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Accurate steam-water equation of state for two-phase flow LMNC model with phase transition*

Stéphane Dellacherie[†] Gloria Faccanoni[‡] Bérénice Grec[§] Yohan Penel[¶]

Abstract

This paper is dedicated to the design of incomplete equations of state for a two-phase flow with phase transition that are specific to the low Mach number regime thanks to the fact that the thermodynamic pressure can be considered constant. These equations of state supplement the 2D LMNC model introduced in previous works.

This innovative strategy relies on tabulated values and is proven to satisfy crucial thermodynamic requirements such as positivity, monotonicity, continuity. In particular, saturation values are exact. This procedure is assessed by means of analytical steady solutions and comparisons with standard analytical equations of state which shows a great improvement in accuracy.

Keywords. Thermohydraulics; Low Mach number regime; Phase transition; Tabulated equations of state; Thermodynamic consistency; Numerical simulations.

1 Introduction

The present paper belongs to a series of articles and conference proceedings [4, 5, 13, 14] devoted to the modelling and the numerical simulation of the coolant fluid in a nuclear reactor of *Pressurised Water Reactor* (PWR) type. Such a complex flow is characterised by a small Mach number which transcribes the fact that the material velocity is much smaller than the speed of sound. This property strongly influences both theoretical and numerical approaches in the modelling process. Indeed, it has been proven in numerous works that issues may occur in a compressible framework when the Mach number \mathcal{M} is quite small (see for instance [12, 25] and references therein). The pressure variable is mainly held responsible for such instabilities due to the singularity of corresponding terms with respect to \mathcal{M} and due to the propagation of initial perturbations. Let us recall some results:

- First, solutions of a compressible model formally converge to incompressible solutions when \mathcal{M} goes to 0 only under suitable conditions upon the initial conditions (with correct orders of magnitude, *i.e. well-prepared*), the domain where the fluid flows and the equation of state modelling water properties (see for instance [16, 19, 32, 33, 47] and references therein).
- Second, even with *well-prepared* initial conditions, classical numerical schemes may lead to inaccurate solutions due to the discrepancy between equilibrium states at the continuous (PDE) and discrete levels [12, 24, 25].
- Third, in addition to accuracy issues, efficiency is also pointed out. Indeed, time scales induced by the speed of sound and the flow velocity are quite different in that context. An explicit stability condition would take the speed of sound into account and over-constrain the computations. In contrast, an implicit scheme would fail to reproduce fast physical phenomena.

For all these reasons, three main strategies have been considered in the literature:

- ① From the numerical point of view, splitting strategies are introduced relying on a decomposition of the compressible equations between non-stiff terms (handled explicitly) and stiff terms (handled implicitly) and mostly

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inspired from the incompressible framework. Several decompositions have been proposed making implicit the pressure [7, 31, 43] or more generally acoustic terms [26, 30, 48]. Others are based on a Lagrange-projection [8] or on a Hodge decomposition [10]. In [41, 50], preconditioning techniques are used to deal with diffusion terms.

- ② Given the analysis of the inaccuracy of classical schemes by means of asymptotic expansions with respect to the Mach number, some modifications are proposed to adapt those schemes to make them able to deal with the low Mach regime [15, 34]. It comprises corrected Godunov-type schemes based on the matching between kernels of the continuous operator and its discrete counterpart.
- ③ Another strategy consists in deriving models of reduced complexity which are specific the low Mach regime [1, 2, 13, 38, 39]. These models are derived from asymptotic expansions with respect to the Mach number which lead to a hierarchy of intermediate (simpler) models between compressible and incompressible systems. However, applications are restricted to cases where acoustics can be neglected. Indeed, the truncation of the asymptotic expansion amounts to filtering out acoustic waves.

The present paper is based on Approach ③ as acoustics phenomena are not predominant in the context we are investigating. Newly derived models have several advantages from both theoretical and numerical points of view. In particular their mathematical structure is much simpler than compressible systems. For the Euler equations, the hyperbolic operator becomes a mixed hyperbolic-elliptic operator that transcribes the weak compressibility. It comprises a divergence constraint whose source term is not 0 as it would be in the incompressible limit. Moreover, unlike incompressible models, low Mach number systems account for large heat transfer (through variations of the density).

The asymptotic expansion leads to a decoupling of the pressure field into a *dynamic* pressure appearing in the momentum equation and a *thermodynamic* pressure – involved in the equation of state – which turns out to be independent from the space variable. This property is crucial and has major consequences:

- First, it enables to compute explicit analytical solutions in the 1D case [4, 23] which provides reliable tools to assess compressible solvers in the low Mach regime.
- Second, it leads to simple robust and efficient numerical schemes as it turns the acoustic stability condition (for explicit schemes) into a purely convective stability condition. Time steps are thus larger and the computational cost due to the PDEs dramatically decreases.
- Third, it lowers the number of evaluations of the equation of state (only once per time iteration) which also reduces the computational cost due to the modelling. The latter consequence is the key-point of the paper.

The model investigated in the present paper is named LMNC for *Low Mach Nuclear Core* model and was designed in [13]. It is derived from the Navier-Stokes equations keeping terms of order 0 in the asymptotic expansion. It originally concerned monophasic flows.

In addition to the PDE system, we want to describe water properties thanks to thermodynamics. Classically, PDEs are supplemented with an equation of state (EOS) which is a relation between pressure, density and entropy variables. This relation is expected to transcribe the behaviour of water over a specific range of physical conditions (temperature and pressure). Analytical EOS have been proposed in the literature such as the *ideal gas* law or the *stiffened gas* (SG) law [35, 40]. They have the advantage of a simple formulation leading to explicit computations. In particular, the SG law corresponds to a rational function involving five parameters to tune depending on the range of temperature to be investigated. However, it turns out that any tuning provides inaccurate results for the values at stake in our context, namely a pressure of 15.5 MPa and high temperatures close to the saturation. A similar 6-parameter law (NASG) has been recently designed [36] to cover an extended range of temperature but the inaccuracy close to saturation still remains. The SG law was mainly used in previous works [4, 5, 14] to take advantage of its simple formulation leading to explicit solutions in dimension 1. It also enabled to assess numerical strategies. Analytical laws such as SG and NASG are academic tools which are quite useful to assess models and numerical strategies but they are known to be restricted to simple cases. Consequently, they do not lead to physically relevant results as we shall show in this paper. Hence, to provide more reliable results in particular for safety evaluations, another strategy is necessary for defining equations of state in the range of temperature of interest.

Physical values of main thermodynamic variables are stored over a wide range of pressure/temperature values for water and available for researchers in [37]. A strategy thus consists in constructing fitting polynomials for each physical variable involved in the model which match tabulated data. It would be an expensive computation in a compressible framework since an equation of state depends on two variables [11, 51, 52]. But in the low Mach number regime, low Mach number models exhibit a constant thermodynamic pressure which makes the equation of state a function of a single variable (which is not the temperature as in classic works but the enthalpy for reasons explained

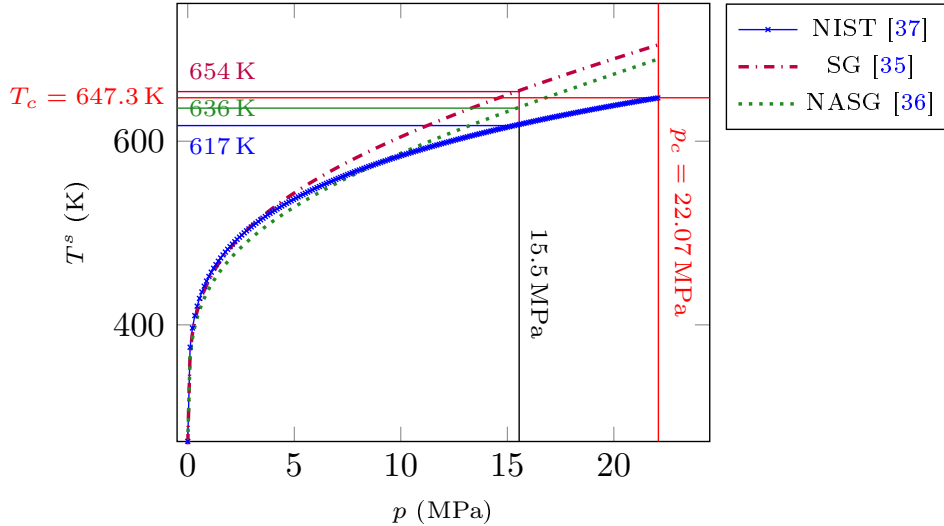


Figure 1: Temperature at saturation as a function of the pressure. Notice that at $p = 15.5$ bar, $T_{SG}^s > T_c > T_{NASG}^s > T_{NIST}^s$.

below). Even if the resulting equation of state has a more complex expression than the SG/NASG laws, it is expected to provide more accurate results in the physical context we are interested in. To this end, fitting polynomials should be very carefully designed [18] to satisfy basic thermodynamic requirements (positivity, monotonicity, ...). We present in the sequel an innovative procedure to construct such relevant EOS for pure phases (liquid and vapour) at low Mach numbers.

The challenge is then more tricky when phase transition occurs. In addition to the design of accurate specific equations of state for water and steam, one has to model equilibria across phase transition and saturation phenomena. In the present work, we consider mechanical and thermodynamic equilibria which amounts to assuming that phases move at the same velocity and that vaporisation, condensation and heat transfer processes are instantaneous. This is why the LMNC model is a system with 3 equations and a single velocity field. LMNC can be seen as the zero Mach limit of the HEM model [9] with source terms. Other equilibria can also be considered and would lead to 4- to 7-equation systems.

Under these equilibrium assumptions, water appears under liquid and vapour phases but also as a mixture of the two. Mixture must thus be modelled through the evolution of thermodynamic and mechanical variables. In particular, temperature is constant within the mixture phase. As the differential model must account for the three phases, temperature cannot be chosen as a main variable. That is why the LMNC model is expressed depending on the enthalpy, the velocity and the dynamic pressure¹. Consequently, equations of state in low Mach diphasic models can be only function of the enthalpy whereas complete equations of state such as SG or NASG are functions of two variables. In the latter case, the saturation is determined by the equality of Gibbs potentials which may result in inaccurate values at saturation. Figure 1 shows the physical unrelevance of these equations of state because the temperature at saturation deduced from any of them is larger than the critical temperature at $p = 15.5$ MPa.

The strength of the procedure described in the sequel is that analytic equations of state are derived from tabulated values and satisfy exactly the saturation. It takes advantage of the fact that the thermodynamic pressure is constant which enables to guarantee crucial properties for the well-posedness of the model as well as for the physical relevance of its solutions. Classically, the density and temperature should be fitted as functions of the enthalpy and then the compressibility coefficient (2) and the heat capacity would be deduced by differentiation. That would raise some consistency and stability issues. By means of thermodynamic relations (see Proposition 3.1), it is possible to apply the reverse way: fitting² the compressibility coefficient and then integrating to deduce the density turns out to be a decisive procedure as it ensures positivity, monotonicity and continuity of the density. Moreover, saturation values exactly match with tabulated ones (Proposition 3.2). The same procedure can be applied independently to the temperature which is deduced from the isobaric heat capacity with the same mathematical properties. Coefficients of the tabulated equation of state are only computed once (as for SG/NASG laws) at the beginning of the algorithm and holds for all iterations. Such an incomplete equation of state is obviously restricted to constant pressure regimes but in the present context, it is a very efficient and stable method to incorporate water physics in the LMNC model.

¹The reader is reminded that the second variable – the thermodynamic pressure – is constant.

