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# Stability in the Energy Space of the Sum of $N$ Peakons for the Degasperis-Procesi Equation

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## Abstract

The Degasperis-Procesi equation possesses well-known peaked solitary waves that are called peakons. Their stability has been established by Lin and Liu in [5]. In this paper, we localize the proof (in some suitable sense detailed in Section 3) of the stability of a single peakon. Thanks to this, we extend the result of stability to the sum of  $N$  peakons traveling to the right with respective speeds  $c_1, \dots, c_N$ , such that the difference between consecutive speeds  $c_i - c_{i-1}$  is large enough, to cancel all interactions.

## 1 Introduction

The Degasperis-Procesi (DP) equation

$$u_t - u_{txx} + 4uu_x = 3u_x u_{xx} + uu_{xxx}, \quad (t, x) \in \mathbb{R}_+^* \times \mathbb{R} \quad (1.1)$$

is completely integrable (see [1]) and possesses among others the following invariants

$$E(u) = \int_{\mathbb{R}} yv \quad \text{and} \quad F(u) = \int_{\mathbb{R}} u^3, \quad (1.2)$$

where  $y = (1 - \partial_x^2)u$  and  $v = (4 - \partial_x^2)^{-1}u$ . Substituting  $u$  by  $4v - v_{xx}$  in (1.2) and using integration by parts, the conservation laws can be rewritten as

$$E(u) = \int_{\mathbb{R}} (4v^2 + 5v_x^2 + v_{xx}^2) \quad \text{and} \quad F(u) = \int_{\mathbb{R}} (-v_{xx}^3 + 12vv_{xx}^2 - 48v^2v_{xx} + 64v^3). \quad (1.3)$$

One can see that the conservation law  $E(\cdot)$  is equivalent to  $\|\cdot\|_{L^2(\mathbb{R})}^2$ . Indeed, using integration by parts

$$\|u\|_{L^2(\mathbb{R})}^2 = \int_{\mathbb{R}} u^2 = \int_{\mathbb{R}} (4v - v_{xx})^2 = \int_{\mathbb{R}} (16v^2 + 8v_x^2 + v_{xx}^2) \leq 4E(u), \quad (1.4)$$

and applying Plancherel-Parseval identity

$$E(u) = \int_{\mathbb{R}} yv = \int_{\mathbb{R}} \frac{1 + \omega^2}{4 + \omega^2} |\widehat{u}(\omega)|^2 \leq \int_{\mathbb{R}} |\widehat{u}(\omega)|^2 = \|u\|_{L^2(\mathbb{R})}^2, \quad (1.5)$$

where  $\widehat{u}$  denotes the Fourier transform of  $u$ . In the sequel we will denote

$$\|u\|_{\mathcal{H}} = \sqrt{E(u)}. \quad (1.6)$$

Note that, by reversing the operator  $(1 - \partial_x^2)$  in (1.1), the DP equation can be rewritten in conservation form as

$$u_t + \frac{1}{2}\partial_x u^2 + \frac{3}{2}(1 - \partial_x^2)^{-1}\partial_x u^2 = 0, \quad (t, x) \in \mathbb{R}_+^* \times \mathbb{R}. \quad (1.7)$$

The DP equation possesses solitary waves called *peakons* (see Figure 1a) and defined by

$$u(t, x) = \varphi_c(x - ct) = c\varphi(x - ct) = ce^{-|x-ct|}, \quad c \in \mathbb{R}, \quad (1.8)$$

but they are not smooth since  $\varphi_c \notin C^1(\mathbb{R})$  (see Figure 1b). The peakons are only global weak solutions of (1.7). It means, for any smooth test function  $\phi \in C^\infty(\mathbb{R}_+ \times \mathbb{R})$ , it holds

$$\begin{aligned} & \int_0^{+\infty} \int_{\mathbb{R}} \varphi_c(x - ct)\phi_t(t, x)dt dx + \frac{1}{2} \int_0^{+\infty} \int_{\mathbb{R}} \varphi_c^2(x - ct)\phi_x(t, x)dt dx \\ & + \frac{3}{2} \int_0^{+\infty} \int_{\mathbb{R}} (1 - \partial_x^2)^{-1}\varphi_c^2(x - ct)\phi_x(t, x)dt dx + \int_{\mathbb{R}} \varphi_c(x)\phi(0, x)dx = 0. \end{aligned}$$

The goal of our work is to prove that ordered trains of peakons are stable under small perturbations in the energy space  $\mathcal{H}$  (equivalent to  $L^2$ ).

**Definition 1.1** (Stability). *Let  $c > 0$  be given. The peakon  $\varphi_c$  is said stable in  $\mathcal{H}$ , if for all  $\delta > 0$ , there exists  $\varepsilon > 0$  such that if*

$$\|u_0 - \varphi_c\|_{\mathcal{H}} \leq \varepsilon, \quad (1.9)$$

*then for all  $t \geq 0$ , there exists  $\xi(t)$  such that*

$$\|u(t, \cdot) - \varphi_c(\cdot - \xi(t))\|_{\mathcal{H}} \leq \delta, \quad (1.10)$$

*where  $u$  is the solution to (1.1) emanating from  $u_0$ .*

Lin and Liu proved in [5] the stability of a single peakon under the additional condition that  $(1 - \partial_x^2)u_0 \in \mathcal{M}^+(\mathbb{R})$ . Using this result and the general strategy introduced by Martel, Merle and Tsai in [7] for the generalized Korteweg-de Vries (gKdV) equation and adapted by Dika and Molinet in [3] and [2] for the Camassa-Holm (CH) equation, we prove here the stability of the sum of  $N$  peakons for the DP equation.

Before stating the main result we introduce the function space where will live our class of solutions to the equation. For  $I$  a finite or infinite time interval of  $\mathbb{R}_+$ , we denote by  $\mathcal{X}(I)$  the function space <sup>1</sup>

$$\mathcal{X}(I) = \{u \in C(I; H^1(\mathbb{R})) \cap L^\infty(I; W^{1,1}(\mathbb{R})), u_x \in L^\infty(I; BV(\mathbb{R}))\}. \quad (1.11)$$

The main result of the present paper is the following theorem.

**Theorem 1.1** (Stability of the sum of  $N$  peakons). *Let be given  $N$  velocities  $c_1, \dots, c_N$  such that  $0 < c_1 < \dots < c_N$ . Let  $u \in \mathcal{X}([0, T])$ , with  $0 < T \leq +\infty$ , be a solution of the DP equation. There exist  $C > 0$ ,  $L_0 > 0$  and  $\varepsilon_0 > 0$  only depending on the speeds  $(c_i)_{i=1}^N$ , such that if*

$$y_0 = (1 - \partial_x^2)u_0 \in \mathcal{M}^+(\mathbb{R}) \quad (1.12)$$

*and*

$$\left\| u_0 - \sum_{i=1}^N \varphi_{c_i}(\cdot - z_i^0) \right\|_{\mathcal{H}} \leq \varepsilon^2, \quad \text{with } 0 < \varepsilon < \varepsilon_0, \quad (1.13)$$

<sup>1</sup> $W^{1,1}(\mathbb{R})$  is the space of  $L^1(\mathbb{R})$  functions with derivatives in  $L^1(\mathbb{R})$  and  $BV(\mathbb{R})$  is the space of function with bounded variation.

for some  $z_1^0, \dots, z_N^0$  satisfying

$$z_1^0 < \dots < z_N^0 \quad \text{and} \quad z_i^0 - z_{i-1}^0 \geq L, \quad \text{with} \quad L > L_0 > 0, \quad i = 2, \dots, N, \quad (1.14)$$

then there exist  $\xi_1^1(t), \dots, \xi_1^N(t)$  such that

$$\left\| u(t) - \sum_{i=1}^N \varphi_{c_i}(\cdot - \xi_1^i(t)) \right\|_{\mathcal{H}} \leq C(\sqrt{\varepsilon} + L^{-1/8}), \quad \forall t \in [0, T[ \quad (1.15)$$

and

$$\xi_1^i(t) - \xi_1^{i-1}(t) > \frac{L}{2}, \quad \forall t \in [0, T[, \quad i = 2, \dots, N, \quad (1.16)$$

where  $\xi_1^1(t), \dots, \xi_1^N(t)$  are defined in Subsection 4.1.

## 2 Preliminaries

In this section we briefly recall the global well-posedness results for the DP equation and its consequences (see [4] and [6] for details).

**Theorem 2.1** (Global weak solution; see [4]). *Assume that  $u_0 \in L^2(\mathbb{R})$  with  $y_0 = (1 - \partial_x^2)u_0 \in \mathcal{M}^+(\mathbb{R})$ . Then the DP equation has a unique global weak solution  $u \in \mathcal{X}(\mathbb{R}_+)$  such that*

$$y(t, \cdot) = (1 - \partial_x^2)u(t, \cdot) \in \mathcal{M}^+(\mathbb{R}), \quad \forall t \in \mathbb{R}_+. \quad (2.1)$$

Moreover  $E(\cdot)$  and  $F(\cdot)$  are conserved by the flow.

**Remark 2.1** (Control of  $L^\infty$  norm by  $L^2$  norm). From (2.1), it holds

$$u(x) = \frac{e^{-x}}{2} \int_{-\infty}^x e^{x'} y(x') dx' + \frac{e^x}{2} \int_x^{+\infty} e^{-x'} y(x') dx'$$

and

$$u_x(x) = -\frac{e^{-x}}{2} \int_{-\infty}^x e^{x'} y(x') dx' + \frac{e^x}{2} \int_x^{+\infty} e^{-x'} y(x') dx',$$

which lead to

$$|u_x(x)| \leq u(x), \quad \forall x \in \mathbb{R}. \quad (2.2)$$

Then, using the Sobolev embedding of  $H^1(\mathbb{R})$  into  $L^\infty(\mathbb{R})$  and (2.2), we infer that there exists a constant  $C_S > 0$  such that

$$\|u\|_{L^\infty(\mathbb{R})} \leq C_S \|u\|_{H^1(\mathbb{R})} \leq 2C_S \|u\|_{L^2(\mathbb{R})}. \quad (2.3)$$

**Lemma 2.1** (Positivity; see [6]). *Let  $u \in H^1(\mathbb{R})$  with  $y = (1 - \partial_x^2)u \in \mathcal{M}^+(\mathbb{R})$ . If  $k_1 \geq 1$ , then we have*

$$(k_1 \pm \partial_x)u(x) \geq 0, \quad \forall x \in \mathbb{R}. \quad (2.4)$$

**Lemma 2.2** (Positivity; see [6]). *Let  $w(x) = (k_1 \pm \partial_x)u(x)$ . Assume that  $u \in H^1(\mathbb{R})$  with  $y = (1 - \partial_x^2)u \in \mathcal{M}^+(\mathbb{R})$ . If  $k_1 \geq 1$  and  $k_2 \geq 2$ , then we have*

$$(k_2 \pm \partial_x)(4 - \partial_x^2)^{-1}w(x) \geq 0, \quad \forall x \in \mathbb{R}. \quad (2.5)$$

### 3 Stability of a single peakon

The proof of Lin and Liu in [5] is not entirely suitable for our work, because it involves all local extrema of the function  $v = (4 - \partial_x^2)^{-1}u$  on  $\mathbb{R}$ , and thus is not local. For our work, we have to localize the estimates. Therefore, we need to modify a little the proof of Lin and Liu. We do this first for a single peakon.

**Theorem 3.1** (Stability of peakons). *Let  $u \in \mathcal{X}([0, T])$ , with  $0 < T \leq +\infty$ , be a solution of the DP equation and  $\varphi_c$  be the peakon defined in (1.8), traveling to the right at the speed  $c > 0$ . There exist  $C > 0$  and  $\varepsilon_0 > 0$  only depending on the speed  $c > 0$ , such that if*

$$y_0 = (1 - \partial_x^2)u_0 \in \mathcal{M}^+(\mathbb{R}) \quad (3.1)$$

and

$$\|u_0 - \varphi_c\|_{\mathcal{H}} \leq \varepsilon^2, \quad \text{with } 0 < \varepsilon < \varepsilon_0, \quad (3.2)$$

then

$$\|u(t, \cdot) - \varphi_c(\cdot - \xi_1(t))\|_{\mathcal{H}} \leq C\sqrt{\varepsilon}, \quad \forall t \in [0, T], \quad (3.3)$$

where  $\xi_1(t) \in \mathbb{R}$  is any point where the function  $v(t, \cdot) = (4 - \partial_x^2)^{-1}u(t, \cdot)$  attains its maximum.

To prove this theorem we first need the following lemma that enables to control the distance of  $E(u)$  and  $F(u)$  to respectively  $E(\varphi_c)$  and  $F(\varphi_c)$ .

**Lemma 3.1** (Control of distances between energies). *Let  $u \in H^1(\mathbb{R})$  with  $y = (1 - \partial_x^2)u \in \mathcal{M}^+(\mathbb{R})$ . If  $\|u - \varphi_c\|_{\mathcal{H}} \leq \varepsilon^2$ , then*

$$|E(u) - E(\varphi_c)| \leq O(\varepsilon^2) \quad (3.4)$$

and

$$|F(u) - F(\varphi_c)| \leq O(\varepsilon^2), \quad (3.5)$$

where  $O(\cdot)$  only depends on  $c$ .

**Proof.** For the first estimate, applying triangular inequality, and using that  $\|u - \varphi_c\|_{\mathcal{H}} \leq \varepsilon^2$  and  $\|\varphi_c\|_{\mathcal{H}} = c/\sqrt{3}$ , we have

$$\begin{aligned} |E(u) - E(\varphi_c)| &= | \|u\|_{\mathcal{H}} - \|\varphi_c\|_{\mathcal{H}} | (\|u\|_{\mathcal{H}} + \|\varphi_c\|_{\mathcal{H}}) \\ &\leq \|u - \varphi_c\|_{\mathcal{H}} (\|u - \varphi_c\|_{\mathcal{H}} + 2\|\varphi_c\|_{\mathcal{H}}) \\ &\leq \varepsilon^2 \left( \varepsilon^2 + \frac{2c}{\sqrt{3}} \right) \\ &\leq O(\varepsilon^2). \end{aligned}$$

For the second estimate, applying Hölder inequality, and using that  $\|u - \varphi_c\|_{\mathcal{H}} \leq \varepsilon^2$  and (2.3), we have

$$\begin{aligned} |F(u) - F(\varphi_c)| &\leq \int_{\mathbb{R}} |u^3 - \varphi_c^3| \\ &\leq \int_{\mathbb{R}} |u - \varphi_c| (u^2 + u\varphi_c + \varphi_c^2) \\ &\leq \|u - \varphi_c\|_{L^2(\mathbb{R})} \left( \int_{\mathbb{R}} (u^2 + u\varphi_c + \varphi_c^2)^2 \right)^{1/2} \\ &= \|u - \varphi_c\|_{L^2(\mathbb{R})} \left( \int_{\mathbb{R}} (u^4 + 2u^3\varphi_c + 3u^2\varphi_c^2 + 2u\varphi_c^3 + \varphi_c^4) \right)^{1/2} \\ &\leq \|u - \varphi_c\|_{L^2(\mathbb{R})} \\ &\quad \cdot \left( 4C_S^2 \|u\|_{L^2(\mathbb{R})}^4 + 4cC_S \|u\|_{L^2(\mathbb{R})}^2 + 3c^2 \|u\|_{L^2(\mathbb{R})}^2 + \frac{8}{3}c^3C_S \|u\|_{L^2(\mathbb{R})} + \frac{1}{2}c^4 \right)^{1/2} \\ &\leq O(\varepsilon^2), \end{aligned}$$

where we also use that the  $L^2$  norm of  $u$  is bounded, and the following measures of peakon:

$$\|\varphi_c\|_{L^\infty(\mathbb{R})} = c, \quad \|\varphi_c\|_{L^3(\mathbb{R})}^3 = \frac{2}{3}c^3 \quad \text{and} \quad \|\varphi_c\|_{L^4(\mathbb{R})}^4 = \frac{1}{2}c^4.$$

This proves the lemma.

Now, to prove Theorem 3.1, by the conservation of  $E(\cdot)$ ,  $F(\cdot)$  and the continuity of the map  $t \mapsto u(t)$  from  $[0, T[$  to  $H^1(\mathbb{R}) \hookrightarrow \mathcal{H}$  (since  $\mathcal{H} \simeq L^2$  and  $\|u\|_{L^2(\mathbb{R})} \leq \|u\|_{H^1(\mathbb{R})}$ ), it suffices to prove that for any function  $u \in H^1(\mathbb{R})$  satisfying (3.1), (3.2), (3.4) and (3.5), if

$$\inf_{\xi \in \mathbb{R}} \|u - \varphi_c(\cdot - \xi)\|_{\mathcal{H}} \leq \varepsilon^{1/4}, \quad (3.6)$$

then

$$\|u(t, \cdot) - \varphi_c(\cdot - \xi_1)\|_{\mathcal{H}} \leq C\sqrt{\varepsilon}, \quad \forall t \in [0, T], \quad (3.7)$$

where  $\xi_1 \in \mathbb{R}$  is any point where the function  $v = (4 - \partial_x^2)^{-1}u$  attains its maximum.

We divide the proof of Theorem 3.1 into a sequence of lemmas. In the sequel we will need to introduce the following *smooth-peakons* defined for all  $x \in \mathbb{R}$  by

$$\rho_c(x) = c\rho(x) = (4 - \partial_x^2)^{-1}\varphi_c(x) = \frac{c}{3}e^{-|x|} - \frac{c}{6}e^{-2|x|}. \quad (3.8)$$

One can check that  $\rho_c \in H^3(\mathbb{R}) \hookrightarrow C^2(\mathbb{R})$  (by Sobolev embedding) since  $\varphi_c \in H^1(\mathbb{R})$ . Indeed, we have

$$\|\rho_c\|_{H^3(\mathbb{R})}^2 = \int_{\mathbb{R}} \frac{(1 + \omega^2)^3}{(4 + \omega^2)^2} |\widehat{\varphi}_c(\omega)|^2 \leq \int_{\mathbb{R}} (4 + \omega^2) |\widehat{\varphi}_c(\omega)|^2 \leq 4\|\varphi_c\|_{H^1(\mathbb{R})}^2 = 8c^2. \quad (3.9)$$

Also, it is a positive even function which decays at infinity, and admits a single maximum  $c/6$  at point 0 (see Figure 1a-1c).

**Lemma 3.2** (Uniform estimates). *Let  $u \in H^1(\mathbb{R})$  with  $y = (1 - \partial_x^2)u \in \mathcal{M}^+(\mathbb{R})$ , and  $\xi \in \mathbb{R}$ . If*

$$\|u - \varphi_c(\cdot - \xi)\|_{\mathcal{H}} \leq \varepsilon^{1/4}, \quad (3.10)$$

then

$$\|u - \varphi_c(\cdot - \xi)\|_{L^\infty(\mathbb{R})} \leq O(\varepsilon^{1/8}) \quad (3.11)$$

and

$$\|v - \rho_c(\cdot - \xi)\|_{L^\infty(\mathbb{R})} \leq O(\varepsilon^{1/4}), \quad (3.12)$$

where  $v = (4 - \partial_x^2)^{-1}u$  and  $\rho_c$  is defined in (3.8).

**Proof.** For the second estimate, applying Hölder inequality and using (3.10), we get for all  $x \in \mathbb{R}$ ,

$$\begin{aligned} |v(x) - \rho_c(x - \xi)| &\leq \frac{1}{4} \int_{\mathbb{R}} e^{-2|x'|} |u(x - x') - \varphi_c[(x - x') - \xi]| dx' \\ &\leq \frac{1}{4} \left( \int_{\mathbb{R}} e^{-4|x'|} dx' \right)^{1/2} \left( \int_{\mathbb{R}} |u(x') - \varphi_c(x' - \xi)|^2 dx' \right)^{1/2} \\ &\leq \frac{1}{2\sqrt{2}} \|u - \varphi_c(\cdot - \xi)\|_{\mathcal{H}} \\ &\leq O(\varepsilon^{1/4}). \end{aligned}$$

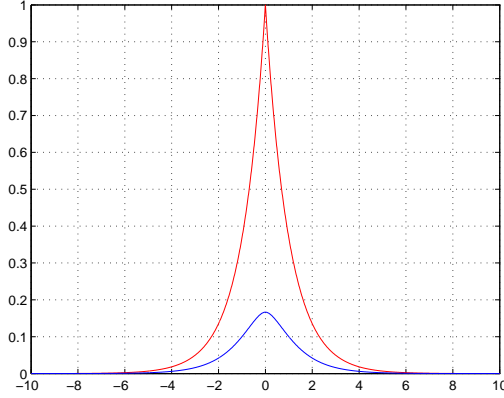
For the first estimate, note that the assumption  $y = (1 - \partial_x^2)u \geq 0$  implies that  $u = (1 - \partial_x^2)^{-1}y \geq 0$  and satisfies (2.2). Then, applying triangular inequality, and using that  $|\varphi'_c| = \varphi_c$  on  $\mathbb{R}$  and (2.3), we have

$$\begin{aligned} \|u - \varphi_c(\cdot - \xi)\|_{H^1(\mathbb{R})} &\leq \|u\|_{H^1(\mathbb{R})} + \|\varphi_c\|_{H^1(\mathbb{R})} \\ &\leq 2\|u\|_{L^2(\mathbb{R})} + 2\|\varphi_c\|_{L^2(\mathbb{R})} \\ &\leq 2\|u - \varphi_c(\cdot - \xi)\|_{L^2(\mathbb{R})} + 4\|\varphi_c\|_{L^2(\mathbb{R})} \\ &\leq O(\varepsilon^{1/4}) + O(1). \end{aligned}$$

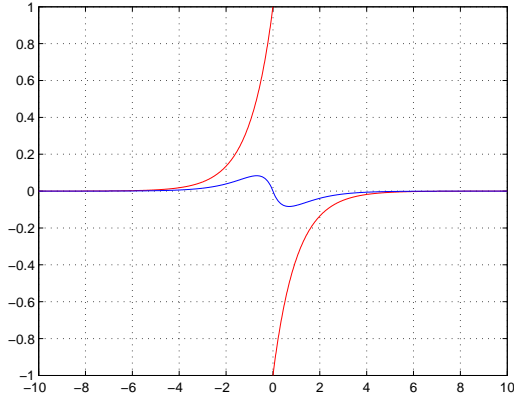
Now, applying Gagliardo-Nirenberg inequality and using (3.10), we obtain

$$\begin{aligned} \|u - \varphi_c(\cdot - \xi)\|_{L^\infty(\mathbb{R})} &\leq C_G \|u - \varphi_c(\cdot - \xi)\|_{L^2(\mathbb{R})}^{1/2} \|u - \varphi_c(\cdot - \xi)\|_{H^1(\mathbb{R})}^{1/2} \\ &\leq O(\varepsilon^{1/8}) \left( O(\varepsilon^{1/8}) + O(1) \right) \\ &\leq O(\varepsilon^{1/8}). \end{aligned}$$

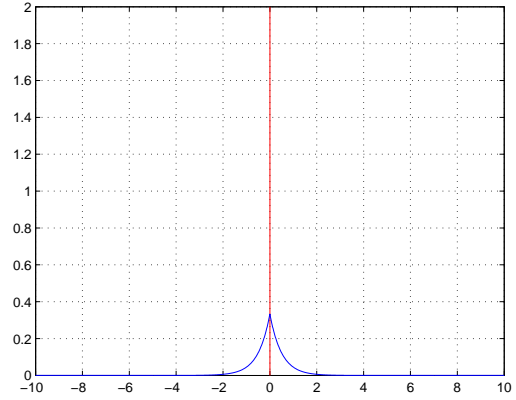
This proves the lemma.



