^2 v g''' + \frac{1}{2}\int_{\mathbb{R}} u^2 v_x g'' + \int_{\mathbb{R}} uhg' \\
&\qquad + \frac{1}{2}\int_{\mathbb{R}} uh_x g'' - \frac{5}{2}\int_{\mathbb{R}} vh_x g'' - 2\int_{\mathbb{R}} v_x h g'' + \frac{1}{2}\int_{\mathbb{R}} vh_x g^{(4)}
\end{aligned} \tag{4.25}$$

where $y = (1 - \partial_x^2)u$, $v = (4 - \partial_x^2)^{-1}u$ and $h = (1 - \partial_x^2)^{-1}u^2$.

The proof of Lemma 4.2 is given in the appendix.

Proof of Proposition 4.1. We first note that, combining (4.6) and (4.8), it holds for $i = 2, \ldots, N$,

$$\begin{aligned}\dot{y}_i(t) &= \frac{\dot{\tilde{x}}_{i-1}(t) + \dot{\tilde{x}}_i(t)}{2} \\ &= \frac{c_{i-1} + c_i}{2} + O(\gamma^{1/2}) \\ &\geq c_{i-1} + O(\gamma^{1/2}) \\ &\geq \frac{c_1}{2}. \end{aligned} \tag{4.26}$$

Recall that the assumption (1.12) ensures that $u \geq 0$ and $v \geq 0$ on $\mathbb{R}$. Now, applying the Virial type identity (4.25) with $g = \psi_{i,K}$ and using (4.26), we get

$$\begin{aligned}\frac{d}{dt}\mathcal{J}_{i,K}(t) &= -\dot{y}_i \int_{\mathbb{R}} \left(4v^2 + 5v_x^2 + v_{xx}^2\right) \psi'_{i,K} + \frac{2}{3}\int_{\mathbb{R}} u^3 \psi'_{i,K} - 4\int_{\mathbb{R}} u^2 v \psi'_{i,K} \\ &\quad - \frac{1}{2}\int_{\mathbb{R}} u^2 v \psi'''_{i,K} + \frac{1}{2}\int_{\mathbb{R}} u^2 v_x \psi''_{i,K} + \int_{\mathbb{R}} uh\psi'_{i,K} + \frac{1}{2}\int_{\mathbb{R}} uh_x \psi''_{i,K} \\ &\quad - \frac{5}{2}\int_{\mathbb{R}} vh_x \psi''_{i,K} - 2\int_{\mathbb{R}} v_x h \psi''_{i,K} + \frac{1}{2}\int_{\mathbb{R}} vh_x \psi^{(4)}_{i,K} \\ &\leq -\dot{y}_i \int_{\mathbb{R}} \left(4v^2 + 5v_x^2 + v_{xx}^2\right) \psi'_{i,K} + \frac{2}{3}\int_{\mathbb{R}} u^3 \psi'_{i,K} - \frac{1}{2}\int_{\mathbb{R}} u^2 v \psi'''_{i,K} \\ &\quad + \frac{1}{2}\int_{\mathbb{R}} u^2 v_x \psi''_{i,K} + \int_{\mathbb{R}} uh\psi'_{i,K} + \frac{1}{2}\int_{\mathbb{R}} uh_x \psi''_{i,K} - \frac{5}{2}\int_{\mathbb{R}} vh_x \psi''_{i,K} \\ &\quad - 2\int_{\mathbb{R}} v_x h \psi''_{i,K} + \frac{1}{2}\int_{\mathbb{R}} vh_x \psi^{(4)}_{i,K} \\ &\leq -\frac{c_1}{2}\int_{\mathbb{R}} \left(4v^2 + 5v_x^2 + v_{xx}^2\right) \psi'_{i,K} + \sum_{k=1}^{8} J_k. \end{aligned} \tag{4.27}$$

We claim that for $k = 1, \ldots, 8$, it holds

$$J_k \leq \frac{c_1}{20}\int_{\mathbb{R}} \left(4v^2 + 5v_x^2 + v_{xx}^2\right) \psi'_{i,K} + \frac{C}{K}\|u_0\|^3_{\mathcal{H}} e^{-\frac{1}{K}(\sigma_0 t + L/8)}. \tag{4.28}$$

We divide $\mathbb{R}$ into two regions D_i and D_i^c with

$$D_i = \left[\tilde{x}_{i-1}(t) + \frac{L}{4}, \tilde{x}_i(t) - \frac{L}{4}\right], \quad i = 2, \ldots, N.$$

Combining (4.7) and (4.8), one can check that for $x \in D_i^c$,

$$\begin{aligned}|x - y_i(t)| &\geq \frac{\tilde{x}_i(t) - \tilde{x}_{i-1}(t)}{2} - \frac{L}{4} \\ &\geq \frac{c_i - c_{i-1}}{4}t + \frac{L}{8} \\ &\geq \sigma_0 t + \frac{L}{8}. \end{aligned} \tag{4.29}$$

Let us begin by an estimate of J_1. Using (2.3), (4.29) and the exponential decay of $\psi'_{i,K}$ on D_i^c, we get

$$\begin{aligned}\frac{2}{3}\int_{\mathbb{R}} u^3 \psi'_{i,K} &= \frac{2}{3}\int_{D_i} u^3 \psi'_{i,K} + \frac{2}{3}\int_{D_i^c} u^3 \psi'_{i,K} \\ &\leq \frac{2}{3}\|u\|_{L^\infty(D_i)} \int_{\mathbb{R}} u^2 \psi'_{i,K} + \frac{2}{3}\|\psi'_{i,K}\|_{L^\infty(D_i^c)} \|u\|_{L^\infty(\mathbb{R})} \|u\|^2_{L^2(\mathbb{R})} \\ &\leq \frac{2}{3}\|u\|_{L^\infty(D_i)} \int_{\mathbb{R}} u^2 \psi'_{i,K} + \frac{C}{K}\|u\|^3_{L^2(\mathbb{R})} e^{-\frac{1}{K}(\sigma_0 t + L/8)}. \end{aligned} \tag{4.30}$$

Note that, using the exponential decay of $|\psi'''_{j,K}|$ on D_i^c, $|\psi'''_{i,K}| \le (10/K^2)\psi'_{i,K}$, with $K \ge 4$, and that $\|u\|_{\mathcal{H}} = \|u_0\|_{\mathcal{H}}$, we have

$$\begin{aligned}
\int_{\mathbb{R}} u^2\psi'_{i,K} &= \int_{\mathbb{R}} (4v - v_{xx})^2\psi'_{i,K} \\
&= 16\int_{\mathbb{R}} v^2\psi'_{i,K} + \int_{\mathbb{R}} v_{xx}^2\psi'_{i,K} - 8\int_{\mathbb{R}} vv_{xx}\psi'_{i,K} \\
&= 16\int_{\mathbb{R}} v^2\psi'_{i,K} + \int_{\mathbb{R}} v_{xx}^2\psi'_{i,K} + 8\int_{\mathbb{R}} v_x^2\psi'_{i,K} + 4\int_{\mathbb{R}} \partial_x(v^2)\psi''_{i,K} \\
&= 16\int_{\mathbb{R}} v^2\psi'_{i,K} + \int_{\mathbb{R}} v_{xx}^2\psi'_{i,K} + 8\int_{\mathbb{R}} v_x^2\psi'_{i,K} - 4\int_{\mathbb{R}} v^2\psi'''_{i,K} \\
&\le \int_{\mathbb{R}} \left(16v^2 + 8v_x^2 + v_{xx}^2\right)\psi'_{i,K} + 4\int_{D_i} v^2|\psi'''_{j,K}| + 4\int_{D_i^c} v^2|\psi'''_{j,K}| \\
&\le \int_{\mathbb{R}} \left(16v^2 + 8v_x^2 + v_{xx}^2\right)\psi'_{i,K} + \frac{40}{K^2}\int_{\mathbb{R}} v^2\psi'_{i,K} + \frac{C}{K^3}\|u_0\|_{\mathcal{H}}^2 e^{-\frac{1}{K}(\sigma_0 t + L/8)} \\
&\le 5\int_{\mathbb{R}} \left(4v^2 + 5v_x^2 + v_{xx}^2\right)\psi'_{i,K} + \frac{C}{K^3}\|u_0\|_{\mathcal{H}}^2 e^{-\frac{1}{K}(\sigma_0 t + L/8)}. \qquad (4.31)
\end{aligned}$$

Now, using the exponential decay of φ_{c_i} on D_i, (4.4), and proceeding as for the estimate (3.11) (see Lemma 3.2), it holds

$$\begin{aligned}
\|u\|_{L^\infty(D_i)} &\le \left\| u - \sum_{j=1}^{N} \varphi_{c_j}(\cdot - \tilde{x}_j(t)) \right\|_{L^\infty(D_i)} + \sum_{j=1}^{N} \left\|\varphi_{c_j}(\cdot - \tilde{x}_j(t))\right\|_{L^\infty(D_i)} \\
&\le O(\gamma^{1/2}) + O(e^{-L/8}). \qquad (4.32)
\end{aligned}$$

Therefore, for $0 < \gamma < \gamma_0$ and $L > L_0 > 0$, with $\gamma_0 \ll 1$ and $L_0 \gg 1$, combining (4.30)-(4.31), we obtain

$$J_1 \le \frac{c_1}{20}\int_{\mathbb{R}} \left(4v^2 + 5v_x^2 + v_{xx}^2\right)\psi'_{i,K} + \frac{C}{K}\|u\|_{L^2(\mathbb{R})}^3 e^{-\frac{1}{K}(\sigma_0 t + L/8)}.$$

Next, the estimate of J_2 on D_i^c gives us

$$\left|\frac{1}{2}\int_{D_i^c} u^2 v\psi'''_{i,K}\right| \le \frac{1}{2}\|\psi'''_{i,K}\|_{L^\infty(D_i^c)}\|v\|_{L^\infty(\mathbb{R})}\|u\|_{L^2(\mathbb{R})}^2.$$

Note that, applying the Hölder inegality, we have for all $x \in \mathbb{R}$,

$$\begin{aligned}
v(x) &= \frac{1}{4}\int_{\mathbb{R}} e^{-2|x-x'|}u(x')dx' \\
&\le \frac{1}{4}\left(\int_{\mathbb{R}} e^{-4|x-x'|}dx'\right)^{1/2}\left(\int_{\mathbb{R}} |u(x')|^2 dx'\right)^{1/2} \\
&= \frac{1}{4\sqrt{2}}\|u\|_{L^2(\mathbb{R})} \qquad (4.33)
\end{aligned}$$

and thus, using (4.33) and the exponential decay of $|\psi'''_{i,K}|$ on D_i^c, it holds

$$\left|\frac{1}{2}\int_{D_i^c} u^2 v\psi'''_{i,K}\right| \le \frac{1}{8\sqrt{2}K^3}\|u\|_{L^2(\mathbb{R})}^3 e^{-\frac{1}{K}(\sigma_0 t + L/8)}. \qquad (4.34)$$

Using that $|\psi_{i,K}'''| \leq (10/K^2)\psi_{i,K}'$, the estimate of J_2 on D_i leads to

$$\left|\frac{1}{2}\int_{D_i} u^2 v\psi_{i,K}'''\right| \leq \frac{5}{K^2}\|u\|_{L^\infty(D_i)}\int_{\mathbb{R}} uv\psi_{i,K}'. \tag{4.35}$$

Also, one can notice that, using the exponential decay of $|\psi_{j,K}'''|$ on D_i^c, $|\psi_{i,K}'''| \leq (10/K^2)\psi_{i,K}'$, with $K \geq 4$, and that $\|u\|_{\mathcal{H}} = \|u_0\|_{\mathcal{H}}$, we have

$$\begin{aligned}
\int_{\mathbb{R}} uv\psi_{i,K}' &= \int_{\mathbb{R}} (4v - v_{xx})v\psi_{i,K}' \\
&= 4\int_{\mathbb{R}} v^2\psi_{i,K}' + \int_{\mathbb{R}} \partial_x(v\psi_{i,K}')v_x \\
&= 4\int_{\mathbb{R}} v^2\psi_{j,K}' + \int_{\mathbb{R}} v_x^2\psi_{i,K}' + \int_{\mathbb{R}} vv_x\psi_{i,K}'' \\
&= \int_{\mathbb{R}} \left(4v^2 + v_x^2\right)\psi_{i,K}' - \frac{1}{2}\int_{\mathbb{R}} v^2\psi_{i,K}''' \\
&\leq \int_{\mathbb{R}} \left(4v^2 + v_x^2\right)\psi_{i,K}' + \frac{1}{2}\int_{D_i} v^2|\psi_{i,K}'''| + \frac{1}{2}\int_{D_i^c} v^2|\psi_{i,K}'''| \\
&\leq \int_{\mathbb{R}} \left(4v^2 + v_x^2\right)\psi_{i,K}' + \frac{10}{2K^2}\int_{\mathbb{R}} v^2\psi_{i,K}' + \frac{C}{K^3}\|u_0\|_{\mathcal{H}}^2 e^{-\frac{1}{K}(\sigma_0 t + L/8)} \\
&\leq 2\int_{\mathbb{R}} \left(4v^2 + 5v_x^2 + v_{xx}^2\right)\psi_{i,K}' + \frac{C}{K^3}\|u_0\|_{\mathcal{H}}^2 e^{-\frac{1}{K}(\sigma_0 t + L/8)}.
\end{aligned} \tag{4.36}$$

Therefore, for $0 < \gamma < \gamma_0$ and $L > L_0 > 0$, with $\gamma_0 \ll 1$ and $L_0 \gg 1$, combining (4.32), (4.34)-(4.36), it holds

$$J_2 \leq \frac{c_1}{20}\int_{\mathbb{R}} \left(4v^2 + 5v_x^2 + v_{xx}^2\right)\psi_{i,K}' + \frac{C}{K^3}\|u\|_{L^2(\mathbb{R})}^3 e^{-\frac{1}{K}(\sigma_0 t + L/8)}.$$

In the same way, using that $|v_x| \leq 2v$ on $\mathbb{R}$ (see (3.40)), and the definition of $\psi_{i,K}$ (see (4.20) and (4.21)), we deduce the estimate of J_3.