²The choice of the fitting procedure is left to the reader. The least square method applied in the present work is not exclusive at all.

The paper is organised as follows. We first present in Section 2 the LMNC model which consists of 3 equations involving velocity, enthalpy and dynamic pressure. We mention that viscosity and thermal conductivity are taken into account. Boundary conditions are also specified. The core of this paper is detailed in Section 3: we briefly recall the modelling of mixture (see also [4]) and the expression of the *stiffened gas* and *Noble-Abel – stiffened gas* laws. We then explain the construction of the dedicated equations of state in pure phases based on the fitting of tabulated values [37]. This amounts to specifying the dependency of all thermodynamic and physical variables with respect to the enthalpy. Two strategies are investigated to fit the compressibility coefficient β and the isobaric heat capacity c_p :³ NIST-p which corresponds to high order fitting polynomials and NIST-0 which is a phasewise constant approximation. We state the crucial properties satisfied by the corresponding equations of state. In Section 4, we provide mathematical assumptions on data and on the smoothness of solutions (jump conditions across phase transition) and we compare the accuracy of the aforementioned equations of state upon an analytical steady solution. Section 5 is dedicated to numerical simulations to compare the equations of state on a stiff test case and highlight the influence of the thermal conductivity and of the gravity field.

2 Low Mach two-phase flow model with phase transition and thermal diffusion

2.1 Equations

Fluid flows are often described by the Navier-Stokes equations. When the flow regime is such that the Mach number is small, they can be replaced by simplified models such as the LMNC model derived in [13]. The latter system is mainly investigated under its non-conservative formulation [4, 14]

$$\begin{cases} \nabla \cdot \mathbf{u} = \frac{\beta(h, p_*)}{p_*} [\nabla \cdot (\Lambda(h, p_*) \nabla h) + \Phi], & (1a) \\ \varrho(h, p_*) \times [\partial_t h + \mathbf{u} \cdot \nabla h] = \nabla \cdot (\Lambda(h, p_*) \nabla h) + \Phi, & (t, \mathbf{x}) \in \mathbb{R}^+ \times \Omega^d, & (1b) \\ \varrho(h, p_*) \times [\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u}] - \nabla \cdot \sigma(\mathbf{u}) + \nabla \mathcal{P} = \varrho(h, p_*) \mathbf{g}, & (1c) \end{cases}$$

for some bounded domain⁴ $\Omega^d \subset \mathbb{R}^d$, $d \in \{1, 2\}$. The unknowns involved in this model are the velocity \mathbf{u} , the enthalpy h and the dynamic pressure \mathcal{P} while the data are the constant⁵ thermodynamic pressure p_* , the power density $\Phi(t, \mathbf{x})$ which models the heating of the coolant fluid due to the fission reactions in the nuclear core and the constant gravity field \mathbf{g} . To close the model, we have to give the expression of the density $\varrho = \varrho(h, p_*)$ and the compressibility coefficient $\beta(h, p_*)$ – *i.e.* specify the equation of state, see § 3.1 –, the coefficient $\Lambda = \Lambda(h, p_*)$ which is related to thermal conduction effects – see § 3.3 – and the viscous stress tensor $\sigma(\mathbf{u})$. The present article is dedicated to the modelling of these terms which characterise properties of water through equations of state and physical laws when taking into account phase transition.

More precisely, the compressibility coefficient is the key variable in this work, and is originally defined by

$$\beta(h, p_*) \stackrel{\text{def}}{=} - \frac{p_*}{\varrho(h, p_*)^2} \cdot \frac{\partial \varrho}{\partial h} \Big|_p (h, p_*). \quad (2)$$

Another expression will be designed in Proposition 3.1 which turns out to be crucial for the study.

As for viscous effects, they include the classical internal friction in the fluid as well as the friction on the fluid due to technological devices in the nuclear core (*e.g.* the friction on the fluid due to the fuel rods):

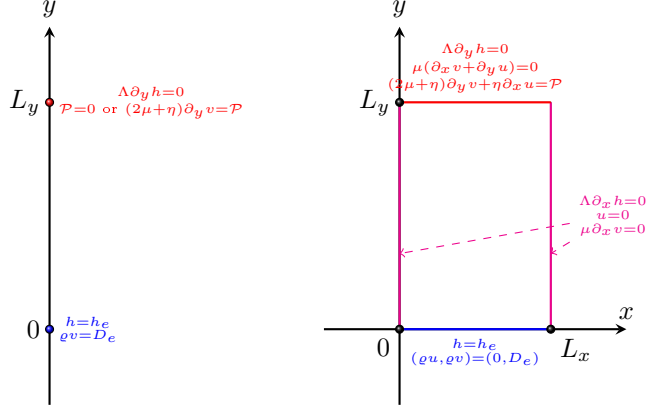
$$\sigma(\mathbf{u}) = \mu(h, p_*) \times (\nabla \mathbf{u} + (\nabla \mathbf{u})^T) + \eta(h, p_*) \times (\nabla \cdot \mathbf{u}) \mathbb{I}.$$

The very expression of the viscous term is not important in the 1D framework as (1c) decouples from the other equations [4]. The computation of \mathcal{P} is then only a postprocessing. This is no more the case in higher dimensions where the latter equation is a real part of the coupled system.

³Unlike the SG/NASG laws where β and c_p are fully determined by the parameters of the laws, in the NIST strategy, they are computed independently from each other.

⁴In 1D, $\Omega^1 = [0, L_y]$, $\mathbf{x} = y$ and $\mathbf{u} = v$, whereas in 2D $\Omega^2 = [0, L_x] \times [0, L_y]$, $\mathbf{x} = (x, y)$ and $\mathbf{u} = (u, v)$. We do not study the 3D case in this paper but the extension is straightforward.

⁵In the low Mach approximation, the thermodynamic pressure p_* does not depend on space but can be considered in general as a function of time [5, 39]. We focus here on applications where it can be assumed to be constant.

(a) Domain Ω^1 (b) Domain Ω^2 Figure 2: Domain Ω^d , $d \in \{1, 2\}$ and boundary conditions

When a monophasic flow is considered, System (1) is then coupled to an equation of state and to physical laws dedicated to a pure phase. To go further by taking into account phase transition, it is necessary to specify the equilibria at stake between phases. In the present case, we assume that phases are in thermodynamic and mechanic equilibria which induces the equality of temperatures, pressures, Gibbs potentials and velocities. As a consequence, this model has a single velocity for all phases under which water appears. As we shall see below, System (1) has a simpler structure than the Navier-Stokes equations which makes it easier to deal with phase transition.

We must also emphasize that System (1) is characterised by two pressure fields, which is classical in low Mach number approaches: the *thermodynamic pressure* p_* is involved in the equation of state while the *dynamic pressure* \mathcal{P} appears in Equation (1c). This pressure decomposition results from an asymptotic expansion with respect to the Mach number which induces the filtering out of the acoustic waves. They are indeed not involved anymore in System (1).

2.2 Boundary conditions

Boundary conditions (BC) are specified in 2D but adapt directly to 1D (see Figure 2). We also make a few remarks about 1D BC prescribed in earlier works.

- The fluid is injected at the bottom $y = 0$ of the core at a given density ϱ_e and at a given vertical flow rate D_e :

$$\varrho(h(t, x, 0), p_*) = \varrho_e(t, x), \quad (3a)$$

$$(\varrho \mathbf{u})(t, x, 0) = (0, D_e(t, x)). \quad (3b)$$

The entrance enthalpy $h_e(t, x)$ is implicitly defined by the equation $\varrho(h_e, p_*) = \varrho_e$. The vertical entrance velocity $v_e(t, x)$ applied at $y = 0$ is deduced from the relation $v_e(t, x) = D_e(t, x)/\varrho_e(t, x)$.

Observe that imposing ϱ_e or h_e is not completely equivalent, since it depends on the choice of the equation of state $\varrho(\cdot, p_*)$. In particular, it implies that steady states for distinct EOS are different (see Proposition 4.1 and Figure 7(a)).

- As for the temperature, we impose adiabatic conditions on all boundaries, except at the entry:

$$\begin{aligned} \Lambda(h, p_*) \nabla h \cdot \mathbf{n}(t, 0, y) &= \Lambda(h, p_*) \nabla h \cdot \mathbf{n}(t, L_x, y) = 0, & [\text{lateral walls}] \\ \Lambda(h, p_*) \nabla h \cdot \mathbf{n}(t, x, L_y) &= 0. & [\text{top}] \end{aligned}$$

- On the lateral walls we consider Robin conditions on the velocity:

$$\begin{aligned} (\mathbf{u} \cdot \mathbf{n})(t, 0, y) &= (\mathbf{u} \cdot \mathbf{n})(t, L_x, y) = 0, \\ \sigma(\mathbf{u}) \mathbf{n} \cdot \boldsymbol{\tau}(t, 0, y) &= \sigma(\mathbf{u}) \mathbf{n} \cdot \boldsymbol{\tau}(t, L_x, y) = 0. \end{aligned}$$

- Finally, at the top of the core, we consider a free outflow

$$[\sigma(\mathbf{u})\mathbf{n} - \mathcal{P}\mathbf{n}](t, x, L_y) = \mathbf{0}. \quad (3c)$$

Let us mention that it corresponds to the following boundary condition for the “parent model” (Navier-Stokes for a compressible fluid)⁶

$$[\sigma(\mathbf{u})\mathbf{n} - p\mathbf{n}](t, x, L_y) = -p_*\mathbf{n}. \quad (3c')$$

Remark 2.1. *When viscosity is involved ($\mu \neq 0$, $\eta \neq 0$), boundary condition (3c) does not match the boundary condition we considered in previous works [4] which was*

$$\mathcal{P}(t, x, L_y) = 0.$$

Indeed, the asymptotic expansion with the Mach number not only filters out the acoustic waves but it also changes the nature of the equations through the generation of two pressure fields. In 1D, equation (1c) decouples from the others and becomes an ODE upon \mathcal{P} while, in higher dimensions, the decoupling does not occur, and equation (1c) is a second-order PDE upon $(\mathbf{u}, \mathcal{P})$.

Nevertheless, we can wonder about the signification of pressure p_ in (3c'). In the primary circuit of a PWR, there is a pressuriser at the exit of the core which maintains the water at high pressure. p_* can thus be interpreted as the imposed pressure except that it is not imposed at the exit of the core but further in the circuit at a position $L_y + \epsilon$. The domain $[L_y, L_y + \epsilon]$ is modelled as an inviscid fluid – see the right hand side in (3c') – without heating process ($\Phi = 0$). We may think that ϵ goes to 0 as the Mach number does.*

3 Accurate steam-water closures for the two-phase flow LMNC model with phase transition

In this section, we present different strategies to close the model by means of equations of state (EOS) and physical laws. These relations between thermodynamic and physical variables are expected to accurately transcribe properties of water over a range of enthalpy values (h_{min}, h_{max}) – to be specified – so that we can reproduce the behaviour of the flow in a realistic way. On the one hand, we mention that the narrower the interval, the more accurate the modelling. On the other hand, we have to take an interval large enough to handle incidental values close to phase transition (hence close to the enthalpy at saturation in liquid water and vapor phase).

An EOS is related to thermodynamic properties of a fluid and consists of an algebraic/differential relation between thermodynamic variables. An EOS must satisfy thermodynamic constraints such as the positivity of the squared speed of sound or the decrease of the density with respect to the enthalpy. The other issue is to derive an EOS that models not only pure liquid and vapour phases of water but also the transition from one phase to another in a relevant and accurate way. Indeed, a bad calibrated EOS may dramatically modify the temperature at saturation in the fluid and cause unphysical phase transition from liquid to vapour.