(a)  $\varphi(x)$  and  $\rho(x)$  profiles.



(b)  $\varphi'(x)$  and  $\rho'(x)$  profiles.



(c)  $(1 - \partial_x^2)\varphi(x) = 2\delta_0$  and  $(1 - \partial_x^2)\rho(x) = (1/3)e^{-2|x|}$  profiles

Figure 1: Peakon and smooth-peakon at initial time with the speed  $c = 1$ .

**Lemma 3.3** (Quadratic identity; see [5]). *For any  $u \in L^2(\mathbb{R})$  and  $\xi \in \mathbb{R}$ , it holds*

$$E(u) - E(\varphi_c) = \|u - \varphi_c(\cdot - \xi)\|_{\mathcal{H}}^2 + 4c \left( v(\xi) - \frac{c}{6} \right), \quad (3.13)$$

where  $v = (4 - \partial_x^2)^{-1}u$  and  $c/6 = \rho_c(0) = \max_{\xi \in \mathbb{R}} \rho_c(\cdot - \xi)$ .

**Sketch of proof.** The proof follows by direct computation, with the aid of two integration by parts, and using that  $(1 - \partial_x^2)\varphi_c(\cdot - \xi) = 2c\delta_\xi$ , where  $\delta_\xi$  denotes the Dirac mass applied at point  $\xi$ .

Let  $u \in H^1(\mathbb{R})$  with  $y = (1 - \partial_x^2)u \in \mathcal{M}^+(\mathbb{R})$ , and assume that there exists  $\xi \in \mathbb{R}$  such that (3.10) holds. We consider now the interval in which the peakon (respectively smooth-peakon) is concentrated, and we will decompose this interval according to the variation of  $v = (4 - \partial_x^2)^{-1}u$  in the following way: we set

$$\alpha = \sup \left\{ x < \xi, v(x) = \frac{c}{2400} \right\} \quad \text{and} \quad \beta = \inf \left\{ x > \xi, v(x) = \frac{c}{2400} \right\}. \quad (3.14)$$

According to Lemma 3.2, we know that  $v$  is close to  $\rho_c(\cdot - \xi)$  in  $L^\infty$  norm with  $\rho_c(0) = c/6$ . Therefore  $v$  must have at least one local maximum on  $[\alpha, \beta]$ . Assume that on  $[\alpha, \beta]$  the function  $v$  admits  $k+1$  points  $(\xi_j)_{j=1}^{k+1}$  with local maximal values for some integer  $k \geq 0$ , where  $\xi_1$  is the first local maximum point and  $\xi_{k+1}$  the last local maximum point<sup>2</sup>. Then Between  $\xi_1$  and  $\xi_{k+1}$ , the function  $v$  admits  $k$  points  $(\eta_j)_{j=1}^k$  with local minimal values. We rename  $\alpha = \eta_0$  and  $\beta = \eta_{k+1}$  so that it holds

$$\eta_0 < \xi_1 < \eta_1 < \dots < \xi_j < \eta_j < \xi_{j+1} < \eta_{j+1} < \dots < \eta_k < \xi_{k+1} < \eta_{k+1}. \quad (3.15)$$

Let

$$M_j = v(\xi_j), \quad j = 1, \dots, k+1, \quad \text{and} \quad m_j = v(\eta_j), \quad j = 1, \dots, k. \quad (3.16)$$

By construction

$$v_x(x) \geq 0, \quad \forall x \in [\eta_{j-1}, \xi_j], \quad j = 1, \dots, k \quad (3.17)$$

and

$$v_x(x) \leq 0, \quad \forall x \in [\xi_j, \eta_j], \quad j = 1, \dots, k+1. \quad (3.18)$$

We claim that

$$v(x) \leq \frac{c}{300}, \quad \forall x \in \mathbb{R} \setminus [\eta_0, \eta_{k+1}], \quad (3.19)$$

$$u(x) \leq \frac{c}{300}, \quad \forall x \in \mathbb{R} \setminus [\eta_0, \eta_{k+1}], \quad (3.20)$$

and there exists  $C_0 > 0$  such that

$$[\eta_0, \eta_{k+1}] \subset [\xi - C_0, \xi + C_0]. \quad (3.21)$$

Indeed, from (3.12) we have

$$\rho_c(\eta_0 - \xi) = v(\eta_0) + O(\varepsilon^{1/4}) = \frac{c}{2400} + O(\varepsilon^{1/4}) \leq \frac{c}{350},$$

please note that we abuse notation by writing that the difference between  $v$  and  $\rho_c(\cdot - \xi)$  is equal to  $O(\varepsilon^{1/4})$ . Therefore, using that  $\rho_c(\cdot - \xi)$  is increasing on  $] - \infty, \xi]$ , it holds for all  $x \in ] - \infty, \eta_0[$ ,

$$v(x) = \rho_c(x - \xi) + O(\varepsilon^{1/4}) \leq \frac{c}{350} + O(\varepsilon^{1/4}) \leq \frac{c}{300}.$$

Proceeding in the same way for  $x \in ]\eta_{k+1}, +\infty[$ , we obtain (3.19).

One can remark that for all  $x \in \mathbb{R}$ ,

$$\varphi_c(x) - 6\rho_c(x) = ce^{-|x|} - 6 \left( \frac{c}{3}e^{-|x|} - \frac{c}{6}e^{-2|x|} \right) = -ce^{-|x|} + ce^{-2|x|} \leq 0. \quad (3.22)$$

Thus, combining (3.11), (3.22) and proceeding as for the estimate (3.19), we infer (3.20).

Finally, from (3.12) we have

$$\rho_c(\eta_0 - \xi) = v(\eta_0) + O(\varepsilon^{1/4}) = \frac{c}{2400} + O(\varepsilon^{1/4}) \geq \frac{c}{3000}.$$

Therefore, since  $\rho_c = (c/3)e^{-|x|} - (c/6)e^{-2|x|}$  and that  $x \mapsto (1/3)e^{-|x|} - (1/6)e^{-2|x|}$  is a positive even function decreasing to 0 on  $\mathbb{R}_+$  (see Figure 1a, there exists a universal constant  $C_0 > 0$  such that (3.21) holds.

We now are ready to establish the connection between the conservation laws. Please note that, we will change the order of the extrema of  $v = (4 - \partial_x^2)^{-1}u$  while keeping the same notations as in (3.16).

<sup>2</sup>In the case of a infinite countable number of local maximal values, the proof is exactly the same.



**Lemma 3.4** (Connection between  $E(\cdot)$  and the local extrema of  $v$ ). *Let  $u \in H^1(\mathbb{R})$  with  $u \geq 0$ . Define the function  $g$  by*

$$g(x) = \begin{cases} 2v + v_{xx} - 3v_x, & x < \xi_1, \\ 2v + v_{xx} + 3v_x, & \xi_j < x < \eta_j, \\ 2v + v_{xx} - 3v_x, & \eta_j < x < \xi_{j+1}, \\ 2v + v_{xx} + 3v_x, & x > \xi_{k+1}, \end{cases} \quad j = 1, \dots, k. \quad (3.23)$$

Then it holds

$$\int_{\mathbb{R}} g^2(x) dx = E(u) - 12 \left( \sum_{j=0}^k M_{j+1}^2 - \sum_{j=1}^k m_j^2 \right). \quad (3.24)$$

**Proof.** We have

$$\int_{\mathbb{R}} g^2(x) dx = \int_{-\infty}^{\xi_1} g^2(x) dx + \sum_{j=1}^k \int_{\xi_j}^{\xi_{j+1}} g^2(x) dx + \int_{\xi_{k+1}}^{+\infty} g^2(x) dx. \quad (3.25)$$

For  $j = 1, \dots, k$ ,

$$\begin{aligned} \int_{\xi_j}^{\xi_{j+1}} g^2(x) dx &= \int_{\xi_j}^{\eta_j} (2v + v_{xx} + 3v_x)^2 + \int_{\eta_j}^{\xi_{j+1}} (2v + v_{xx} - 3v_x)^2 \\ &= J + I. \end{aligned}$$

Let us compute  $I$ ,

$$\begin{aligned} I &= \int_{\eta_j}^{\xi_{j+1}} (4v^2 + v_{xx}^2 + 9v_x^2 + 4vv_{xx} - 12vv_x - 6v_x v_{xx}) \\ &= \int_{\eta_j}^{\xi_{j+1}} (4v^2 + v_{xx}^2 + 9v_x^2) + 4 \int_{\eta_j}^{\xi_{j+1}} vv_{xx} - 12 \int_{\eta_j}^{\xi_{j+1}} vv_x - 6 \int_{\eta_j}^{\xi_{j+1}} v_x v_{xx} \\ &= \int_{\eta_j}^{\xi_{j+1}} (4v^2 + v_{xx}^2 + 9v_x^2) + I_1 + I_2 + I_3. \end{aligned}$$

Applying integration by parts and using that  $v_x(\xi_j) = v_x(\eta_j) = 0$ , we get

$$I_1 = -4 \int_{\eta_j}^{\xi_{j+1}} v_x^2, \quad I_2 = -6 \int_{\eta_j}^{\xi_{j+1}} \partial_x(v^2) = -6v^2(\xi_{j+1}) + 6v^2(\eta_j) \quad \text{and} \quad I_3 = -3 \int_{\eta_j}^{\xi_{j+1}} \partial_x(v_x^2) = 0.$$

Therefore

$$I = \int_{\eta_j}^{\xi_{j+1}} (4v^2 + 5v_x^2 + v_{xx}^2) - 6v^2(\xi_{j+1}) + 6v^2(\eta_j). \quad (3.26)$$

Similar computations lead to

$$J = \int_{\xi_j}^{\eta_j} (4v^2 + 5v_x^2 + v_{xx}^2) - 6v^2(\xi_j) + 6v^2(\eta_j), \quad \int_{-\infty}^{\xi_1} g^2(x) dx = \int_{-\infty}^{\xi_1} (4v^2 + 5v_x^2 + v_{xx}^2) - 6v^2(\xi_1) \quad (3.27)$$

and

$$\int_{\xi_{k+1}}^{+\infty} g^2(x) dx = \int_{\xi_{k+1}}^{+\infty} (4v^2 + 5v_x^2 + v_{xx}^2) - 6v^2(\xi_{k+1}). \quad (3.28)$$

Adding  $I$  and  $J$ , and summing over  $j \in \{1, \dots, k\}$ , we obtain

$$\int_{\xi_1}^{\xi_{k+1}} g^2(x) dx = \int_{\xi_1}^{\xi_{k+1}} (4v^2 + 5v_x^2 + v_{xx}^2) - 6 \sum_{j=1}^k v^2(\xi_{j+1}) - 6 \sum_{j=1}^k v^2(\xi_j) + 12 \sum_{j=1}^k v^2(\eta_j). \quad (3.29)$$

The lemma follows by combining (3.25) and (3.27)-(3.29).

**Lemma 3.5** (Connection between  $F(\cdot)$  and the local extrema of  $v$ ). *Let  $u \in H^1(\mathbb{R})$  with  $u \geq 0$ . Define the function  $h$  by*

$$h(x) = \begin{cases} -v_{xx} - 6v_x + 16v, & x < \xi_1, \\ -v_{xx} + 6v_x + 16v, & \xi_j < x < \eta_j, \\ -v_{xx} - 6v_x + 16v, & \eta_j < x < \xi_{j+1}, \\ -v_{xx} + 6v_x + 16v, & x > \xi_{k+1}, \end{cases} \quad j = 1, \dots, k. \quad (3.30)$$

Then it holds

$$\int_{\mathbb{R}} h(x)g^2(x)dx = F(u) - 144 \left( \sum_{j=0}^k M_{j+1}^3 - \sum_{j=1}^k m_j^3 \right). \quad (3.31)$$

**Proof.** We have

$$\int_{\mathbb{R}} h(x)g^2(x)dx = \int_{-\infty}^{\xi_1} h(x)g^2(x)dx + \sum_{j=1}^k \int_{\xi_j}^{\xi_{j+1}} h(x)g^2(x)dx + \int_{\xi_{k+1}}^{+\infty} h(x)g^2(x)dx. \quad (3.32)$$

For  $j = 1, \dots, k$ ,

$$\begin{aligned} \int_{\xi_j}^{\xi_{j+1}} h(x)g^2(x)dx &= \int_{\xi_j}^{\eta_j} (-v_{xx} - 6v_x + 16v)(2v + v_{xx} - 3v_x)^2 \\ &\quad + \int_{\eta_j}^{\xi_{j+1}} (-v_{xx} + 6v_x + 16v)(2v + v_{xx} + 3v_x)^2 \\ &= J + I. \end{aligned}$$

Let us compute  $I$ ,

$$\begin{aligned} I &= \int_{\eta_j}^{\xi_{j+1}} (-v_{xx}^3 + 12vv_{xx}^2 + 64v^3 + 60v^2v_{xx}) - 54 \int_{\eta_j}^{\xi_{j+1}} v_x^3 + 27 \int_{\eta_j}^{\xi_{j+1}} v_x^2 v_{xx} \\ &\quad - 108 \int_{\eta_j}^{\xi_{j+1}} vv_x v_{xx} - 216 \int_{\eta_j}^{\xi_{j+1}} v^2 v_x + 216 \int_{\eta_j}^{\xi_{j+1}} vv_x^2 \\ &= \int_{\eta_j}^{\xi_{j+1}} (-v_{xx}^3 + 12vv_{xx}^2 + 64v^3 + 60v^2v_{xx}) \\ &\quad - 54 \int_{\eta_j}^{\xi_{j+1}} v_x^3 + I_1 + I_2 + I_3 + I_4. \end{aligned}$$

Applying integration by parts and using that  $v_x(\xi_j) = v_x(\eta_j) = 0$ , we get

$$I_1 = 9 \int_{\eta_j}^{\xi_{j+1}} \partial_x(v_x^3) = 0, \quad I_2 = 54 \int_{\eta_j}^{\xi_{j+1}} v_x^3, \quad I_3 = -72 \int_{\eta_j}^{\xi_{j+1}} \partial_x(v^3) = -72v^3(\xi_{j+1}) + 72v^3(\eta_j)$$

and

$$I_4 = 108 \int_{\eta_j}^{\xi_{j+1}} \partial_x(v^2)v_x = -108 \int_{\eta_j}^{\xi_{j+1}} v^2 v_{xx}.$$

Therefore

$$I = \int_{\eta_j}^{\xi_{j+1}} (-v_{xx}^3 + 12vv_{xx}^2 + 64v^3 + 60v^2v_{xx}) - 72v^3(\xi_{j+1}) + 72v^3(\eta_j). \quad (3.33)$$

Similar computations lead to

$$J = \int_{\xi_j}^{\eta_j} (-v_x^3 + 12vv_{xx}^2 + 64v^3 + 60v^2v_{xx}) - 72v^3(\xi_j) + 72v^3(\eta_j), \quad (3.34)$$

$$\int_{-\infty}^{\xi_1} h(x)g^2(x)dx = \int_{-\infty}^{\xi_1} (-v_x^3 + 12vv_{xx}^2 + 64v^3 + 60v^2v_{xx}) - 72v^3(\xi_1) \quad (3.35)$$

and

$$\int_{\xi_{k+1}}^{+\infty} h(x)g^2(x)dx = \int_{\xi_{k+1}}^{+\infty} (-v_x^3 + 12vv_{xx}^2 + 64v^3 + 60v^2v_{xx}) - 72v^3(\xi_{k+1}). \quad (3.36)$$

Adding (3.26) and (3.34), and summing over  $j \in \{1, \dots, k\}$ , we obtain

$$\begin{aligned} \int_{\xi_1}^{\xi_{k+1}} h(x)g^2(x)dx &= \int_{\xi_1}^{\xi_{k+1}} (-v_x^3 + 12vv_{xx}^2 + 64v^3 + 60v^2v_{xx}) - 72 \sum_{j=1}^k v^3(\xi_{j+1}) \\ &\quad - 72 \sum_{j=1}^k v^3(\xi_j) + 72 \sum_{j=1}^k v^3(\eta_j). \end{aligned} \quad (3.37)$$

The lemma follows by combining (3.32) and (3.35)-(3.37).

**Lemma 3.6** (Connection between  $E(\cdot)$  and  $F(\cdot)$ ). *Let  $u \in H^1(\mathbb{R})$ , with  $y = (1 - \partial_x^2)u \in \mathcal{M}^+(\mathbb{R})$ , that satisfies (3.10) for some  $\xi \in \mathbb{R}$ . Assume that  $v = (4 - \partial_x^2)^{-1}u$  satisfies (3.14)-(3.21), with local extrema on  $[\eta_0, \eta_{k+1}]$  arranged in decreasing order in the following way:*

$$M_1 \geq M_2 \geq \dots \geq M_{k+1} \geq 0, \quad m_1 \geq m_2 \geq \dots \geq m_k \geq 0, \quad M_{j+1} \geq m_j, \quad j = 1, \dots, k. \quad (3.38)$$

Then it holds,

$$M_1^3 - \frac{1}{4}E(u)M_1 + \frac{1}{72}F(u) \leq 0. \quad (3.39)$$

**Proof.** The key is to show that  $h \leq 18M_1$  on  $\mathbb{R}$ . Note that by (3.10) we know that  $18M_1 \geq c/4$ . We rewrite the function  $h$  as

$$h(x) = \begin{cases} -v_{xx} - 6v_x + 16v, & x < \eta_0, \\ -(\partial_x^2 + 3\partial_x + 2)v - 3v_x + 18v, & \eta_0 < x < \xi_1, \\ -(\partial_x^2 - 3\partial_x + 2)v + 3v_x + 18v, & \xi_j < x < \eta_j, \\ -(\partial_x^2 + 3\partial_x + 2)v - 3v_x + 18v, & \eta_j < x < \xi_{j+1}, \\ -(\partial_x^2 - 3\partial_x + 2)v + 3v_x + 18v, & \xi_{k+1} < x < \eta_{k+1}, \\ -v_{xx} + 6v_x + 16v, & x > \eta_{k+1}, \end{cases} \quad j = 1, \dots, k.$$

First, one can remark that for all  $x \in \mathbb{R}$ ,

$$v(x) = \frac{e^{-2x}}{4} \int_{-\infty}^x e^{2x'} u(x') dx' + \frac{e^{2x}}{4} \int_x^{+\infty} e^{-2x'} u(x') dx'$$

and

$$v_x(x) = -\frac{e^{-2x}}{2} \int_{-\infty}^x e^{2x'} u(x') dx' + \frac{e^{2x}}{2} \int_x^{+\infty} e^{-2x'} u(x') dx'.$$

Then using that  $u = (1 - \partial_x^2)^{-1}y \geq 0$  on  $\mathbb{R}$ , we get

$$|v_x(x)| \leq 2v(x), \quad \forall x \in \mathbb{R}. \quad (3.40)$$

Next, if  $x \in \mathbb{R} \setminus [\eta_0, \eta_{k+1}]$ , using that  $v_{xx} = 4v - u$ , (3.19), (3.20) and (3.40), it holds

$$h \leq |v_{xx}| + 6|v_x| + 16v \leq u + 32v \leq \frac{c}{9}.$$

If  $\eta_0 < x < \xi_1$ , then  $v_x \geq 0$ , and using that  $y = (1 - \partial_x^2)u \geq 0$ , it follows from Lemma 2.2 that

$$\begin{aligned} h &= -(\partial_x^2 + 3\partial_x + 2)v - 3v_x + 18v \\ &= -(2 + \partial_x)(4 - \partial_x^2)^{-1}(1 + \partial_x)u - 3v_x + 18v \\ &\leq 18v. \end{aligned}$$

If  $\xi_j < x < \eta_j$ , then  $v_x \leq 0$ , and similarly using that  $y = (1 - \partial_x^2)u \geq 0$ , it follows from Lemma 2.2 that

$$\begin{aligned} h &= -(\partial_x^2 - 3\partial_x + 2)v + 3v_x + 18v \\ &= -(2 - \partial_x)(4 - \partial_x^2)^{-1}(1 - \partial_x)u + 3v_x + 18v \\ &\leq 18v. \end{aligned}$$

Therefore, it holds

$$h(x) \leq 18 \max_{x \in \mathbb{R}} v(x) = 18M_1, \quad \forall x \in \mathbb{R}. \quad (3.41)$$

Now, combining (3.24), (3.31) and (3.41), we get

$$\begin{aligned} F(u) - 144 \left( \sum_{j=0}^k M_{j+1}^3 - \sum_{j=1}^k m_j^3 \right) &= \int_{\mathbb{R}} h(x)g^2(x)dx \\ &\leq \|h\|_{L^\infty(\mathbb{R})} \int_{\mathbb{R}} g^2(x)dx \\ &\leq 18M_1 \left[ E(u) - 12 \left( \sum_{j=0}^k M_{j+1}^2 - \sum_{j=1}^k m_j^2 \right) \right]. \end{aligned}$$

For  $j = 1, \dots, k$ , we set

$$A_j = M_{j+1}^3 - m_j^3 \quad \text{and} \quad B_j = M_{j+1}^2 - m_j^2,$$

and our inequality becomes

$$M_1^3 - \frac{1}{4}E(u)M_1 + \frac{1}{72}F(u) \leq 2 \sum_{j=1}^k \left( A_j - \frac{3}{2}M_1 B_j \right). \quad (3.42)$$

On the other hand, using that  $M_{j+1} \geq m_j$ , we have

$$A_j - \frac{3}{2}M_1 B_j = -\frac{1}{2}(M_{j+1} - m_j)(3M_1 m_j + 3M_1 M_{j+1} - 2M_{j+1} m_j - 2M_{j+1}^2 - 2m_j^2) \leq 0. \quad (3.43)$$

Finally, combining (3.42) and (3.43), we obtain the lemma.