Let us tackle now the estimate of J_4. On D_i^c we have

$$\begin{aligned}
\int_{D_i^c} uh\psi_{i,K}' &\leq \|\psi_{i,K}'\|_{L^\infty(D_i^c)}\int_{\mathbb{R}} uh \\
&= \|\psi_{i,K}'\|_{L^\infty(D_i^c)}\int_{\mathbb{R}} u[(1-\partial_x^2)^{-1}u^2] \\
&= \|\psi_{i,K}'\|_{L^\infty(D_i^c)}\int_{\mathbb{R}} u^2[(1-\partial_x^2)^{-1}u] \\
&\leq \|\psi_{i,K}'\|_{L^\infty(D_j^c)}\|(1-\partial_x^2)^{-1}u\|_{L^\infty(\mathbb{R})}\|u\|_{L^2(\mathbb{R})}^2.
\end{aligned}$$

Remark that, applying the Hölder inegality, we have for all $x \in \mathbb{R}$,

$$\begin{aligned}
(1-\partial_x^2)^{-1}u(x) &\leq \frac{1}{2}\int_{\mathbb{R}} e^{-|x-x'|}u(x')dx' \\
&\leq \frac{1}{2}\left(\int_{\mathbb{R}} e^{-2|x-x'|}dx'\right)^{1/2}\left(\int_{\mathbb{R}} |u(x')|^2 dx'\right)^{1/2} \\
&= \frac{1}{2}\|u\|_{L^2(\mathbb{R})}
\end{aligned} \tag{4.37}$$

and thus, using (4.37) and the exponential decay of $\psi_{i,K}'$ on D_i^c, it holds

$$\int_{D_i^c} uh\psi_{i,K}' \leq \frac{1}{2K}\|u\|_{L^2(\mathbb{R})}^3 e^{-\frac{1}{K}(\sigma_0 t + L/8)}. \tag{4.38}$$

The estimate of J_4 on D_i leads to

$$\begin{aligned}\int_{D_i} uh\psi'_{i,K} &\le \|u\|_{L^\infty(D_i)} \int_{\mathbb{R}} \psi'_{i,K}[(1-\partial_x^2)^{-1}u^2] \\ &= \|u\|_{L^\infty(D_i)} \int_{\mathbb{R}} u^2[(1-\partial_x^2)^{-1}\psi'_{i,K}]. \end{aligned} \tag{4.39}$$

On the other hand, using that $|\psi'''_{i,K}| \le (10/K^2)\psi'_{i,K}$, we have

$$(1-\partial_x^2)\psi'_{i,K}(x) = \psi'_{i,K}(x) - \frac{1}{K^2}\psi'''_{i,K}(x) \ge \left(1-\frac{10}{K^2}\right)\psi'_{i,K}(x), \quad \forall x \in \mathbb{R},$$

and since $K \ge 4$, it holds

$$(1-\partial_x^2)^{-1}\psi'_{i,K}(x) \le \left(1-\frac{10}{K^2}\right)^{-1}\psi'_{i,K}(x), \quad \forall x \in \mathbb{R}. \tag{4.40}$$

Therefore, for $0<\gamma<\gamma_0$ and $L>L_0>0$, with $\gamma_0 \ll 1$ and $L_0 \gg 1$, combining (4.32), (4.38)-(4.40), it holds

$$J_4 \le \frac{c_1}{20}\int_{\mathbb{R}} \left(4v^2+5v_x^2+v_{xx}^2\right)\psi'_{i,K} + \frac{C}{K}\|u\|^3_{L^2(\mathbb{R})} e^{-\frac{1}{K}(\sigma_0 t + L/8)}.$$

Noticing that for all $x \in \mathbb{R}$,

$$h(x) = \frac{e^{-x}}{2}\int_{-\infty}^{x} e^{x'}u^2(x')dx' + \frac{e^{x}}{2}\int_{x}^{+\infty} e^{-x'}u^2(x')dx'$$

and

$$h_x(x) = -\frac{e^{-x}}{2}\int_{-\infty}^{x} e^{x'}u^2(x')dx' + \frac{e^{x}}{2}\int_{x}^{+\infty} e^{-x'}u^2(x')dx',$$

we infer that

$$|h_x(x)| \le h(x), \quad \forall x \in \mathbb{R}. \tag{4.41}$$

Then, combining (4.20), (4.41), and proceeding as for the estimate of J_4, we deduce the estimate of J_5.

Now, combining (4.33) and (4.37), we have for all $x \in \mathbb{R}$,

$$\begin{aligned}(1-\partial_x^2)^{-1}v(x) &= \frac{1}{3}(1-\partial_x^2)^{-1}u(x) - \frac{1}{3}v(x) \\ &\le \frac{1}{3}\left\|(1-\partial_x^2)^{-1}u\right\|_{L^\infty(\mathbb{R})} + \frac{1}{3}\|v\|_{L^\infty(\mathbb{R})} \\ &\le \frac{4+\sqrt{2}}{24}\|u\|_{L^2(\mathbb{R})}, \end{aligned} \tag{4.42}$$

and using the exponential decay of ρ_{c_i} on D_i and (4.5), it holds

$$\begin{aligned}\|v\|_{L^\infty(D_i)} &\le \left\| v - \sum_{j=1}^{N}\rho_{c_j}(\cdot - \tilde{x}_j(t))\right\|_{L^\infty(D_i)} + \sum_{j=1}^{N}\left\|\rho_{c_j}(\cdot-\tilde{x}_j(t))\right\|_{L^\infty(D_i)} \\ &\le O(\gamma) + O(e^{-L/8}). \end{aligned} \tag{4.43}$$

Therefore, combining (3.40), (4.20), (4.41)-(4.43), and proceeding as for the estimate of J_4, we deduce the estimates of the remaining terms.

Finally, combining (4.27), (4.28) and using that $\|u\|_{L^2(\mathbb{R})} \sim \|u_0\|_{\mathcal{H}}$, it holds actually

$$\frac{d}{dt}\mathcal{J}_{i,K}(t) \le \frac{C}{K}\|u_0\|^3_{\mathcal{H}} e^{-\frac{1}{K}(\sigma_0 t + L/8)}.$$

Integrating between 0 and t, we obtain

$$\begin{aligned}\mathcal{J}_{i,K}(t)-\mathcal{J}_{i,K}(0) &\le \frac{C}{K}\|u_0\|^3_{\mathcal{H}}\left(-\frac{K}{\sigma_0}e^{-\frac{1}{K}(\sigma_0 t+L/8)}+\frac{K}{\sigma_0}e^{-\frac{L}{8K}}\right)\\ &\le \frac{C}{\sigma_0}\|u_0\|^3_{\mathcal{H}}e^{-\frac{L}{8K}},\end{aligned}$$

and this proves the proposition for smooth initial solutions.

For $u\in\mathcal{X}([0,T[)$, we will use that for any $T_0>0$ and any sequence $(u_{0,n})_{n\ge1}\subset L^2(\mathbb{R})$ such that $(u_{0,n}-\partial_x^2 u_{0,n})_{n\ge1}\subset\mathcal{M}^+(\mathbb{R})$ and $u_{0,n}\to u_0$ in $L^2(\mathbb{R})$, the sequence of emanating global weak solutions $(u_n)_{n\ge1}$ to the DP equation satisfies

$$u_n \underset{n\to+\infty}{\longrightarrow} u \quad\text{in}\quad C\left([0,T_0];L^2(\mathbb{R})\right), \tag{4.44}$$

where u is the global weak solution emanating from u_0. This fact can be easily deduced from the proof of the existence of the global weak solutions in [4]. Indeed, by the same arguments developed in this proof, we obtain that, up to a subsequence, $(u_n)_{n\ge1}$ converges in $C\left([0,T_0];L^2(\mathbb{R})\right)$ towards a solution of the DP equation emanating from u_0. (4.44) then follows by the uniqueness result. Combining (4.44) and Remark 4.1, it follows that

$$v_n \underset{n\to+\infty}{\longrightarrow} v \quad\text{in}\quad C\left([0,T_0];W^{1,\infty}(\mathbb{R})\right), \tag{4.45}$$

where $v=(4-\partial_x^2)^{-1}u$. For all $t\in[0,T[$, we set

$$\mathcal{J}^n_{i,K}(t)=\mathcal{J}_{i,K}(u_n(t))=\int_{\mathbb{R}}\left(4v_n^2+5v_{n,x}^2+v_{n,xx}^2\right)\psi_{i,K}(t), \tag{4.46}$$

and we claim that

$$\lim_{n\to+\infty}\sup_{0\le t<T}\left|\mathcal{J}^n_{i,K}(t)-\mathcal{J}_{i,K}(t)\right|=0. \tag{4.47}$$

Let $t\in[0,T[$ be fixed, we compute

$$\begin{aligned}\mathcal{J}^n_{i,K}(t)-\mathcal{J}_{i,K}(t) &= 4\int_{\mathbb{R}}(v_n^2-v^2)\psi_{i,K}(t)+5\int_{\mathbb{R}}(v_{n,x}^2-v_x^2)\psi_{i,K}(t)+\int_{\mathbb{R}}(v_{n,xx}^2-v_{xx}^2)\psi_{i,K}(t)\\ &= K_1^n(t)+K_2^n(t)+K_3^n(t).\end{aligned} \tag{4.48}$$

Then it is easy to check that

$$\begin{aligned}|K_1^n(t)| &\le 4\int_{\mathbb{R}}|v_n-v|(v_n+v)\psi_{i,K}(t)\\ &= 4\|v_n-v\|_{L^\infty(\mathbb{R})}\|v_n+v\|_{L^\infty(\mathbb{R})}\|\psi_{i,K}\|_{L^1(\mathbb{R})}\\ &\le O(\|v_n-v\|_{L^\infty(\mathbb{R})})\\ &\to 0 \quad\text{as}\quad n\to+\infty\end{aligned} \tag{4.49}$$

and

$$\begin{aligned}|K_2^n(t)| &\le 5\int_{\mathbb{R}}|v_{n,x}-v_x|(v_{n,x}+v_x)\psi_{i,K}(t)\\ &= 4\|v_{n,x}-v_x\|_{L^\infty(\mathbb{R})}\|v_{n,x}+v_x\|_{L^\infty(\mathbb{R})}\|\psi_{i,K}\|_{L^1(\mathbb{R})}\\ &\le O(\|v_{n,x}-v_x\|_{L^\infty(\mathbb{R})})\\ &\to 0 \quad\text{as}\quad n\to+\infty.\end{aligned} \tag{4.50}$$

Recalling that $v_{xx} = 4v - u$ and thus $v_{xx}^2 = 16v^2 + u^2 - 8uv$, we also get

$$
\begin{aligned}
|K_3^n(t)| &\leq 16 \int_{\mathbb{R}} |v_n - v|(v_n + v)\psi_{i,K}(t) + \int_{\mathbb{R}} |u_n - u|(u_n + u)\psi_{i,K}(t) \\
&\quad + 8 \int_{\mathbb{R}} u_n|v_n - v|\psi_{i,K}(t) + 8 \int_{\mathbb{R}} v|u_n - u|\psi_{i,K}(t) \\
&\leq 16\|v_n - v\|_{L^\infty(\mathbb{R})}\|v_n + v\|_{L^\infty(\mathbb{R})}\|\psi_{i,K}\|_{L^1(\mathbb{R})} \\
&\quad + \|u_n - u\|_{L^2(\mathbb{R})}\|u_n + u\|_{L^2(\mathbb{R})}\|\psi_{i,K}\|_{L^\infty(\mathbb{R})} \\
&\qquad + 8\|u_n\|_{L^\infty(\mathbb{R})}\|v_n - v\|_{L^\infty(\mathbb{R})}\|\psi_{i,K}\|_{L^1(\mathbb{R})} \\
&\qquad\quad + 8\|v\|_{L^2(\mathbb{R}}\|u_n - u\|_{L^2(\mathbb{R})}\|\psi_{i,K}\|_{L^\infty(\mathbb{R})} \\
&\leq O\left(\|u_n - u\|_{L^2(\mathbb{R})}\right) + O(\|v_n - v\|_{L^\infty(\mathbb{R})}) \\
&\to 0 \quad \text{as} \quad n \to +\infty.
\end{aligned} \tag{4.51}
$$

Combining (4.48)-(4.51), we obtain (4.47).