Since the fluid is considered a continuum (see *e.g.* [44, 49] for a review of multiphase flow modelling), we consider here that water can appear under liquid phase (referred to by the index $\kappa = \ell$), vapour phase ($\kappa = g$), or a mixture of liquid and vapour ($\kappa = m$). The overall problem could be treated by means of distinct models for each phase supplemented with transmission conditions when phase transition occurs as it is the case for immiscible fluids. Our model relies on the assumption of local kinematic and thermodynamic equilibria between phases. This means that phases are assumed to evolve at the same velocity, and that vaporisation, condensation and heat transfer processes are assumed to be instantaneous. As a result, we model the two-phase flow by means of a single system of PDEs (with a single velocity field) governing physical variables no matter which phase they are related to. This process only holds provided the EOS has a wide domain of validity including phase transition. For more details on the construction of these EOS and their applications to CFD for compressible fluid flows, see for instance [9, 21, 28, 29, 46].

Variables to be defined as functions of (h, p_*) are ρ , β , Λ , μ and η involved in (1) as well as c_p and T for postprocessing. In classical thermodynamics, two variables are sufficient to represent a thermodynamic state for a pure single-phase fluid through the EOS. In the literature, there exist numerous EOS specific to each fluid under investigation. The case of liquid-vapour phase transitions is schematically depicted on Figure 3 and represented by means of experimental data extracted from [37] on Figure 4. When phase transition is taken into account, a mixture zone appears where the two phases co-exist: it is called the *saturation zone* (Figs. 3a and 4a). This region is bounded by two saturation curves connected at the critical point $(1/\rho_c, p_c)$ which also belongs to the critical isotherm $T = T_c$. Since phase transition appears at constant pressure and temperature, the physical isotherm is horizontal.

⁶The physical pressure field p is recovered from pressure in the low Mach number model by $p = p_* + \mathcal{P}$. Thus $\mathcal{P} = \mathcal{O}(\mathcal{M}^2)$ is a perturbation around the mean value p_* .

Once an EOS is provided for both liquid and vapour phases, the equality of Gibbs potentials leads to the definition of the temperature at saturation and then to enthalpies at saturation $h_l^s(p_*)$ and $h_g^s(p_*)$ [6, 40]. To model phase transition, we introduce a phasewise decomposition for any variable $\zeta \in \{\varrho, \beta, T, \Lambda, c_p, \mu, \eta\}$:

$$\zeta(h, p_*) = \begin{cases} \zeta_l(h, p_*), & \text{if } h \leq h_l^s(p_*), \\ \zeta_m(h, p_*), & \text{if } h_l^s(p_*) < h < h_g^s(p_*), \\ \zeta_g(h, p_*), & \text{if } h \geq h_g^s(p_*). \end{cases} \quad (4)$$

From now on, we shall ignore the dependency with respect to p_* for thermodynamic variables as the thermodynamic pressure is constant throughout this study.

To construct the global EOS, we aim at constructing explicit functions $h \mapsto \tilde{\zeta}_\kappa(h)$, $\kappa \in \{\ell, g\}$, approximating $h \mapsto \zeta_\kappa(h, p_*)$ so that each pure phase has its own thermodynamics described in an accurate way. Then, the modelling of the mixture $\tilde{\zeta}_m$ is carried out by means of an analytical formula based on thermodynamic principles [4, § 3.2].

Notations. *In the sequel, we shall adopt the following notations:*

- ζ_κ refers to the thermodynamic variable in phase κ ;
- $\tilde{\zeta}_\kappa$ is an approximation function of ζ_κ : the core of this paper is to present different approaches to design such approximation functions;
- $(\hat{\zeta}_\kappa)_i$ are the tabulated values extracted from the NIST tables [37];
- In case when ζ_κ can be considered constant, the corresponding constant value is denoted by $\bar{\zeta}_\kappa$.

A natural approach is to try to design a complete EOS from which $T(h, p_*)$ and $\varrho(h, p_*)$ are constructed in the first place, and then $\beta(h, p_*)$ and $c_p(h, p_*)$ are deduced from (2) and (23). However, as we shall see in the sequel, for stability and consistency reasons we prefer constructing an incomplete EOS based on $\beta(h, p_*)$.

In the sequel we detail several strategies we investigated to derive an EOS for pure phases. We shall consider analytical EOS such as ❶ the *Stiffened Gas* (SG) law (see for instance [35, 40]) or ❷ the *Noble-Abel – Stiffened Gas* (NASG) law [36]. We can also think of another strategy based on experimental measurements gathered in tables [37, 52]. We shall investigate some fitting strategies which will be referred to as ❸ NIST-p for high-degree fitting polynomials $h \mapsto \tilde{\zeta}(h)$ and ❹ NIST-0 for constant approximations. We underline that this is a strength of the low Mach number approach: in a compressible framework, one would have to derive a function $(h, p) \mapsto \tilde{\zeta}(h, p)$ that fits a surface of tabulated values whereas in the low Mach framework, the procedure reduces to fitting a curve as the polynomial only depends on h ($p \equiv p_*$ here).

Since these fitting functions cannot model the behaviour of water for all values of h , we focus on a restricted interval of values, prescribed in the following Hypothesis.

Hypothesis 3.1. *Functions ϱ , β , T , Λ , c_p , μ and η are defined over the interval (h_{min}, h_{max}) . Moreover:*

1. ϱ_e involved in the boundary condition (3a) is such that there exists a unique $h_e \in (h_{min}, h_{max})$ satisfying $\varrho(h_e, p_*) = \varrho_e$;

2. $h_{min} > h_l^s - \frac{p_*}{\varrho_l^s \max_{h \in [h_{min}, h_l^s]} \beta(h, p_*)}$.

Hypothesis 3.1-2. is needed in order to ensure the positivity of ϱ in $[h_{min}, h_l^s]$ for our chosen definition (11) of $1/\varrho$.

3.1 Compressibility coefficient and density

Let us now detail how we construct the global approximated function $\tilde{\beta}$ defined phasewise as in (4) via the EOS $\tilde{\beta}_\kappa$ in each phase $\kappa \in \{\ell, m, g\}$.

3.1.1 Compressibility coefficient and density in the mixture phase

As detailed in [4, § 3.2], the compressibility coefficient in the mixture is phasewise constant and given by

$$\bar{\beta}_m \stackrel{\text{def}}{=} p_* \frac{1/\varrho_g^s - 1/\varrho_l^s}{h_g^s - h_l^s}. \quad (5a)$$

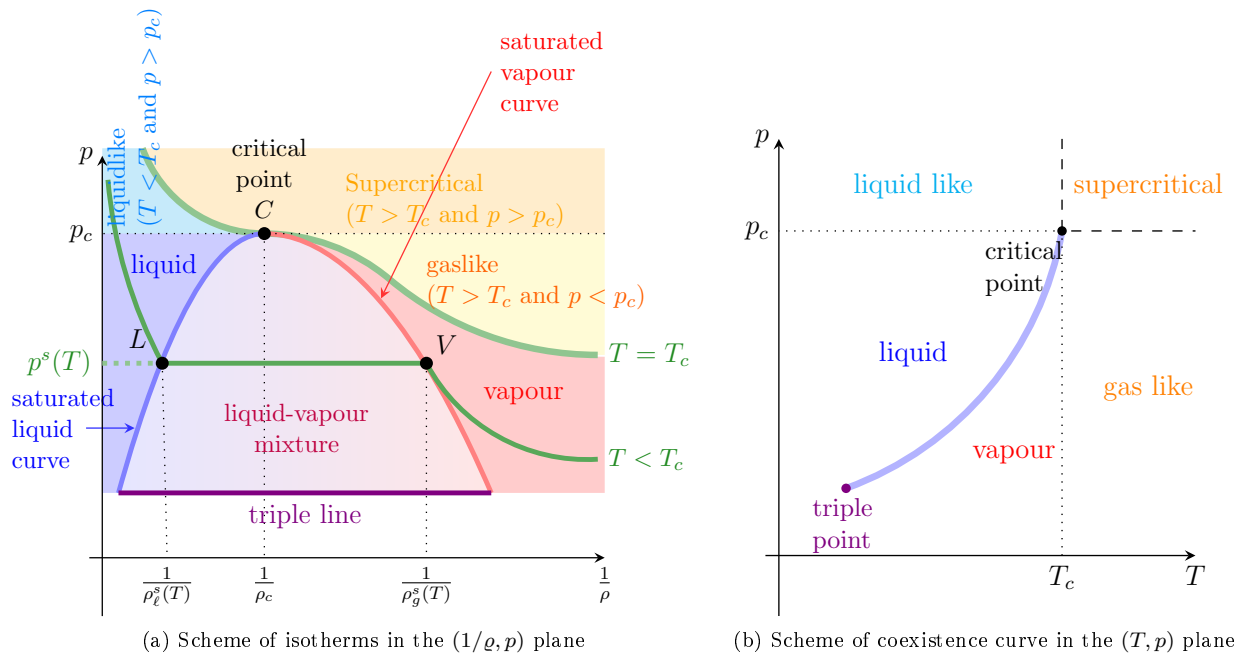


Figure 3: Schematic saturation and coexistence curves.

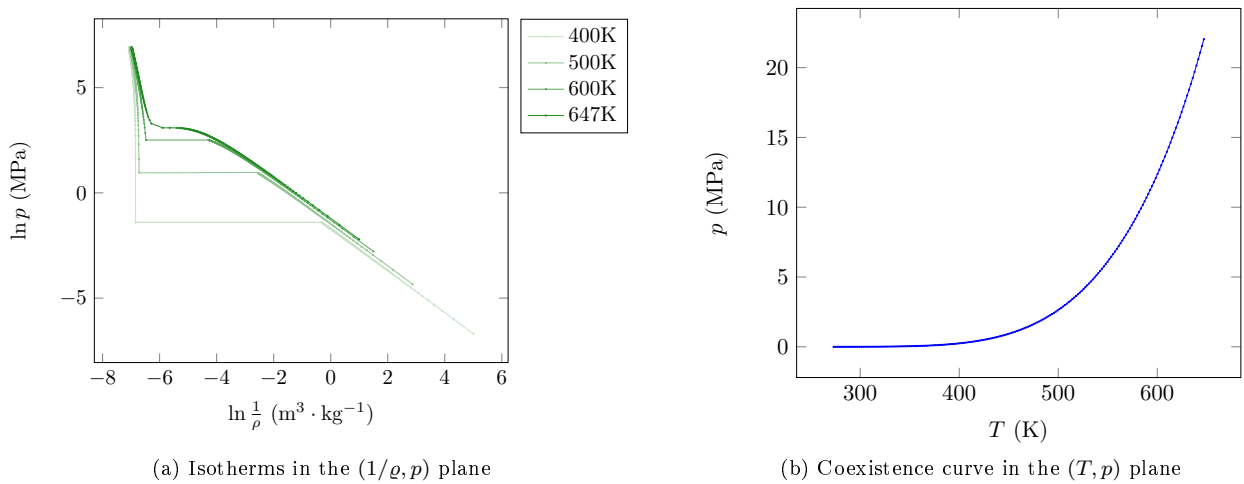


Figure 4: Saturation and coexistence curves using experimental data from [37]