**Proof of Theorem 3.1.** As noticed after the statement of the theorem, it suffices to prove (3.7) assuming that  $u \in H^1(\mathbb{R})$  satisfies (3.1), (3.2) and (3.4)-(3.6). We set  $M_1 = v(\xi_1) = \max_{x \in \mathbb{R}} v(x)$  and  $\delta = c/6 - M_1$ . We first remark that if  $\delta \leq 0$ , combining (3.4) and (3.13), it holds

$$\|u - \varphi_c(\cdot - \xi_1)\|_{\mathcal{H}} \leq |E(u_0) - E(\varphi_c)|^{1/2} \leq O(\varepsilon),$$

that yields the desired result. Now suppose that  $\delta > 0$ , that is the maximum of the function  $v$  is less than the maximum of  $\rho_c$ . Combining (3.4), (3.5) and (3.39), we get

$$M_1^3 - \frac{1}{4}E(\varphi_c)M_1 + \frac{1}{72}F(\varphi_c) \leq O(\varepsilon^2).$$

Using that  $E(\varphi_c) = c^2/3$  and  $F(\varphi_c) = 2c^3/3$ , our inequality becomes

$$\left(M_1 - \frac{c}{6}\right)^2 \left(M_1 + \frac{c}{3}\right) \leq O(\varepsilon^2).$$

Substituting  $M_1$  by  $c/6 - \delta$  and using that  $(M_1 + c/3)^{-1} < 3/c$ , it holds

$$\delta^2 \leq O(\varepsilon^2) \Rightarrow \delta \leq O(\varepsilon). \quad (3.44)$$

Finally, combining (3.4), (3.13) and (3.44), we obtain

$$\|u - \varphi_c(\cdot - \xi_1)\|_{\mathcal{H}} \leq C\sqrt{\varepsilon},$$

where  $C > 0$  only depends on the speed  $c$ . This completes the proof of the stability of a single peakon.

## 4 Stability of the train of peakons

For  $\gamma > 0$  and  $L > 0$ , we define the following neighborhood of all the sums of  $N$  peakons of speed  $c_1, \dots, c_N$  with spatial shifts  $z_i$  that satisfied  $z_i - z_{i-1} \geq L$ ,

$$U(\gamma, L) = \left\{ u \in H^1(\mathbb{R}), \inf_{z_i - z_{i-1} > L} \left\| u - \sum_{i=1}^N \varphi_{c_i}(\cdot - z_i) \right\|_{\mathcal{H}} < \gamma \right\}. \quad (4.1)$$

By the continuity of the map  $t \mapsto u(t)$  from  $[0, T[$  into  $H^1(\mathbb{R}) \hookrightarrow \mathcal{H}$ , to prove Theorem 1.1 it suffices to prove that there exist  $A > 0$ ,  $\varepsilon_0 > 0$  and  $L_0 > 0$  such that for all  $L > L_0$  and  $0 < \varepsilon < \varepsilon_0$ , if  $u_0$  satisfies (1.12)-(1.14), and if for some  $0 < t_0 < T$ ,

$$u(t) \in U\left(A(\sqrt{\varepsilon} + L^{-1/8}), \frac{L}{2}\right), \quad \forall t \in [0, t_0], \quad (4.2)$$

then

$$u(t_0) \in U\left(\frac{A}{2}(\sqrt{\varepsilon} + L^{-1/8}), \frac{2L}{3}\right). \quad (4.3)$$

Therefore, in the sequel of this section we will assume (4.2) for some  $0 < \varepsilon < \varepsilon_0$  and  $L > L_0$ , with  $A$ ,  $\varepsilon_0$  and  $L_0$  to be specified later, and we will prove (4.3).

**Remark 4.1** (Distance between  $v$  and the sum of  $N$  smooth-peakons). From the definition of  $E(\cdot)$  and  $\mathcal{H}$  (see (1.3) and (1.6)), one can clearly see that  $\|u\|_{\mathcal{H}}$  is equivalent to  $\|v\|_{H^2(\mathbb{R})}$ , where  $v = (4 - \partial_x^2)^{-1}u$ . Therefore if  $u(t) \in U(\gamma, L/2)$  on  $[0, t_0]$ , then  $v(t) \in H^3(\mathbb{R}) \hookrightarrow C^2(\mathbb{R})$  stays close to  $\sum_{i=1}^N \rho_{c_i}(\cdot - z_i)$  in the  $H^2$  norm for all  $t \in [0, t_0]$ , where  $\rho_{c_i}$  is defined in (3.8).

### 4.1 Control of the distance between the peakons

In this subsection we want to prove that the different bumps of  $u$  that are individually close to a peakon get away from each others as time is increasing. This is crucial in our analysis since we do not know how to manage strong interactions.

**Lemma 4.1** (Decomposition of the solution). *Let  $u_0$  satisfying (1.12)-(1.14). There exist  $\gamma_0 > 0$ ,  $L_0 > 0$  and  $C_0 > 0$  such that for all  $0 < \gamma < \gamma_0$  and  $0 < L_0 < L$ , if  $u(t) \in U(\gamma, L/2)$  on  $[0, t_0]$  for some  $0 < t_0 < T$ , then there exist  $N$   $C^1$  functions  $\tilde{x}_1, \dots, \tilde{x}_N$  defined on  $[0, t_0]$  such that*

$$\left\| u(t) - \sum_{i=1}^N \varphi_{c_i}(\cdot - \tilde{x}_i(t)) \right\|_{\mathcal{H}} \leq O(\gamma), \quad (4.4)$$

$$\left\| v(t) - \sum_{i=1}^N \rho_{c_i}(\cdot - \tilde{x}_i(t)) \right\|_{C^1(\mathbb{R})} \leq O(\gamma), \quad (4.5)$$

$$|\dot{\tilde{x}}_i(t) - c_i| \leq c_1^{-2} \left( O(\gamma) + O(e^{-L/4}) \right), \quad i = 1, \dots, N, \quad (4.6)$$

and

$$\tilde{x}_i(t) - \tilde{x}_{i-1}(t) \geq \frac{3L}{4} + \frac{(c_i - c_{i-1})t}{2}, \quad i = 2, \dots, N. \quad (4.7)$$

Moreover, for  $i = 1, \dots, N$ , setting  $J_i = [y_i(t), y_{i+1}(t)]$ , with

$$\begin{cases} y_1 = -\infty, \\ y_i(t) = \frac{\tilde{x}_{i-1}(t) + \tilde{x}_i(t)}{2}, \quad i = 2, \dots, N, \\ y_{N+1} = +\infty, \end{cases} \quad (4.8)$$

it holds

$$|\xi_1^i(t) - \tilde{x}_i(t)| \leq \frac{L}{12}, \quad i = 1, \dots, N, \quad (4.9)$$

where  $\xi_1^1(t), \dots, \xi_1^N(t)$  are any point such that

$$v(t, \xi_1^i(t)) = \max_{x \in J_i} v(t, x), \quad i = 1, \dots, N, \quad (4.10)$$

and where  $v = (4 - \partial_x^2)^{-1}u$  and  $O(\cdot)$  only depends on  $(c_i)_{i=1}^N$ .

**Proof.** We will slightly modify the construction done in [3]. One can remark that the functions  $\varphi_{c_i}(\cdot - c_i t)$  and  $\rho_{c_i}(\cdot - c_i t)$  travel at the same speed  $c_i$ , thanks to this, we will do our construction with  $v = (4 - \partial_x^2)^{-1}u$  instead of  $u$ . We do that because the  $\mathcal{H}$  (equivalent to  $L^2$ ) approximation (4.2) does not permit us to construct a  $C^1$  function, which is crucial for application of the implicit function theorem. We note that the same approach can also be used for the CH equation.

For  $Z = (z_1, \dots, z_N) \in \mathbb{R}^N$  fixed such that  $|z_i - z_{i-1}| > L/2$ , we set

$$R_Z(\cdot) = \sum_{i=1}^N \rho_{c_i}(\cdot - z_i) \quad \text{and} \quad S_Z(\cdot) = \sum_{i=1}^N \varphi_{c_i}(\cdot - z_i). \quad (4.11)$$

For  $0 < \gamma < \gamma_0$ , we define the function

$$\begin{aligned} \mathcal{Y} : (-\gamma, \gamma)^N \times B_{H^2}(R_Z, \gamma) &\rightarrow \mathbb{R}^N, \\ (y_1, \dots, y_N, v) &\mapsto (\mathcal{Y}^1(y_1, \dots, y_N, v), \dots, \mathcal{Y}^N(y_1, \dots, y_N, v)) \end{aligned}$$

with

$$\mathcal{Y}^i(y_1, \dots, y_N, v) = \int_{\mathbb{R}} \left( v - \sum_{j=1}^N \rho_{c_j}(\cdot - z_j - y_j) \right) \partial_x \rho_{c_i}(\cdot - z_i - y_i).$$

$Y$  is clearly of class  $C^1$ . For  $i = 1, \dots, N$ ,

$$\frac{\partial \mathcal{Y}^i}{\partial y_i}(y_1, \dots, y_N, v) = - \int_{\mathbb{R}} \left( v - \sum_{\substack{1 \leq j \leq N \\ j \neq i}} \rho_{c_j}(\cdot - z_j - y_j) \right) \partial_x^2 \rho_{c_i}(\cdot - z_i - y_i)$$

and for  $j \neq i$ ,

$$\frac{\partial \mathcal{Y}^i}{\partial y_j}(y_1, \dots, y_N, v) = \int_{\mathbb{R}} \partial_x \rho_{c_j}(\cdot - z_j - y_j) \partial_x \rho_{c_i}(\cdot - z_i - y_i).$$

Hence

$$\frac{\partial \mathcal{Y}^i}{\partial y_i}(0, \dots, 0, R_Z) = \|\partial_x \rho_{c_i}\|_{L^2(\mathbb{R})}^2 = \frac{c_i^2}{54} \geq \frac{c_1^2}{54}$$

and for  $j \neq i$ , using the exponential decay of  $\varphi_{c_i}$  and that  $|z_i - z_{i-1}| > L/2$ , for  $L > L_0 > 0$  with  $L_0 \gg 1$ , it holds

$$\begin{aligned} \left| \frac{\partial \mathcal{Y}^i}{\partial y_j}(0, \dots, 0, R_Z) \right| &= \left| \int_{\mathbb{R}} \partial_x \rho_{c_j}(\cdot - z_j) \partial_x \rho_{c_i}(\cdot - z_i) \right| \\ &= \left| \int_{\mathbb{R}} \rho_{c_j}(\cdot - z_j) \partial_x^2 \rho_{c_i}(\cdot - z_i) \right| \\ &\leq \frac{1}{9} c_i c_j \left\{ \int_{\mathbb{R}} e^{-|x-z_j|-|x-z_i|} dx + 2 \int_{\mathbb{R}} e^{-|x-z_j|-2|x-z_i|} dx \right. \\ &\quad \left. + 2 \int_{\mathbb{R}} e^{-2|x-z_j|-|x-z_i|} dx + 4 \int_{\mathbb{R}} e^{-2|x-z_j|-2|x-z_i|} dx \right\} \\ &\leq O(e^{-L/4}). \end{aligned}$$

We deduce that, for  $L > 0$  large enough,  $D_{(y_1, \dots, y_N)} \mathcal{Y}(0, \dots, 0, R_Z) = D + P$  where  $D$  is an invertible diagonal matrix with  $\|D^{-1}\| \leq (c_1/3\sqrt{6})^{-2}$  and  $\|P\| \leq O(e^{-L/4})$ . Hence there exists  $L_0 > 0$  such that for  $L > L_0$ ,  $D_{(y_1, \dots, y_N)} \mathcal{Y}(0, \dots, 0, R_Z)$  is invertible with an inverse matrix of norm smaller than  $2(c_1/3\sqrt{6})^{-2}$ . From the implicit function theorem we deduce that there exists  $\beta_0 > 0$  and  $C^1$  functions  $(y_1, \dots, y_N)$  from  $B_{H^2}(R_Z, \beta_0)$  to a neighborhood of  $(0, \dots, 0)$  which are uniquely determined such that

$$\mathcal{Y}(y_1(v), \dots, y_N(v), v) = 0, \quad \forall v \in B_{H^2}(R_Z, \beta_0).$$

In particular, there exists  $C_0 > 0$  such that if  $v \in B_{C^0}(R_Z, \beta)$ , with  $0 < \beta \leq \beta_0$ , then

$$\sum_{i=1}^N |y_i(v)| \leq C_0 \beta. \quad (4.12)$$

Note that  $\beta_0$  and  $C_0$  only depend on  $c_1$  and  $L_0$  and not on the point  $(z_1, \dots, z_N)$ . For  $v \in B_{H^2}(R_Z, \beta_0)$  we set  $\tilde{x}_i(v) = z_i + y_i(v)$ . Assuming that  $\beta_0 \leq L_0/8C_0$ ,  $(\tilde{x}_1(v), \dots, \tilde{x}_N(v))$  are thus  $C^1$  functions on  $B_{H^2}(R_Z, \beta)$  satisfying

$$\tilde{x}_i(v) - \tilde{x}_{i-1}(v) = z_i - z_{i-1} + y_i(v) - y_{i-1}(v) > \frac{L}{2} - 2C_0\beta > \frac{L}{4}. \quad (4.13)$$

For  $L > L_0$  and  $0 < \gamma < \gamma_0 < \beta_0/2$  to be chosen later, we define the modulation of  $v$  in the following way: we cover the trajectory of  $v$  by a finite number of open balls in the following way:

$$\{v(t), t \in [0, t_0]\} \subset \bigcup_{k=1, \dots, M} B_{H^2}(R_{Z^k}, 2\gamma).$$

This is possible thanks to Remark 4.1. It is worth noticing that, since  $0 < \gamma < \gamma_0 < \beta_0/2$ , the functions  $\tilde{x}_i(v)$  are uniquely determined for  $v \in B_{H^2}(R_{Z^k}, 2\gamma) \cap B_{H^2}(R_{Z^{k'}}, 2\gamma)$ . We can thus define the functions  $t \mapsto \tilde{x}_i(t)$  on  $[0, t_0]$  by setting  $\tilde{x}_i(t) = \tilde{x}_i(v(t))$ . By construction

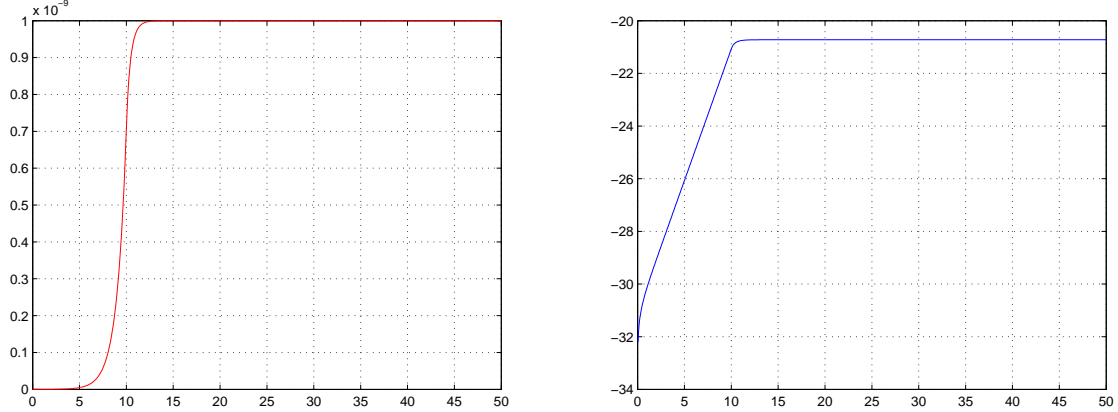
$$\int_{\mathbb{R}} \left( v(t, \cdot) - \sum_{j=1}^N \rho_{c_j}(\cdot - \tilde{x}_j(t)) \right) \partial_x \rho_{c_i}(\cdot - \tilde{x}_i(t)) = 0. \quad (4.14)$$

For  $0 < \gamma < \gamma_0$ , with  $\gamma_0 \ll 1$ , using that  $u \in U(\gamma, L/2)$  and (4.12), we have

$$\begin{aligned} & \|u(t) - S_{\tilde{X}(t)}\|_{\mathcal{H}} \\ & \leq \|u(t) - S_Z(t)\|_{\mathcal{H}} + \sum_{i=1}^N \|\varphi_{c_i}(\cdot - z_i) - \varphi_{c_i}(\cdot - z_i - y_i(v(t)))\|_{L^2(\mathbb{R})} \\ & \leq \gamma + \sqrt{2} \sum_{i=1}^N \left( \int_{\mathbb{R}} \varphi_{c_i}^2(x) dx - \int_{\mathbb{R}} \varphi_{c_i}(x - z_i) \varphi_{c_i}(x - z_i - y_i(v(t))) dx \right)^{1/2} \\ & = \gamma + \sqrt{2} \sum_{i=1}^N c_i \left( 1 - e^{-|y_i(v(t))|} - |y_i(v(t))| e^{-|y_i(v(t))|} \right)^{1/2} \\ & \leq \gamma + \sum_{i=1}^N O(|y_i(v(t))|) \\ & \leq O(\gamma), \end{aligned}$$

where we apply two time the mean value theorem with the function  $\varphi$  on  $[0, |y_i(v(t))|]$  for substituting  $(1 - e^{-|y_i(v(t))|})$  by  $|y_i(v(t))| e^{-\theta}$ , with  $\theta \in ]0, |y_i(v(t))|$ , and this proves (4.4) (see Figures 2a-2b).

The estimate (4.5) follows directly by using (4.4), Remark 4.1 and the Sobolev embedding of  $H^2(\mathbb{R})$  into  $C^1(\mathbb{R})$ .



(a)  $(\int_0^x |\varphi(x-10) - \varphi(x-10-10^{-9})|^2)^{1/2} \approx 10^{-9}$       (b)  $\log(\int_0^x |\varphi(x-10) - \varphi(x-10-10^{-9})|^2)^{1/2} \approx -20.7233$

Figure 2: Distance between two very close peaks (at time  $t = 10$  with respective speeds 1 and  $1+10^{-10}$ ).