Thanks to (4.47), the monotonicity formula (4.24) holds for any $u \in \mathcal{X}([0, T[)$, with $0 < T \leq +\infty$.

4.3 A localized and a global estimate

Let $K = \sqrt{L}/8$ and define the function $\phi_i = \phi_i(t, x)$ (see Figure 4) by

$$
\begin{cases}
\phi_1 = 1 - \psi_{2,K} = 1 - \psi_K(\cdot - y_2(t)), \\
\phi_i = \psi_{i,K} - \psi_{i+1,K} = \psi_K(\cdot - y_i(t)) - \psi_K(\cdot - y_{i+1}(t)), \quad i = 2, \dots, N-1, \\
\phi_N = \psi_{N,K} = \psi_K(\cdot - y_N(t)),
\end{cases} \tag{4.52}
$$

where $\psi_{i,K}$'s and y_i's are defined in Subsection 4.2. One can see that the ϕ_i's are positive functions and that $\sum_{i=1}^N \phi_i = 1$. We take $L/K > 4$ so that ϕ_i satisfies for $i = 1, \dots, N$,

$$
|1 - \phi_i| \leq 2e^{-\frac{L}{8K}} \quad \text{on} \quad \left]y_i + \frac{L}{8}, y_{i+1} - \frac{L}{8}\right[\tag{4.53}
$$

and

$$
|\phi_i| \leq 2e^{-\frac{L}{8K}} \quad \text{on} \quad \mathbb{R} \setminus \left]y_i - \frac{L}{8}, y_{i+1} + \frac{L}{8}\right[. \tag{4.54}
$$

We will use the following localized version of the conservation laws defined for $i = 1, \dots, N$ by

$$
E_i(t) = \int_{\mathbb{R}} \left(4v^2 + 5v_x^2 + v_{xx}^2\right)\phi_i(t) \quad \text{and} \quad F_i(t) = \int_{\mathbb{R}} \left(-v_{xx}^3 + 12vv_{xx}^2 - 48v^2v_{xx} + 64v^3\right)\phi_i(t). \tag{4.55}
$$

One can remark that the functionals $E_i(\cdot)$ and $F_i(\cdot)$ do not depend on time in the statement below since we fix $-\infty = y_1 < y_2 < \dots < y_N < y_{N+1} = +\infty$.

For $i = 1, \dots, N$, we set $\Omega_i =]y_i - L/8, y_{i+1} + L/8[$. First, one can notice that

$$
\sum_{j=1}^N \rho_{c_j}(x - \tilde{x}_j) = \rho_{c_i}(x - \tilde{x}_i) + O(e^{-L/4}), \quad \forall x \in \Omega_i, \tag{4.56}
$$

we abuse notation by writing $\rho_{c_i,\alpha}(x - \tilde{x}_i) = O(e^{-L/4})$ for all $x \in \mathbb{R} \setminus \Omega_i$. We will now decompose this interval according to the variation of $v = (4 - \partial_x^2)^{-1}u$ in the same way as in Section 3. We set

$$
\alpha_i = \sup\left\{x < \tilde{x}_i, \ v(x) = \frac{c_i}{2400}\right\} \quad \text{and} \quad \beta_i = \inf\left\{x > \tilde{x}_i, \ v(x) = \frac{c_i}{2400}\right\}. \tag{4.57}
$$

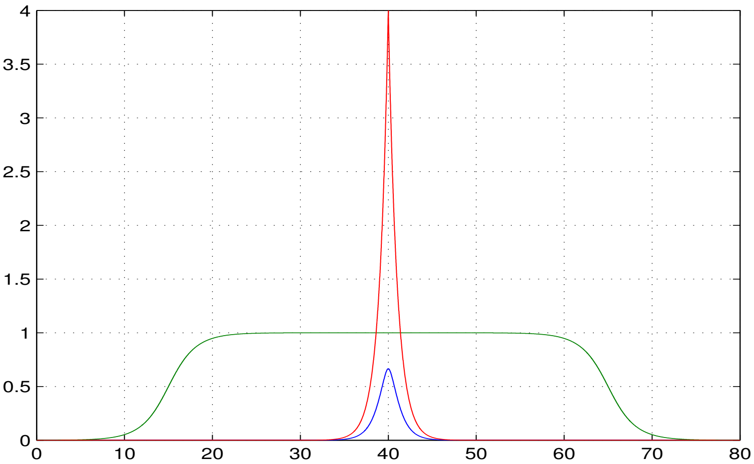


Figure 4: Localization-function $\phi_{green}(x) = \psi(x-15) - \psi(x-65)$ (at time $t = 10$) profile. Also, the peakon $4\varphi(x-40)$ and the smooth-peakon $4\rho(x-40)$ (at time $t = 10$ with speed $c = 4$) profiles. In this example, one can see that ϕ_{green} is close to 1 on $]25, 55[$, and decays exponentially to 0 on $\mathbb{R}\backslash]10, 70[$.

According to Lemma 4.1, we know that v is close to $\sum_{i=1}^{N} \rho_c(\cdot - \tilde{x}_i)$ in L^∞ norm with $\rho_{c_i}(0) = c_i/6$. Therefore v must have at least one local maximum on $[\alpha_i, \beta_i]$. Assume that on $[\alpha_i, \beta_i]$ the function v admits $k_i + 1$ points $(\xi_j^i)_{j=1}^{k_i+1}$ with local maximal values for some integer $k_i \geq 0$, where ξ_1^i is the first local maximum point and $\xi_{k_i+1}^i$ the last local maximum point[3]. Then between ξ_1^i and $\xi_{k_i+1}^i$, the function v admits k_i points $(\eta_j^i)_{j=1}^{k_i}$ with local minimal values. We rename $\alpha_i = \eta_0^i$ and $\beta_i = \eta_{k+1}^i$ so that it holds

$$\eta_0^i < \xi_1^i < \eta_1^i < \ldots < \xi_j^i < \eta_j^i < \xi_{j+1}^i < \eta_{j+1}^i < \ldots < \eta_{k_i}^i < \xi_{k_i+1}^i < \eta_{k_i+1}^i. \tag{4.58}$$

Let

$$M_j^i = v(\xi_j^i), \quad j = 1, \ldots, k_i + 1, \quad \text{and} \quad m_j^i = v(\eta_j^i), \quad j = 1, \ldots, k_i. \tag{4.59}$$

By construction

$$v_x(x) \geq 0, \quad \forall x \in [\eta_{j-1}^i, \xi_j^i], \quad j = 1, \ldots, k_i \tag{4.60}$$

and

$$v_x(x) \leq 0, \quad \forall x \in [\xi_j^i, \eta_j^i], \quad j = 1, \ldots, k_i + 1. \tag{4.61}$$

Proceeding as for (3.19)-(3.21), we also have

$$v(x) \leq \frac{c_i}{300}, \quad \forall x \in \Omega_i \setminus [\eta_0^i, \eta_{k_i+1}^i], \tag{4.62}$$

$$u(x) \leq \frac{c_i}{300}, \quad \forall x \in \Omega_i \setminus [\eta_0^i, \eta_{k_i+1}^i], \tag{4.63}$$

and taking $L > L_0 > 8C_0$, it holds

$$[\eta_0^i, \eta_{k_i+1}^i] \subset [\tilde{x}_i - C_0, \tilde{x}_i + C_0] \subset \left] y_i + \frac{L}{8}, y_{i+1} - \frac{L}{8} \right[, \tag{4.64}$$

where $C_0 > 0$ is the universal constant appearing in (3.21).

We now derive versions of Lemma 3.4, Lemma 3.5 and Lemma 3.6 where the global functional $E(\cdot)$ and $F(\cdot)$ are replaced by their localized versions $E_i(\cdot)$ and $F_i(\cdot)$. Please note that, we will change the order of the extrema of $v = (4 - \partial_x^2)^{-1} u$ while keeping the same notations as in (4.59).

[3] In the case of an infinite countable number of local maximal values, the proof is exactly the same.

Lemma 4.3 (Connection between $E_i(\cdot)$ and the local extrema of v)**.** *Let $u \in H^1(\mathbb{R})$ with $u \geq 0$. For $i = 1, \dots, N$, define the function g_i by*

$$g_i(x) = \begin{cases} 2v + v_{xx} - 3v_x, & x < \xi_1^i, \\ 2v + v_{xx} + 3v_x, & \xi_j^i < x < \eta_j^i, \\ 2v + v_{xx} - 3v_x, & \eta_j^i < x < \xi_{j+1}^i, \\ 2v + v_{xx} + 3v_x, & x > \xi_{k_i+1}^i, \end{cases} \quad j = 1, \dots, k_i. \tag{4.65}$$

Then it holds

$$\int_{\mathbb{R}} g_i^2(x)\phi_i(x) = E_i(u) - 12\left(\sum_{j=0}^{k_i}(M_{j+1}^i)^2\phi_i(\xi_{j+1}^i) - \sum_{j=1}^{k_i}(m_j^i)^2\phi_i(\eta_j)\right) + \|u\|_{\mathcal{H}}^2 O(L^{-1/2}). \tag{4.66}$$

Proof. We have

$$\int_{\mathbb{R}} g_i^2(x)\phi_i(x)dx = \int_{-\infty}^{\xi_1^i} g_i^2(x)\phi_i(x)dx + \sum_{j=1}^{k_i}\int_{\xi_j^i}^{\xi_{j+1}^i} g_i^2(x)\phi_i(x)dx + \int_{\xi_{k_i+1}^i}^{+\infty} g_i^2(x)\phi_i(x)dx. \tag{4.67}$$

For $j = 1, \dots, k_i$,

$$\begin{aligned}\int_{\xi_j^i}^{\xi_{j+1}^i} g_i^2(x)\phi_i(x)dx &= \int_{\xi_j^i}^{\eta_j^i} (2v + v_{xx} + 3v_x)^2\phi_i(x)dx + \int_{\eta_j^i}^{\xi_{j+1}^i} (2v + v_{xx} - 3v_x)^2\phi_i(x)dx \\ &= J + I.\end{aligned}$$