κ	\widehat{h}_κ [kJ · kg ⁻¹]	\widehat{q}_κ [kg · m ⁻³]	\widehat{T}_κ [K]	\widehat{c}_κ^* [m · s ⁻¹]	$\widehat{c}_{p\kappa}$ [J · kg ⁻¹ · K ⁻¹]	$\widehat{c}_{v\kappa}$ [J · kg ⁻¹ · K ⁻¹]	$\widehat{\lambda}_\kappa$ [W · m ⁻¹ · K ⁻¹]	$\widehat{\mu}_\kappa$ [Pa · s]
ℓ	978.702	842.783	500.000	1293.67	4561.5	3218.0	0.657	1.209 × 10 ⁻⁴
ℓ	980.223	842.359	500.336	1292.50	4563.5	3217.0	0.657	1.207 × 10 ⁻⁴
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
ℓ	1627.450	595.733	617.667	624.66	8871.0	3098.1	0.459	6.850 × 10 ⁻⁵
ℓ	\widehat{h}_ℓ^s	\widehat{q}_ℓ^s	\widehat{T}^s	621.43	8950.0	3101.0	0.458	6.833 × 10 ⁻⁵
g	\widehat{h}_g^s	\widehat{q}_g^s	\widehat{T}^s	433.40	14000.6	3633.1	0.121	2.311 × 10 ⁻⁵
g	2596.965	101.816	618.00	433.69	13931.7	3627.6	0.121	2.310 × 10 ⁻⁵
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
g	3066.962	60.540	699.667	573.70	3670.7	2173.2	0.0803	2.615 × 10 ⁻⁵
g	3068.184	60.473	700.000	574.02	3664.7	2171.5	0.0803	2.616 × 10 ⁻⁵

Table 1: Sample of water values [37] at $p_* = 15.5$ MPa with data at saturation $\widehat{h}_\ell^s = 1629.880$ kJ · kg⁻¹; $\widehat{h}_g^s = 2596.119$ kJ · kg⁻¹; $\widehat{q}_\ell^s = 594.379$ kg · m⁻³; $\widehat{q}_g^s = 101.930$ kg · m⁻³; $\widehat{T}^s = 617.939$ K

The derivation is much simpler than in the fully compressible case [20].

Let us mention that the expression of $\overline{\beta}_m$ does not depend on the EOS in pure phases but only on values at saturation which may differ from one strategy to another (see Table 4). As for the density, it reads

$$\varrho_m(h) \stackrel{\text{def}}{=} \frac{\varrho_g^s \varrho_\ell^s (h_g^s - h_\ell^s)}{\varrho_g^s h_g^s - \varrho_\ell^s h_\ell^s - (\varrho_g^s - \varrho_\ell^s) h}. \quad (5b)$$

Let us also notice that given these two definitions, we can write

$$\frac{1}{\varrho_m(h)} = \frac{\overline{\beta}_m}{p_*} (h - \overline{q}_m)$$

where

$$\overline{q}_m \stackrel{\text{def}}{=} \frac{\varrho_g^s h_g^s - \varrho_\ell^s h_\ell^s}{\varrho_g^s - \varrho_\ell^s}. \quad (5c)$$

Hence this process which results from thermodynamic arguments makes the mixture EOS look like the *Stiffened Gas* law (7).

3.1.2 Compressibility coefficient and density in a pure phase

Let us then introduce the construction of $\widetilde{\beta}_\ell$ and $\widetilde{\beta}_g$, as well as \widetilde{q}_ℓ and \widetilde{q}_g , for the different approaches we consider.

1-2 SG-NASG: The *Stiffened Gas* (SG) law and the *Noble-Abel Stiffened Gas* (NASG) law result from the same process of construction, namely a complete equation of state (after Menikoff and Plohr [40]) which is a smooth relation between the specific entropy \mathcal{S} , the specific volume $\tau \stackrel{\text{def}}{=} 1/\varrho$ and the specific energy ε

$$(\tau, \varepsilon) \mapsto \mathcal{S}(\tau, \varepsilon) = \bar{c}_v [\ln(\varepsilon - \overline{Q} - \overline{\pi}(\tau - \bar{b})) + (\overline{\gamma} - 1) \ln(\tau - \bar{b})] + \overline{\mathcal{S}}_0, \quad (6)$$

where six constant parameters have to be tuned: \bar{c}_v , \overline{Q} , $\overline{\pi}$, $\overline{\gamma}$, $\overline{\mathcal{S}}_0$ and \bar{b} ($\bar{b} = 0$ for the SG law) depending on the range of enthalpy values and on the phase. More details on these complete EOS are given in Appendix A.

Consequently, SG and NASG laws have the same expression for the density

$$\frac{1}{\widetilde{q}(h)} = \frac{\widetilde{\beta}(h)}{p_*} (h - \widetilde{q}(h)) \quad (7a)$$

taking into account (5b). The phasewise constant compressibility coefficient is defined by

$$\widetilde{\beta}(h) = \begin{cases} \overline{\beta}_\ell, & \text{if } h \leq h_\ell^s, \\ \overline{\beta}_m, & \text{if } h_\ell^s < h < h_g^s, \\ \overline{\beta}_g, & \text{if } h \geq h_g^s, \end{cases} \quad (7b)$$

where $\bar{\beta}_m$ is given by (5a) with data at saturation in the corresponding column of Table 4 and

$$\tilde{q}(h) = \begin{cases} \bar{q}_l, & \text{if } h \leq h_l^s, \\ \bar{q}_m, & \text{if } h_l^s < h < h_g^s, \\ \bar{q}_g, & \text{if } h \geq h_g^s, \end{cases} \quad (7c)$$

with \bar{q}_m defined in (5c). SG [35] and NASG [36] EOS only differ through the values of constant parameters $\bar{\beta}_\kappa$ and \bar{q}_κ which are given in Table 5. Let us mention that Hypothesis 3.1-2 reduces in that case to $h_{min} > \bar{q}_l$ which transcribes the fact that those analytical EOS are defined over $(0, +\infty)$.

- **NIST-p**: A sample of tabulated values for the range of interest is given in Table 1 over the range (h_{min}, h_{max}) with $h_{min} = 978.702 \text{ kJ} \cdot \text{kg}^{-1}$ and $h_{max} = 3068.184 \text{ kJ} \cdot \text{kg}^{-1}$.

We remark that β does not belong to the set of variables referenced in NIST tables. Consequently, we have to find out a strategy to express the compressibility coefficient. One may have made use of Formula (2) that defines β from the a fitting function $h \mapsto \tilde{\rho}(h)$. Deriving the fitting function would raise some stability issues as well as positivity problems. One may also think of a finite-difference approximation of (2) but that would violate the consistency of thermodynamics (see [18] for more details). That is why we propose here an innovative idea which consists in deriving another expression for β involving tabulated variables but no derivative. From such an expression of β , we shall then deduce a consistent expression for ϱ . We shall see how this idea solves all the aforementioned issues.

Proposition 3.1. *In a pure phase, the compressibility coefficient β satisfies the relation*

$$\beta = \frac{p}{\varrho c^* \sqrt{T}} \sqrt{\frac{1}{c_v} - \frac{1}{c_p}} > 0, \quad (8)$$

where p is the pressure, ϱ the density, T the temperature, c^* the speed of sound, c_v the isochoric heat capacity and c_p the isobaric heat capacity.

The proof of the proposition can be found in Appendix B. Let us observe that, by construction, β is positive. Moreover, this relation is satisfied by the *Stiffened Gas/Noble-Abel – Stiffened Gas* laws.

Proposition 3.1 is crucial insofar as it enables to provide tabulated values (indexed by i) $(\hat{\beta}_\kappa)_i$ given pointwise values $(\hat{\varrho}_\kappa)_i$, $(\hat{T}_\kappa)_i$, $(\hat{c}_\kappa^*)_i$, $(\hat{c}_{v_\kappa})_i$ and $(\hat{c}_{p_\kappa})_i$ for each phase $\kappa \in \{l, g\}$. Consequently, we can determine the functions $h \mapsto \tilde{\beta}_\kappa(h)$, $\kappa \in \{l, g\}$ as adequate fitting polynomials over the intervals $\mathcal{H}_l \stackrel{\text{def}}{=} (h_{min}, \hat{h}_l^s)$ and $\mathcal{H}_g \stackrel{\text{def}}{=} (\hat{h}_g^s, h_{max})$ matching the resulting values $(\hat{\beta}_\kappa)_i$ in each phase

$$h \mapsto \tilde{\beta}_\kappa(h) = \sum_{j=0}^{d_{\beta, \kappa}} b_{\kappa, j} \left(\frac{h}{10^6} \right)^j. \quad (9)$$

Any numerical method can be used to determine the fitting polynomials (degrees $d_{\beta, \kappa}$ and coefficients $b_{\kappa, j}$) providing that it ensures the positivity of both functions $\tilde{\beta}_l$ and $\tilde{\beta}_g$, since the values $(\hat{\beta}_\kappa)_i$ are positive according to (8).

In the present work, we apply a least-square regression technique which leads to the coefficients given in Table 2(a) and the curves in Figure 5(a) for (9) at $p_* = 15.5 \text{ MPa}$. Degrees of the polynomial were increased until an expected accuracy was reached, *i.e.* when the relative errors defined by

$$\frac{\max_i \left| \tilde{\beta}_\kappa(h_i) - (\hat{\beta}_\kappa)_i \right|}{\max_i (\hat{\beta}_\kappa)_i}$$

are small enough. The values of these relative errors are also given in Table 2(a). It can be noticed on Figure 5(a) that for the least-square method we used here, $\tilde{\beta}_l$ and $\tilde{\beta}_g$ are positive. There is a large variety of fitting procedures in the literature which are a lot more accurate than the least-square method. However, we do not need here this level of accuracy, since the relative errors we obtain are satisfactory for our applications.

Hence the global expression of $\tilde{\beta}$ is given by

$$\tilde{\beta}(h) = \begin{cases} \tilde{\beta}_l(h), & \text{if } h \leq \hat{h}_l^s, \\ \tilde{\beta}_m(h) = \bar{\beta}_m, & \text{if } \hat{h}_l^s < h < \hat{h}_g^s, \\ \tilde{\beta}_g(h), & \text{if } h \geq \hat{h}_g^s, \end{cases} \quad (10)$$

where $\bar{\beta}_m$ is given by (5a) with $(h_\kappa^s, \varrho_\kappa^s)$ replaced by $(\hat{h}_\kappa^s, \hat{\varrho}_\kappa^s)$ (see corresponding data in Table 4). Orders of magnitude of $\tilde{\beta}_\kappa$ are given in the third column of Table 5.

Observe that the second item in Hypothesis 3.1 is satisfied by the chosen range of enthalpies, since

$$h_{min} = 9.787 \times 10^5 \text{ J} \cdot \text{kg}^{-1} > \hat{h}_l^s - \frac{p_*}{\hat{\varrho}_l^s \tilde{\beta}(\hat{h}_l^s)} = 5.646 \times 10^5 \text{ J} \cdot \text{kg}^{-1}.$$

Let us now focus on a consistent construction of the density variable by integrating $\tilde{\beta}_\kappa$ using (2):

$$\frac{1}{\tilde{\varrho}(h)} = \begin{cases} \frac{1}{\tilde{\varrho}_l(h)} \stackrel{\text{def}}{=} \frac{1}{\hat{\varrho}_l^s} + \int_{\hat{h}_l^s}^h \frac{\tilde{\beta}_l(\mathfrak{h})}{p_*} d\mathfrak{h}, & \text{if } h \leq \hat{h}_l^s, \\ \frac{1}{\varrho_m(h)}, & \text{if } \hat{h}_l^s < h < \hat{h}_g^s, \\ \frac{1}{\tilde{\varrho}_g(h)} \stackrel{\text{def}}{=} \frac{1}{\hat{\varrho}_g^s} + \int_{\hat{h}_g^s}^h \frac{\tilde{\beta}_g(\mathfrak{h})}{p_*} d\mathfrak{h}, & \text{if } h \geq \hat{h}_g^s. \end{cases} \quad (11)$$

with ϱ_m defined by (5b).⁷ We write in a global way

$$\frac{1}{\tilde{\varrho}_\kappa(h)} = \sum_{j=0}^{d_{\varrho,\kappa}} r_{\kappa,j} \left(\frac{h}{10^6} \right)^j \quad (12)$$

where the coefficients are stored in Table 2(b) and the curves are plotted on Figure 5(b). The values of the relative errors are also given in Table 2(b). Obviously $d_{\varrho,\kappa} = d_{\beta,\kappa} + 1$.