To prove that the speed of  $\tilde{x}_i(\cdot)$  stays close to  $c_i$ , we set

$$S_j(t) = \varphi_{c_j}(\cdot - \tilde{x}_j(t)), \quad \varepsilon_1(t) = u(t) - \sum_{j=1}^N S_j(t)$$



and

$$R_j(t) = \rho_{c_j}(\cdot - \tilde{x}_j(t)), \quad \varepsilon_2(t) = v(t) - \sum_{j=1}^N R_j(t).$$

One can notice that

$$\partial_x^2 R_i = 4R_i - S_i, \tag{4.15}$$

and using the Fourier transformation

$$\begin{aligned} (1 - \partial_x^2)^{-1}(4 - \partial_x^2)^{-1}(\cdot) &= \mathcal{F}^{-1} \left[ \frac{1}{(1 - \omega^2)(4 - \omega^2)} \right] (\cdot) \\ &= \mathcal{F}^{-1} \left[ \frac{1}{3(1 - \omega^2)} - \frac{1}{3(4 - \omega^2)} \right] (\cdot) \\ &= \frac{1}{3}(1 - \partial_x^2)^{-1}(\cdot) - \frac{1}{3}(4 - \partial_x^2)^{-1}(\cdot). \end{aligned} \tag{4.16}$$

Differentiating (4.14) with respect to time and using (4.15), we get

$$\int_{\mathbb{R}} \partial_t \varepsilon_2 \partial_x R_i = \dot{\tilde{x}}_i(t) \left( 4 \int_{\mathbb{R}} \varepsilon_2 R_i - \int_{\mathbb{R}} \varepsilon_2 S_i \right)$$

and thus

$$\begin{aligned} \left| \int_{\mathbb{R}} \partial_t \varepsilon_2 \partial_x R_i \right| &\leq |\dot{\tilde{x}}_i(t)| (4 \|\varepsilon_2\|_{L^\infty(\mathbb{R})} \|R_i\|_{L^1(\mathbb{R})} + \|\varepsilon_2\|_{L^\infty(\mathbb{R})} \|S_i\|_{L^1(\mathbb{R})}) \\ &\leq |\dot{\tilde{x}}_i(t) - c_i| O(\gamma) + O(\gamma). \end{aligned} \tag{4.17}$$

Substituting  $u$  by  $\varepsilon_1 + \sum_{j=1}^N S_j$  in (1.7) and using that  $S_j$  satisfies

$$\partial_t S_j = -(\dot{\tilde{x}}_j(t) - c_j) \partial_x S_j - \frac{1}{2} \partial_x S_j^2 - \frac{3}{2} (1 - \partial_x^2)^{-1} \partial_x S_j^2,$$

we infer that  $\varepsilon_1$  satisfies on  $[0, t_0]$ ,

$$\begin{aligned} \partial_t \varepsilon_1 - \sum_{j=1}^N (\dot{\tilde{x}}_j(t) - c_j) \partial_x S_j \\ &= -\frac{1}{2} \partial_x \left[ \left( \varepsilon_1 + \sum_{j=1}^N S_j \right)^2 - \sum_{j=1}^N S_j^2 \right] \\ &\quad - \frac{3}{2} \partial_x (1 - \partial_x^2)^{-1} \left[ \left( \varepsilon_1 + \sum_{j=1}^N S_j \right)^2 - \sum_{j=1}^N S_j^2 \right]. \end{aligned}$$

Multiplying by  $(4 - \partial_x^2)^{-1}$  and using (4.16), we get

$$\partial_t \varepsilon_2 - \sum_{j=1}^N (\dot{\tilde{x}}_j(t) - c_j) \partial_x R_j = -\frac{1}{2} \partial_x (1 - \partial_x^2)^{-1} \left[ \left( \varepsilon_1 + \sum_{j=1}^N S_j \right)^2 - \sum_{j=1}^N S_j^2 \right].$$

Taking the  $L^2$  scalar product with  $\partial_x R_i$ , integrating by parts, we find

$$\begin{aligned}
& -(\dot{x}_i(t) - c_i) \int_{\mathbb{R}} (\partial_x R_i)^2 \\
&= - \int_{\mathbb{R}} \partial_t \varepsilon_2 \partial_x R_i + \sum_{\substack{1 \leq j \leq N \\ j \neq i}} (\dot{x}_j(t) - c_j) \int_{\mathbb{R}} (\partial_x R_i)(\partial_x R_j) \\
&\quad + \frac{1}{2} \int_{\mathbb{R}} (1 - \partial_x^2)^{-1} \left[ \left( \varepsilon_1 + \sum_{j=1}^N S_j \right)^2 - \sum_{j=1}^N S_j^2 \right] \partial_x^2 R_i.
\end{aligned} \tag{4.18}$$

We set

$$Q = \left( \varepsilon_1 + \sum_{j=1}^N S_j \right)^2 - \sum_{j=1}^N S_j^2 = \varepsilon_1^2 + 2\varepsilon_1 \left( \sum_{j=1}^N S_j \right) + \sum_{\substack{1 \leq i, j \leq N \\ j \neq i}} S_i S_j,$$

then

$$\begin{aligned}
(1 - \partial_x^2)^{-1} Q &= (1 - \partial_x^2)^{-1} \varepsilon_1^2 + 2 \sum_{j=1}^N (1 - \partial_x^2)^{-1} (\varepsilon_1 S_j) + \sum_{\substack{1 \leq i, j \leq N \\ j \neq i}} (1 - \partial_x^2)^{-1} (S_i S_j) \\
&= I + J + K.
\end{aligned}$$

We have the following estimates

$$I = \frac{1}{2} \int_{\mathbb{R}} e^{-|x-x'|} \varepsilon_1^2(x') \leq \frac{1}{2} \|e^{-|\cdot|}\|_{L^\infty(\mathbb{R})} \|\varepsilon_1\|_{L^2(\mathbb{R})} \leq \frac{1}{2} \|\varepsilon_1\|_{L^2(\mathbb{R})},$$

$$J = \sum_{j=1}^N \int_{\mathbb{R}} e^{-|x-x'|} \varepsilon_1(x') S_j(x') \leq \|e^{-|\cdot|}\|_{L^\infty(\mathbb{R})} \|\varepsilon_1\|_{L^2(\mathbb{R})} \sum_{j=1}^N \|S_j\|_{L^2(\mathbb{R})} \leq \left( \sum_{j=1}^N c_j \right) \|\varepsilon_1\|_{L^2(\mathbb{R})}$$

and

$$K = \frac{1}{2} \sum_{\substack{1 \leq i, j \leq N \\ j \neq i}} \int_{\mathbb{R}} e^{-|x-x'|} S_j(x') S_i(x') \leq \frac{1}{2} \sum_{\substack{1 \leq i, j \leq N \\ j \neq i}} \int_{\mathbb{R}} S_j(x') S_i(x').$$

Thus, using (4.4) and the exponential decay of  $S_j$ , it holds

$$\|(1 - \partial_x^2)^{-1} Q\|_{L^\infty(\mathbb{R})} \leq O(\gamma) + O(e^{-L/4})$$

and then

$$\begin{aligned}
\left| \frac{1}{2} \int_{\mathbb{R}} [(1 - \partial_x^2)^{-1} Q] \partial_x^2 R_i \right| &\leq \frac{1}{2} \|(1 - \partial_x^2)^{-1} Q\|_{L^\infty(\mathbb{R})} \|\partial_x^2 R_i\|_{L^1(\mathbb{R})} \\
&\leq O(\gamma) + O(e^{-L/4}).
\end{aligned} \tag{4.19}$$

Now, combining (4.17), (4.19), and using the exponential decay of  $\partial_x R_i$ , it holds

$$|\dot{x}_i(t) - c_i| \|\partial_x R_i\|_{L^2(\mathbb{R})}^2 \leq |\dot{x}_i(t) - c_i| O(\gamma) + O(\gamma) + O(e^{-L/4}),$$

then

$$|\dot{x}_i(t) - c_i| \left( \frac{c_i^2}{54} - O(\gamma) \right) \leq O(\gamma) + O(e^{-L/4}),$$

which yields (4.6).

Taking  $0 < \gamma < \gamma_0$  and  $L > L_0 > 0$  with  $\gamma_0 \ll 1$  and  $L_0 \gg 1$ , combining (1.12)-(1.14), (4.6) and (4.13), we deduce that

$$\begin{aligned}\tilde{x}_i(t) - \tilde{x}_{i-1}(t) &= \tilde{x}_i(0) - \tilde{x}_{i-1}(0) + (c_i - c_{i-1})t \\ &\geq L - 2C_0\gamma_0 + \frac{(c_i - c_{i-1})t}{2} \\ &\geq \frac{3L}{4} + \frac{(c_i - c_{i-1})t}{2},\end{aligned}$$

this proves (4.7).

From (4.5), we infer that

$$v(x) = \sum_{j=1}^N \rho_{c_j}(x - \tilde{x}_j) + O(\gamma), \quad \forall x \in \mathbb{R},$$

please note that we abuse notation by writing  $\varepsilon_2(x) = O(\gamma)$ . Applying this formula with  $x = \xi_1^i$  and  $v(\xi_1^i) = \max_{x \in J_i} v(x)$ , and using (4.7), it holds

$$\begin{aligned}v(\xi_1^i) &= \max_{x \in J_i} \left\{ \sum_{j=1}^N \rho_{c_j}(x - \tilde{x}_j) \right\} + O(\gamma) \\ &= \frac{c_i}{6} + O(e^{-L/4}) + O(\gamma) \\ &\geq \frac{c_i}{7}.\end{aligned}$$

On the other hand, for  $x \in J_i \setminus [\tilde{x}_i(t) - L/12, \tilde{x}_i(t) + L/12]$ , we get

$$v(x) \leq \frac{c_i}{3} e^{-L/12} + O(e^{-L/4}) + O(\gamma) \leq \frac{c_i}{8}.$$

This ensures that  $\xi_1^i \in [\tilde{x}_i(t) - L/12, \tilde{x}_i(t) + L/12]$ , and this concluded the proof of the lemma.

## 4.2 Monotonicity property

Thanks to the preceding lemma, for  $\varepsilon_0 > 0$  small enough and  $L_0 > 0$  large enough, one can construct  $N$   $C^1$  functions  $\tilde{x}_1, \dots, \tilde{x}_N$  defined on  $[0, t_0]$  such that (4.4)-(4.8) are satisfied. In this subsection we state the almost monotonicity of functionals that are very close to the energy at the right of  $i$ th bump,  $i = 1, \dots, N-1$  of  $u$ . Let  $\psi$  be a  $C^\infty$  test-function (see Figure 3) such that

$$\begin{cases} 0 < \psi(x) < 1, \quad \psi'(x) > 0, & x \in \mathbb{R}, \\ |\psi^{(q)}(x)| \leq 10\psi'(x), \quad q = 2, 3, 4, 5, & x \in [-10, 10], \end{cases} \quad (4.20)$$

and

$$\psi(x) = \begin{cases} e^{-|x|}, & x < -10, \\ 1 - e^{-|x|}, & x > 10. \end{cases} \quad (4.21)$$

Setting  $\psi_K = \psi(\cdot/K)$ , we introduce for  $i = 2, \dots, N$ ,

$$\mathcal{J}_{i,K}(t) = \int_{\mathbb{R}} (4v^2 + 5v_x^2 + v_{xx}^2) \psi_{i,K}(t), \quad (4.22)$$

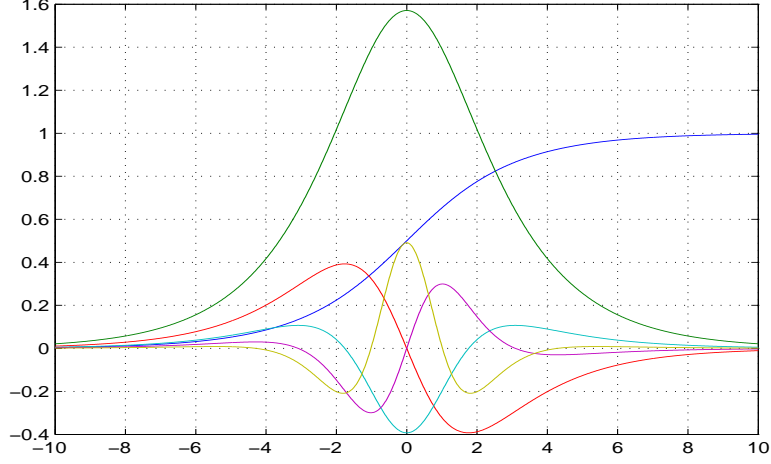


Figure 3:  $\psi^{(q)}(x)$ ,  $q = 0, 1, 2, 3, 4, 5$ , profiles.

where  $\psi_{i,K}(t, x) = \psi_K(x - y_i(t))$  with  $y_i$ 's as in (4.8). Note that  $\mathcal{J}_{i,K}(t)$  is close to  $\|u(t)\|_{\mathcal{H}(x > y_i(t))}^2$  and thus measures the energy at the right of the  $(i - 1)$ th bump of  $u$ . Finally, we set

$$\sigma_0 = \frac{1}{4} \min\{c_1, c_2 - c_1, \dots, c_N - c_{N-1}\}. \quad (4.23)$$

We have the following monotonicity result.

**Proposition 4.1** (Exponential decay of the functional  $\mathcal{J}_{i,K}(t)$ ). *Let  $u \in \mathcal{X}([0, T])$ , with  $0 < T \leq +\infty$ , be a solution of equation (1.1) that satisfies (1.12)-(1.14) and (4.4)-(4.5). There exist  $\gamma_0 > 0$  and  $L_0 > 0$  only depending on  $c_1$  such that if  $0 < \gamma < \gamma_0$  and  $L > L_0 > 0$ , then for any  $4 \leq K \lesssim \sqrt{L}$ ,*

$$\mathcal{J}_{i,K}(t) - \mathcal{J}_{i,K}(0) \leq O(e^{-\frac{t}{8K}}), \quad \forall t \in [0, t_0], \quad i = 2, \dots, N. \quad (4.24)$$

The proof of Proposition 4.1 relies on the following Virial type identity.

**Lemma 4.2** (Virial type identity). *Let  $u \in \mathcal{X}([0, T])$ , with  $0 < T \leq +\infty$ , be a solution of equation (1.1) that satisfies (1.12)-(1.14). For any smooth space function  $g : \mathbb{R} \mapsto \mathbb{R}$ , it holds*

$$\begin{aligned} & \frac{d}{dt} \int_{\mathbb{R}} (4v^2 + 5v_x^2 + v_{xx}^2) g \\ &= \frac{2}{3} \int_{\mathbb{R}} u^3 g' - 4 \int_{\mathbb{R}} u^2 v g' - \frac{1}{2} \int_{\mathbb{R}} u^2 v g''' + \frac{1}{2} \int_{\mathbb{R}} u^2 v_x g'' + \int_{\mathbb{R}} u h g' \\ & \quad + \frac{1}{2} \int_{\mathbb{R}} u h_x g'' - \frac{5}{2} \int_{\mathbb{R}} v h_x g'' - 2 \int_{\mathbb{R}} v_x h g'' + \frac{1}{2} \int_{\mathbb{R}} v h_x g^{(4)} \end{aligned} \quad (4.25)$$

where  $y = (1 - \partial_x^2)u$ ,  $v = (4 - \partial_x^2)^{-1}u$  and  $h = (1 - \partial_x^2)^{-1}u^2$ .

The proof of Lemma 4.2 is given in the appendix.

**Proof of Proposition 4.1.** We first note that, combining (4.6) and (4.8), it holds for  $i = 2, \dots, N$ ,

$$\begin{aligned} \dot{y}_i(t) &= \frac{\tilde{x}_{i-1}(t) + \tilde{x}_i(t)}{2} \\ &= \frac{c_{i-1} + c_i}{2} + O(\gamma^{1/2}) \\ &\geq c_{i-1} + O(\gamma^{1/2}) \\ &\geq \frac{c_1}{2}. \end{aligned} \tag{4.26}$$

Recall that the assumption (1.12) ensures that  $u \geq 0$  and  $v \geq 0$  on  $\mathbb{R}$ . Now, applying the Virial type identity (4.25) with  $g = \psi_{i,K}$  and using (4.26), we get

$$\begin{aligned} \frac{d}{dt} \mathcal{J}_{i,K}(t) &= -\dot{y}_i \int_{\mathbb{R}} (4v^2 + 5v_x^2 + v_{xx}^2) \psi'_{i,K} + \frac{2}{3} \int_{\mathbb{R}} u^3 \psi'_{i,K} - 4 \int_{\mathbb{R}} u^2 v \psi'_{i,K} \\ &\quad - \frac{1}{2} \int_{\mathbb{R}} u^2 v \psi'''_{i,K} + \frac{1}{2} \int_{\mathbb{R}} u^2 v_x \psi''_{i,K} + \int_{\mathbb{R}} u h \psi'_{i,K} + \frac{1}{2} \int_{\mathbb{R}} u h_x \psi''_{i,K} \\ &\quad - \frac{5}{2} \int_{\mathbb{R}} v h_x \psi''_{i,K} - 2 \int_{\mathbb{R}} v_x h \psi''_{i,K} + \frac{1}{2} \int_{\mathbb{R}} v h_x \psi^{(4)}_{i,K} \\ &\leq -\dot{y}_i \int_{\mathbb{R}} (4v^2 + 5v_x^2 + v_{xx}^2) \psi'_{i,K} + \frac{2}{3} \int_{\mathbb{R}} u^3 \psi'_{i,K} - \frac{1}{2} \int_{\mathbb{R}} u^2 v \psi'''_{i,K} \\ &\quad + \frac{1}{2} \int_{\mathbb{R}} u^2 v_x \psi''_{i,K} + \int_{\mathbb{R}} u h \psi'_{i,K} + \frac{1}{2} \int_{\mathbb{R}} u h_x \psi''_{i,K} - \frac{5}{2} \int_{\mathbb{R}} v h_x \psi''_{i,K} \\ &\quad - 2 \int_{\mathbb{R}} v_x h \psi''_{i,K} + \frac{1}{2} \int_{\mathbb{R}} v h_x \psi^{(4)}_{i,K} \\ &\leq -\frac{c_1}{2} \int_{\mathbb{R}} (4v^2 + 5v_x^2 + v_{xx}^2) \psi'_{i,K} + \sum_{k=1}^8 J_k. \end{aligned} \tag{4.27}$$

We claim that for  $k = 1, \dots, 8$ , it holds

$$J_k \leq \frac{c_1}{20} \int_{\mathbb{R}} (4v^2 + 5v_x^2 + v_{xx}^2) \psi'_{i,K} + \frac{C}{K} \|u_0\|_{\mathcal{H}}^3 e^{-\frac{1}{K}(\sigma_0 t + L/8)}. \tag{4.28}$$

We divide  $\mathbb{R}$  into two regions  $D_i$  and  $D_i^c$  with

$$D_i = \left[ \tilde{x}_{i-1}(t) + \frac{L}{4}, \tilde{x}_i(t) - \frac{L}{4} \right], \quad i = 2, \dots, N.$$

Combining (4.7) and (4.8), one can check that for  $x \in D_i^c$ ,

$$\begin{aligned} |x - y_i(t)| &\geq \frac{\tilde{x}_i(t) - \tilde{x}_{i-1}(t)}{2} - \frac{L}{4} \\ &\geq \frac{c_i - c_{i-1}}{4} t + \frac{L}{8} \\ &\geq \sigma_0 t + \frac{L}{8}. \end{aligned} \tag{4.29}$$

Let us begin by an estimate of  $J_1$ . Using (2.3), (4.29) and the exponential decay of  $\psi'_{i,K}$  on  $D_i^c$ , we get

$$\begin{aligned} \frac{2}{3} \int_{\mathbb{R}} u^3 \psi'_{i,K} &= \frac{2}{3} \int_{D_i} u^3 \psi'_{i,K} + \frac{2}{3} \int_{D_i^c} u^3 \psi'_{i,K} \\ &\leq \frac{2}{3} \|u\|_{L^\infty(D_i)} \int_{\mathbb{R}} u^2 \psi'_{i,K} + \frac{2}{3} \|\psi'_{i,K}\|_{L^\infty(D_i^c)} \|u\|_{L^\infty(\mathbb{R})} \|u\|_{L^2(\mathbb{R})}^2 \\ &\leq \frac{2}{3} \|u\|_{L^\infty(D_i)} \int_{\mathbb{R}} u^2 \psi'_{i,K} + \frac{C}{K} \|u\|_{L^2(\mathbb{R})}^3 e^{-\frac{1}{K}(\sigma_0 t + L/8)}. \end{aligned} \tag{4.30}$$

Note that, using the exponential decay of  $|\psi_{j,K}'''|$  on  $D_i^c$ ,  $|\psi_{i,K}'''| \leq (10/K^2)\psi'_{i,K}$ , with  $K \geq 4$ , and that  $\|u\|_{\mathcal{H}} = \|u_0\|_{\mathcal{H}}$ , we have

$$\begin{aligned}
\int_{\mathbb{R}} u^2 \psi'_{i,K} &= \int_{\mathbb{R}} (4v - v_{xx})^2 \psi'_{i,K} \\
&= 16 \int_{\mathbb{R}} v^2 \psi'_{i,K} + \int_{\mathbb{R}} v_{xx}^2 \psi'_{i,K} - 8 \int_{\mathbb{R}} v v_{xx} \psi'_{i,K} \\
&= 16 \int_{\mathbb{R}} v^2 \psi'_{i,K} + \int_{\mathbb{R}} v_{xx}^2 \psi'_{i,K} + 8 \int_{\mathbb{R}} v_x^2 \psi'_{i,K} + 4 \int_{\mathbb{R}} \partial_x(v^2) \psi''_{i,K} \\
&= 16 \int_{\mathbb{R}} v^2 \psi'_{i,K} + \int_{\mathbb{R}} v_{xx}^2 \psi'_{i,K} + 8 \int_{\mathbb{R}} v_x^2 \psi'_{i,K} - 4 \int_{\mathbb{R}} v^2 \psi'''_{i,K} \\
&\leq \int_{\mathbb{R}} (16v^2 + 8v_x^2 + v_{xx}^2) \psi'_{i,K} + 4 \int_{D_i} v^2 |\psi'''_{j,K}| + 4 \int_{D_i^c} v^2 |\psi'''_{j,K}| \\
&\leq \int_{\mathbb{R}} (16v^2 + 8v_x^2 + v_{xx}^2) \psi'_{i,K} + \frac{40}{K^2} \int_{\mathbb{R}} v^2 \psi'_{i,K} + \frac{C}{K^3} \|u_0\|_{\mathcal{H}}^2 e^{-\frac{1}{K}(\sigma_0 t + L/8)} \\
&\leq 5 \int_{\mathbb{R}} (4v^2 + 5v_x^2 + v_{xx}^2) \psi'_{i,K} + \frac{C}{K^3} \|u_0\|_{\mathcal{H}}^2 e^{-\frac{1}{K}(\sigma_0 t + L/8)}. \tag{4.31}
\end{aligned}$$

Now, using the exponential decay of  $\varphi_{c_i}$  on  $D_i$ , (4.4), and proceeding as for the estimate (3.11) (see Lemma 3.2), it holds

$$\begin{aligned}
\|u\|_{L^\infty(D_i)} &\leq \left\| u - \sum_{j=1}^N \varphi_{c_j}(\cdot - \tilde{x}_j(t)) \right\|_{L^\infty(D_i)} + \sum_{j=1}^N \|\varphi_{c_j}(\cdot - \tilde{x}_j(t))\|_{L^\infty(D_i)} \\
&\leq O(\gamma^{1/2}) + O(e^{-L/8}). \tag{4.32}
\end{aligned}$$

Therefore, for  $0 < \gamma < \gamma_0$  and  $L > L_0 > 0$ , with  $\gamma_0 \ll 1$  and  $L_0 \gg 1$ , combining (4.30)-(4.31), we obtain

$$J_1 \leq \frac{c_1}{20} \int_{\mathbb{R}} (4v^2 + 5v_x^2 + v_{xx}^2) \psi'_{i,K} + \frac{C}{K} \|u\|_{L^2(\mathbb{R})}^3 e^{-\frac{1}{K}(\sigma_0 t + L/8)}.$$

Next, the estimate of  $J_2$  on  $D_i^c$  gives us

$$\left| \frac{1}{2} \int_{D_i^c} u^2 v \psi'''_{i,K} \right| \leq \frac{1}{2} \|\psi'''_{i,K}\|_{L^\infty(D_i^c)} \|v\|_{L^\infty(\mathbb{R})} \|u\|_{L^2(\mathbb{R})}^2.$$