Computing I , we obtain

$$\begin{aligned} I &= \int_{\eta_j^i}^{\xi_{j+1}^i} \left(4v^2 + v_{xx}^2 + 9v_x^2 + 4vv_{xx} - 12vv_x - 6v_xv_{xx}\right)\phi_i \\ &= \int_{\eta_j^i}^{\xi_{j+1}^i} \left(4v^2 + v_{xx}^2 + 9v_x^2\right)\phi_i + 4\int_{\eta_j^i}^{\xi_{j+1}^i} vv_{xx}\phi_i - 12\int_{\eta_j^i}^{\xi_{j+1}^i} vv_x\phi_i - 6\int_{\eta_j^i}^{\xi_{j+1}^i} v_xv_{xx}\phi_i \\ &= \int_{\eta_j^i}^{\xi_{j+1}^i} \left(4v^2 + v_{xx}^2 + 9v_x^2\right)\phi_i + I_1 + I_2 + I_3 \end{aligned} \tag{4.68}$$

with

$$\begin{aligned} I_1 &= -4\int_{\eta_j^i}^{\xi_{j+1}^i} \partial_x(v\phi_i)v_x = -4\int_{\eta_j^i}^{\xi_{j+1}^i} v_x^2\phi_i - 4\int_{\eta_j^i}^{\xi_{j+1}^i} vv_x\phi_i' = -4\int_{\eta_j^i}^{\xi_{j+1}^i} v_x^2\phi_i - 2\int_{\eta_j^i}^{\xi_{j+1}^i} (v^2)_x\phi_i' \\ &= -2v^2(\xi_{j+1}^i)\phi_i'(\xi_{j+1}^i) + 2v^2(\eta_j^i)\phi_i'(\eta_j^i) - 4\int_{\eta_j^i}^{\xi_{j+1}^i} v_x^2\phi_i + 2\int_{\eta_j^i}^{\xi_{j+1}^i} v^2\phi_i'' \end{aligned} \tag{4.69}$$

$$I_2 = -6\int_{\eta_j^i}^{\xi_{j+1}^i} \partial_x(v^2)\phi_i = -6v^2(\xi_{j+1}^i)\phi_i(\xi_{j+1}^i) + 6v^2(\eta_j^i)\phi_i(\eta_j^i) + 6\int_{\eta_j^i}^{\xi_{j+1}^i} v^2\phi_i' \tag{4.70}$$

and

$$I_3 = -3\int_{\eta_j^i}^{\xi_{j+1}^i} \partial_x(v_x^2)\phi_i = 3\int_{\eta_j^i}^{\xi_{j+1}^i} v_x^2\phi_i'. \tag{4.71}$$

Adding (4.68)-(4.71), we get

$$I = \int_{\eta_j^i}^{\xi_{j+1}^i} \left(4v^2 + 5v_x^2 + v_{xx}^2\right) \phi_i - 6v^2(\xi_{j+1}^i)\phi_i(\xi_{j+1}^i) + 6v^2(\eta_j^i)\phi_i(\eta_j^i)$$
$$- 2v^2(\xi_{j+1}^i)\phi_i'(\xi_{j+1}^i) + 2v^2(\eta_j^i)\phi_i'(\eta_j^i) + R_1, \tag{4.72}$$

where using that $K = \sqrt{L}/8$, we have

$$|R_1| \leq 6(\|\phi_i'\|_{L^\infty(\mathbb{R})} + \|\phi_i''\|_{L^\infty(\mathbb{R})}) \int_{\eta_j^i}^{\xi_{j+1}^i} (v^2 + v_x^2) \leq O(L^{-1/2}) \int_{\eta_j^i}^{\xi_{j+1}^i} (v^2 + v_x^2).$$

Similar computations lead to

$$J = \int_{\xi_j^i}^{\eta_j^i} \left(4v^2 + 5v_x^2 + v_{xx}^2\right) \phi_i - 6v^2(\xi_j^i)\phi_i(\xi_j^i) + 6v^2(\eta_j^i)\phi_i(\eta_j^i)$$
$$+ 2v^2(\xi_j^i)\phi_i'(\xi_j^i) - 2v^2(\eta_j^i)\phi_i'(\eta_j^i) + R_2, \tag{4.73}$$

$$\int_{-\infty}^{\xi_1^i} g^2(x)\phi_i(x)\,dx = \int_{-\infty}^{\xi_1^i} \left(4v^2 + 5v_x^2 + v_{xx}^2\right) \phi_i - 6v^2(\xi_1^i)\phi_i(\xi_1^i) - 2v^2(\xi_1^i)\phi_i'(\xi_1^i) + R_3 \tag{4.74}$$

and

$$\int_{\xi_{k_i+1}^i}^{+\infty} g^2(x)\phi_i(x)\,dx = \int_{\xi_{k_i+1}^i}^{+\infty} \left(4v^2 + 5v_x^2 + v_{xx}^2\right) \phi_i - 6v^2(\xi_1^i)\phi_i(\xi_{k_i+1}^i) + 2v^2(\xi_{k_i+1}^i)\phi_i'(\xi_{k_i+1}^i) + R_4, \tag{4.75}$$

with

$$|R_2| \leq O(L^{-1/2}) \int_{\xi_j^i}^{\eta_j^i} (v^2 + v_x^2), \quad |R_3| \leq O(L^{-1/2}) \int_{-\infty}^{\xi_1^i} (v^2 + v_x^2) \;\text{ and }\; |R_4| \leq O(L^{-1/2}) \int_{\xi_{k_i+1}^i}^{+\infty} (v^2 + v_x^2).$$

Then, adding (4.72) and (4.73), and summing over $j \in \{1, \ldots, k_i\}$, we infer that

$$\int_{\xi_1^i}^{\xi_{k_i+1}^i} g_i^2(x)\phi_i(x)dx = \int_{\xi_1^i}^{\xi_{k_i+1}^i} \left(4v^2 + 5v_x^2 + v_{xx}^2\right) \phi_i - 6\sum_{j=1}^{k_i} v^2(\xi_{j+1}^i)\phi_i(\xi_{j+1}^i) - 6\sum_{j=1}^{k_i} v^2(\xi_j^i)\phi_i(\xi_j^i)$$
$$+ 12\sum_{j=1}^{k_i} v^2(\eta_j^i)\phi_i(\eta_j^i) - 2\sum_{j=1}^{k_i} v^2(\xi_{j+1}^i)\phi_i'(\xi_{j+1}^i) + 2\sum_{j=1}^{k_i} v^2(\xi_j^i)\phi_i'(\xi_j^i) + R, \tag{4.76}$$

with

$$|R| \leq O(L^{-1/2}) \int_{\xi_1^i}^{\xi_{k_i+1}^i} (v^2 + v_x^2).$$

Finally, adding (4.74)-(4.76), and recalling that $\|v\|_{H^1} \leq \|u\|_{\mathcal{H}}$, we obtain the lemma.

Lemma 4.4 (Connection between $F_i(\cdot)$ and the local extrema of v)**.** *Let $u \in H^1(\mathbb{R})$ with $u \geq 0$. For $i = 1, \ldots, N$, define the function h_i by*

$$h_i(x) = \begin{cases} -v_{xx} - 6v_x + 16v, & x < \xi_1^i, \\ -v_{xx} + 6v_x + 16v, & \xi_j^i < x < \eta_j^i, \\ -v_{xx} - 6v_x + 16v, & \eta_j^i < x < \xi_{j+1}^i, \\ -v_{xx} + 6v_x + 16v, & x > \xi_{k^i+1}^i, \end{cases} \quad j = 1, \ldots, k_i. \tag{4.77}$$

Then it holds

$$\int_{\mathbb{R}} h_i(x)g_i^2(x)\phi_i(x) = F_i(u) - 144\left(\sum_{j=0}^{k_i}(M_{j+1}^i)^3\phi_i(\xi_{j+1}^i) - \sum_{j=1}^{k_i}(m_j^i)^3\phi_i(\eta_j^i)\right) + \|u\|_{\mathcal{H}}^3 O(L^{-1/2}). \quad (4.78)$$

Proof. We have

$$\int_{\mathbb{R}} h_i(x)g_i^2(x)\phi_i(x)dx = \int_{-\infty}^{\xi_1^i} h_i(x)g_i^2(x)\phi_i(x)dx + \sum_{j=1}^{k_i}\int_{\xi_j^i}^{\xi_{j+1}^i} h_i(x)g_i^2(x)\phi_i(x)dx$$
$$+ \int_{\xi_{k_i+1}^i}^{+\infty} h_i(x)g_i^2(x)\phi_i(x)dx. \quad (4.79)$$

For $j = 1, \ldots, k_i$,

$$\begin{aligned}\int_{\xi_j^i}^{\xi_{j+1}^i} h_i(x)g_i^2(x)\phi_i(x)dx &= \int_{\xi_j^i}^{\eta_j^i}(-v_{xx} - 6v_x + 16v)(2v + v_{xx} - 3v_x)^2\phi_i \\ &\quad + \int_{\eta_j^i}^{\xi_{j+1}^i}(-v_{xx} + 6v_x + 16v)(2v + v_{xx} + 3v_x)^2\phi_i \\ &= J + I.\end{aligned}$$

Computing I, we obtain

$$\begin{aligned}I &= \int_{\eta_j^i}^{\xi_{j+1}^i}\left(-v_{xx}^3 + 12vv_{xx}^2 + 64v^3 + 60v^2v_{xx}\right)\phi_i - 54\int_{\eta_j^i}^{\xi_{j+1}^i} v_x^3\phi_i + 27\int_{\eta_j^i}^{\xi_{j+1}^i} v_x^2v_{xx}\phi_i \\ &\quad - 108\int_{\eta_j^i}^{\xi_{j+1}^i} vv_xv_{xx}\phi_i - 216\int_{\eta_j^i}^{\xi_{j+1}^i} v^2v_x\phi_i + 216\int_{\eta_j^i}^{\xi_{j+1}^i} vv_x^2\phi_i \\ &= \int_{\eta_j^i}^{\xi_{j+1}^i}\left(-v_{xx}^3 + 12vv_{xx}^2 + 64v^3 + 60v^2v_{xx}\right)\phi_i - 54\int_{\eta_j^i}^{\xi_{j+1}^i} v_x^3\phi_i + I_1 + I_2 + I_3 + I_4 \quad (4.80)\end{aligned}$$

with

$$I_1 = 9\int_{\eta_j^i}^{\xi_{j+1}^i}\partial_x(v_x^3)\phi_i = -9\int_{\eta_j^i}^{\xi_{j+1}^i} v_x^3\phi_i', \quad (4.81)$$

$$I_2 = -54\int_{\eta_j^i}^{\xi_{j+1}^i} v\partial_x(v_x^2)\phi_i = 54\int_{\eta_j^i}^{\xi_{j+1}^i}\partial_x(v\phi_i)v_x^2 = 54\int_{\eta_j^i}^{\xi_{j+1}^i} v_x^3\phi_i + 54\int_{\eta_j^i}^{\xi_{j+1}^i} vv_x^2\phi_i', \quad (4.82)$$

$$I_3 = -72\int_{\eta_j^i}^{\xi_{j+1}^i}\partial_x(v^3)\phi_i = -72v^3(\xi_{j+1}^i)\phi_i(\xi_{j+1}^i) + 72v^3(\eta_j^i)\phi_i(\eta_j^i) + 72\int_{\eta_j^i}^{\xi_{j+1}^i} v^3\phi_i', \quad (4.83)$$

and

$$\begin{aligned}I_4 &= 108\int_{\eta_j^i}^{\xi_{j+1}^i}\partial_x(v^2)v_x\phi_i = -108\int_{\eta_j^i}^{\xi_{j+1}^i} v^2\partial_x(v_x\phi_i) = -108\int_{\eta_j^i}^{\xi_{j+1}^i} v^2v_{xx}\phi_i - 108\int_{\eta_j^i}^{\xi_{j+1}^i} v^2v_x\phi_i' \\ &= -108\int_{\eta_j^i}^{\xi_{j+1}^i} v^2v_{xx}\phi_i - 36\int_{\eta_j^i}^{\xi_{j+1}^i}\partial_x(v^3)\phi_i' \\ &= -36v^3(\xi_{j+1}^i)\phi_i'(\xi_{j+1}^i) + 36v^3(\eta_j^i)\phi_i'(\eta_j^i) - 108\int_{\eta_j^i}^{\xi_{j+1}^i} v^2v_{xx}\phi_i + 36\int_{\eta_j^i}^{\xi_{j+1}^i} v^3\phi_i''. \quad (4.84)\end{aligned}$$