By construction, the following properties are satisfied:

Proposition 3.2. *Provided that $\tilde{\beta}_l$ and $\tilde{\beta}_g$ are positive, the density $\tilde{\varrho}$ defined by (11) is such that*

- (a) *Under Hypothesis 3.1-2., $h \mapsto \tilde{\varrho}(h)$ is positive over (h_{min}, h_{max}) ;*
- (b) *$h \mapsto \tilde{\varrho}(h)$ is continuous over (h_{min}, h_{max}) and satisfies exactly the tabulated values at saturation; in particular, $\tilde{\varrho}_\kappa(\hat{h}_\kappa^s) = \varrho_m(\hat{h}_\kappa^s) = \hat{\varrho}_\kappa^s$, $\kappa \in \{l, g\}$;*
- (c) *$h \mapsto \tilde{\varrho}(h)$ is decreasing over (h_{min}, h_{max}) ;*
- (d) *Relation (2) exactly holds.*

These properties are of great importance in thermodynamics and they would not have been fulfilled necessarily by a fitting of the density using only the values $(\hat{\varrho}_\kappa)_i$.

A straightforward consequence of this proposition is that $h \mapsto \tilde{\varrho}(h)$ is invertible, which in particular enables to determine a unique h_e such that $\tilde{\varrho}(h_e) = \varrho_e$ (cf. Hypothesis 3.1-1.).

- ④ **NIST-0:** In the previous strategy, degrees $d_{\beta,\kappa}$ of the fitting polynomials were chosen large enough to make the approximation error as small as possible. However it is also possible to choose $d_{\beta,\kappa} = 0$ to construct a phasewise approximation like (10) but with constant phase functions $\tilde{\beta}_\kappa(h) \equiv \bar{\beta}_\kappa$. We choose here the average between the minimal and the maximal value in each phase

$$\bar{\beta}_\kappa \stackrel{\text{def}}{=} \frac{\min_i (\hat{\beta}_\kappa)_i + \max_i (\hat{\beta}_\kappa)_i}{2} > 0, \quad \kappa \in \{l, g\}, \quad (13a)$$

and $\bar{\beta}_m$ as in (5a) with $(h_\kappa^s, \varrho_\kappa^s)$ replaced by $(\hat{h}_\kappa^s, \hat{\varrho}_\kappa^s)$. Resulting values ϱ_e are given in the fourth column of Table 5. We then apply the same construction (11) for the density as in the NIST-p strategy. In this phasewise constant case, we notice that this construction procedure leads to an expression of the density which is similar to the (NA)SG law (7), namely

$$\frac{1}{\tilde{\varrho}(h)} = \frac{\tilde{\beta}(h)}{p_*} (h - \tilde{q}(h)), \quad (13b)$$

⁷In particular, the definition of ϱ_m ensures that $\frac{1}{\varrho_m(h)} = \frac{1}{\hat{\varrho}_l^s} + \int_{\hat{h}_l^s}^h \frac{\bar{\beta}_m}{p_*} d\mathfrak{h} = \frac{1}{\hat{\varrho}_g^s} + \int_{\hat{h}_g^s}^h \frac{\bar{\beta}_m}{p_*} d\mathfrak{h}$.

	$\kappa = \ell$	$\kappa = g$		$\kappa = \ell$	$\kappa = g$
$d_{\beta,\kappa}$	6	3	$d_{\varrho,\kappa}$	7	4
$b_{\kappa,0}$	$6.129\,697\,412 \times 10^{-2}$	$-5.395\,575\,381 \times 10^{-1}$	$r_{\kappa,0}$	$-0.155\,983\,180 \times 10^{-3}$	$2.916\,351\,682 \times 10^{-2}$
$b_{\kappa,1}$	$-1.378\,946\,053 \times 10^{-1}$	$3.591\,468\,350 \times 10^{-1}$	$r_{\kappa,1}$	$3.954\,643\,492 \times 10^{-3}$	$-3.481\,016\,375 \times 10^{-2}$
$b_{\kappa,2}$	$8.329\,391\,618 \times 10^{-2}$	$9.759\,257\,397 \times 10^{-3}$	$r_{\kappa,2}$	$-4.448\,213\,074 \times 10^{-3}$	$1.158\,538\,178 \times 10^{-2}$
$b_{\kappa,3}$	$6.045\,509\,171 \times 10^{-3}$	$1.465\,520\,061 \times 10^{-2}$	$r_{\kappa,3}$	$1.791\,267\,015 \times 10^{-3}$	$2.098\,765\,032 \times 10^{-4}$
$b_{\kappa,4}$	$2.341\,456\,057 \times 10^{-2}$		$r_{\kappa,4}$	$9.750\,821\,242 \times 10^{-5}$	$-2.363\,742\,034 \times 10^{-4}$
$b_{\kappa,5}$	$-4.623\,916\,770 \times 10^{-2}$		$r_{\kappa,5}$	$3.021\,233\,622 \times 10^{-4}$	
$b_{\kappa,6}$	$1.637\,507\,485 \times 10^{-2}$		$r_{\kappa,6}$	$-4.971\,953\,517 \times 10^{-4}$	
			$r_{\kappa,7}$	$1.509\,223\,489 \times 10^{-4}$	
Error	$6.718\,624\,276 \times 10^{-3}$	$3.328\,803\,232 \times 10^{-3}$	Error	$8.627\,504\,234 \times 10^{-5}$	$9.467\,155\,751 \times 10^{-5}$

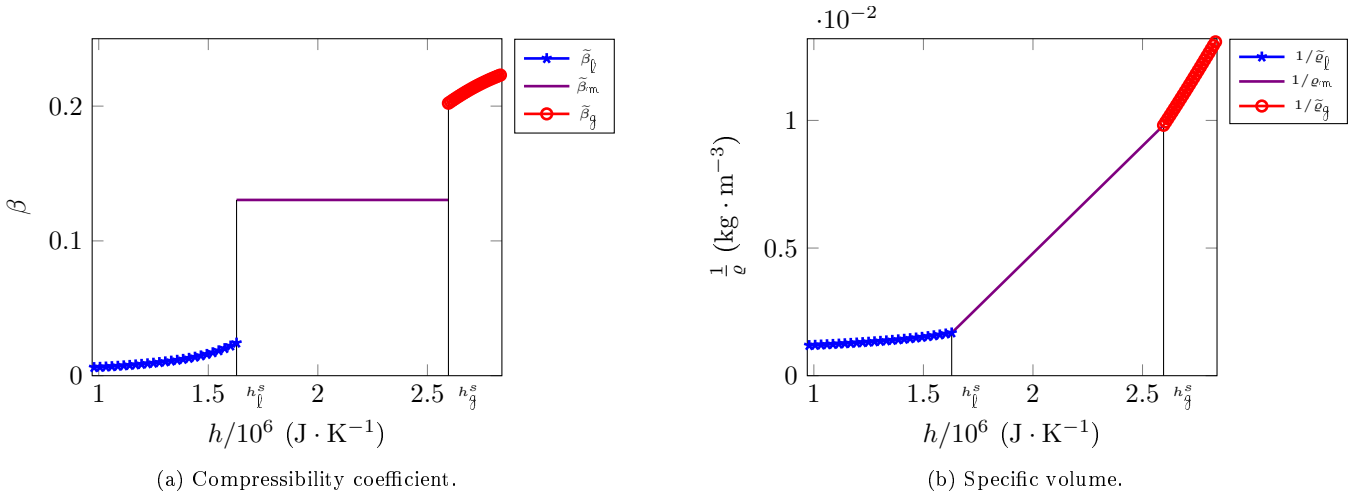
(a) Coefficients for $\tilde{\beta}_\kappa$.(b) Coefficients for $1/\tilde{\varrho}_\kappa$.Table 2: Coefficients of fitting polynomials in the NIST-p strategy at $p_* = 15.5$ MPa.

Figure 5: Fitting polynomials for compressibility coefficient (Tab. 2a) and specific volume (Tab. 2b).

	$\kappa = \ell$	$\kappa = g$
$d_{\varrho, \kappa}$	1	1
$r_{\kappa, 0}$	7.416×10^{-5}	-2.643×10^{-2}
$r_{\kappa, 1}$	9.867×10^{-10}	1.396×10^{-8}

Table 3: Coefficients for $1/\tilde{\varrho}_\kappa$ in the NIST-0 strategy

	❶ SG	❷ NASG	❸-❹ NIST [37]
h_ℓ^s	$1.627 \times 10^6 \text{ J} \cdot \text{kg}^{-1}$	$1.596 \times 10^6 \text{ J} \cdot \text{kg}^{-1}$	\widehat{h}_ℓ^s
h_g^s	$3.004 \times 10^6 \text{ J} \cdot \text{kg}^{-1}$	$2.861 \times 10^6 \text{ J} \cdot \text{kg}^{-1}$	\widehat{h}_g^s
ϱ_ℓ^s	$632.663 \text{ kg} \cdot \text{m}^{-3}$	$737.539 \text{ kg} \cdot \text{m}^{-3}$	$\widehat{\varrho}_\ell^s$
ϱ_g^s	$52.937 \text{ kg} \cdot \text{m}^{-3}$	$55.486 \text{ kg} \cdot \text{m}^{-3}$	$\widehat{\varrho}_g^s$
T^s	654.65 K	636.47 K	\widehat{T}^s

Table 4: Values at saturation for $p_* = 15.5 \text{ MPa}$ found in the NIST tables [37], or deduced from the parameters of [35] for SG and of [36] for NASG.

where

$$\tilde{q}(h) = \begin{cases} \bar{q}_\ell \stackrel{\text{def}}{=} \widehat{h}_\ell^s - \frac{1}{\widehat{\varrho}_\ell^s} \frac{p_*}{\beta_\ell}, & \text{if } h \leq \widehat{h}_\ell^s, \\ \bar{q}_m, & \text{if } \widehat{h}_\ell^s < h < \widehat{h}_g^s, \\ \bar{q}_g \stackrel{\text{def}}{=} \widehat{h}_g^s - \frac{1}{\widehat{\varrho}_g^s} \frac{p_*}{\beta_g}, & \text{if } h \geq \widehat{h}_g^s \end{cases} \quad (13c)$$

and \bar{q}_m is defined by (5c). The values of \bar{q}_κ are set in Table 5, those of $\widehat{\varrho}_\kappa^s$ and \widehat{h}_κ^s in Table 4, and coefficients of the polynomials $1/\tilde{\varrho}_\kappa$ in Table 3. It is straightforward that Proposition 3.2 still holds for the NIST-0 strategy.