Note that, applying Hölder inequality, we have for all  $x \in \mathbb{R}$ ,

$$\begin{aligned}
v(x) &= \frac{1}{4} \int_{\mathbb{R}} e^{-2|x-x'|} u(x') dx' \\
&\leq \frac{1}{4} \left( \int_{\mathbb{R}} e^{-4|x-x'|} dx' \right)^{1/2} \left( \int_{\mathbb{R}} |u(x')|^2 dx' \right)^{1/2} \\
&= \frac{1}{4\sqrt{2}} \|u\|_{L^2(\mathbb{R})} \tag{4.33}
\end{aligned}$$

and thus, using (4.33) and the exponential decay of  $|\psi'''_{i,K}|$  on  $D_i^c$ , it holds

$$\left| \frac{1}{2} \int_{D_i^c} u^2 v \psi'''_{i,K} \right| \leq \frac{1}{8\sqrt{2}K^3} \|u\|_{L^2(\mathbb{R})}^3 e^{-\frac{1}{K}(\sigma_0 t + L/8)}. \tag{4.34}$$

Using that  $|\psi'''_{i,K}| \leq (10/K^2)\psi'_{i,K}$ , the estimate of  $J_2$  on  $D_i$  leads to

$$\left| \frac{1}{2} \int_{D_i} u^2 v \psi'''_{i,K} \right| \leq \frac{5}{K^2} \|u\|_{L^\infty(D_i)} \int_{\mathbb{R}} uv \psi'_{i,K}. \quad (4.35)$$

Also, one can notice that, using the exponential decay of  $|\psi'''_{j,K}|$  on  $D_i^c$ ,  $|\psi'''_{i,K}| \leq (10/K^2)\psi'_{i,K}$ , with  $K \geq 4$ , and that  $\|u\|_{\mathcal{H}} = \|u_0\|_{\mathcal{H}}$ , we have

$$\begin{aligned} \int_{\mathbb{R}} uv \psi'_{i,K} &= \int_{\mathbb{R}} (4v - v_{xx}) v \psi'_{i,K} \\ &= 4 \int_{\mathbb{R}} v^2 \psi'_{i,K} + \int_{\mathbb{R}} \partial_x (v \psi'_{i,K}) v_x \\ &= 4 \int_{\mathbb{R}} v^2 \psi'_{i,K} + \int_{\mathbb{R}} v_x^2 \psi'_{i,K} + \int_{\mathbb{R}} v v_x \psi''_{i,K} \\ &= \int_{\mathbb{R}} (4v^2 + v_x^2) \psi'_{i,K} - \frac{1}{2} \int_{\mathbb{R}} v^2 \psi'''_{i,K} \\ &\leq \int_{\mathbb{R}} (4v^2 + v_x^2) \psi'_{i,K} + \frac{1}{2} \int_{D_i} v^2 |\psi'''_{i,K}| + \frac{1}{2} \int_{D_i^c} v^2 |\psi'''_{i,K}| \\ &\leq \int_{\mathbb{R}} (4v^2 + v_x^2) \psi'_{i,K} + \frac{10}{2K^2} \int_{\mathbb{R}} v^2 \psi'_{i,K} + \frac{C}{K^3} \|u_0\|_{\mathcal{H}}^2 e^{-\frac{1}{K}(\sigma_0 t + L/8)} \\ &\leq 2 \int_{\mathbb{R}} (4v^2 + 5v_x^2 + v_{xx}^2) \psi'_{i,K} + \frac{C}{K^3} \|u_0\|_{\mathcal{H}}^2 e^{-\frac{1}{K}(\sigma_0 t + L/8)}. \end{aligned} \quad (4.36)$$

Therefore, for  $0 < \gamma < \gamma_0$  and  $L > L_0 > 0$ , with  $\gamma_0 \ll 1$  and  $L_0 \gg 1$ , combining (4.32), (4.34)-(4.36), it holds

$$J_2 \leq \frac{c_1}{20} \int_{\mathbb{R}} (4v^2 + 5v_x^2 + v_{xx}^2) \psi'_{i,K} + \frac{C}{K^3} \|u\|_{L^2(\mathbb{R})}^3 e^{-\frac{1}{K}(\sigma_0 t + L/8)}.$$

In the same way, using that  $|v_x| \leq 2v$  on  $\mathbb{R}$  (see (3.40)), and the definition of  $\psi_{i,K}$  (see (4.20) and (4.21)), we deduce the estimate of  $J_3$ .

Let us tackle now the estimate of  $J_4$ . On  $D_i^c$  we have

$$\begin{aligned} \int_{D_i^c} uh \psi'_{i,K} &\leq \|\psi'_{i,K}\|_{L^\infty(D_i^c)} \int_{\mathbb{R}} uh \\ &= \|\psi'_{i,K}\|_{L^\infty(D_i^c)} \int_{\mathbb{R}} u [(1 - \partial_x^2)^{-1} u^2] \\ &= \|\psi'_{i,K}\|_{L^\infty(D_i^c)} \int_{\mathbb{R}} u^2 [(1 - \partial_x^2)^{-1} u] \\ &\leq \|\psi'_{i,K}\|_{L^\infty(D_i^c)} \|(1 - \partial_x^2)^{-1} u\|_{L^\infty(\mathbb{R})} \|u\|_{L^2(\mathbb{R})}^2. \end{aligned}$$

Remark that, applying Hölder inequality, we have for all  $x \in \mathbb{R}$ ,

$$\begin{aligned} (1 - \partial_x^2)^{-1} u(x) &\leq \frac{1}{2} \int_{\mathbb{R}} e^{-|x-x'|} |u(x')| dx' \\ &\leq \frac{1}{2} \left( \int_{\mathbb{R}} e^{-2|x-x'|} dx' \right)^{1/2} \left( \int_{\mathbb{R}} |u(x')|^2 dx' \right)^{1/2} \\ &= \frac{1}{2} \|u\|_{L^2(\mathbb{R})} \end{aligned} \quad (4.37)$$

and thus, using (4.37) and the exponential decay of  $\psi'_{i,K}$  on  $D_i^c$ , it holds

$$\int_{D_i^c} uh \psi'_{i,K} \leq \frac{1}{2K} \|u\|_{L^2(\mathbb{R})}^3 e^{-\frac{1}{K}(\sigma_0 t + L/8)}. \quad (4.38)$$

The estimate of  $J_4$  on  $D_i$  leads to

$$\begin{aligned} \int_{D_i} u h \psi'_{i,K} &\leq \|u\|_{L^\infty(D_i)} \int_{\mathbb{R}} \psi'_{i,K} [(1 - \partial_x^2)^{-1} u^2] \\ &= \|u\|_{L^\infty(D_i)} \int_{\mathbb{R}} u^2 [(1 - \partial_x^2)^{-1} \psi'_{i,K}]. \end{aligned} \quad (4.39)$$

On the other hand, using that  $|\psi'''_{i,K}| \leq (10/K^2)\psi'_{i,K}$ , we have

$$(1 - \partial_x^2) \psi'_{i,K}(x) = \psi'_{i,K}(x) - \frac{1}{K^2} \psi'''_{i,K}(x) \geq \left(1 - \frac{10}{K^2}\right) \psi'_{i,K}(x), \quad \forall x \in \mathbb{R},$$

and since  $K \geq 4$ , it holds

$$(1 - \partial_x^2)^{-1} \psi'_{i,K}(x) \leq \left(1 - \frac{10}{K^2}\right)^{-1} \psi'_{i,K}(x), \quad \forall x \in \mathbb{R}. \quad (4.40)$$

Therefore, for  $0 < \gamma < \gamma_0$  and  $L > L_0 > 0$ , with  $\gamma_0 \ll 1$  and  $L_0 \gg 1$ , combining (4.32), (4.38)-(4.40), it holds

$$J_4 \leq \frac{c_1}{20} \int_{\mathbb{R}} (4v^2 + 5v_x^2 + v_{xx}^2) \psi'_{i,K} + \frac{C}{K} \|u\|_{L^2(\mathbb{R})}^3 e^{-\frac{1}{K}(\sigma_0 t + L/8)}.$$

Noticing that for all  $x \in \mathbb{R}$ ,

$$h(x) = \frac{e^{-x}}{2} \int_{-\infty}^x e^{x'} u^2(x') dx' + \frac{e^x}{2} \int_x^{+\infty} e^{-x'} u^2(x') dx'$$

and

$$h_x(x) = -\frac{e^{-x}}{2} \int_{-\infty}^x e^{x'} u^2(x') dx' + \frac{e^x}{2} \int_x^{+\infty} e^{-x'} u^2(x') dx',$$

we infer that

$$|h_x(x)| \leq h(x), \quad \forall x \in \mathbb{R}. \quad (4.41)$$

Then, combining (4.20), (4.41), and proceeding as for the estimate of  $J_4$ , we deduce the estimate of  $J_5$ .

Now, combining (4.33) and (4.37), we have for all  $x \in \mathbb{R}$ ,

$$\begin{aligned} (1 - \partial_x^2)^{-1} v(x) &= \frac{1}{3} (1 - \partial_x^2)^{-1} u(x) - \frac{1}{3} v(x) \\ &\leq \frac{1}{3} \|(1 - \partial_x^2)^{-1} u\|_{L^\infty(\mathbb{R})} + \frac{1}{3} \|v\|_{L^\infty(\mathbb{R})} \\ &\leq \frac{4 + \sqrt{2}}{24} \|u\|_{L^2(\mathbb{R})}, \end{aligned} \quad (4.42)$$

and using the exponential decay of  $\rho_{c_i}$  on  $D_i$  and (4.5), it holds

$$\begin{aligned} \|v\|_{L^\infty(D_i)} &\leq \left\| v - \sum_{j=1}^N \rho_{c_j}(\cdot - \tilde{x}_j(t)) \right\|_{L^\infty(D_i)} + \sum_{j=1}^N \|\rho_{c_j}(\cdot - \tilde{x}_j(t))\|_{L^\infty(D_i)} \\ &\leq O(\gamma) + O(e^{-L/8}). \end{aligned} \quad (4.43)$$

Therefore, combining (3.40), (4.20), (4.41)-(4.43), and proceeding as for the estimate of  $J_4$ , we deduce the estimates of the remaining terms.

Finally, combining (4.27), (4.28) and using that  $\|u\|_{L^2(\mathbb{R})} \sim \|u_0\|_{\mathcal{H}}$ , it holds actually

$$\frac{d}{dt} \mathcal{J}_{i,K}(t) \leq \frac{C}{K} \|u_0\|_{\mathcal{H}}^3 e^{-\frac{1}{K}(\sigma_0 t + L/8)}.$$



Integrating between 0 and  $t$ , we obtain

$$\begin{aligned}\mathcal{J}_{i,K}(t) - \mathcal{J}_{i,K}(0) &\leq \frac{C}{K} \|u_0\|_{\mathcal{H}}^3 \left( -\frac{K}{\sigma_0} e^{-\frac{1}{K}(\sigma_0 t + L/8)} + \frac{K}{\sigma_0} e^{-\frac{L}{8K}} \right) \\ &\leq \frac{C}{\sigma_0} \|u_0\|_{\mathcal{H}}^3 e^{-\frac{L}{8K}},\end{aligned}$$

and this proves the proposition for smooth initial solutions.

For  $u \in \mathcal{X}([0, T[)$ , we will use that for any  $T_0 > 0$  and any sequence  $(u_{0,n})_{n \geq 1} \subset L^2(\mathbb{R})$  such that  $(u_{0,n} - \partial_x^2 u_{0,n})_{n \geq 1} \subset \mathcal{M}^+(\mathbb{R})$  and  $u_{0,n} \rightarrow u_0$  in  $L^2(\mathbb{R})$ , the sequence of emanating global weak solutions  $(u_n)_{n \geq 1}$  to the DP equation satisfies

$$u_n \xrightarrow[n \rightarrow +\infty]{} u \text{ in } C([0, T_0]; L^2(\mathbb{R})), \quad (4.44)$$

where  $u$  is the global weak solution emanating from  $u_0$ . This fact can be easily deduced from the proof of the existence of the global weak solutions in [4]. Indeed, by the same arguments developed in this proof, we obtain that, up to a subsequence,  $(u_n)_{n \geq 1}$  converges in  $C([0, T_0]; L^2(\mathbb{R}))$  towards a solution of the DP equation emanating from  $u_0$ . (4.44) then follows by the uniqueness result. Combining (4.44) and Remark 4.1, it follows that

$$v_n \xrightarrow[n \rightarrow +\infty]{} v \text{ in } C([0, T_0]; W^{1,\infty}(\mathbb{R})), \quad (4.45)$$

where  $v = (4 - \partial_x^2)^{-1}u$ . For all  $t \in [0, T[$ , we set

$$\mathcal{J}_{i,K}^n(t) = \mathcal{J}_{i,K}(u_n(t)) = \int_{\mathbb{R}} (4v_n^2 + 5v_{n,x}^2 + v_{n,xx}^2) \psi_{i,K}(t), \quad (4.46)$$

and we claim that

$$\lim_{n \rightarrow +\infty} \sup_{0 \leq t < T} |\mathcal{J}_{i,K}^n(t) - \mathcal{J}_{i,K}(t)| = 0. \quad (4.47)$$

Let  $t \in [0, T[$  be fixed, we compute

$$\begin{aligned}\mathcal{J}_{i,K}^n(t) - \mathcal{J}_{i,K}(t) &= 4 \int_{\mathbb{R}} (v_n^2 - v^2) \psi_{i,K}(t) + 5 \int_{\mathbb{R}} (v_{n,x}^2 - v_x^2) \psi_{i,K}(t) + \int_{\mathbb{R}} (v_{n,xx}^2 - v_{xx}^2) \psi_{i,K}(t) \\ &= K_1^n(t) + K_2^n(t) + K_3^n(t).\end{aligned} \quad (4.48)$$

Then it is easy to check that

$$\begin{aligned}|K_1^n(t)| &\leq 4 \int_{\mathbb{R}} |v_n - v| (v_n + v) \psi_{i,K}(t) \\ &= 4 \|v_n - v\|_{L^\infty(\mathbb{R})} \|v_n + v\|_{L^\infty(\mathbb{R})} \|\psi_{i,K}\|_{L^1(\mathbb{R})} \\ &\leq O(\|v_n - v\|_{L^\infty(\mathbb{R})}) \\ &\rightarrow 0 \text{ as } n \rightarrow +\infty\end{aligned} \quad (4.49)$$

and

$$\begin{aligned}|K_2^n(t)| &\leq 5 \int_{\mathbb{R}} |v_{n,x} - v_x| (v_{n,x} + v_x) \psi_{i,K}(t) \\ &= 4 \|v_{n,x} - v_x\|_{L^\infty(\mathbb{R})} \|v_{n,x} + v_x\|_{L^\infty(\mathbb{R})} \|\psi_{i,K}\|_{L^1(\mathbb{R})} \\ &\leq O(\|v_{n,x} - v_x\|_{L^\infty(\mathbb{R})}) \\ &\rightarrow 0 \text{ as } n \rightarrow +\infty.\end{aligned} \quad (4.50)$$

Recalling that  $v_{xx} = 4v - u$  and thus  $v_{xx}^2 = 16v^2 + u^2 - 8uv$ , we also get

$$\begin{aligned}
|K_3^n(t)| &\leq 16 \int_{\mathbb{R}} |v_n - v|(v_n + v)\psi_{i,K}(t) + \int_{\mathbb{R}} |u_n - u|(u_n + u)\psi_{i,K}(t) \\
&\quad + 8 \int_{\mathbb{R}} u_n |v_n - v| \psi_{i,K}(t) + 8 \int_{\mathbb{R}} v |u_n - u| \psi_{i,K}(t) \\
&\leq 16 \|v_n - v\|_{L^\infty(\mathbb{R})} \|v_n + v\|_{L^\infty(\mathbb{R})} \|\psi_{i,K}\|_{L^1(\mathbb{R})} \\
&\quad + \|u_n - u\|_{L^2(\mathbb{R})} \|u_n + u\|_{L^2(\mathbb{R})} \|\psi_{i,K}\|_{L^\infty(\mathbb{R})} \\
&\quad + 8 \|u_n\|_{L^\infty(\mathbb{R})} \|v_n - v\|_{L^\infty(\mathbb{R})} \|\psi_{i,K}\|_{L^1(\mathbb{R})} \\
&\quad + 8 \|v\|_{L^2(\mathbb{R})} \|u_n - u\|_{L^2(\mathbb{R})} \|\psi_{i,K}\|_{L^\infty(\mathbb{R})} \\
&\leq O(\|u_n - u\|_{L^2(\mathbb{R})}) + O(\|v_n - v\|_{L^\infty(\mathbb{R})}) \\
&\rightarrow 0 \text{ as } n \rightarrow +\infty.
\end{aligned} \tag{4.51}$$

Combining (4.48)-(4.51), we obtain (4.47).

Thanks to (4.47), the monotonicity formula (4.24) holds for any  $u \in \mathcal{X}([0, T])$ , with  $0 < T \leq +\infty$ .

### 4.3 A localized and a global estimate

Let  $K = \sqrt{L}/8$  and define the function  $\phi_i = \phi_i(t, x)$  (see Figure 4) by

$$\begin{cases} \phi_1 = 1 - \psi_{2,K} = 1 - \psi_K(\cdot - y_2(t)), \\ \phi_i = \psi_{i,K} - \psi_{i+1,K} = \psi_K(\cdot - y_i(t)) - \psi_K(\cdot - y_{i+1}(t)), \quad i = 2, \dots, N-1, \\ \phi_N = \psi_{N,K} = \psi_K(\cdot - y_N(t)), \end{cases} \tag{4.52}$$

where  $\psi_{i,K}$ 's and  $y_i$ 's are defined in Subsection 4.2. One can see that the  $\phi_i$ 's are positive functions and that  $\sum_{i=1}^N \phi_i = 1$ . We take  $L/K > 4$  so that  $\phi_i$  satisfies for  $i = 1, \dots, N$ ,

$$|1 - \phi_i| \leq 2e^{-\frac{L}{8K}} \text{ on } \left] y_i + \frac{L}{8}, y_{i+1} - \frac{L}{8} \right[ \tag{4.53}$$

and

$$|\phi_i| \leq 2e^{-\frac{L}{8K}} \text{ on } \mathbb{R} \setminus \left] y_i - \frac{L}{8}, y_{i+1} + \frac{L}{8} \right[. \tag{4.54}$$

We will use the following localized version of the conservation laws defined for  $i = 1, \dots, N$  by

$$E_i(t) = \int_{\mathbb{R}} (4v^2 + 5v_x^2 + v_{xx}^2) \phi_i(t) \text{ and } F_i(t) = \int_{\mathbb{R}} (-v_{xx}^3 + 12vv_{xx}^2 - 48v^2v_{xx} + 64v^3) \phi_i(t). \tag{4.55}$$

One can remark that the functionals  $E_i(\cdot)$  and  $F_i(\cdot)$  do not depend on time in the statement below since we fix  $-\infty = y_1 < y_2 < \dots < y_N < y_{N+1} = +\infty$ .

For  $i = 1, \dots, N$ , we set  $\Omega_i = ]y_i - L/8, y_{i+1} + L/8[$ . First, one can notice that

$$\sum_{j=1}^N \rho_{c_j}(x - \tilde{x}_j) = \rho_{c_i}(x - \tilde{x}_i) + O(e^{-L/4}), \quad \forall x \in \Omega_i, \tag{4.56}$$

we abuse notation by writing  $\rho_{c_i, \alpha}(x - \tilde{x}_i) = O(e^{-L/4})$  for all  $x \in \Omega_j$  when  $j \neq i$ . We will now decompose this interval according to the variation of  $v = (4 - \partial_x^2)^{-1}u$  in the same way as in Section 3. We set

$$\alpha_i = \sup \left\{ x < \tilde{x}_i, v(x) = \frac{c_i}{2400} \right\} \text{ and } \beta_i = \inf \left\{ x > \tilde{x}_i, v(x) = \frac{c_i}{2400} \right\}. \tag{4.57}$$

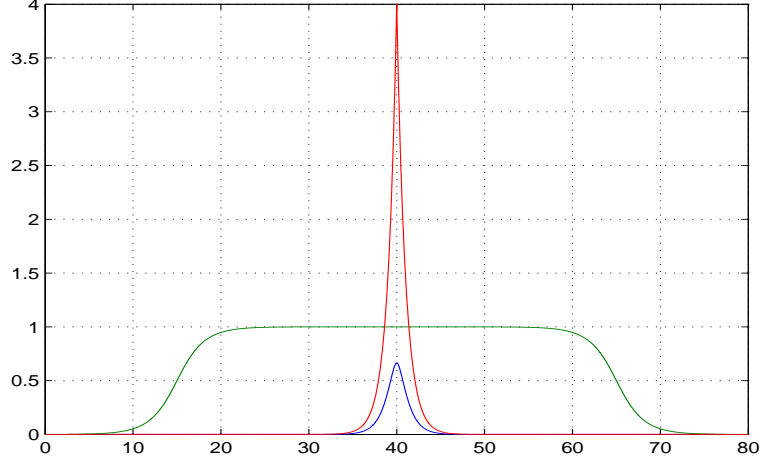


Figure 4: Localization-function  $\phi_{green}(x) = \psi(x - 15) - \psi(x - 65)$  (at time  $t = 10$ ) profile. Also, the peakon  $4\varphi(x - 40)$  and the smooth-peakon  $4\rho(x - 40)$  (at time  $t = 10$  with speed  $c = 4$ ) profiles. In this example, one can see that  $\phi_{green}$  is close to 1 on  $]25, 55[$ , and decays exponentially to 0 on  $\mathbb{R} \setminus ]10, 70[$ .