Adding (4.80)-(4.84), we get

$$I = \int_{\eta_j^i}^{\xi_{j+1}^i} \left(-v_{xx}^3 + 12vv_{xx}^2 + 64v^3 - 48v^2v_{xx}\right)\phi_i - 72v^3(\xi_{j+1}^i)\phi_i(\xi_{j+1}^i) + 72v^3(\eta_j^i)\phi_i(\eta_j^i)$$
$$- 36v^3(\xi_{j+1}^i)\phi_i'(\xi_{j+1}^i) + 36v^3(\eta_j^i)\phi_i'(\eta_j^i) + R, \tag{4.85}$$

where using that $\|v\|_{C^1(\mathbb{R})} \leq C_S'\|v\|_{H^2(\mathbb{R})}$ (with C_S' the constant of Sobolev), and $\|v\|_{H^2(\mathbb{R})} \sim \|u\|_{\mathcal{H}}$, the estimate of R leads to

$$|R| \leq (\|\phi_i'\|_{L^\infty(\mathbb{R})} + \|\phi_i''\|_{L^\infty(\mathbb{R})})(\|v\|_{L^\infty(\mathbb{R})} + \|v_x\|_{L^\infty(\mathbb{R})}) \int_{\eta_j^i}^{\xi_{j+1}^i} (v^2 + v_x^2)$$
$$\leq O(L^{-1/2})\|u\|_{\mathcal{H}} \int_{\eta_j^i}^{\xi_{j+1}^i} (v^2 + v_x^2).$$

Similar computations lead to

$$J = \int_{\xi_j^i}^{\eta_j^i} \left(-v_x^3 + 12vv_{xx}^2 + 64v^3 - 48v^2v_{xx}\right)\phi_i - 72v^3(\xi_j^i)\phi_i(\xi_j^i) + 72v^3(\eta_j^i)\phi_i(\eta_j^i)$$
$$+ 36v^3(\xi_j^i)\phi_i'(\xi_j^i) - 36v^3(\eta_j^i)\phi_i'(\eta_j^i) + O(L^{-1/2})\|u\|_{\mathcal{H}} \int_{\xi_j^i}^{\eta_j^i} (v^2 + v_x^2), \tag{4.86}$$

$$\int_{-\infty}^{\xi_1^i} h_i(x)g_i^2(x)\phi_i(x) = \int_{-\infty}^{\xi_1^i} \left(-v_{xx}^3 + 12vv_{xx}^2 + 64v^3 - 48v^2v_{xx}\right)\phi_i - 72v^3(\xi_1^i)\phi_i(\xi_1^i)$$
$$- 36v^3(\xi_1^i)\phi_i'(\xi_1^i) + O(L^{-1/2})\|u\|_{\mathcal{H}} \int_{-\infty}^{\xi_1^i} (v^2 + v_x^2) \tag{4.87}$$

and

$$\int_{\xi_{k_i+1}^i}^{+\infty} h_i(x)g_i^2(x)\phi_i(x) = \int_{\xi_{k_i+1}^i}^{+\infty} \left(-v_{xx}^3 + 12vv_{xx}^2 + 64v^3 - 48v^2v_{xx}\right)\phi_i - 72v^3(\xi_{k_i+1}^i)\phi_i(\xi_{k_i+1}^i)$$
$$+ 36v^3(\xi_{k_i+1}^i)\phi_i'(\xi_{k_i+1}^i) + O(L^{-1/2})\|u\|_{\mathcal{H}} \int_{\xi_{k_i+1}^i}^{+\infty} (v^2 + v_x^2). \tag{4.88}$$

Adding (4.85) and (4.86), and summing over $j \in \{1, \ldots, k_i\}$, we get

$$\int_{\xi_1^i}^{\xi_{k_i+1}^i} h_i(x)g_i^2(x)\phi_i(x)dx$$
$$= \int_{\xi_1^i}^{\xi_{k_i+1}^i} \left(-v_x^3 + 12vv_{xx}^2 + 64v^3 - 48v^2v_{xx}\right)\phi_i - 72\sum_{j=1}^{k_i} v^3(\xi_{j+1}^i)\phi_i(\xi_{j+1}^i)$$
$$- 72\sum_{j=1}^{k_i} v^3(\xi_j^i)\phi_i(\xi_j^i) + 144\sum_{j=1}^{k_i} v^3(\eta_j^i)\phi_i(\eta_j^i) - 36\sum_{j=1}^{k_i} v^3(\xi_{j+1}^i)\phi_i'(\xi_{j+1}^i)$$
$$+ 36\sum_{j=1}^{k_i} v^3(\xi_j^i)\phi_i'(\xi_j^i) + O(L^{-1/2})\|u\|_{\mathcal{H}} \int_{\xi_1^i}^{\xi_{k_i+1}^i} (v^2 + v_x^2). \tag{4.89}$$

Finally, adding (4.87)-(4.89), we obtain the lemma.

Lemma 4.5 (Connection between $E_i(\cdot)$ and $F_i(\cdot)$)**.** *Let $u \in H^1(\mathbb{R})$, with $y = (1-\partial_x^2)u \in \mathcal{M}^+(\mathbb{R})$, that satisfies (4.2). Let be given $N-1$ real numbers $-\infty = y_1 < y_2 < \ldots < y_N < y_{N+1} = +\infty$ with $y_i - y_{i-1} \geq 2L/3$. For $i = 1, \ldots, N$, assume that $v = (4-\partial_x^2)^{-1}u$ satisfies (4.57)-(4.64), with local extrema on $[\eta_0^i, \eta_{k_i+1}^i]$ arranged in decreasing order in the following way:*

$$M_1^i \geq M_2^i \geq \ldots \geq M_{k_i+1}^i \geq 0, \quad m_1^i \geq m_2^i \geq \ldots \geq m_{k_i}^i \geq 0, \quad M_{j+1}^i \geq m_j^i, \quad j = 1, \ldots, k_i. \tag{4.90}$$

Then, defining the functional $E_i(\cdot)$'s and $F_i(\cdot)$'s as in (4.52)-(4.55), it holds

$$F_i(u) \leq 18M_1^i E_i(u) - 72(M_1^i)^3 + \|u\|_{\mathcal{H}}^3 O(L^{-1/2}), \quad i = 1, \ldots, N. \tag{4.91}$$

Proof. Combining (4.53), (4.64) and (4.66) with $K = \sqrt{L}/8$, we get

$$\int_{\mathbb{R}} g_i^2(x)\phi_i(x) = E_i(u) - 12\left(\sum_{j=0}^{k_i}(M_{j+1}^i)^2 - \sum_{j=1}^{k_i}(m_j^i)^2\right) + \|u\|_{\mathcal{H}}^2 O(L^{-1/2}). \tag{4.92}$$

Similarly, combining (4.53), (4.64) and (4.78), we get

$$\int_{\mathbb{R}} h_i(x)g_i^2(x)\phi_i(x)dx = F_i(u) - 144\left(\sum_{j=0}^{k_i}(M_{j+1}^i)^3 - \sum_{j=1}^{k_i}(m_j^i)^3\right) + \|u\|_{\mathcal{H}}^3 O(L^{-1/2}). \tag{4.93}$$

Now, let us show that $h_i \leq 18M_1^i$ on Ω_i. Note that by (4.5) and (4.56), one can check that $18M_1^i \geq c_i/4$. We rewrite the function h_i as

$$h_i(x) = \begin{cases} -v_{xx} - 6v_x + 16v, \quad x < \eta_0^i, \\ -\left(\partial_x^2 + 3\partial_x + 2\right)v - 3v_x + 18v, \quad \eta_0^i < x < \xi_1^i, \\ -\left(\partial_x^2 - 3\partial_x + 2\right)v + 3v_x + 18v, \quad \xi_j^i < x < \eta_j^i, \\ -\left(\partial_x^2 + 3\partial_x + 2\right)v - 3v_x + 18v, \quad \eta_j^i < x < \xi_{j+1}^i, \\ -\left(\partial_x^2 - 3\partial_x + 2\right)v + 3v_x + 18v, \quad \xi_{k_i+1}^i < x < \eta_{k_i+1}^i, \\ -v_{xx} + 6v_x + 16v, \quad x > \eta_{k_i+1}^i, \end{cases} \qquad j = 1, \ldots, k_i.$$

Then, if $x \in \Omega_i \setminus [\eta_0^i, \eta_{k_i+1}^i]$, using that $v_{xx} = 4v - u$, (3.40), (4.64) and (4.65), it holds

$$h_i \leq |v_{xx}| + 6|v_x| + 16v \leq u + 32v \leq \frac{c_i}{9}.$$

If $\eta_0^i < x < \xi_1^i$, then $v_x \geq 0$, and using that $y = (1-\partial_x^2)u \geq 0$, it follows from Lemma 2.2 that

$$\begin{aligned} h_i &= -(\partial_x^2 + 3\partial_x + 2)v - 3v_x + 18v \\ &= -(2+\partial_x)(4-\partial_x^2)^{-1}(1+\partial_x)u - 3v_x + 18v \\ &\leq 18v. \end{aligned}$$

If $\xi_j^i < x < \eta_j^i$, then $v_x \leq 0$, and similarly using that $y = (1-\partial_x^2)u \geq 0$, it follows from Lemma 2.2 that

$$\begin{aligned} h_i &= -(\partial_x^2 - 3\partial_x + 2)v + 3v_x + 18v \\ &= -(2-\partial_x)(4-\partial_x^2)^{-1}(1-\partial_x)u + 3v_x + 18v \\ &\leq 18v. \end{aligned}$$

Therefore, it holds

$$h_i(x) \leq 18\max_{x\in\Omega_i} v(x) = 18M_1^i, \quad \forall x \in \Omega_i. \tag{4.94}$$

Now, taking $\phi_i \equiv 1$ on $\mathbb{R}$ in (4.66), we have $\|g_i\|_{L^2(\mathbb{R})} \leq \|u\|_{\mathcal{H}}$. Also, from the definition of h_i, and using (2.3) and (4.5), we have $\|h_i\|_{L^\infty(\mathbb{R})} \leq \|u\|_{L^\infty(\mathbb{R})} + 32\|v\|_{L^\infty(\mathbb{R})} \leq O(\|u\|_{\mathcal{H}})$. Then, combining (4.92)-(4.94), we obtain

$$
\begin{aligned}
&F_i(u) - 144\left(\sum_{j=0}^{k_i}(M_{j+1}^i)^3 - \sum_{j=1}^{k_i}(m_j^i)^3\right) \\
&\quad = \int_{\mathbb{R}} h_i(x)g_i^2(x)\phi_i(x)dx + \|u\|_{\mathcal{H}}^3 O(L^{-1/2}) \\
&\quad = \int_{\Omega_i} h_i(x)g_i^2(x)\phi_i(x)dx + \int_{\Omega_i^c} h_i(x)g_i^2(x)\phi_i(x)dx + \|u\|_{\mathcal{H}}^3 O(L^{-1/2}) \\
&\quad \leq 18M_1^i \int_{\Omega_i} g_i^2(x)\phi_i(x)dx + \|h_i\|_{L^\infty(\mathbb{R})}\|g_i\|_{L^2(\mathbb{R})}^2\|\phi_i\|_{L^\infty(\Omega_i^c)} + \|u\|_{\mathcal{H}}^3 O(L^{-1/2}) \\
&\quad \leq 18M_1^i\left[E_i(u) - 12\left(\sum_{j=0}^{k_i}(M_{j+1}^i)^2 - \sum_{j=1}^{k_i}(m_j^i)^2\right)\right] + \|u\|_{\mathcal{H}}^3 O(L^{-1/2}).
\end{aligned}
$$

Therefore, using that $M_{j+1}^i \geq m_j^i$ and proceeding as in Lemma 3.6 (see (3.43)), we infer that

$$
\begin{aligned}
F_i(u) &\leq 18M_1^i E_i(u) - 72(M_1^i)^3 + 144\sum_{j=1}^{k_i}\left\{\left[(M_{j+1}^i)^3 - (m_j^i)^3\right] - \frac{3}{2}M_1^i\left[(M_{j+1}^i)^2 - (m_j^i)^2\right]\right\} \\
&\quad + \|u\|_{\mathcal{H}}^3 O(L^{-1/2}) \\
&\leq 18M_1^i E_i(u) - 72(M_1^i)^3 + \|u\|_{\mathcal{H}}^3 O(L^{-1/2}).
\end{aligned}
$$

This proves the lemma.

The lemma below is the generalization of Lemma 3.3.