With our construction, the saturation is exactly satisfied unlike EOS obtained with parameters of [35] for SG or with parameters of [36] for NASG. Nevertheless, observe that the NIST-0 EOS is not a complete equation of state, so that we cannot use this law in a compressible model, but it is the simplest and most accurate analytical law to obtain an exact analytical solution not only for the steady 1D LMNC model but also for transient versions. The four strategies will be compared below (see § 4.3). The comparison shows that the constant property of β is not a restriction but the most crucial point is to satisfy accurately the saturation.

In conclusion, we summarise the definitions $\tilde{\beta}$ and \tilde{q} depending on the EOS in the Table 6.

3.2 Viscosity coefficients

We also have to specify how to compute the viscosity coefficients in the stress tensor involved in (1c). Let μ and η be respectively the dynamic and bulk viscosities. As previously, we split the modelling into three pieces corresponding

	❶ SG	❷ NASG	❸ NIST-p	❹ NIST-0
$\tilde{\beta}_\ell$	8.769×10^{-3}	4.803×10^{-3}	6.143×10^{-3} to 2.437×10^{-2}	1.529×10^{-2}
$\tilde{\beta}_m$	1.949×10^{-1}	2.042×10^{-1}	1.303×10^{-1}	1.303×10^{-1}
$\tilde{\beta}_g$	3.007×10^{-1}	4.872×10^{-1}	2.022×10^{-1} to 2.310×10^{-1}	2.163×10^{-1}
\bar{q}_ℓ	$-1.167 \times 10^6 \text{ J} \cdot \text{kg}^{-1}$	$-2.779 \times 10^6 \text{ J} \cdot \text{kg}^{-1}$	N/A	$-7.654 \times 10^4 \text{ J} \cdot \text{kg}^{-1}$
\bar{q}_m	$1.501 \times 10^6 \text{ J} \cdot \text{kg}^{-1}$	$1.493 \times 10^6 \text{ J} \cdot \text{kg}^{-1}$	$1.429 \times 10^6 \text{ J} \cdot \text{kg}^{-1}$	$1.429 \times 10^6 \text{ J} \cdot \text{kg}^{-1}$
\bar{q}_g	$2.030 \times 10^6 \text{ J} \cdot \text{kg}^{-1}$	$2.287 \times 10^6 \text{ J} \cdot \text{kg}^{-1}$	N/A	$1.893 \times 10^6 \text{ J} \cdot \text{kg}^{-1}$

Table 5: Orders of magnitude of the compressibility coefficient β and the equivalent binding energy q at $p_* = 15.5 \text{ MPa}$. For pure phases: values obtained thanks to (8) using NIST tables [37], or using formulae introduced in [4] for SG and NASG. For values in the mixture: values given by (5a) and (5c).

Law	❶ SG and ❷ NASG	❸ NIST-p	❹ NIST-0
$\tilde{\beta}$	(7b)-(5a)	(10)-(9)-(5a)	(10)-(13a)-(5a)
$\tilde{\rho}$	(7)-(5)	(11)-(12)-(5)	(13)-(5)

Table 6: Definitions $\tilde{\beta}$ and $\tilde{\rho}$ depending on the EOS

to each phase. More precisely, we set:

$$\tilde{\mu}(h) = \begin{cases} \tilde{\mu}_l(h), & \text{if } h \leq h_l^s, \\ \tilde{\mu}_m(h) \stackrel{\text{def}}{=} \tilde{\alpha}(h)\tilde{\mu}_g(h_g^s) + (1 - \tilde{\alpha}(h))\tilde{\mu}_l(h_l^s), & \text{if } h_l^s < h < h_g^s, \\ \tilde{\mu}_g(h), & \text{if } h \geq h_g^s. \end{cases}$$

As for the bulk viscosity, we take $\tilde{\eta}(h) = -\frac{2}{3}\tilde{\mu}(h)$ under the Stokes hypothesis. The expression in the mixture is defined by means of a linear approximation involving the volume fraction α , given by

$$\tilde{\alpha}(h) = \begin{cases} 0, & \text{if } h \leq h_l^s, \\ \frac{(h - h_l^s)\varrho_l^s}{\varrho_g^s h_g^s - \varrho_l^s h_l^s - (\varrho_g^s - \varrho_l^s)h}, & \text{if } h_l^s < h < h_g^s, \\ 1, & \text{if } h \geq h_g^s. \end{cases}$$

As for other NIST data, the viscosity could be approximated by a polynomial expression $\tilde{\mu}_\kappa(h)$ by means of a least-square regression technique fitting values. However, due to the weak variability of the viscosity, we approximate μ by a phasewise constant function, also computed from the NIST data, as the average between the minimal and the maximal value in each phase

$$\tilde{\mu}_l(h) \equiv \bar{\mu}_l = 9.462 \times 10^{-5} \text{ Pa} \cdot \text{s}, \quad \tilde{\mu}_g(h) \equiv \bar{\mu}_g = 2.464 \times 10^{-5} \text{ Pa} \cdot \text{s}.$$

3.3 Thermal effects

Low Mach number models allow to deal with large heat transfers unlike incompressible models. It is thus interesting to investigate the influence of the thermal conductivity.

Since the temperature is constant within the mixture phase, T is not an appropriate unknown for the modelling of phase transition at this scale. This is why we rather expressed the model in terms of enthalpy which is not constant. Let us denote by λ the thermal conductivity of water. We mention that this physical variable is referenced in the NIST tables (see Table 1). Then the classical term $\nabla \cdot (\lambda \nabla T)$ which models heat conduction (Fourier's law) becomes in a pure phase $\nabla \cdot (\Lambda \nabla h)$ where Λ is defined by

$$\Lambda \stackrel{\text{def}}{=} \frac{\lambda}{c_p}.$$

As previously, c_p is the isochoric heat capacity (defined by (20) or (23)) which is involved in the classical thermodynamic equality $dh = c_p dT$. Consequently, we define Λ over the whole range of enthalpy values by

$$\tilde{\Lambda}(h) = \begin{cases} \tilde{\lambda}_l(h) \times \left(\frac{1}{c_p}\right)_l(h), & \text{if } h \leq \hat{h}_l^s, \\ 0, & \text{if } \hat{h}_l^s < h < \hat{h}_g^s, \\ \tilde{\lambda}_g(h) \times \left(\frac{1}{c_p}\right)_g(h), & \text{if } h \geq \hat{h}_g^s. \end{cases} \quad (14)$$

The coefficient is set to 0 for $h \in (h_l^s, h_g^s)$ because of the fact that the temperature is constant within the mixture phase and so there is no thermal conduction therein.

Isobaric heat coefficient To construct functions $\left(\frac{1}{c_p}\right)_\kappa$ ($\kappa \in \{l, g\}$), we may use the same strategies as in § 3.1.2 for the computation of β , namely ❶-❷ for the SG/NASG EOS, ❸ for high-order fitting polynomials and ❹ for constant approximations. In the cases ❶-❷, the heat coefficients are constant and given by $\bar{c}_{p\kappa} = \bar{\gamma}_\kappa \bar{c}_{v\kappa}$, $\kappa \in \{l, m\}$, with values of parameters $\bar{\gamma}_\kappa$ and $\bar{c}_{v\kappa}$ recalled in Table 8.

	$\kappa = \ell$	$\kappa = g$		$\kappa = \ell$	$\kappa = g$
$d_{c_p, \kappa}$	7	6	$d_{T, \kappa}$	8	7
$c_{\kappa, 0}$	$1.689\,602\,951 \times 10^{-4}$	$5.382\,374\,281 \times 10^{-4}$	$t_{\kappa, 0}$	293.445 144 7	300.584 284 2
$c_{\kappa, 1}$	$1.232\,572\,675 \times 10^{-4}$	$-2.539\,666\,705 \times 10^{-4}$	$t_{\kappa, 1}$	168.960 295 1	538.237 428 1
$c_{\kappa, 2}$	$2.380\,465\,421 \times 10^{-5}$	$-2.690\,479\,385 \times 10^{-4}$	$t_{\kappa, 2}$	61.628 633 75	-126.983 335 2
$c_{\kappa, 3}$	$-1.311\,465\,424 \times 10^{-4}$	$1.079\,811\,811 \times 10^{-4}$	$t_{\kappa, 3}$	7.934 884 737	-89.682 646 17
$c_{\kappa, 4}$	$-1.531\,310\,649 \times 10^{-5}$	$1.163\,418\,815 \times 10^{-5}$	$t_{\kappa, 4}$	-32.786 635 60	26.995 295 28
$c_{\kappa, 5}$	$6.360\,562\,483 \times 10^{-5}$	$-3.873\,402\,400 \times 10^{-7}$	$t_{\kappa, 5}$	-3.062 621 298	2.326 837 630
$c_{\kappa, 6}$	$-7.958\,443\,754 \times 10^{-6}$	$-1.196\,179\,285 \times 10^{-6}$	$t_{\kappa, 6}$	10.600 937 47	$-6.455\,670\,667 \times 10^{-2}$
$c_{\kappa, 7}$	$-7.446\,866\,275 \times 10^{-6}$		$t_{\kappa, 7}$	-1.136 920 536	$-1.708\,827\,550 \times 10^{-1}$
			$t_{\kappa, 8}$	-0.930 858 284 4	
Error	$2.840\,610\,126 \times 10^{-3}$	$1.731\,914\,106 \times 10^{-3}$	Error	$7.340\,188\,470 \times 10^{-6}$	$4.820\,515\,518 \times 10^{-6}$

(a) Coefficients for $\left(\frac{1}{c_p}\right)_\kappa$. (b) Coefficients for \tilde{T}_κ .

Table 7: Coefficients of fitting polynomials in the NIST-p strategy at $p_* = 15.5$ MPa.

For strategies **3-4**, a least-square regression technique is used to fit values from the NIST tables [37]:

$$h \mapsto \left(\frac{1}{c_p}\right)_\kappa(h) = \sum_{j=0}^{d_{c_p, \kappa}} c_{\kappa, j} \left(\frac{h}{10^6}\right)^j.$$

Note that we decided to fit $1/c_p$ instead of c_p since it appears as such in the definitions of Λ and of T (see below).

Let us notice that strategies for the computation of β and c_p are decoupled as we are considering incomplete equations of state. It corresponds to numerical techniques to approximate thermodynamic variables using the most suitable way. In particular, a NIST-0 (for β) – NIST-0 (for $1/c_p$) strategy would mimic the SG-NASG EOS in terms of algebraic formulation (constant) except that constant values are not related to each other. No matter which numerical method is applied to fit the NIST data, it must be ensured that the resulting function is positive over (h_{min}, h_{max}) . In the present case, we decided to apply a NIST-p strategy whose coefficients are given in Table 7(a), and the curves can be seen on Figure 6(a).