According to Lemma 4.1, we know that  $v$  is close to  $\sum_{i=1}^N \rho_c(\cdot - \tilde{x}_i)$  in  $L^\infty$  norm with  $\rho_c(0) = c_i/6$ . Therefore  $v$  must have at least one local maximum on  $[\alpha_i, \beta_i]$ . Assume that on  $[\alpha_i, \beta_i]$  the function  $v$  admits  $k_i + 1$  points  $(\xi_j^i)_{j=1}^{k_i+1}$  with local maximal values for some integer  $k_i \geq 0$ , where  $\xi_1^i$  is the first local maximum point and  $\xi_{k_i+1}^i$  the last local maximum point<sup>3</sup>. Then between  $\xi_1^i$  and  $\xi_{k_i+1}^i$ , the function  $v$  admits  $k_i$  points  $(\eta_j^i)_{j=1}^{k_i}$  with local minimal values. We rename  $\alpha_i = \eta_0^i$  and  $\beta_i = \eta_{k_i+1}^i$  so that it holds

$$\eta_0^i < \xi_1^i < \eta_1^i < \dots < \xi_j^i < \eta_j^i < \xi_{j+1}^i < \eta_{j+1}^i < \dots < \eta_{k_i}^i < \xi_{k_i+1}^i < \eta_{k_i+1}^i. \quad (4.58)$$

Let

$$M_j^i = v(\xi_j^i), \quad j = 1, \dots, k_i + 1, \quad \text{and} \quad m_j^i = v(\eta_j^i), \quad j = 1, \dots, k_i. \quad (4.59)$$

By construction

$$v_x(x) \geq 0, \quad \forall x \in [\eta_{j-1}^i, \xi_j^i], \quad j = 1, \dots, k_i \quad (4.60)$$

and

$$v_x(x) \leq 0, \quad \forall x \in [\xi_j^i, \eta_j^i], \quad j = 1, \dots, k_i + 1. \quad (4.61)$$

Proceeding as for (3.19)-(3.21), we also have

$$v(x) \leq \frac{c_i}{300}, \quad \forall x \in \Omega_i \setminus [\eta_0^i, \eta_{k_i+1}^i], \quad (4.62)$$

$$u(x) \leq \frac{c_i}{300}, \quad \forall x \in \Omega_i \setminus [\eta_0^i, \eta_{k_i+1}^i], \quad (4.63)$$

and taking  $L > L_0 > 8C_0$ , it holds

$$[\eta_0^i, \eta_{k_i+1}^i] \subset [\tilde{x}_i - C_0, \tilde{x}_i + C_0] \subset \left] y_i + \frac{L}{8}, y_{i+1} - \frac{L}{8} \right[ , \quad (4.64)$$

where  $C_0 > 0$  is the universal constant appearing in (3.21).

We now derive versions of Lemmas 3.4, Lemma 3.5 and Lemma 3.6 where the global functional  $E(\cdot)$  and  $F(\cdot)$  are replaced by their localized versions  $E_i(\cdot)$  and  $F_i(\cdot)$ . Please note that, we will change the order of the extrema of  $v = (4 - \partial_x^2)^{-1}u$  while keeping the same notations as in (4.59).

<sup>3</sup>In the case of a infinite countable number of local maximal values, the proof is exactly the same.

**Lemma 4.3** (Connection between  $E_i(\cdot)$  and the local extrema of  $v$ ). *Let  $u \in H^1(\mathbb{R})$  with  $u \geq 0$ . For  $i = 1, \dots, N$ , define the function  $g_i$  by*

$$g_i(x) = \begin{cases} 2v + v_{xx} - 3v_x, & x < \xi_1^i, \\ 2v + v_{xx} + 3v_x, & \xi_j^i < x < \eta_j^i, \\ 2v + v_{xx} - 3v_x, & \eta_j^i < x < \xi_{j+1}^i, \\ 2v + v_{xx} + 3v_x, & x > \xi_{k_i+1}^i, \end{cases} \quad j = 1, \dots, k_i. \quad (4.65)$$

Then it holds

$$\int_{\mathbb{R}} g_i^2(x) \phi_i(x) dx = E_i(u) - 12 \left( \sum_{j=0}^{k_i} (M_{j+1}^i)^2 \phi_i(\xi_{j+1}^i) - \sum_{j=1}^{k_i} (m_j^i)^2 \phi_i(\eta_j^i) \right) + \|u\|_{\mathcal{H}}^2 O(L^{-1/2}). \quad (4.66)$$

**Proof.** We have

$$\int_{\mathbb{R}} g_i^2(x) \phi_i(x) dx = \int_{-\infty}^{\xi_1^i} g_i^2(x) \phi_i(x) dx + \sum_{j=1}^{k_i} \int_{\xi_j^i}^{\xi_{j+1}^i} g_i^2(x) \phi_i(x) dx + \int_{\xi_{k_i+1}^i}^{+\infty} g_i^2(x) \phi_i(x) dx. \quad (4.67)$$

For  $j = 1, \dots, k_i$ ,

$$\begin{aligned} \int_{\xi_j^i}^{\xi_{j+1}^i} g_i^2(x) \phi_i(x) dx &= \int_{\xi_j^i}^{\eta_j^i} (2v + v_{xx} + 3v_x)^2 \phi_i(x) dx + \int_{\eta_j^i}^{\xi_{j+1}^i} (2v + v_{xx} - 3v_x)^2 \phi_i(x) dx \\ &= J + I. \end{aligned}$$

Computing  $I$ , we obtain

$$\begin{aligned} I &= \int_{\eta_j^i}^{\xi_{j+1}^i} (4v^2 + v_{xx}^2 + 9v_x^2 + 4vv_{xx} - 12vv_x - 6v_x v_{xx}) \phi_i \\ &= \int_{\eta_j^i}^{\xi_{j+1}^i} (4v^2 + v_{xx}^2 + 9v_x^2) \phi_i + 4 \int_{\eta_j^i}^{\xi_{j+1}^i} vv_{xx} \phi_i - 12 \int_{\eta_j^i}^{\xi_{j+1}^i} vv_x \phi_i - 6 \int_{\eta_j^i}^{\xi_{j+1}^i} v_x v_{xx} \phi_i \\ &= \int_{\eta_j^i}^{\xi_{j+1}^i} (4v^2 + v_{xx}^2 + 9v_x^2) \phi_i + I_1 + I_2 + I_3 \end{aligned} \quad (4.68)$$

with

$$\begin{aligned} I_1 &= -4 \int_{\eta_j^i}^{\xi_{j+1}^i} \partial_x(v \phi_i) v_x = -4 \int_{\eta_j^i}^{\xi_{j+1}^i} v_x^2 \phi_i - 4 \int_{\eta_j^i}^{\xi_{j+1}^i} vv_x \phi_i' = -4 \int_{\eta_j^i}^{\xi_{j+1}^i} v_x^2 \phi_i - 2 \int_{\eta_j^i}^{\xi_{j+1}^i} (v^2)_x \phi_i' \\ &= -2v^2(\xi_{j+1}^i) \phi_i'(\xi_{j+1}^i) + 2v^2(\eta_j^i) \phi_i'(\eta_j^i) - 4 \int_{\eta_j^i}^{\xi_{j+1}^i} v_x^2 \phi_i + 2 \int_{\eta_j^i}^{\xi_{j+1}^i} v^2 \phi_i'' \end{aligned} \quad (4.69)$$

$$I_2 = -6 \int_{\eta_j^i}^{\xi_{j+1}^i} \partial_x(v^2) \phi_i = -6v^2(\xi_{j+1}^i) \phi_i(\xi_{j+1}^i) + 6v^2(\eta_j^i) \phi_i(\eta_j^i) + 6 \int_{\eta_j^i}^{\xi_{j+1}^i} v^2 \phi_i' \quad (4.70)$$

and

$$I_3 = -3 \int_{\eta_j^i}^{\xi_{j+1}^i} \partial_x(v_x^2) \phi_i = 3 \int_{\eta_j^i}^{\xi_{j+1}^i} v_x^2 \phi_i'. \quad (4.71)$$

Adding (4.68)-(4.71), we get

$$I = \int_{\eta_j^i}^{\xi_{j+1}^i} (4v^2 + 5v_x^2 + v_{xx}^2) \phi_i - 6v^2(\xi_{j+1}^i)\phi_i(\xi_{j+1}^i) + 6v^2(\eta_j^i)\phi_i(\eta_j^i) - 2v^2(\xi_{j+1}^i)\phi_i'(\xi_{j+1}^i) + 2v^2(\eta_j^i)\phi_i'(\eta_j^i) + R_1, \quad (4.72)$$

where using that  $K = \sqrt{L}/8$ , we have

$$|R_1| \leq 6(\|\phi_i'\|_{L^\infty(\mathbb{R})} + \|\phi_i''\|_{L^\infty(\mathbb{R})}) \int_{\eta_j^i}^{\xi_{j+1}^i} (v^2 + v_x^2) \leq O(L^{-1/2}) \int_{\eta_j^i}^{\xi_{j+1}^i} (v^2 + v_x^2).$$

Similar computations lead to

$$J = \int_{\xi_j^i}^{\eta_j^i} (4v^2 + 5v_x^2 + v_{xx}^2) \phi_i - 6v^2(\xi_j^i)\phi_i(\xi_j^i) + 6v^2(\eta_j^i)\phi_i(\eta_j^i) + 2v^2(\xi_j^i)\phi_i'(\xi_j^i) - 2v^2(\eta_j^i)\phi_i'(\eta_j^i) + R_2, \quad (4.73)$$

$$\int_{-\infty}^{\xi_1^i} g^2(x)\phi_i(x) dx = \int_{-\infty}^{\xi_1^i} (4v^2 + 5v_x^2 + v_{xx}^2) \phi_i - 6v^2(\xi_1^i)\phi_i(\xi_1^i) - 2v^2(\xi_1^i)\phi_i'(\xi_1^i) + R_3 \quad (4.74)$$

and

$$\int_{\xi_{k_i+1}^i}^{+\infty} g^2(x)\phi_i(x) dx = \int_{\xi_{k_i+1}^i}^{+\infty} (4v^2 + 5v_x^2 + v_{xx}^2) \phi_i - 6v^2(\xi_{k_i+1}^i)\phi_i(\xi_{k_i+1}^i) + 2v^2(\xi_{k_i+1}^i)\phi_i'(\xi_{k_i+1}^i) + R_4, \quad (4.75)$$

with

$$|R_2| \leq O(L^{-1/2}) \int_{\xi_j^i}^{\eta_j^i} (v^2 + v_x^2), \quad |R_3| \leq O(L^{-1/2}) \int_{-\infty}^{\xi_1^i} (v^2 + v_x^2) \quad \text{and} \quad |R_4| \leq O(L^{-1/2}) \int_{\xi_{k_i+1}^i}^{+\infty} (v^2 + v_x^2).$$

Then, adding (4.72) and (4.73), and summing over  $j \in \{1, \dots, k_i\}$ , we infer that

$$\begin{aligned} \int_{\xi_1^i}^{\xi_{k_i+1}^i} g_i^2(x)\phi_i(x) dx &= \int_{\xi_1^i}^{\xi_{k_i+1}^i} (4v^2 + 5v_x^2 + v_{xx}^2) \phi_i - 6 \sum_{j=1}^{k_i} v^2(\xi_{j+1}^i)\phi_i(\xi_{j+1}^i) - 6 \sum_{j=1}^{k_i} v^2(\xi_j^i)\phi_i(\xi_j^i) \\ &\quad + 12 \sum_{j=1}^{k_i} v^2(\eta_j^i)\phi_i(\eta_j^i) - 2 \sum_{j=1}^{k_i} v^2(\xi_{j+1}^i)\phi_i'(\xi_{j+1}^i) + 2 \sum_{j=1}^{k_i} v^2(\xi_j^i)\phi_i'(\xi_j^i) + R, \end{aligned} \quad (4.76)$$

with

$$|R| \leq O(L^{-1/2}) \int_{\xi_1^i}^{\xi_{k_i+1}^i} (v^2 + v_x^2).$$

Finally, adding (4.74)-(4.76), and recalling that  $\|v\|_{H^1} \leq \|u\|_{\mathcal{H}}$ , we obtain the lemma.

**Lemma 4.4** (Connection between  $F_i(\cdot)$  and the local extrema of  $v$ ). *Let  $u \in H^1(\mathbb{R})$  with  $u \geq 0$ . For  $i = 1, \dots, N$ , define the function  $h_i$  by*

$$h_i(x) = \begin{cases} -v_{xx} - 6v_x + 16v, & x < \xi_1^i, \\ -v_{xx} + 6v_x + 16v, & \xi_j^i < x < \eta_j^i, \\ -v_{xx} - 6v_x + 16v, & \eta_j^i < x < \xi_{j+1}^i, \\ -v_{xx} + 6v_x + 16v, & x > \xi_{k_i+1}^i, \end{cases} \quad j = 1, \dots, k_i. \quad (4.77)$$

Then it holds

$$\int_{\mathbb{R}} h_i(x) g_i^2(x) \phi_i(x) dx = F_i(u) - 144 \left( \sum_{j=0}^{k_i} (M_{j+1}^i)^3 \phi_i(\xi_{j+1}^i) - \sum_{j=1}^{k_i} (m_j^i)^3 \phi_i(\eta_j^i) \right) + \|u\|_{\mathcal{H}}^3 O(L^{-1/2}). \quad (4.78)$$

**Proof.** We have

$$\begin{aligned} \int_{\mathbb{R}} h_i(x) g_i^2(x) \phi_i(x) dx &= \int_{-\infty}^{\xi_1^i} h_i(x) g_i^2(x) \phi_i(x) dx + \sum_{j=1}^{k_i} \int_{\xi_j^i}^{\xi_{j+1}^i} h_i(x) g_i^2(x) \phi_i(x) dx \\ &\quad + \int_{\xi_{k_i+1}^i}^{+\infty} h_i(x) g_i^2(x) \phi_i(x) dx. \end{aligned} \quad (4.79)$$

For  $j = 1, \dots, k_i$ ,

$$\begin{aligned} \int_{\xi_j^i}^{\xi_{j+1}^i} h_i(x) g_i^2(x) \phi_i(x) dx &= \int_{\xi_j^i}^{\eta_j^i} (-v_{xx} - 6v_x + 16v) (2v + v_{xx} - 3v_x)^2 \phi_i \\ &\quad + \int_{\eta_j^i}^{\xi_{j+1}^i} (-v_{xx} + 6v_x + 16v) (2v + v_{xx} + 3v_x)^2 \phi_i \\ &= J + I. \end{aligned}$$

Computing  $I$ , we obtain

$$\begin{aligned} I &= \int_{\eta_j^i}^{\xi_{j+1}^i} (-v_{xx}^3 + 12v v_{xx}^2 + 64v^3 + 60v^2 v_{xx}) \phi_i - 54 \int_{\eta_j^i}^{\xi_{j+1}^i} v_x^3 \phi_i + 27 \int_{\eta_j^i}^{\xi_{j+1}^i} v_x^2 v_{xx} \phi_i \\ &\quad - 108 \int_{\eta_j^i}^{\xi_{j+1}^i} v v_x v_{xx} \phi_i - 216 \int_{\eta_j^i}^{\xi_{j+1}^i} v^2 v_x \phi_i + 216 \int_{\eta_j^i}^{\xi_{j+1}^i} v v_x^2 \phi_i \\ &= \int_{\eta_j^i}^{\xi_{j+1}^i} (-v_{xx}^3 + 12v v_{xx}^2 + 64v^3 + 60v^2 v_{xx}) \phi_i - 54 \int_{\eta_j^i}^{\xi_{j+1}^i} v_x^3 \phi_i + I_1 + I_2 + I_3 + I_4 \end{aligned} \quad (4.80)$$

with

$$I_1 = 9 \int_{\eta_j^i}^{\xi_{j+1}^i} \partial_x(v_x^3) \phi_i = -9 \int_{\eta_j^i}^{\xi_{j+1}^i} v_x^3 \phi_i', \quad (4.81)$$

$$I_2 = -54 \int_{\eta_j^i}^{\xi_{j+1}^i} v \partial_x(v_x^2) \phi_i = 54 \int_{\eta_j^i}^{\xi_{j+1}^i} \partial_x(v \phi_i) v_x^2 = 54 \int_{\eta_j^i}^{\xi_{j+1}^i} v_x^3 \phi_i + 54 \int_{\eta_j^i}^{\xi_{j+1}^i} v v_x^2 \phi_i', \quad (4.82)$$

$$I_3 = -72 \int_{\eta_j^i}^{\xi_{j+1}^i} \partial_x(v^3) \phi_i = -72 v^3(\xi_{j+1}^i) \phi_i(\xi_{j+1}^i) + 72 v^3(\eta_j^i) \phi_i(\eta_j^i) + 72 \int_{\eta_j^i}^{\xi_{j+1}^i} v^3 \phi_i', \quad (4.83)$$

and

$$\begin{aligned} I_4 &= 108 \int_{\eta_j^i}^{\xi_{j+1}^i} \partial_x(v^2) v_x \phi_i = -108 \int_{\eta_j^i}^{\xi_{j+1}^i} v^2 \partial_x(v_x \phi_i) = -108 \int_{\eta_j^i}^{\xi_{j+1}^i} v^2 v_{xx} \phi_i - 108 \int_{\eta_j^i}^{\xi_{j+1}^i} v^2 v_x \phi_i' \\ &= -108 \int_{\eta_j^i}^{\xi_{j+1}^i} v^2 v_{xx} \phi_i - 36 \int_{\eta_j^i}^{\xi_{j+1}^i} \partial_x(v^3) \phi_i' \\ &= -36 v^3(\xi_{j+1}^i) \phi_i'(\xi_{j+1}^i) + 36 v^3(\eta_j^i) \phi_i'(\eta_j^i) - 108 \int_{\eta_j^i}^{\xi_{j+1}^i} v^2 v_{xx} \phi_i + 36 \int_{\eta_j^i}^{\xi_{j+1}^i} v^3 \phi_i''. \end{aligned} \quad (4.84)$$

Adding (4.80)-(4.84), we get

$$\begin{aligned}
I &= \int_{\eta_j^i}^{\xi_{j+1}^i} (-v_{xx}^3 + 12vv_{xx}^2 + 64v^3 - 48v^2v_{xx}) \phi_i - 72v^3(\xi_{j+1}^i)\phi_i(\xi_{j+1}^i) + 72v^3(\eta_j^i)\phi_i(\eta_j^i) \\
&\quad - 36v^3(\xi_{j+1}^i)\phi_i'(\xi_{j+1}^i) + 36v^3(\eta_j^i)\phi_i'(\eta_j^i) + R,
\end{aligned} \tag{4.85}$$

where using that  $\|v\|_{C^1(\mathbb{R})} \leq C'_S \|v\|_{H^2(\mathbb{R})}$ , and  $\|v\|_{H^2(\mathbb{R})} \sim \|u\|_{\mathcal{H}}$ , the estimate of  $R$  leads to

$$\begin{aligned}
|R| &\leq (\|\phi_i'\|_{L^\infty(\mathbb{R})} + \|\phi_i''\|_{L^\infty(\mathbb{R})})(\|v\|_{L^\infty(\mathbb{R})} + \|v_x\|_{L^\infty(\mathbb{R})}) \int_{\eta_j^i}^{\xi_{j+1}^i} (v^2 + v_x^2) \\
&\leq O(L^{-1/2})\|u\|_{\mathcal{H}} \int_{\eta_j^i}^{\xi_{j+1}^i} (v^2 + v_x^2).
\end{aligned}$$

Similar computations lead to

$$\begin{aligned}
J &= \int_{\xi_j^i}^{\eta_j^i} (-v_x^3 + 12vv_{xx}^2 + 64v^3 - 48v^2v_{xx}) \phi_i - 72v^3(\xi_j^i)\phi_i(\xi_j^i) + 72v^3(\eta_j^i)\phi_i(\eta_j^i) \\
&\quad + 36v^3(\xi_j^i)\phi_i'(\xi_j^i) - 36v^3(\eta_j^i)\phi_i'(\eta_j^i) + O(L^{-1/2})\|u\|_{\mathcal{H}} \int_{\xi_j^i}^{\eta_j^i} (v^2 + v_x^2),
\end{aligned} \tag{4.86}$$

$$\begin{aligned}
\int_{-\infty}^{\xi_1^i} h_i(x)g_i^2(x)\phi_i(x) &= \int_{-\infty}^{\xi_1^i} (-v_{xx}^3 + 12vv_{xx}^2 + 64v^3 - 48v^2v_{xx}) \phi_i - 72v^3(\xi_1^i)\phi_i(\xi_1^i) \\
&\quad - 36v^3(\xi_1^i)\phi_i'(\xi_1^i) + O(L^{-1/2})\|u\|_{\mathcal{H}} \int_{-\infty}^{\xi_1^i} (v^2 + v_x^2)
\end{aligned} \tag{4.87}$$

and

$$\begin{aligned}
\int_{\xi_{k_i+1}^i}^{+\infty} h_i(x)g_i^2(x)\phi_i(x) &= \int_{\xi_{k_i+1}^i}^{+\infty} (-v_{xx}^3 + 12vv_{xx}^2 + 64v^3 - 48v^2v_{xx}) \phi_i - 72v^3(\xi_{k_i+1}^i)\phi_i(\xi_{k_i+1}^i) \\
&\quad + 36v^3(\xi_{k_i+1}^i)\phi_i'(\xi_{k_i+1}^i) + O(L^{-1/2})\|u\|_{\mathcal{H}} \int_{\xi_{k_i+1}^i}^{+\infty} (v^2 + v_x^2).
\end{aligned} \tag{4.88}$$

Adding (4.85) and (4.86), and summing over  $j \in \{1, \dots, k_i\}$ , we get

$$\begin{aligned}
&\int_{\xi_1^i}^{\xi_{k_i+1}^i} h_i(x)g_i^2(x)\phi_i(x)dx \\
&= \int_{\xi_1^i}^{\xi_{k_i+1}^i} (-v_x^3 + 12vv_{xx}^2 + 64v^3 - 48v^2v_{xx}) \phi_i - 72 \sum_{j=1}^{k_i} v^3(\xi_{j+1}^i)\phi_i(\xi_{j+1}^i) \\
&\quad - 72 \sum_{j=1}^{k_i} v^3(\xi_j^i)\phi_i(\xi_j^i) + 144 \sum_{j=1}^{k_i} v^3(\eta_j^i)\phi_i(\eta_j^i) - 36 \sum_{j=1}^{k_i} v^3(\xi_{j+1}^i)\phi_i'(\xi_{j+1}^i) \\
&\quad + 36 \sum_{j=1}^{k_i} v^3(\xi_j^i)\phi_i'(\xi_j^i) + O(L^{-1/2})\|u\|_{\mathcal{H}} \int_{\xi_1^i}^{\xi_{k_i+1}^i} (v^2 + v_x^2).
\end{aligned} \tag{4.89}$$

Finally, adding (4.87)-(4.89), we obtain the lemma.