Lemma 4.6 (Quadratic identity). *Let $Z = (z_i)_{i=1}^N \in \mathbb{R}^N$ with $|z_i - z_{i-1}| \geq L/2$, and $u \in L^2(\mathbb{R})$. It holds*

$$
E(u) - \sum_{i=1}^N E(\varphi_{c_i}) = \|u - S_Z\|_{\mathcal{H}}^2 + 4\sum_{i=1}^N c_i\left(v(z_i) - \frac{c_i}{6}\right) + O(e^{-L/4}), \tag{4.95}
$$

where S_Z is defined in (4.11) *and $O(\cdot)$ only depends on $(c_i)_{i=1}^N$.*

Proof. Let us compute

$$
\begin{aligned}
\|u - S_Z\|_{\mathcal{H}}^2 &= \int_{\mathbb{R}}[(1-\partial_x^2)(u - S_Z)][(4-\partial_x^2)^{-1}(u - S_Z)] \\
&= \|u\|_{\mathcal{H}}^2 + \|S_Z\|_{\mathcal{H}}^2 - 2\int_{\mathbb{R}}[(1-\partial_x^2)S_Z][(4-\partial_x^2)^{-1}u] \\
&= \|u\|_{\mathcal{H}}^2 + \|S_Z\|_{\mathcal{H}}^2 - 2\sum_{i=1}^N c_i\int_{\mathbb{R}}[(1-\partial_x^2)\varphi_{c_i}(x - z_i)]v \\
&= \|u\|_{\mathcal{H}}^2 + \|S_Z\|_{\mathcal{H}}^2 - 4\sum_{i=1}^N c_i v(z_i),
\end{aligned} \tag{4.96}
$$

where we use that

$$
(1-\partial_x^2)\varphi_{c_i}(\cdot - z_i) = 2c_i\delta_{z_i}
$$

with δ_{z_i} the Dirac mass applied at point z_i. We also have

$$
\begin{aligned}
\|S_Z\|_{\mathcal{H}}^2 &= \int_{\mathbb{R}} [(1-\partial_x^2)S_Z][(4-\partial_x^2)^{-1}S_Z] \\
&= \sum_{i=1}^N \int_{\mathbb{R}} [(1-\partial_x^2)\varphi_{c_i}(x-z_i)][(4-\partial_x^2)^{-1}S_Z] \\
&= 2\sum_{i=1}^N c_i \left\langle \delta_{z_i}, (4-\partial_x^2)^{-1}S_Z \right\rangle_{H^{-1},H^1} \\
&= 2\sum_{1\le i,j\le N} c_i \left\langle \delta_{z_i}, (4-\partial_x^2)^{-1}\varphi_{c_j}(\cdot - z_j) \right\rangle_{H^{-1},H^1} \\
&= 2\sum_{i=1}^N c_i (4-\partial_x^2)^{-1}\varphi_{c_i}(0) + 2\sum_{\substack{1\le i,j\le N \\ i\neq j}} c_i c_j (4-\partial_x^2)^{-1} e^{-|z_i-z_j|},
\end{aligned}
\tag{4.97}
$$

where $\langle \cdot,\cdot\rangle_{H^{-1},H^1}$ denote the duality H^{-1}/H^1, and we recall that $\delta_{z_i} \in \mathcal{L}\left(H^{-1}(\mathbb{R}), H^1(\mathbb{R})\right)$ since $\|\delta_{z_i}\|_{H^{-1}(\mathbb{R})} \le C_S$, with C_S the constant appearing in (2.3) . Now, using that $|z_i - z_{i-1}| \ge L/2$,

$$
\begin{aligned}
(4-\partial_x^2)^{-1} e^{-|z_i-z_j|} &= \frac{1}{4}\int_{\mathbb{R}} e^{-2|x'-(z_i-z_j)|} e^{-|x'|} dx' \\
&= \frac{1}{3} e^{-|z_i-z_j|} - \frac{1}{6} e^{-2|z_i-z_j|} \\
&= O(e^{-L/4}),
\end{aligned}
\tag{4.98}
$$

and combining (4.97) and (4.98), for $L > L_0 > 0$ with $L_0 \gg 1$, we get

$$
\|S_Z\|_{\mathcal{H}}^2 = \sum_{i=1}^N E(\varphi_{c_i}) + O(e^{-L/4}). \tag{4.99}
$$

Finally, combining (4.96) and (4.99), we obtain the lemma.

The last lemma is the localized version of Lemma 3.1.

Lemma 4.7 (Control of the distances between local and global energies at $t=0$)**.** *Let* $u_0 \in H^1(\mathbb{R})$ *satisfying* (1.12)-(1.14). *Then it holds*

$$
\left| E(u_0) - \sum_{i=1}^N E(\varphi_{c_i}) \right| \le O(\varepsilon^2) + O(e^{-L/4}), \tag{4.100}
$$

$$
|E_i(u_0) - E(\varphi_{c_i})| \le O(\varepsilon^2) + O(e^{-\sqrt{L}}), \quad i = 1,\dots,N, \tag{4.101}
$$

and

$$
|F_i(u_0) - F(\varphi_{c_i})| \le O(\varepsilon^2) + O(e^{-\sqrt{L}}), \quad i = 1,\dots,N, \tag{4.102}
$$

where $O(\cdot)$ *only depend on* $(c_i)_{i=1}^N$.

Proof. For the first estimate, applying triangular inequality and (1.14), we have

$$
\begin{aligned}
|E(u_0) - E(S_{Z^0})| &= |\|u_0\|_{\mathcal{H}} - \|S_{Z^0}\|_{\mathcal{H}}| \left(\|u_0\|_{\mathcal{H}} + \|S_{Z^0}\|_{\mathcal{H}}\right) \\
&\le \|u_0 - S_{Z^0}\|_{\mathcal{H}} \left(\|u_0 - S_{Z^0}\|_{\mathcal{H}} + 2\|S_{Z^0}\|_{\mathcal{H}}\right) \\
&\le \varepsilon^2 \left(\varepsilon^2 + \frac{2}{\sqrt{3}}\sum_{i=1}^N c_i\right).
\end{aligned}
\tag{4.103}
$$

Thus, combining (4.99) and (4.103), it holds

$$
\begin{aligned}
\left|E(u_0) - \sum_{i=1}^{N} E(\varphi_{c_i})\right| &\le |E(u_0) - E(S_{Z^0})| + \left|E(S_{Z^0}) - \sum_{i=1}^{N} E(\varphi_{c_i})\right| \\
&\le \varepsilon^2(\varepsilon^2 + O(1)) + O(e^{-L/4}) \\
&\le O(\varepsilon^2) + O(e^{-L/4}).
\end{aligned}
$$

For the second estimate, using the exponential decay of φ_{c_i}'s and the ϕ_i's, and the definition of $E_i(\cdot)$, we have

$$
\begin{aligned}
&|E_i(u_0) - E(\varphi_{c_i})| \\
&\quad \le \left| \|u_0\|^2_{\mathcal{H}(\Omega_i)} - \|\varphi_{c_i}\|^2_{\mathcal{H}(\Omega_i)} \right| + O(e^{-\sqrt{L}}) \\
&\quad = \left| \|u_0\|_{\mathcal{H}(\Omega_i)} - \|\varphi_{c_i}\|_{\mathcal{H}(\Omega_i)} \right| \left(\|u_0\|_{\mathcal{H}(\Omega_i)} + \|\varphi_{c_i}\|_{\mathcal{H}(\Omega_i)} \right) + O(e^{-\sqrt{L}}) \\
&\quad \le \left(\|u_0 - S_{Z^0}\|_{\mathcal{H}(\Omega_i)} + \sum_{\substack{1\le j\le N \\ j\neq i}} \|\varphi_{c_j}\|_{\mathcal{H}(\Omega_i)} \right) \left(\|u_0 - S_{Z^0}\|_{\mathcal{H}} + \frac{2}{\sqrt{3}} \sum_{j=1}^{N} c_j \right) + O(e^{-\sqrt{L}}) \\
&\quad \le \left(\varepsilon^2 + O(e^{-L/8})\right) \left(\varepsilon^2 + O(1)\right) + O(e^{-\sqrt{L}}) \\
&\quad \le O(\varepsilon^2) + O(e^{-\sqrt{L}}).
\end{aligned}
$$

Similarly, for the third estimate, using the exponential decay of φ_{c_i}'s and the ϕ_i's, and the definition of $F_i(\cdot)$, we have

$$
\begin{aligned}
&|F_i(u_0) - F(\varphi_{c_i})| \\
&\quad \le \left| \int_{\Omega_i} \left(u_0^3 - \varphi_{c_i}^3\right) \right| + O(e^{-\sqrt{L}}) \\
&\quad \le \int_{\Omega_i} |u_0 - \varphi_{c_i}| \left(u_0^2 + u_0\varphi_{c_i} + \varphi_{c_i}^2\right) + O(e^{-\sqrt{L}}) \\
&\quad \le \|u_0 - \varphi_{c_i}\|_{L^2(\Omega_i)} \left(\int_{\Omega_i} \left(u_0^2 + u_0\varphi_{c_i} + \varphi_{c_i}^2\right)^2 \right)^{1/2} + O(e^{-\sqrt{L}}) \\
&\quad \le \left(\|u_0 - S_{Z^0}\|_{L^2(\Omega_i)} + \sum_{\substack{1\le j\le N \\ j\neq i}} \|\varphi_{c_j}\|_{L^2(\Omega_i)} \right) \cdot O(1) + O(e^{-\sqrt{L}}) \\
&\quad \le \left(\varepsilon^2 + O(e^{-L/8})\right) \cdot O(1) + O(e^{-\sqrt{L}}) \\
&\quad \le O(\varepsilon^2) + O(e^{-\sqrt{L}}).
\end{aligned}
$$

This proves the lemma.

4.4 End of the proof of Theorem 1.1

Let $u \in \mathcal{X}([0,T[)$, with $0 < T \le +\infty$, be a solution of the DP equation satisfying (1.12)-(1.14) and (4.2) for some $t_0 \in]0,T[$. Let $M_1^i = v(t_0, \xi_1^i(t_0)) = \max_{x\in J_i} v(t_0,x)$, with J_i's as in (4.8), and $\delta_i = c_i/6 - M_1^i$. First, from (4.7) and (4.9), we know that for $i = 2,\dots,N$,

$$
\xi_1^i(t_0) - \xi_1^{i-1}(t_0) \ge 2L/3 > L/2\,.
$$

Applying (4.95) and (4.100) with $u(t_0)$, we get

$$\left\| u(t_0) - \sum_{i=1}^{N} \varphi_{c_i}(\cdot - \xi_1^i(t_0)) \right\|_{\mathcal{H}}^2 \leq 4 \sum_{i=1}^{N} c_i \delta_i + O(\varepsilon^2) + O(e^{-L/4}). \tag{4.104}$$

In the same way, from (4.91) we get

$$F_i(u(t_0)) \leq 18 M_1^i E_i(u(t_0)) - 72 (M_1^i)^3 + O(L^{-1/2}),$$

which leads to

$$F(u(t_0)) = \sum_{i=1}^{N} F_i(u(t_0)) \leq 18 \sum_{i=1}^{N} M_1^i E_i(u(t_0)) - 72 \sum_{i=1}^{N} (M_1^i)^3 + O(L^{-1/2}), \tag{4.105}$$

by summing over $i \in \{1, \ldots, N\}$.