Temperature Although the temperature is not involved in the LMNC system (1), it is useful to compute it in the purpose of analysis of numerical results and for safety evaluations. For the **1-2** strategies, the temperature reads

$$\tilde{T}(h) = \begin{cases} \frac{h - \bar{q}_\ell}{\bar{c}_{p_\ell}}, & \text{if } h \leq h_\ell^s, \\ T^s, & \text{if } h_\ell^s < h < h_g^s, \\ \frac{h - \bar{q}_g}{\bar{c}_{p_g}}, & \text{if } h \geq h_g^s, \end{cases}$$

where \bar{c}_{p_κ} was introduced above and \bar{q}_κ is stored in Table 5. Values at saturation T^s and h_κ^s are recalled in Table 4. For **3-4** strategies, similarly to the way we defined the density ρ from the compressibility coefficient β , we deduce the expression of the temperature T by integrating $1/c_p$ defined by (23). Hence

$$\tilde{T}(h) = \begin{cases} \tilde{T}_\ell(h) \stackrel{\text{def}}{=} \hat{T}^s + \int_{\hat{h}_\ell^s}^h \left(\frac{1}{c_p}\right)_\ell(h) dh, & \text{if } h \leq \hat{h}_\ell^s, \\ \tilde{T}_m(h) \stackrel{\text{def}}{=} \hat{T}^s, & \text{if } \hat{h}_\ell^s < h < \hat{h}_g^s, \\ \tilde{T}_g(h) \stackrel{\text{def}}{=} \hat{T}^s + \int_{\hat{h}_g^s}^h \left(\frac{1}{c_p}\right)_g(h) dh, & \text{if } h \geq \hat{h}_g^s. \end{cases} \quad (15)$$

We write in a global way

$$\tilde{T}_\kappa(h) = \sum_{j=0}^{d_{T, \kappa}} t_{\kappa, j} \left(\frac{h}{10^6}\right)^j \quad (16)$$

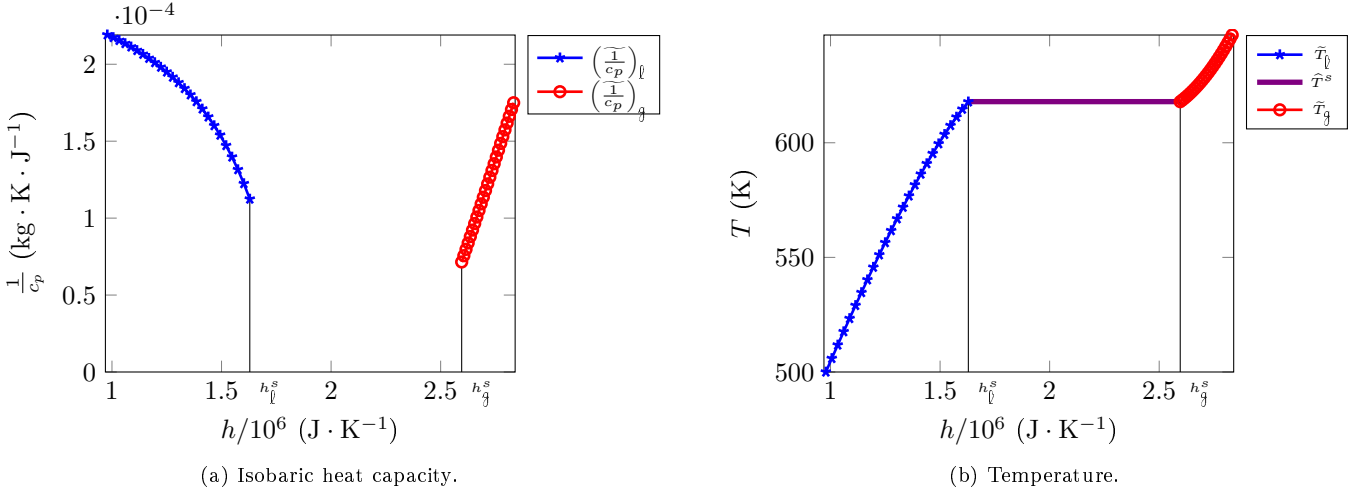


Figure 6: Fitting polynomials for isobaric heat capacity (Table 7a) and temperature (Table 7b).

where the coefficients are stored in Table 7(b) and the curves are plotted on Figure 6(b). The values of the relative errors are also given in Table 7(b). Obviously $d_{T,\kappa} = d_{c_p,\kappa} + 1$.

This construction automatically ensures the following properties (see Figure 6(b)):

Proposition 3.3. *Provided that $\tilde{\beta}_\ell$ and $\tilde{\beta}_g$ are positive, the temperature \tilde{T} defined by (15) satisfies:*

- $h \mapsto \tilde{T}(h)$ is continuous and positive over (h_{min}, h_{max}) ;
- $h \mapsto \tilde{T}(h)$ is increasing over (h_{min}, h_{max}) ,⁸
- The differential relation $h \mapsto \left(\frac{1}{c_p}\right)'(h) = \tilde{T}'(h)$ exactly holds.

Thermal conductivity Similarly to other NIST data, the thermal conductivity could have been approximated by a polynomial expression $\tilde{\lambda}_\kappa(h)$ by means of a least-square regression technique. However, due to the weak variability of the thermal conductivity, we approximate λ by a phasewise constant function, also computed from the NIST data as the average between the minimal and the maximal value in each phase:

$$\tilde{\lambda}_\ell(h) \equiv \bar{\lambda}_\ell = 5.578 \times 10^{-1} \text{ W} \cdot \text{m}^{-1} \cdot \text{K}^{-1}, \quad \tilde{\lambda}_g(h) \equiv \bar{\lambda}_g = 1.008 \times 10^{-1} \text{ W} \cdot \text{m}^{-1} \cdot \text{K}^{-1}.$$

4 Analytical results on the model and comparison of the EOS

The model we actually address in the sequel is thus

$$\begin{cases} \nabla \cdot \mathbf{u} = \frac{\tilde{\beta}(h)}{p_*} [\nabla \cdot (\tilde{\Lambda}(h) \nabla h) + \Phi], & (17a) \end{cases}$$

$$\begin{cases} \tilde{\varrho}(h) \times [\partial_t h + \mathbf{u} \cdot \nabla h] = \nabla \cdot (\tilde{\Lambda}(h) \nabla h) + \Phi, & (t, \mathbf{x}) \in \mathbb{R}^+ \times \Omega^d, & (17b) \end{cases}$$

$$\begin{cases} \tilde{\varrho}(h) \times [\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u}] - \nabla \cdot [\tilde{\mu}(h) (\nabla \mathbf{u} + (\nabla \mathbf{u})^T)] - \nabla [\tilde{\eta}(h) \nabla \cdot \mathbf{u}] + \nabla P = \tilde{\varrho}(h) \mathbf{g}, & (17c) \end{cases}$$

where the expressions of $\tilde{\beta}$ and $\tilde{\varrho}$ are detailed in § 3.1, those of $\tilde{\mu}$ and $\tilde{\eta}$ in § 3.2 and that of $\tilde{\Lambda}$ in § 3.3.

4.1 Jump conditions

Taking into account thermal diffusion modifies the nature of the differential problem in depth. Indeed, as detailed in § 3.3, $\tilde{\Lambda}$ is defined by (14) and is thus discontinuous over the domain. Then Equation (17b) – which governs the evolution of the enthalpy – is a degenerate parabolic equation: it reduces to a nonlinear transport equation in the mixture phase. This requires to increase the regularity of h in the variational approach (see § 5.1).

⁸ $h \mapsto \tilde{T}(h)$ is constant over (h_ℓ^s, h_g^s) .

Moreover, the diffusion coefficient is discontinuous across phase transition curves. Consequently, we have to define rigorously the conservative second-order differential term in (17b).

Let us introduce some notations. We set

$$\Omega_{\ell}^d(t) \stackrel{\text{def}}{=} \{\mathbf{x} \in \Omega^d : h(t, \mathbf{x}) < h_{\ell}^s\}, \quad \Omega_m^d(t) \stackrel{\text{def}}{=} \{\mathbf{x} \in \Omega^d : h_{\ell}^s < h(t, \mathbf{x}) < h_g^s\} \quad \text{and} \quad \Omega_g^d(t) \stackrel{\text{def}}{=} \{\mathbf{x} \in \Omega^d : h(t, \mathbf{x}) > h_g^s\}$$

the domains occupied by each phase. Each of them may become an empty set depending on the evolution of h .

Phase domains are separated by level sets of the enthalpy

$$\Gamma_{\ell}(t) = \{\mathbf{x} \in \Omega^d : h(t, \mathbf{x}) = h_{\ell}^s\} \quad \text{and} \quad \Gamma_g(t) = \{\mathbf{x} \in \Omega^d : h(t, \mathbf{x}) = h_g^s\}.$$

From now on, a solution $(h, \mathbf{u}, \mathcal{P})$ to System (17) must be understood as follows:

- In each phase domain $\Omega_{\kappa}^d(t)$, $(h, \mathbf{u}, \mathcal{P})(t, \cdot)$ is a strong solution.
- Where phase transition occurs, we impose the following jump conditions over $\Gamma_{\ell}(t)$ and $\Gamma_g(t)$

$$\left[\tilde{\Lambda}(h) \nabla h \right] \cdot \mathbf{n} = 0, \tag{18a}$$

$$\left[\tilde{\mu}(h) (\nabla \mathbf{u} + (\nabla \mathbf{u})^T) + \tilde{\eta}(h) (\nabla \cdot \mathbf{u}) \mathcal{I}_d - \mathcal{P} \mathcal{I}_d \right] \mathbf{n} = 0. \tag{18b}$$

- Moreover, h satisfies the continuity condition

$$[[h]] = 0 \tag{18c}$$

only where the flow goes into the mixture phase, *i.e.* over $\bigcup_{\kappa \in \{\ell, g\}} \{\mathbf{x} \in \Gamma_{\kappa}(t) : \mathbf{u}(t, \mathbf{x}) \cdot \mathbf{n}_{\kappa}(t, \mathbf{x}) > 0\}$, where \mathbf{n}_{κ} is the normal vector pointing towards the mixture phase.

This problem can be considered a coupling issue between two models: one of hyperbolic nature, one of parabolic nature. Such a coupling has been investigated in the literature (*e.g.* [17, 22]) but in the case of fixed boundaries. An additional obstacle in the present case is that the discontinuity location moves with time (level set of the enthalpy). The theoretical analysis of the model (17-3-18) is left to future works.

4.2 Assumptions

For the problem to be well-posed, we impose some assumptions upon the data. We first suppose that physical data satisfy:

Hypothesis 4.1.

1. For all $(t, \mathbf{x}) \in \mathbb{R}^+ \times \Omega^d$, $\Phi(t, \mathbf{x})$ is non-negative.
2. For all $(t, x) \in \mathbb{R}^+ \times [0, L_x]$, $D_e(t, x)$ is non-negative.
3. p_* is a positive constant.

The first assumption characterizes the fact that we study a nuclear core where the coolant fluid is heated. In a nuclear power plant of PWR or BWR type, the flow is *upward*, which is the meaning of the second assumption.

We make the following modelling hypothesis which somehow restricts the range of experiments as we cannot predict the orientation of the output velocity field.

Hypothesis 4.2. The velocity field is outgoing, *i.e.* $\mathbf{u} \cdot \mathbf{n}(t, \mathbf{x})|_{y=L_y} > 0$ for all $(t, \mathbf{x}) \in \mathbb{R}^+ \times \Omega^d$.

This hypothesis is necessary as no Dirichlet boundary conditions is imposed at the exit of the core upon the enthalpy. It thus induces that no downward flow (recirculation) occurs at the exit. In dimension 1 when $\tilde{\Lambda} \equiv 0$, Hypothesis 4.2 is a direct consequence of Hypothesis 4.1 and of the positivity of β ; in dimensions 2 and 3, Hypothesis 4.2 must stand.

4.3 Influence of the equation of state on steady states

In dimension 1 and without thermal diffusion, we can explicitly compute steady solutions of the LMNC model no matter what the equation of state.