**Lemma 4.5** (Connection between  $E_i(\cdot)$  and  $F_i(\cdot)$ ). Let  $u \in H^1(\mathbb{R})$ , with  $y = (1 - \partial_x^2)u \in \mathcal{M}^+(\mathbb{R})$ , that satisfies (4.2). Let be given  $N - 1$  real numbers  $-\infty = y_1 < y_2 < \dots < y_N < y_{N+1} = +\infty$  with  $y_i - y_{i-1} \geq 2L/3$ . For  $i = 1, \dots, N$ , assume that  $v = (4 - \partial_x^2)^{-1}u$  satisfies (4.57)-(4.64), with local extrema on  $[\eta_0^i, \eta_{k_i+1}^i]$  arranged in decreasing order in the following way:

$$M_1^i \geq M_2^i \geq \dots \geq M_{k_i+1}^i \geq 0, \quad m_1^i \geq m_2^i \geq \dots \geq m_{k_i}^i \geq 0, \quad M_{j+1}^i \geq m_j^i, \quad j = 1, \dots, k_i. \quad (4.90)$$

Then, defining the functional  $E_i(\cdot)$ 's and  $F_i(\cdot)$ 's as in (4.52)-(4.55), it holds

$$F_i(u) \leq 18M_1^i E_i(u) - 72(M_1^i)^3 + \|u\|_{\mathcal{H}}^3 O(L^{-1/2}), \quad i = 1, \dots, N. \quad (4.91)$$

**Proof.** Combining (4.53), (4.64) and (4.66) with  $K = \sqrt{L}/8$ , we get

$$\int_{\mathbb{R}} g_i^2(x) \phi_i(x) = E_i(u) - 12 \left( \sum_{j=0}^{k_i} (M_{j+1}^i)^2 - \sum_{j=1}^{k_i} (m_j^i)^2 \right) + \|u\|_{\mathcal{H}}^2 O(L^{-1/2}). \quad (4.92)$$

Similarly, combining (4.53), (4.64) and (4.78), we get

$$\int_{\mathbb{R}} h_i(x) g_i^2(x) \phi_i(x) dx = F_i(u) - 144 \left( \sum_{j=0}^{k_i} (M_{j+1}^i)^3 - \sum_{j=1}^{k_i} (m_j^i)^3 \right) + \|u\|_{\mathcal{H}}^3 O(L^{-1/2}). \quad (4.93)$$

Now, let us show that  $h_i \leq 18M_1^i$  on  $\Omega_i$ . Note that by (4.2) and (4.56), one can check that  $18M_1^i \geq c_i/4$ . We rewrite the function  $h_i$  as

$$h_i(x) = \begin{cases} -v_{xx} - 6v_x + 16v, & x < \eta_0^i, \\ -(\partial_x^2 + 3\partial_x + 2)v - 3v_x + 18v, & \eta_0^i < x < \xi_1^i, \\ -(\partial_x^2 - 3\partial_x + 2)v + 3v_x + 18v, & \xi_j^i < x < \eta_j^i, \\ -(\partial_x^2 + 3\partial_x + 2)v - 3v_x + 18v, & \eta_j^i < x < \xi_{j+1}^i, \\ -(\partial_x^2 - 3\partial_x + 2)v + 3v_x + 18v, & \xi_{k_i+1}^i < x < \eta_{k_i+1}^i, \\ -v_{xx} + 6v_x + 16v, & x > \eta_{k_i+1}^i, \end{cases} \quad j = 1, \dots, k_i.$$

Then, if  $x \in \Omega_i \setminus [\eta_0^i, \eta_{k_i+1}^i]$ , using that  $v_{xx} = 4v - u$ , (3.40), (4.64) and (4.65), it holds

$$h_i \leq |v_{xx}| + 6|v_x| + 16v \leq u + 32v \leq \frac{c_i}{9}.$$

If  $\eta_0^i < x < \xi_1^i$ , then  $v_x \geq 0$ , and using that  $y = (1 - \partial_x^2)u \geq 0$ , it follows from Lemma 2.2 that

$$\begin{aligned} h_i &= -(\partial_x^2 + 3\partial_x + 2)v - 3v_x + 18v \\ &= -(2 + \partial_x)(4 - \partial_x^2)^{-1}(1 + \partial_x)u - 3v_x + 18v \\ &\leq 18v. \end{aligned}$$

If  $\xi_j^i < x < \eta_j^i$ , then  $v_x \leq 0$ , and similarly using that  $y = (1 - \partial_x^2)u \geq 0$ , it follows from Lemma 2.2 that

$$\begin{aligned} h_i &= -(\partial_x^2 - 3\partial_x + 2)v + 3v_x + 18v \\ &= -(2 - \partial_x)(4 - \partial_x^2)^{-1}(1 - \partial_x)u + 3v_x + 18v \\ &\leq 18v. \end{aligned}$$

Therefore, it holds

$$h_i(x) \leq 18 \max_{x \in \Omega_i} v(x) = 18M_1^i, \quad \forall x \in \Omega_i. \quad (4.94)$$



Finally, combining (3.24), (4.33), (4.54) and (4.92)-(4.94), we obtain

$$\begin{aligned}
F_i(u) &= 144 \left( \sum_{j=0}^{k_i} (M_{j+1}^i)^3 - \sum_{j=1}^{k_i} (m_j^i)^3 \right) \\
&= \int_{\mathbb{R}} h_i(x) g_i^2(x) \phi_i(x) dx + \|u\|_{\mathcal{H}}^3 O(L^{-1/2}) \\
&= \int_{\Omega_i} h_i(x) g_i^2(x) \phi_i(x) dx + \int_{\Omega_i^c} h_i(x) g_i^2(x) \phi_i(x) dx + \|u\|_{\mathcal{H}}^3 O(L^{-1/2}) \\
&\leq 18M_1^i \int_{\Omega_i} g_i^2(x) \phi_i(x) dx + 18\|h_i\|_{L^\infty(\mathbb{R})} \|g_i\|_{L^2(\mathbb{R})}^2 \|\phi_i\|_{L^\infty(\Omega_i^c)} + \|u\|_{\mathcal{H}}^3 O(L^{-1/2}) \\
&\leq 18M_1^i \left[ E_i(u) - 12 \left( \sum_{j=0}^{k_i} (M_{j+1}^i)^2 - \sum_{j=1}^{k_i} (m_j^i)^2 \right) \right] + \|u\|_{\mathcal{H}}^3 O(L^{-1/2}).
\end{aligned}$$

Therefore, using that  $M_{j+1}^i \geq m_j^i$  and proceeding as in Lemma 3.6 (see (3.43)), we infer that

$$\begin{aligned}
F_i(u) &\leq 18M_1^i E_i(u) - 72(M_1^i)^3 + 144 \sum_{j=1}^{k_i} \left\{ [(M_{j+1}^i)^3 - (m_j^i)^3] - \frac{3}{2} M_1^i [(M_{j+1}^i)^2 - (m_j^i)^2] \right\} \\
&\quad + \|u\|_{\mathcal{H}}^3 O(L^{-1/2}) \\
&\leq 18M_1^i E_i(u) - 72(M_1^i)^3 + \|u\|_{\mathcal{H}}^3 O(L^{-1/2}).
\end{aligned}$$

This proves the lemma.

The lemma below is the generalization of Lemma 3.3.

**Lemma 4.6** (Quadratic identity). *Let  $Z = (z_i)_{i=1}^N \in \mathbb{R}^N$  with  $|z_i - z_{i-1}| \geq L/2$ , and  $u \in L^2(\mathbb{R})$ . It holds*

$$E(u) - \sum_{i=1}^N E(\varphi_{c_i}) = \|u - S_Z\|_{\mathcal{H}}^2 + 4 \sum_{i=1}^N c_i \left( v(z_i) - \frac{c_i}{6} \right) + O(e^{-L/4}), \quad (4.95)$$

where  $S_Z$  is defined in (4.11) and  $O(\cdot)$  only depends on  $(c_i)_{i=1}^N$ .

**Proof.** Let us compute

$$\begin{aligned}
\|u - S_Z\|_{\mathcal{H}}^2 &= \int_{\mathbb{R}} [(1 - \partial_x^2)(u - S_Z)][(4 - \partial_x^2)^{-1}(u - S_Z)] \\
&= \|u\|_{\mathcal{H}}^2 + \|S_Z\|_{\mathcal{H}}^2 - 2 \int_{\mathbb{R}} [(1 - \partial_x^2)S_Z][(4 - \partial_x^2)^{-1}u] \\
&= \|u\|_{\mathcal{H}}^2 + \|S_Z\|_{\mathcal{H}}^2 - 2 \sum_{i=1}^N c_i \int_{\mathbb{R}} [(1 - \partial_x^2)\varphi_{c_i}(x - z_i)]v \\
&= \|u\|_{\mathcal{H}}^2 + \|S_Z\|_{\mathcal{H}}^2 - 4 \sum_{i=1}^N c_i v(z_i), \quad (4.96)
\end{aligned}$$

where we use that

$$(1 - \partial_x^2)\varphi_{c_i}(\cdot - z_i) = 2c_i \delta_{z_i}$$

with  $\delta_{z_i}$  the Dirac mass applied at point  $z_i$ . We also have

$$\begin{aligned}
\|S_Z\|_{\mathcal{H}}^2 &= \int_{\mathbb{R}} [(1 - \partial_x^2)S_Z][(4 - \partial_x^2)^{-1}S_Z] \\
&= \sum_{i=1}^N \int_{\mathbb{R}} [(1 - \partial_x^2)\varphi_{c_i}(x - z_i)][(4 - \partial_x^2)^{-1}S_Z] \\
&= 2 \sum_{i=1}^N c_i \langle \delta_{z_i}, (4 - \partial_x^2)^{-1}S_Z \rangle_{H^{-1}, H^1} \\
&= 2 \sum_{1 \leq i, j \leq N} c_i \langle \delta_{z_i}, (4 - \partial_x^2)^{-1}\varphi_{c_j}(\cdot - z_j) \rangle_{H^{-1}, H^1} \\
&= 2 \sum_{i=1}^N c_i (4 - \partial_x^2)^{-1}\varphi_{c_i}(0) + 2 \sum_{\substack{1 \leq i, j \leq N \\ i \neq j}} c_i c_j (4 - \partial_x^2)^{-1}e^{-|z_i - z_j|}, \tag{4.97}
\end{aligned}$$

where  $\langle \cdot, \cdot \rangle_{H^{-1}, H^1}$  denote the duality  $H^{-1}/H^1$ , and we recall that  $\delta_{z_i} \in \mathcal{L}(H^{-1}(\mathbb{R}), H^1(\mathbb{R}))$  since  $\|\delta_{z_i}\|_{H^{-1}(\mathbb{R})} \leq C_S$ , with  $C_S$  the constant appearing in (2.3). Now, using that  $|z_i - z_{i-1}| \geq L/2$ ,

$$\begin{aligned}
(4 - \partial_x^2)^{-1}e^{-|z_i - z_j|} &= \frac{1}{4} \int_{\mathbb{R}} e^{-2|x' - (z_i - z_j)|} e^{-|x'|} dx' \\
&= \frac{1}{3}e^{-|z_i - z_j|} - \frac{1}{6}e^{-2|z_i - z_j|} \\
&= O(e^{-L/4}), \tag{4.98}
\end{aligned}$$

and combining (4.97) and (4.98), for  $L > L_0 > 0$  with  $L_0 \gg 1$ , we get

$$\|S_Z\|_{\mathcal{H}}^2 = \sum_{i=1}^N E(\varphi_{c_i}) + O(e^{-L/4}). \tag{4.99}$$

Finally, combining (4.96) and (4.99), we obtain the lemma.

The last lemma is the localized version of Lemma 3.1.

**Lemma 4.7** (Control of the distances between local and global energies at  $t = 0$ ). *Let  $u_0 \in H^1(\mathbb{R})$  satisfying (1.12)-(1.14). Then it holds*

$$\left| E(u_0) - \sum_{i=1}^N E(\varphi_{c_i}) \right| \leq O(\varepsilon^2) + O(e^{-L/4}), \tag{4.100}$$

$$|E_i(u_0) - E(\varphi_{c_i})| \leq O(\varepsilon^2) + O(e^{-\sqrt{L}}), \quad i = 1, \dots, N, \tag{4.101}$$

and

$$|F_i(u_0) - F(\varphi_{c_i})| \leq O(\varepsilon^2) + O(e^{-\sqrt{L}}), \quad i = 1, \dots, N, \tag{4.102}$$

where  $O(\cdot)$  only depend on  $(c_i)_{i=1}^N$ .

**Proof.** For the first estimate, applying triangular inequality and (1.14), we have

$$\begin{aligned}
|E(u_0) - E(S_{Z^0})| &= | \|u_0\|_{\mathcal{H}} - \|S_{Z^0}\|_{\mathcal{H}} | (\|u_0\|_{\mathcal{H}} + \|S_{Z^0}\|_{\mathcal{H}}) \\
&\leq \|u_0 - S_{Z^0}\|_{\mathcal{H}} (\|u_0\|_{\mathcal{H}} + 2\|S_{Z^0}\|_{\mathcal{H}}) \\
&\leq \varepsilon^2 \left( \varepsilon^2 + \frac{2}{\sqrt{3}} \sum_{i=1}^N c_i \right). \tag{4.103}
\end{aligned}$$

Thus, combining (4.99) and (4.103), it holds

$$\begin{aligned} \left| E(u_0) - \sum_{i=1}^N E(\varphi_{c_i}) \right| &\leq |E(u_0) - E(S_{Z^0})| + \left| E(S_{Z^0}) - \sum_{i=1}^N E(\varphi_{c_i}) \right| \\ &\leq \varepsilon^2(\varepsilon^2 + O(1)) + O(e^{-L/4}) \\ &\leq O(\varepsilon^2) + O(e^{-L/4}). \end{aligned}$$

For the second estimate, using the exponential decay of  $\varphi_{c_i}$ 's and the  $\phi_i$ 's, and the definition of  $E_i(\cdot)$ , we have

$$\begin{aligned} &|E_i(u_0) - E(\varphi_{c_i})| \\ &\leq \left| \|u_0\|_{\mathcal{H}(\Omega_i)}^2 - \|\varphi_{c_i}\|_{\mathcal{H}(\Omega_i)}^2 \right| + O(e^{-\sqrt{L}}) \\ &= \left| \|u_0\|_{\mathcal{H}(\Omega_i)} - \|\varphi_{c_i}\|_{\mathcal{H}(\Omega_i)} \right| (\|u_0\|_{\mathcal{H}(\Omega_i)} + \|\varphi_{c_i}\|_{\mathcal{H}(\Omega_i)}) + O(e^{-\sqrt{L}}) \\ &\leq \left( \|u_0 - S_{Z^0}\|_{\mathcal{H}(\Omega_i)} + \sum_{\substack{1 \leq j \leq N \\ j \neq i}} \|\varphi_{c_j}\|_{\mathcal{H}(\Omega_i)} \right) \left( \|u_0 - S_{Z^0}\|_{\mathcal{H}} + \frac{2}{\sqrt{3}} \sum_{j=1}^N c_j \right) + O(e^{-\sqrt{L}}) \\ &\leq (\varepsilon^2 + O(e^{-L/8})) (\varepsilon^2 + O(1)) + O(e^{-\sqrt{L}}) \\ &\leq O(\varepsilon^2) + O(e^{-\sqrt{L}}). \end{aligned}$$

Similarly, for the third estimate, using the exponential decay of  $\varphi_{c_i}$ 's and the  $\phi_i$ 's, and the definition of  $F_i(\cdot)$ , we have

$$\begin{aligned} &|F_i(u_0) - F(\varphi_{c_i})| \\ &\leq \left| \int_{\Omega_i} (u_0^3 - \varphi_{c_i}^3) \right| + O(e^{-\sqrt{L}}) \\ &\leq \int_{\Omega_i} |u_0 - \varphi_{c_i}| (u_0^2 + u_0\varphi_{c_i} + \varphi_{c_i}^2) + O(e^{-\sqrt{L}}) \\ &\leq \|u_0 - \varphi_{c_i}\|_{L^2(\Omega_i)} \left( \int_{\Omega_i} (u_0^2 + u_0\varphi_{c_i} + \varphi_{c_i}^2)^2 \right)^{1/2} + O(e^{-\sqrt{L}}) \\ &\leq \left( \|u_0 - S_{Z^0}\|_{L^2(\Omega_i)} + \sum_{\substack{1 \leq j \leq N \\ j \neq i}} \|\varphi_{c_j}\|_{L^2(\Omega_i)} \right) \cdot O(1) + O(e^{-\sqrt{L}}) \\ &\leq (\varepsilon^2 + O(e^{-L/8})) \cdot O(1) + O(e^{-\sqrt{L}}) \\ &\leq O(\varepsilon^2) + O(e^{-\sqrt{L}}). \end{aligned}$$

This proves the lemma.

#### 4.4 End of the proof of Theorem 1.1

Let  $u \in \mathcal{X}([0, T])$ , with  $0 < T \leq +\infty$ , be a solution of the DP equation satisfying (1.12)-(1.14) and (4.2) for some  $t_0 \in ]0, T[$ . Let  $M_1^i = v(t_0, \xi_1^i(t_0)) = \max_{x \in J_i} v(t_0, x)$ , with  $J_i$ 's as in (4.8), and  $\delta_i = c_i/6 - M_1^i$ . First, from (4.7) and (4.9), we know that for  $i = 2, \dots, N$ ,

$$\xi_1^i(t_0) - \xi_1^{i-1}(t_0) \geq 2L/3 > L/2.$$

Applying (4.95) and (4.100) with  $u(t_0)$ , we get

$$\left\| u(t_0) - \sum_{i=1}^N \varphi_{c_i}(\cdot - \xi_1^i(t_0)) \right\|_{\mathcal{H}}^2 \leq 4 \sum_{i=1}^N c_i \delta_i + O(\varepsilon^2) + O(e^{-L/4}). \quad (4.104)$$

In the same way, from (4.91) we get

$$F_i(u(t_0)) \leq 18M_1^i E_i(u(t_0)) - 72(M_1^i)^3 + O(L^{-1/2}),$$

which leads to

$$F(u(t_0)) = \sum_{i=1}^N F_i(u(t_0)) \leq 18 \sum_{i=1}^N M_1^i E_i(u(t_0)) - 72 \sum_{i=1}^N (M_1^i)^3 + O(L^{-1/2}), \quad (4.105)$$

by summing over  $i \in \{1, \dots, N\}$ .