Now, we will use the following notation: for a function $f : \mathbb{R}_+ \mapsto \mathbb{R}$, we set

$$\Delta_0^{t_0} f = f(t_0) - f(0). \tag{4.106}$$

From (4.105) and the fact that $E(\cdot)$ and $F(\cdot)$ are conservation laws for u, we obtain

$$\begin{aligned} 0 = \Delta_0^{t_0} F(u) = \sum_{i=1}^{N} \Delta_0^{t_0} F_i(u) &\leq 18 \sum_{i=1}^{N} M_1^i \Delta_0^{t_0} E_i(u) \\ &+ \sum_{i=1}^{N} \left[-72 (M_1^i)^3 + 18 M_1^i E_i(u_0) - F_i(u_0) \right] + O(L^{-1/2}). \end{aligned} \tag{4.107}$$

Note that, from (4.101) and (4.102), for $0 < \varepsilon < \varepsilon_0$ and $L > L_0 > 0$ with $\varepsilon_0 \ll 1$ and $L_0 \gg 1$, it holds

$$\sum_{i=1}^{N} \left[-72 (M_1^i)^3 + 18 M_1^i E_i(u_0) - F_i(u_0) \right] = -72 \sum_{i=1}^{N} \delta_i^2 \left(M_1^i + \frac{c_i}{3} \right) + O(\varepsilon^2) + O(e^{-\sqrt{L}}). \tag{4.108}$$

Combining (4.107) and (4.108), we get

$$\sum_{i=1}^{N} \delta_i^2 \left(M_1^i + \frac{c_i}{3} \right) \leq \frac{1}{4} \sum_{i=1}^{N} M_1^i \Delta_0^{t_0} E_i(u) + O(\varepsilon^2) + O(L^{-1/2}),$$

and using the Abel transformation with $M_1^0 = 0$, we obtain

$$\sum_{i=1}^{N} \delta_i^2 \left(M_1^i + \frac{c_i}{3} \right) \leq \frac{1}{4} \sum_{i=1}^{N} (M_1^i - M_1^{i-1}) \Delta_0^t \mathcal{J}_{i,K} + O(\varepsilon^2) + O(L^{-1/2}), \tag{4.109}$$

where $\mathcal{J}_{i,K}(t)$ is defined in (4.22). From (4.2) we know that $u(t_0) \in U(\gamma, L/2)$, on account of Lemma 4.1 there exists $\tilde{X} = (\tilde{x}_1, \ldots, \tilde{x}_N)$ with $\tilde{x}_i \in J_i$ such that $E\left(u(t_0) - S_{\tilde{X}}\right) \leq O(\gamma^2)$, where $S_{\tilde{X}}$ is defined in (4.11). Recalling that $v(t_0, \xi_1^i(t_0)) = \max_{x \in J_i} v(t_0, x)$ and using (4.95), we obtain $E\left(u(t_0) - S_{\xi_1}\right) \leq O(\gamma^2) + O(e^{-L/4})$, with $\xi_1 = (\xi_1^1, \ldots, \xi_1^N)$. From (4.5), we deduce that

$$\left\| v(t_0) - \sum_{j=1}^{N} \rho_{c_j}(\cdot - \xi_1^j(t_0)) \right\|_{L^\infty(\mathbb{R})} \leq O(\gamma) + O(e^{-L/8}).$$

Thus, we infer that

$$v(x) = \sum_{j=1}^{N} \rho_{c_j}(\cdot - \xi_1^j(t_0)) + O(\gamma) + O(e^{-L/8}), \quad \forall x \in \mathbb{R},$$

and applying this formula with $x = \xi_1^i(t_0)$ and using that $\xi_1^j(t_0) - \xi_1^{j-1}(t_0) > L/2$, we get

$$\begin{aligned} v(\xi_1^i(t_0)) &= \sum_{j=1}^{N} \rho_{c_j}(\xi_1^i(t_0) - \xi_1^j(t_0)) + O(\gamma) + O(e^{-L/8}) \\ &= \frac{c_i}{6} + \sum_{\substack{1 \le j \le N \\ j \ne i}} \rho_{c_j}(\xi_1^i(t_0) - \xi_1^j(t_0)) + O(\gamma) + O(e^{-L/8}) \\ &= \frac{c_i}{6} + O(\gamma) + O(e^{-L/8}). \end{aligned}$$

We take $\gamma = A(\sqrt{\varepsilon} + L^{-1/8})$, then $M_1^i = c_i/6 + O(\sqrt{\varepsilon}) + O(L^{-1/8})$. Therefore, for $0 < \varepsilon < \varepsilon_0$ and $L > L_0 > 0$, with $\varepsilon_0 \ll 1$ and $L_0 \gg 1$, it holds

$$0 < M_1^1 < M_1^2 < \ldots < M_1^N. \tag{4.110}$$

Combining (4.109), (4.110) and using the monotonicity estimate (4.24), it holds

$$\sum_{i=1}^{N} \delta_i^2 \left(M_1^i + \frac{c_i}{3} \right) \le O(\varepsilon^2) + O(L^{-1/8}).$$

Therefore, using that $(M_1^i + c_i/3)^{-1} < 3/c_i$, there exists $C > 0$ only depending on $(c_i)_{i=1}^N$ such that

$$\delta_i \le C(\varepsilon + L^{-1/4}), \quad i = 1, \ldots, N. \tag{4.111}$$

Now, combining (4.104) and (4.111), we obtain

$$\left\| u(t_0) - \sum_{i=1}^{N} \varphi_{c_i}(\cdot - \xi_1^i(t_0)) \right\|_{\mathcal{H}} \le C(\sqrt{\varepsilon} + L^{-1/8}),$$

and the theorem follows by choosing $A = 2C$.

Remark 4.2 (The role of the number of extrema)**.** In the case where $v = (4 - \partial_x^2)^{-1} u$ admits a countable infinite number of local maximal values on some $[\alpha_i, \beta_i]$ (see (4.57)), with $i \in \{1, \ldots, N\}$, it suffices to change the finite sums over j by infinite sums in Lemmas 3.4-3.5 and Lemmas 4.3-4.4.

Appendix. Proof of Lemma 4.2

The aim of this subsection is to prove Lemma 4.2. Let us first assume that u is smooth solution. The case $u \in \mathcal{X}([0, T[)$ will follow by a density argument.

We compute the time variation of the following energy:

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}} y v g &= \int_{\mathbb{R}} y_t v g + \int_{\mathbb{R}} y v_t g \\ &= I + J. \end{aligned}$$

Applying the operator $(1 - \partial_x^2)$ on both sides of equation (1.7), we get

$$y_t = -\frac{1}{2}(1 - \partial_x^2)\partial_x u^2 - \frac{3}{2}\partial_x u^2$$

and substituting y_t by this value, I becomes

$$\begin{aligned} I &= -\frac{1}{2}\int_{\mathbb{R}} \left[(1-\partial_x^2)\partial_x u^2\right] vg - \frac{3}{2}\int_{\mathbb{R}} \left[\partial_x(u^2)\right] vg \\ &= I_1 + I_2. \end{aligned}$$

By computing

$$I_2 = \frac{3}{2}\int_{\mathbb{R}} u^2\partial_x(vg) = \frac{3}{2}\int_{\mathbb{R}} u^2 v_x g + \frac{3}{2}\int_{\mathbb{R}} u^2 v g' \tag{4.112}$$

and

$$\begin{aligned} I_1 &= \frac{1}{2}\int_{\mathbb{R}} \left[(1-\partial_x^2)u^2\right] \partial_x(vg) \\ &= \frac{1}{2}\int_{\mathbb{R}} \left[(1-\partial_x^2)u^2\right] v_x g + \frac{1}{2}\int_{\mathbb{R}} \left[(1-\partial_x^2)u^2\right] vg' \\ &= I_3 + I_4 \end{aligned}$$

with

$$\begin{aligned} I_3 &= \frac{1}{2}\int_{\mathbb{R}} u^2 v_x g - \frac{1}{2}\int_{\mathbb{R}} \partial_x^2(u^2) v_x g \\ &= \frac{1}{2}\int_{\mathbb{R}} u^2 v_x g + \frac{1}{2}\int_{\mathbb{R}} \partial_x(u^2)\partial_x(v_x g) \\ &= \frac{1}{2}\int_{\mathbb{R}} u^2 v_x g + \frac{1}{2}\int_{\mathbb{R}} \partial_x(u^2) v_{xx} g + \frac{1}{2}\int_{\mathbb{R}} \partial_x(u^2) v_x g' \\ &= \frac{1}{2}\int_{\mathbb{R}} u^2 v_x g - \frac{1}{2}\int_{\mathbb{R}} u^2\partial_x(v_{xx} g) - \frac{1}{2}\int_{\mathbb{R}} u^2\partial_x(v_x g') \\ &= \frac{1}{2}\int_{\mathbb{R}} u^2 v_x g - \frac{1}{2}\int_{\mathbb{R}} u^2 v_{xxx} g - \int_{\mathbb{R}} u^2 v_{xx} g' - \frac{1}{2}\int_{\mathbb{R}} u^2 v_x g'' \end{aligned} \tag{4.113}$$

and

$$\begin{aligned} I_4 &= \frac{1}{2}\int_{\mathbb{R}} u^2 v g' - \frac{1}{2}\int_{\mathbb{R}} \partial_x^2(u^2) v g' \\ &= \frac{1}{2}\int_{\mathbb{R}} u^2 v g' + \frac{1}{2}\int_{\mathbb{R}} \partial_x(u^2)\partial_x(v g') \\ &= \frac{1}{2}\int_{\mathbb{R}} u^2 v g' + \frac{1}{2}\int_{\mathbb{R}} \partial_x(u^2) v_x g' + \frac{1}{2}\int_{\mathbb{R}} \partial_x(u^2) v g'' \\ &= \frac{1}{2}\int_{\mathbb{R}} u^2 v g' - \frac{1}{2}\int_{\mathbb{R}} u^2\partial_x(v_x g') - \frac{1}{2}\int_{\mathbb{R}} u^2\partial_x(v g'') \\ &= \frac{1}{2}\int_{\mathbb{R}} u^2 v g' - \frac{1}{2}\int_{\mathbb{R}} u^2 v_{xx} g' - \int_{\mathbb{R}} u^2 v_x g'' - \frac{1}{2}\int_{\mathbb{R}} u^2 v g'''. \end{aligned} \tag{4.114}$$

Adding (4.113) and (4.114), we get

$$\begin{aligned} I_1 &= \frac{1}{2}\int_{\mathbb{R}} u^2 v_x g - \frac{1}{2}\int_{\mathbb{R}} u^2 v_{xxx} g - \int_{\mathbb{R}} u^2 v_{xx} g' - \frac{1}{2}\int_{\mathbb{R}} u^2 v_x g'' \\ &\qquad + \frac{1}{2}\int_{\mathbb{R}} u^2 v g' - \frac{1}{2}\int_{\mathbb{R}} u^2 v_{xx} g' - \int_{\mathbb{R}} u^2 v_x g'' - \frac{1}{2}\int_{\mathbb{R}} u^2 v g''' \\ &= \frac{1}{2}\int_{\mathbb{R}} u^2 v_x g - \frac{1}{2}\int_{\mathbb{R}} u^2 v_{xxx} g - \frac{3}{2}\int_{\mathbb{R}} u^2 v_{xx} g' \\ &\qquad + \frac{1}{2}\int_{\mathbb{R}} u^2 v g' - \frac{3}{2}\int_{\mathbb{R}} u^2 v_x g'' - \frac{1}{2}\int_{\mathbb{R}} u^2 v g''' \end{aligned} \tag{4.115}$$

and adding (4.112) and (4.115), we get

$$
\begin{aligned}
I &= 2\int_{\mathbb{R}} u^2 v_x g - \frac{1}{2}\int_{\mathbb{R}} u^2 v_{xxx} g - \frac{3}{2}\int_{\mathbb{R}} u^2 v_{xx} g' \\
&\quad + 2\int_{\mathbb{R}} u^2 v g' - \frac{3}{2}\int_{\mathbb{R}} u^2 v_x g'' - \frac{1}{2}\int_{\mathbb{R}} u^2 v g''' \\
&= \frac{1}{2}\int_{\mathbb{R}} u^2 \left[(4-\partial_x^2)v_x\right] g + \frac{3}{2}\int_{\mathbb{R}} u^2 \left[\left(\frac{4}{3}-\partial_x^2\right) v\right] g' \\
&\quad - \frac{3}{2}\int_{\mathbb{R}} u^2 v_x g'' - \frac{1}{2}\int_{\mathbb{R}} u^2 v g'''.
\end{aligned}
$$

The first two integrals give us

$$
\frac{1}{2}\int_{\mathbb{R}} u^2 \left[(4-\partial_x^2)v_x\right] g = \frac{1}{2}\int_{\mathbb{R}} u^2 u_x g = \frac{1}{6}\int_{\mathbb{R}} \partial_x(u^3) g = -\frac{1}{6}\int_{\mathbb{R}} u^3 g'
$$

and

$$
\frac{3}{2}\int_{\mathbb{R}} u^2 \left[\left(\frac{4}{3}-\partial_x^2\right) v\right] g' = \frac{3}{2}\int_{\mathbb{R}} u^2 \left[(4-\partial_x^2)v - \frac{8}{3}v\right] g' = \frac{3}{2}\int_{\mathbb{R}} u^3 g' - 4\int_{\mathbb{R}} u^2 v g'.
$$

Finally, we obtain

$$
I = \frac{4}{3}\int_{\mathbb{R}} u^3 g' - 4\int_{\mathbb{R}} u^2 v g' - \frac{3}{2}\int_{\mathbb{R}} u^2 v_x g'' - \frac{1}{2}\int_{\mathbb{R}} u^2 v g'''. \tag{4.116}
$$

We set $h = (1-\partial_x^2)^{-1}u^2$. Applying the operator $(4-\partial_x^2)^{-1}$ on both sides of equation (1.7) and using (4.16), we get

$$
v_t = -\frac{1}{2}(1-\partial_x^2)^{-1}\partial_x u^2 = -\frac{1}{2}h_x. \tag{4.117}
$$