Now, we will use the following notation: for a function  $g : \mathbb{R}_+ \mapsto \mathbb{R}$ , we set

$$\Delta_0^{t_0} g = g(t_0) - g(0). \quad (4.106)$$

From (4.105) and the fact that  $E(\cdot)$  and  $F(\cdot)$  are conservation laws for  $u$ , we obtain

$$\begin{aligned} 0 = \Delta_0^{t_0} F(u) &= \sum_{i=1}^N \Delta_0^{t_0} F_i(u) \leq 18 \sum_{i=1}^N M_1^i \Delta_0^{t_0} E_i(u) \\ &+ \sum_{i=1}^N [-72(M_1^i)^3 + 18M_1^i E_i(u_0) - F_i(u_0)] + O(L^{-1/2}). \end{aligned} \quad (4.107)$$

Note that, from (4.101) and (4.102), for  $0 < \varepsilon < \varepsilon_0$  and  $L > L_0 > 0$  with  $\varepsilon_0 \ll 1$  and  $L_0 \gg 1$ , it holds

$$\sum_{i=1}^N [-72(M_1^i)^3 + 18M_1^i E_i(u_0) - F_i(u_0)] = -72 \sum_{i=1}^N \delta_i^2 \left( M_1^i + \frac{c_i}{3} \right) + O(\varepsilon^2) + O(e^{-\sqrt{L}}). \quad (4.108)$$

Combining (4.107) and (4.108), we get

$$\sum_{i=1}^N \delta_i^2 \left( M_1^i + \frac{c_i}{3} \right) \leq \frac{1}{4} \sum_{i=1}^N M_1^i \Delta_0^{t_0} E_i(u) + O(\varepsilon^2) + O(L^{-1/2}),$$

and using the Abel transformation with  $M_1^0 = 0$ , we obtain

$$\sum_{i=1}^N \delta_i^2 \left( M_1^i + \frac{c_i}{3} \right) \leq \frac{1}{4} \sum_{i=1}^N (M_1^i - M_1^{i-1}) \Delta_0^t \mathcal{J}_{i,K} + O(\varepsilon^2) + O(L^{-1/2}), \quad (4.109)$$

where  $\mathcal{J}_{i,K}(t)$  is defined in (4.22). From (4.2) we know that  $u(t_0) \in U(\gamma, L/2)$ , on account of Lemma 4.1 there exists  $\tilde{X} = (\tilde{x}_1, \dots, \tilde{x}_N)$  with  $\tilde{x}_i \in J_i$  such that  $E(u(t_0) - S_{\tilde{X}}) \leq O(\gamma^2)$ , where  $S_{\tilde{X}}$  is defined in (4.11). Recalling that  $v(t_0, \xi_1^i(t_0)) = \max_{x \in J_i} v(t_0, x)$  and using (4.95), we obtain  $E(u(t_0) - S_{\xi_1}) \leq O(\gamma^2) + O(e^{-L/4})$ , with  $\xi_1 = (\xi_1^1, \dots, \xi_1^N)$ . From (4.5), we deduce that

$$\left\| v(t_0) - \sum_{j=1}^N \rho_{c_j}(\cdot - \xi_1^j(t_0)) \right\|_{L^\infty(\mathbb{R})} \leq O(\gamma) + O(e^{-L/8}).$$

Thus, we infer that

$$v(x) = \sum_{j=1}^N \rho_{c_j}(\cdot - \xi_1^j(t_0)) + O(\gamma) + O(e^{-L/8}), \quad \forall x \in \mathbb{R},$$

and applying this formula with  $x = \xi_1^i(t_0)$  and using that  $\xi_1^j(t_0) - \xi_1^{j-1}(t_0) > L/2$ , we get

$$\begin{aligned} v(\xi_1^i(t_0)) &= \sum_{j=1}^N \rho_{c_j}(\xi_1^i(t_0) - \xi_1^j(t_0)) + O(\gamma) + O(e^{-L/8}) \\ &= \frac{c_i}{6} + \sum_{\substack{1 \leq j \leq N \\ j \neq i}} \rho_{c_j}(\xi_1^i(t_0) - \xi_1^j(t_0)) + O(\gamma) + O(e^{-L/8}) \\ &= \frac{c_i}{6} + O(\gamma) + O(e^{-L/8}). \end{aligned}$$

We take  $\gamma = A(\sqrt{\varepsilon} + L^{-1/8})$ , then  $M_1^i = c_i/6 + O(\sqrt{\varepsilon}) + O(L^{-1/8})$ . Therefore, for  $0 < \varepsilon < \varepsilon_0$  and  $L > L_0 > 0$ , with  $\varepsilon_0 \ll 1$  and  $L_0 \gg 1$ , it holds

$$0 < M_1^1 < M_1^2 < \dots < M_1^N. \quad (4.110)$$

Combining (4.109), (4.110) and using the monotonicity estimate (4.24), it holds

$$\sum_{i=1}^N \delta_i^2 \left( M_1^i + \frac{c_i}{3} \right) \leq O(\varepsilon^2) + O(L^{-1/8}).$$

Therefore, using that  $(M_1^i + c_i/3)^{-1} < 3/c_i$ , there exists  $C > 0$  only depending on  $(c_i)_{i=1}^N$  such that

$$\delta_i \leq C(\varepsilon + L^{-1/4}), \quad i = 1, \dots, N. \quad (4.111)$$

Now, combining (4.104) and (4.111), we obtain

$$\left\| u(t_0) - \sum_{i=1}^N \varphi_{c_i}(\cdot - \xi_1^i(t_0)) \right\|_{\mathcal{H}} \leq C(\sqrt{\varepsilon} + L^{-1/8}),$$

and the theorem follows by choosing  $A = 2C$ .

**Remark 4.2** (The role of the number of extrema). In the case where  $v = (4 - \partial_x^2)^{-1}u$  admits a countable infinite number of local maximal values on some  $[\alpha_i, \beta_i]$  (see (4.57)), with  $i \in \{1, \dots, N\}$ , it suffices to change the finite sums over  $j$  by infinite sums in Lemma 3.4-3.5 and Lemma 4.3-4.4.

## Appendix. Proof of Lemma 4.2

The aim of this subsection is to prove Lemma 4.2. Let us first assume that  $u$  is smooth solution. The case  $u \in \mathcal{X}([0, T])$  will follow by a density argument.

We compute the time variation of the following energy:

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}} yvg &= \int_{\mathbb{R}} y_t v g + \int_{\mathbb{R}} y v_t g \\ &= I + J. \end{aligned}$$

Applying the operator  $(1 - \partial_x^2)$  on both sides of equation (1.7), we get

$$y_t = -\frac{1}{2}(1 - \partial_x^2)\partial_x u^2 - \frac{3}{2}\partial_x u^2$$

and substituting  $y_t$  by this value,  $I$  becomes

$$\begin{aligned} I &= -\frac{1}{2} \int_{\mathbb{R}} [(1 - \partial_x^2) \partial_x u^2] v g - \frac{3}{2} \int_{\mathbb{R}} [\partial_x(u^2)] v g \\ &= I_1 + I_2. \end{aligned}$$

By computing

$$I_2 = \frac{3}{2} \int_{\mathbb{R}} u^2 \partial_x(vg) = \frac{3}{2} \int_{\mathbb{R}} u^2 v_x g + \frac{3}{2} \int_{\mathbb{R}} u^2 v g' \quad (4.112)$$

and

$$\begin{aligned} I_1 &= \frac{1}{2} \int_{\mathbb{R}} [(1 - \partial_x^2) u^2] \partial_x(vg) \\ &= \frac{1}{2} \int_{\mathbb{R}} [(1 - \partial_x^2) u^2] v_x g + \frac{1}{2} \int_{\mathbb{R}} [(1 - \partial_x^2) u^2] v g' \\ &= I_3 + I_4 \end{aligned}$$

with

$$\begin{aligned} I_3 &= \frac{1}{2} \int_{\mathbb{R}} u^2 v_x g - \frac{1}{2} \int_{\mathbb{R}} \partial_x^2(u^2) v_x g \\ &= \frac{1}{2} \int_{\mathbb{R}} u^2 v_x g + \frac{1}{2} \int_{\mathbb{R}} \partial_x(u^2) \partial_x(v_x g) \\ &= \frac{1}{2} \int_{\mathbb{R}} u^2 v_x g + \frac{1}{2} \int_{\mathbb{R}} \partial_x(u^2) v_{xx} g + \frac{1}{2} \int_{\mathbb{R}} \partial_x(u^2) v_x g' \\ &= \frac{1}{2} \int_{\mathbb{R}} u^2 v_x g - \frac{1}{2} \int_{\mathbb{R}} u^2 \partial_x(v_{xx} g) - \frac{1}{2} \int_{\mathbb{R}} u^2 \partial_x(v_x g') \\ &= \frac{1}{2} \int_{\mathbb{R}} u^2 v_x g - \frac{1}{2} \int_{\mathbb{R}} u^2 v_{xxx} g - \int_{\mathbb{R}} u^2 v_{xx} g' - \frac{1}{2} \int_{\mathbb{R}} u^2 v_x g'' \end{aligned} \quad (4.113)$$

and

$$\begin{aligned} I_4 &= \frac{1}{2} \int_{\mathbb{R}} u^2 v g' - \frac{1}{2} \int_{\mathbb{R}} \partial_x^2(u^2) v g' \\ &= \frac{1}{2} \int_{\mathbb{R}} u^2 v g' + \frac{1}{2} \int_{\mathbb{R}} \partial_x(u^2) \partial_x(v g') \\ &= \frac{1}{2} \int_{\mathbb{R}} u^2 v g' + \frac{1}{2} \int_{\mathbb{R}} \partial_x(u^2) v_x g' + \frac{1}{2} \int_{\mathbb{R}} \partial_x(u^2) v g'' \\ &= \frac{1}{2} \int_{\mathbb{R}} u^2 v g' - \frac{1}{2} \int_{\mathbb{R}} u^2 \partial_x(v_x g') - \frac{1}{2} \int_{\mathbb{R}} u^2 \partial_x(v g'') \\ &= \frac{1}{2} \int_{\mathbb{R}} u^2 v g' - \frac{1}{2} \int_{\mathbb{R}} u^2 v_{xx} g' - \int_{\mathbb{R}} u^2 v_x g'' - \frac{1}{2} \int_{\mathbb{R}} u^2 v g''' \end{aligned} \quad (4.114)$$

Adding (4.113) and (4.114), we get

$$\begin{aligned} I_1 &= \frac{1}{2} \int_{\mathbb{R}} u^2 v_x g - \frac{1}{2} \int_{\mathbb{R}} u^2 v_{xxx} g - \int_{\mathbb{R}} u^2 v_{xx} g' - \frac{1}{2} \int_{\mathbb{R}} u^2 v_x g'' \\ &\quad + \frac{1}{2} \int_{\mathbb{R}} u^2 v g' - \frac{1}{2} \int_{\mathbb{R}} u^2 v_{xx} g' - \int_{\mathbb{R}} u^2 v_x g'' - \frac{1}{2} \int_{\mathbb{R}} u^2 v g''' \\ &= \frac{1}{2} \int_{\mathbb{R}} u^2 v_x g - \frac{1}{2} \int_{\mathbb{R}} u^2 v_{xxx} g - \frac{3}{2} \int_{\mathbb{R}} u^2 v_{xx} g' \\ &\quad + \frac{1}{2} \int_{\mathbb{R}} u^2 v g' - \frac{3}{2} \int_{\mathbb{R}} u^2 v_x g'' - \frac{1}{2} \int_{\mathbb{R}} u^2 v g''' \end{aligned} \quad (4.115)$$

and adding (4.112) and (4.115), we get

$$\begin{aligned}
I &= 2 \int_{\mathbb{R}} u^2 v_x g - \frac{1}{2} \int_{\mathbb{R}} u^2 v_{xxx} g - \frac{3}{2} \int_{\mathbb{R}} u^2 v_{xx} g' \\
&\quad + 2 \int_{\mathbb{R}} u^2 v g' - \frac{3}{2} \int_{\mathbb{R}} u^2 v_x g'' - \frac{1}{2} \int_{\mathbb{R}} u^2 v g''' \\
&= \frac{1}{2} \int_{\mathbb{R}} u^2 [(4 - \partial_x^2) v_x] g + \frac{3}{2} \int_{\mathbb{R}} u^2 \left[ \left( \frac{4}{3} - \partial_x^2 \right) v \right] g' \\
&\quad - \frac{3}{2} \int_{\mathbb{R}} u^2 v_x g'' - \frac{1}{2} \int_{\mathbb{R}} u^2 v g'''.
\end{aligned}$$

The first two integrals give us

$$\frac{1}{2} \int_{\mathbb{R}} u^2 [(4 - \partial_x^2) v_x] g = \frac{1}{2} \int_{\mathbb{R}} u^2 u_x g = \frac{1}{6} \int_{\mathbb{R}} \partial_x (u^3) g = -\frac{1}{6} \int_{\mathbb{R}} u^3 g'$$

and

$$\frac{3}{2} \int_{\mathbb{R}} u^2 \left[ \left( \frac{4}{3} - \partial_x^2 \right) v \right] g' = \frac{3}{2} \int_{\mathbb{R}} u^2 \left[ (4 - \partial_x^2) v - \frac{8}{3} v \right] g' = \frac{3}{2} \int_{\mathbb{R}} u^3 g' - 4 \int_{\mathbb{R}} u^2 v g'.$$

Finally, we obtain

$$I = \frac{4}{3} \int_{\mathbb{R}} u^3 g' - 4 \int_{\mathbb{R}} u^2 v g' - \frac{3}{2} \int_{\mathbb{R}} u^2 v_x g'' - \frac{1}{2} \int_{\mathbb{R}} u^2 v g'''. \quad (4.116)$$

We set  $h = (1 - \partial_x^2)^{-1} u^2$ . Applying the operator  $(4 - \partial_x^2)^{-1}$  on both sides of equation (1.7) and using (4.16), we get

$$v_t = -\frac{1}{2} (1 - \partial_x^2)^{-1} \partial_x u^2 = -\frac{1}{2} h_x. \quad (4.117)$$

Substituting  $v_t$  by this value,  $J$  becomes

$$\begin{aligned}
J &= -\frac{1}{2} \int_{\mathbb{R}} y h_x g \\
&= \frac{1}{2} \int_{\mathbb{R}} y_x h g + \frac{1}{2} \int_{\mathbb{R}} y h g' \\
&= J_1 + J_2.
\end{aligned}$$

By computing

$$\begin{aligned}
J_2 &= \frac{1}{2} \int_{\mathbb{R}} (u - u_{xx}) h g' \\
&= \frac{1}{2} \int_{\mathbb{R}} u h g' - \frac{1}{2} \int_{\mathbb{R}} u_{xx} h g' \\
&= \frac{1}{2} \int_{\mathbb{R}} u h g' + \frac{1}{2} \int_{\mathbb{R}} u_x \partial_x (h g') \\
&= \frac{1}{2} \int_{\mathbb{R}} u h g' + \frac{1}{2} \int_{\mathbb{R}} u_x h_x g' + \frac{1}{2} \int_{\mathbb{R}} u_x h g'' \\
&= \frac{1}{2} \int_{\mathbb{R}} u h g' - \frac{1}{2} \int_{\mathbb{R}} u \partial_x (h_x g') - \frac{1}{2} \int_{\mathbb{R}} u \partial_x (h g'') \\
&= \frac{1}{2} \int_{\mathbb{R}} u h g' - \frac{1}{2} \int_{\mathbb{R}} u h_{xx} g' - \int_{\mathbb{R}} u h_x g'' - \frac{1}{2} \int_{\mathbb{R}} u h g'''
\end{aligned} \quad (4.118)$$

and

$$\begin{aligned}
J_1 &= \frac{1}{2} \int_{\mathbb{R}} (u_x - u_{xxx})hg \\
&= \frac{1}{2} \int_{\mathbb{R}} u_x hg - \frac{1}{2} \int_{\mathbb{R}} u_{xxx}hg \\
&= \frac{1}{2} \int_{\mathbb{R}} u_x hg + \frac{1}{2} \int_{\mathbb{R}} u_{xx} \partial_x (hg) \\
&= \frac{1}{2} \int_{\mathbb{R}} u_x hg + \frac{1}{2} \int_{\mathbb{R}} u_{xx} h_x g + \frac{1}{2} \int_{\mathbb{R}} u_{xx} h g' \\
&= J_3 + J_4 + J_5
\end{aligned}$$

with

$$J_3 = -\frac{1}{2} \int_{\mathbb{R}} u \partial_x (hg) = -\frac{1}{2} \int_{\mathbb{R}} u h_x g - \frac{1}{2} \int_{\mathbb{R}} u h g', \quad (4.119)$$

$$\begin{aligned}
J_5 &= -\frac{1}{2} \int_{\mathbb{R}} u_x \partial_x (h g') \\
&= -\frac{1}{2} \int_{\mathbb{R}} u_x h_x g' - \frac{1}{2} \int_{\mathbb{R}} u_x h g'' \\
&= \frac{1}{2} \int_{\mathbb{R}} u \partial_x (h_x g') + \frac{1}{2} \int_{\mathbb{R}} u \partial_x (h g'') \\
&= \frac{1}{2} \int_{\mathbb{R}} u h_{xx} g' + \int_{\mathbb{R}} u h_x g'' + \frac{1}{2} \int_{\mathbb{R}} u h g'''
\end{aligned} \quad (4.120)$$

and

$$\begin{aligned}
J_4 &= -\frac{1}{2} \int_{\mathbb{R}} u_x \partial_x (h_x g) \\
&= -\frac{1}{2} \int_{\mathbb{R}} u_x h_{xx} g - \frac{1}{2} \int_{\mathbb{R}} u_x h_x g' \\
&= \frac{1}{2} \int_{\mathbb{R}} u \partial_x (h_{xx} g) + \frac{1}{2} \int_{\mathbb{R}} u \partial_x (h_x g') \\
&= \frac{1}{2} \int_{\mathbb{R}} u h_{xxx} g + \int_{\mathbb{R}} u h_{xx} g' + \frac{1}{2} \int_{\mathbb{R}} u h_x g''.
\end{aligned} \quad (4.121)$$

Adding (4.118)-(4.121), we get

$$J = -\frac{1}{2} \int_{\mathbb{R}} u h_x g + \frac{1}{2} \int_{\mathbb{R}} u h_{xxx} g + \int_{\mathbb{R}} u h_{xx} g' + \frac{1}{2} \int_{\mathbb{R}} u h_x g''.$$

Using that  $h_{xx} = -u^2 + h$  and  $h_{xxx} = -2uu_x + h_x$ , we have

$$\int_{\mathbb{R}} u h_{xx} g' = \int_{\mathbb{R}} u(-u^2 + h)g' = -\int_{\mathbb{R}} u^3 g' + \int_{\mathbb{R}} u h g'$$

and

$$\begin{aligned}
\frac{1}{2} \int_{\mathbb{R}} u h_{xxx} g &= \frac{1}{2} \int_{\mathbb{R}} u(-2uu_x + h_x)g \\
&= -\int_{\mathbb{R}} u^2 u_x g + \frac{1}{2} \int_{\mathbb{R}} u h_x g \\
&= -\frac{1}{3} \int_{\mathbb{R}} \partial_x (u^3) g + \frac{1}{2} \int_{\mathbb{R}} u h_x g \\
&= \frac{1}{3} \int_{\mathbb{R}} u^3 g' + \frac{1}{2} \int_{\mathbb{R}} u h_x g.
\end{aligned}$$



At this stage it is worth noticing that the term  $\int_{\mathbb{R}} uh_x g$  cancels with the one in  $J$ . Finally, we obtain

$$J = -\frac{2}{3} \int_{\mathbb{R}} u^3 g' + \int_{\mathbb{R}} uhg' + \frac{1}{2} \int_{\mathbb{R}} uh_x g''. \quad (4.122)$$

Combining (4.116) and (4.122), we get

$$\frac{d}{dt} \int_{\mathbb{R}} yvg = \frac{2}{3} \int_{\mathbb{R}} u^3 g' - 4 \int_{\mathbb{R}} u^2 v g' - \frac{3}{2} \int_{\mathbb{R}} u^2 v_x g'' - \frac{1}{2} \int_{\mathbb{R}} u^2 v g''' + \int_{\mathbb{R}} uhg' + \frac{1}{2} \int_{\mathbb{R}} uh_x g''. \quad (4.123)$$

Now, substituting  $u$  by  $4v - v_{xx}$  and using intergration by parts, we rewrite the energy as

$$\begin{aligned} \int_{\mathbb{R}} yvg &= \int_{\mathbb{R}} v [(1 - \partial_x^2)(4v - v_{xx})] g \\ &= 4 \int_{\mathbb{R}} v^2 g - 5 \int_{\mathbb{R}} v v_{xx} g + \int_{\mathbb{R}} v (\partial_x^4 v) g \\ &= 4 \int_{\mathbb{R}} v^2 g + K_1 + K_2. \end{aligned}$$

By computing

$$\begin{aligned} K_1 &= 5 \int_{\mathbb{R}} \partial_x(vg)v_x \\ &= 5 \int_{\mathbb{R}} v_x^2 g + 5 \int_{\mathbb{R}} v v_x g' \\ &= 5 \int_{\mathbb{R}} v_x^2 g + \frac{5}{2} \int_{\mathbb{R}} \partial_x(v^2)g' \\ &= 5 \int_{\mathbb{R}} v_x^2 g - \frac{5}{2} \int_{\mathbb{R}} v^2 g'' \end{aligned} \quad (4.124)$$

and

$$\begin{aligned} K_2 &= - \int_{\mathbb{R}} \partial_x(vg)v_{xxx} \\ &= - \int_{\mathbb{R}} v_x v_{xxx} g - \int_{\mathbb{R}} v v_{xxx} g' \\ &= K_3 + K_4 \end{aligned} \quad (4.125)$$

with

$$\begin{aligned} K_3 &= \int_{\mathbb{R}} \partial_x(v_x g)v_{xx} \\ &= \int_{\mathbb{R}} v_{xx}^2 g + \int_{\mathbb{R}} v_x v_{xx} g' \\ &= \int_{\mathbb{R}} v_{xx}^2 g + \frac{1}{2} \int_{\mathbb{R}} \partial_x(v_x^2)g' \\ &= \int_{\mathbb{R}} v_{xx}^2 g - \frac{1}{2} \int_{\mathbb{R}} v_x^2 g'' \end{aligned} \quad (4.126)$$

and

$$\begin{aligned}
K_4 &= \int_{\mathbb{R}} \partial_x(vg')v_{xx} \\
&= \int_{\mathbb{R}} v_x v_{xx} g' + \int_{\mathbb{R}} v v_{xx} g'' \\
&= \frac{1}{2} \int_{\mathbb{R}} \partial_x(v_x^2)g' - \int_{\mathbb{R}} \partial_x(vg'')v_x \\
&= -\frac{1}{2} \int_{\mathbb{R}} v_x^2 g'' - \int_{\mathbb{R}} v_x^2 g'' - \int_{\mathbb{R}} v v_x g''' \\
&= -\frac{3}{2} \int_{\mathbb{R}} v_x^2 g'' - \frac{1}{2} \int_{\mathbb{R}} \partial_x(v^2)g''' \\
&= -\frac{3}{2} \int_{\mathbb{R}} v_x^2 g'' + \frac{1}{2} \int_{\mathbb{R}} v^2 g^{(4)}. \tag{4.127}
\end{aligned}$$

Combining (4.124)-(4.127), we get

$$\int_{\mathbb{R}} yvg = \int_{\mathbb{R}} (4v^2 + 5v_x^2 + v_{xx}^2)g + \frac{1}{2} \int_{\mathbb{R}} v^2(g^{(4)} - 5g'') - 2 \int_{\mathbb{R}} v_x^2 g''$$

and then

$$\frac{d}{dt} \int_{\mathbb{R}} yvg = \frac{d}{dt} \int_{\mathbb{R}} (4v^2 + 5v_x^2 + v_{xx}^2)g + L_1 + L_2. \tag{4.128}$$

Using (4.117), we have

$$L_1 = \int_{\mathbb{R}} v v_t (g^{(4)} - 5g'') = \frac{5}{2} \int_{\mathbb{R}} v h_x g'' - \frac{1}{2} \int_{\mathbb{R}} v h_x g^{(4)} \tag{4.129}$$

and

$$L_2 = -4 \int_{\mathbb{R}} v_x v_{tx} g'' = 2 \int_{\mathbb{R}} v_x h_{xx} g'' = -2 \int_{\mathbb{R}} u^2 v_x g'' + 2 \int_{\mathbb{R}} v_x h g''. \tag{4.130}$$

Lemma 4.2 follows by combining (4.123) and (4.128)-(4.130).

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