Substituting v_t by this value, J becomes

$$
\begin{aligned}
J &= -\frac{1}{2}\int_{\mathbb{R}} y h_x g \\
&= \frac{1}{2}\int_{\mathbb{R}} y_x h g + \frac{1}{2}\int_{\mathbb{R}} y h g' \\
&= J_1 + J_2.
\end{aligned}
$$

By computing

$$
\begin{aligned}
J_2 &= \frac{1}{2}\int_{\mathbb{R}} (u-u_{xx}) h g' \\
&= \frac{1}{2}\int_{\mathbb{R}} u h g' - \frac{1}{2}\int_{\mathbb{R}} u_{xx} h g' \\
&= \frac{1}{2}\int_{\mathbb{R}} u h g' + \frac{1}{2}\int_{\mathbb{R}} u_x \partial_x(h g') \\
&= \frac{1}{2}\int_{\mathbb{R}} u h g' + \frac{1}{2}\int_{\mathbb{R}} u_x h_x g' + \frac{1}{2}\int_{\mathbb{R}} u_x h g'' \\
&= \frac{1}{2}\int_{\mathbb{R}} u h g' - \frac{1}{2}\int_{\mathbb{R}} u \partial_x(h_x g') - \frac{1}{2}\int_{\mathbb{R}} u \partial_x(h g'') \\
&= \frac{1}{2}\int_{\mathbb{R}} u h g' - \frac{1}{2}\int_{\mathbb{R}} u h_{xx} g' - \int_{\mathbb{R}} u h_x g'' - \frac{1}{2}\int_{\mathbb{R}} u h g''' \qquad (4.118)
\end{aligned}
$$

and

$$
\begin{aligned}
J_1 &= \frac{1}{2}\int_{\mathbb{R}} (u_x - u_{xxx})hg \\
&= \frac{1}{2}\int_{\mathbb{R}} u_x hg - \frac{1}{2}\int_{\mathbb{R}} u_{xxx} hg \\
&= \frac{1}{2}\int_{\mathbb{R}} u_x hg + \frac{1}{2}\int_{\mathbb{R}} u_{xx}\partial_x(hg) \\
&= \frac{1}{2}\int_{\mathbb{R}} u_x hg + \frac{1}{2}\int_{\mathbb{R}} u_{xx} h_x g + \frac{1}{2}\int_{\mathbb{R}} u_{xx} hg' \\
&= J_3 + J_4 + J_5
\end{aligned}
$$

with

$$
J_3 = -\frac{1}{2}\int_{\mathbb{R}} u\partial_x(hg) = -\frac{1}{2}\int_{\mathbb{R}} uh_x g - \frac{1}{2}\int_{\mathbb{R}} uhg' \,, \tag{4.119}
$$

$$
\begin{aligned}
J_5 &= -\frac{1}{2}\int_{\mathbb{R}} u_x\partial_x(hg') \\
&= -\frac{1}{2}\int_{\mathbb{R}} u_x h_x g' - \frac{1}{2}\int_{\mathbb{R}} u_x hg'' \\
&= \frac{1}{2}\int_{\mathbb{R}} u\partial_x(h_x g') + \frac{1}{2}\int_{\mathbb{R}} u\partial_x(hg'') \\
&= \frac{1}{2}\int_{\mathbb{R}} uh_{xx}g' + \int_{\mathbb{R}} uh_x g'' + \frac{1}{2}\int_{\mathbb{R}} uhg'''
\end{aligned} \tag{4.120}
$$

and

$$
\begin{aligned}
J_4 &= -\frac{1}{2}\int_{\mathbb{R}} u_x\partial_x(h_x g) \\
&= -\frac{1}{2}\int_{\mathbb{R}} u_x h_{xx} g - \frac{1}{2}\int_{\mathbb{R}} u_x h_x g' \\
&= \frac{1}{2}\int_{\mathbb{R}} u\partial_x(h_{xx} g) + \frac{1}{2}\int_{\mathbb{R}} u\partial_x(h_x g') \\
&= \frac{1}{2}\int_{\mathbb{R}} uh_{xxx} g + \int_{\mathbb{R}} uh_{xx} g' + \frac{1}{2}\int_{\mathbb{R}} uh_x g''.
\end{aligned} \tag{4.121}
$$

Adding (4.118)-(4.121), we get

$$
J = -\frac{1}{2}\int_{\mathbb{R}} uh_x g + \frac{1}{2}\int_{\mathbb{R}} uh_{xxx} g + \int_{\mathbb{R}} uh_{xx} g' + \frac{1}{2}\int_{\mathbb{R}} uh_x g''.
$$

Using that $h_{xx} = -u^2 + h$ and $h_{xxx} = -2uu_x + h_x$, we have

$$
\int_{\mathbb{R}} uh_{xx} g' = \int_{\mathbb{R}} u(-u^2 + h)g' = -\int_{\mathbb{R}} u^3 g' + \int_{\mathbb{R}} uhg'
$$

and

$$
\begin{aligned}
\frac{1}{2}\int_{\mathbb{R}} uh_{xxx} g &= \frac{1}{2}\int_{\mathbb{R}} u(-2uu_x + h_x)g \\
&= -\int_{\mathbb{R}} u^2 u_x g + \frac{1}{2}\int_{\mathbb{R}} uh_x g \\
&= -\frac{1}{3}\int_{\mathbb{R}} \partial_x(u^3) g + \frac{1}{2}\int_{\mathbb{R}} uh_x g \\
&= \frac{1}{3}\int_{\mathbb{R}} u^3 g' + \frac{1}{2}\int_{\mathbb{R}} uh_x g.
\end{aligned}
$$

At this stage it is worth noticing that the term $\int_{\mathbb{R}} uh_x g$ cancels with the one in J. Finally, we obtain

$$J = -\frac{2}{3}\int_{\mathbb{R}} u^3 g' + \int_{\mathbb{R}} uhg' + \frac{1}{2}\int_{\mathbb{R}} uh_x g''. \tag{4.122}$$

Combining (4.116) and (4.122), we get

$$\frac{d}{dt}\int_{\mathbb{R}} yvg = \frac{2}{3}\int_{\mathbb{R}} u^3 g' - 4\int_{\mathbb{R}} u^2 vg' - \frac{3}{2}\int_{\mathbb{R}} u^2 v_x g'' - \frac{1}{2}\int_{\mathbb{R}} u^2 vg''' + \int_{\mathbb{R}} uhg' + \frac{1}{2}\int_{\mathbb{R}} uh_x g''. \tag{4.123}$$

Now, substituting u by $4v - v_{xx}$ and using integration by parts, we rewrite the energy as

$$\begin{aligned}
\int_{\mathbb{R}} yvg &= \int_{\mathbb{R}} v\left[(1-\partial_x^2)(4v - v_{xx})\right] g \\
&= 4\int_{\mathbb{R}} v^2 g - 5\int_{\mathbb{R}} vv_{xx} g + \int_{\mathbb{R}} v(\partial_x^4 v) g \\
&= 4\int_{\mathbb{R}} v^2 g + K_1 + K_2.
\end{aligned}$$

By computing

$$\begin{aligned}
K_1 &= 5\int_{\mathbb{R}} \partial_x(vg) v_x \\
&= 5\int_{\mathbb{R}} v_x^2 g + 5\int_{\mathbb{R}} vv_x g' \\
&= 5\int_{\mathbb{R}} v_x^2 g + \frac{5}{2}\int_{\mathbb{R}} \partial_x(v^2) g' \\
&= 5\int_{\mathbb{R}} v_x^2 g - \frac{5}{2}\int_{\mathbb{R}} v^2 g''
\end{aligned} \tag{4.124}$$

and

$$\begin{aligned}
K_2 &= -\int_{\mathbb{R}} \partial_x(vg) v_{xxx} \\
&= -\int_{\mathbb{R}} v_x v_{xxx} g - \int_{\mathbb{R}} vv_{xxx} g' \\
&= K_3 + K_4
\end{aligned} \tag{4.125}$$

with

$$\begin{aligned}
K_3 &= \int_{\mathbb{R}} \partial_x(v_x g) v_{xx} \\
&= \int_{\mathbb{R}} v_{xx}^2 g + \int_{\mathbb{R}} v_x v_{xx} g' \\
&= \int_{\mathbb{R}} v_{xx}^2 g + \frac{1}{2}\int_{\mathbb{R}} \partial_x(v_x^2) g' \\
&= \int_{\mathbb{R}} v_{xx}^2 g - \frac{1}{2}\int_{\mathbb{R}} v_x^2 g''
\end{aligned} \tag{4.126}$$

and

$$
\begin{aligned}
K_4 &= \int_{\mathbb{R}} \partial_x(vg')v_{xx} \\
&= \int_{\mathbb{R}} v_x v_{xx} g' + \int_{\mathbb{R}} v v_{xx} g'' \\
&= \frac{1}{2}\int_{\mathbb{R}} \partial_x(v_x^2)g' - \int_{\mathbb{R}} \partial_x(vg'')v_x \\
&= -\frac{1}{2}\int_{\mathbb{R}} v_x^2 g'' - \int_{\mathbb{R}} v_x^2 g'' - \int_{\mathbb{R}} v v_x g''' \\
&= -\frac{3}{2}\int_{\mathbb{R}} v_x^2 g'' - \frac{1}{2}\int_{\mathbb{R}} \partial_x(v^2)g''' \\
&= -\frac{3}{2}\int_{\mathbb{R}} v_x^2 g'' + \frac{1}{2}\int_{\mathbb{R}} v^2 g^{(4)}.
\end{aligned} \tag{4.127}
$$

Combining (4.124)-(4.127), we get

$$
\int_{\mathbb{R}} yvg = \int_{\mathbb{R}} \left(4v^2 + 5v_x^2 + v_{xx}^2\right) g + \frac{1}{2}\int_{\mathbb{R}} v^2(g^{(4)} - 5g'') - 2\int_{\mathbb{R}} v_x^2 g''
$$

and then

$$
\frac{d}{dt}\int_{\mathbb{R}} yvg = \frac{d}{dt}\int_{\mathbb{R}} \left(4v^2 + 5v_x^2 + v_{xx}^2\right) g + L_1 + L_2. \tag{4.128}
$$

Using (4.117), we have

$$
L_1 = \int_{\mathbb{R}} v v_t (g^{(4)} - 5g'') = \frac{5}{2}\int_{\mathbb{R}} v h_x g'' - \frac{1}{2}\int_{\mathbb{R}} v h_x g^{(4)} \tag{4.129}
$$

and

$$
L_2 = -4\int_{\mathbb{R}} v_x v_{tx} g'' = 2\int_{\mathbb{R}} v_x h_{xx} g'' = -2\int_{\mathbb{R}} u^2 v_x g'' + 2\int_{\mathbb{R}} v_x h g''. \tag{4.130}
$$

Lemma 4.2 follows by combining (4.123) and (4.128)-(4.130).

Acknowledgements. The author would like to thank his PhD advisor Luc Molinet for his help and his careful reading of this manuscript.

References

[1] A. Degasperis, D. D. Kholm, and A. N. I. Khon. A new integrable equation with peakon solutions. *Teoret. Mat. Fiz.*, 133(2):170–183, 2002.

[2] Khaled El Dika and Luc Molinet. Stability of multi antipeakon-peakons profile. *Discrete Contin. Dyn. Syst. Ser. B*, 12(3):561–577, 2009.

[3] Khaled El Dika and Luc Molinet. Stability of multipeakons. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 26(4):1517–1532, 2009.

[4] Joachim Escher, Yue Liu, and Zhaoyang Yin. Global weak solutions and blow-up structure for the Degasperis-Procesi equation. *J. Funct. Anal.*, 241(2):457–485, 2006.

[5] Zhiwu Lin and Yue Liu. Stability of peakons for the Degasperis-Procesi equation. *Comm. Pure Appl. Math.*, 62(1):125–146, 2009.

[6] Yue Liu and Zhaoyang Yin. Global existence and blow-up phenomena for the Degasperis-Procesi equation. *Comm. Math. Phys.*, 267(3):801–820, 2006.

[7] Yvan Martel, Frank Merle, and Tai-Peng Tsai. Stability and asymptotic stability in the energy space of the sum of N solitons for subcritical gKdV equations. *Comm. Math. Phys.*, 231(2):347–373, 2002.