Proposition 4.1 (Steady states). *We consider the 1D steady case without thermal diffusion, i.e. ϱ_e , D_e and Φ do not depend on time and $\tilde{\Lambda} \equiv 0$. Then, System (17) admits a unique steady solution given by*

$$h(y) = \tilde{\varrho}^{-1}(\varrho_e) + \frac{1}{D_e} \int_0^y \Phi(z) dz, \quad v(y) = \frac{D_e}{\tilde{\varrho}(h(y))}.$$

Proof. System (17) reduces to

$$\begin{cases} \partial_y v = -\frac{1}{\tilde{\varrho}(h)^2} \tilde{\varrho}'(h) \Phi(y), & v(t, 0) = D_e / \varrho_e > 0, \\ v \partial_y h = \frac{1}{\tilde{\varrho}(h)} \Phi(y), & h(t, 0) = h_e, \end{cases} \quad (19a)$$

$$\quad (19b)$$

with h_e such that $\tilde{\varrho}(h_e) = \varrho_e > 0$. Then

$$\partial_y (\tilde{\varrho}(h)v) = \tilde{\varrho}'(h)v \partial_y h + \tilde{\varrho} \partial_y v \stackrel{(19a)}{=} \tilde{\varrho}'(h)v \partial_y h - \frac{\tilde{\varrho}'(h)}{\tilde{\varrho}(h)} \Phi \stackrel{(19b)}{=} 0.$$

Hence $\tilde{\varrho}(h(y))v(y) = D_e$ and finally $\partial_y h(y) = \frac{1}{\tilde{\varrho}(h(y))v(y)} \Phi(y) = \frac{\Phi(y)}{D_e}$. \square

Remark 4.1.

1. *A distinctive feature of the LMNC model is that the steady enthalpy does not depend on the equation of state except through the computation of h_e . Moreover as Φ is non-negative (Hypothesis 4.1), the enthalpy is monotone increasing.*
2. *There is no proof of existence for an asymptotic state of System (1) (in the sense $t \rightarrow +\infty$) in the general case.⁹ But if such an asymptotic solution exists, it is necessary the one described in Proposition 4.1.*
3. *Given the expression of h in Proposition 4.1, it can be stated whether the steady fluid appears only as a (pure) liquid phase (if $h(L_y) < h_l^s$) or as a mixture ($h_l^s \leq h(L_y) \leq h_g^s$) or also as a (pure) vapour phase ($h(L_y) > h_g^s$).*

To compare the influence of the equation of state on the steady solution, we consider values of a pressurised water reactor with water injected at liquid state:

- thermodynamic pressure: $p_* = 15.5$ MPa,
- height of the core: $L = 4.2$ m,
- power density: $\Phi_* = 170 \times 10^6$ W \cdot m⁻³,
- injection: $v_e = 0.5$ m \cdot s⁻¹ and $\varrho_e = 750$ kg \cdot m⁻³ so that $D_e = 375$ kg \cdot s⁻¹ \cdot m⁻².

Since the injection is water at liquid phase, we compute h_e depending on the equation of state. For NIST-0, SG and NASG, the computation is exact: $h_e = \bar{q}_l + \frac{p_*}{\beta_l} \frac{1}{\varrho_e}$. For NIST-p, we have to solve numerically $\varrho(h_e, p_*) = \varrho_e$. We obtain

	❶ SG	❷ NASG	❸ NIST-p	❹ NIST-0
h_e	1.190×10^6 J \cdot kg ⁻¹	1.523×10^6 J \cdot kg ⁻¹	1.274×10^6 J \cdot kg ⁻¹	1.276×10^6 J \cdot kg ⁻¹

The NIST tables provide an approximate value $\varrho_e = 1.27378 \times 10^6$ J \cdot kg⁻¹.

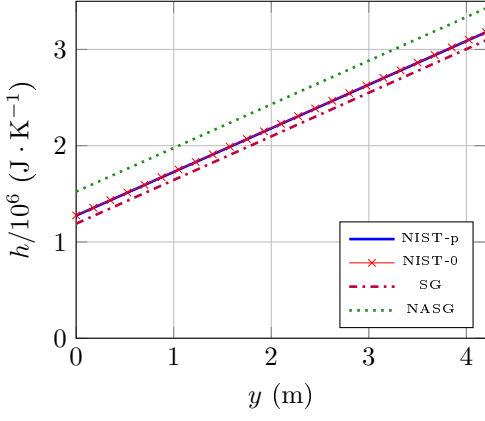
The discrepancy between the four values at $y = 0$ remains the same over the whole domain as the steady solutions $y \mapsto h(y)$ are linear with the same slope (see Figure 7a).

It can also be stated where the fluid is present only as a (pure) liquid phase (if $h(y) < h_l^s$) or as a mixture ($h_l^s \leq h(y) \leq h_g^s$) or also as a (pure) vapour phase ($h(y) > h_g^s$) by solving level sets $h(y) = h_{\kappa}^s \iff y = y_{\kappa}^s$:

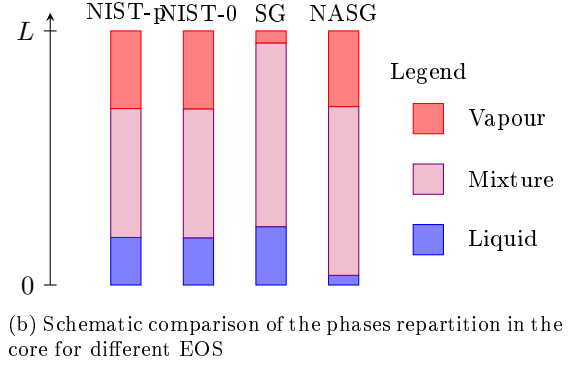
	❶ SG	❷ NASG	❸ NIST-p	❹ NIST-0
y_l^s	0.964 m	0.160 m	0.786 m	0.780 m
y_g^s	4.002 m	2.950 m	2.917 m	2.912 m

The errors upon the enthalpies at saturation and at the entry result in dramatic consequences: the computations provide a late vaporisation as we can see in the schematic comparison of Figure 7b of the distribution of pure liquid, steam-water mixture and vapour phases in the reactor core.

⁹It has been proven for the *Stiffened Gas* law and for simple expressions of Φ in [4].



(a) Enthalpy steady solutions for different EOS



(b) Schematic comparison of the phases repartition in the core for different EOS

Figure 7: Steady solutions and schematic comparison of the phases repartition in the core for different EOS

5 Numerical strategy and simulations

5.1 Numerical scheme

Let us first recall what was investigated from the numerical point of view in our previous works, which all used the SG law for the EOS. In [4], the 1D LMNC model with phase transition is simulated by means of two distinct numerical schemes based on the method of characteristics. Unconditionally stable, they ensure the positivity of the density. In [14], we derive a 2D weak formulation for the monophasic version of the model. It is extended to multiphasic flows in [5] with a weak formulation requiring an interface tracking algorithm.

In this section, we present a new version of the finite element approximation which overcomes the previous issues and deals with tabulated values in the context of phase transition. Moreover, thermal conduction is taken into account.

For the transport terms, we use the method of characteristics. This feature is embedded in the Finite-Element software `FreeFem++` through the function `convect`. For reasons highlighted below in Remark 5.1, we do not consider Equation (17a) as such but under the equivalent form

$$\nabla \cdot \mathbf{u} = \frac{\tilde{\beta}(h)\tilde{\varrho}(h)}{p_*} [\partial_t h + \mathbf{u} \cdot \nabla h]. \quad (17a')$$

Applying the method of characteristics to discretise the convection operators, the weak formulation reads

“At time t^{n+1} , find $(\mathbf{u}^{n+1}, h^{n+1}, \mathcal{P}^{n+1}) \in (\mathbf{u}_e + \mathcal{U}) \times (h_e + \mathcal{H}) \times L^2(\Omega^2)$ such that

$$\begin{aligned} \iint_{\Omega^2} p_{\text{test}} \nabla \cdot \mathbf{u}^{n+1} \, d\mathbf{x} &= \iint_{\Omega^2} \frac{\tilde{\beta}(h^n)\tilde{\varrho}(h^n)}{p_*} \frac{h^{n+1} - h^n(\boldsymbol{\xi}^n)}{\Delta t} p_{\text{test}} \, d\mathbf{x}, & \forall p_{\text{test}} \in L^2, \\ \iint_{\Omega^2} \tilde{\varrho}(h^n) \frac{h^{n+1} - h^n(\boldsymbol{\xi}^n)}{\Delta t} h_{\text{test}} \, d\mathbf{x} &= \iint_{\Omega^2} \Phi(t^{n+1}, \cdot) h_{\text{test}} \, d\mathbf{x} - \iint_{\Omega^2} \tilde{\Lambda}(h^n) \nabla h^{n+1} \cdot \nabla h_{\text{test}} \, d\mathbf{x}, & \forall h_{\text{test}} \in \mathcal{H}, \\ \iint_{\Omega^2} \tilde{\varrho}(h^n) \frac{\mathbf{u}^{n+1} - \mathbf{u}^n(\boldsymbol{\xi}^n)}{\Delta t} \cdot \mathbf{u}_{\text{test}} \, d\mathbf{x} &+ \iint_{\Omega^2} \frac{\tilde{\mu}(h^n)}{2} (\nabla \mathbf{u}^{n+1} + (\nabla \mathbf{u}^{n+1})^T) :: (\nabla \mathbf{u}_{\text{test}} + \nabla \mathbf{u}_{\text{test}}^T) \, d\mathbf{x} \\ &+ \iint_{\Omega^2} \tilde{\eta}(h^n) (\nabla \cdot \mathbf{u}^{n+1})(\nabla \cdot \mathbf{u}_{\text{test}}) \, d\mathbf{x} - \iint_{\Omega^2} \mathcal{P}^{n+1} \nabla \cdot \mathbf{u}_{\text{test}} \, d\mathbf{x} = \iint_{\Omega^2} \tilde{\varrho}(h^n) \mathbf{g} \cdot \mathbf{u}_{\text{test}} \, d\mathbf{x}, & \forall \mathbf{u}_{\text{test}} \in \mathcal{U}. \end{aligned}$$

where

$$\begin{aligned} \mathcal{H} &= \{\theta \in H^1(\Omega^2) : \theta(x, 0) = 0\}, \\ \mathcal{U} &= \{\mathbf{v} \in (H^1(\Omega^2))^2 : \mathbf{v}(x, 0) = \mathbf{0}, \mathbf{v} \cdot \mathbf{n}(0, y) = \mathbf{v} \cdot \mathbf{n}(L_x, y) = 0\}, \end{aligned}$$

and $\boldsymbol{\xi}^n$ is a numerical approximation of $\boldsymbol{\xi}(t^n; t^{n+1}, \mathbf{x})$ where $\boldsymbol{\xi}$ is the solution of the characteristic equation:

$$\begin{cases} \frac{d\boldsymbol{\chi}}{d\tau} = \mathbf{u}(\tau, \boldsymbol{\chi}(\tau; t, \mathbf{x})), \\ \boldsymbol{\chi}(t; t, \mathbf{x}) = \mathbf{x}. \end{cases}$$

Notice that due to discontinuities of most coefficients across (potential) level sets $\Gamma_{\mathfrak{h}}$ and $\Gamma_{\mathfrak{g}}$, there should be some interface terms in the weak formulation but the latter terms vanish due to jump conditions (18). In particular, there is no more need to track the interfaces, which reduces the computational costs.

We do not give more details either about the mathematical content or the numerical treatment since the problem is simulated by means of **FreeFem++** [27], which solves directly the weak formulation given above.

We recall that the parameters $\tilde{\beta}$ and $\tilde{\varrho}$ are given in Table 6.

Remark 5.1. *The replacement of Equation (17a) by (17a') makes the weak formulation easier to solve. Indeed, it would rather read*

$$\iint_{\Omega^2} p_{test} \nabla \cdot \mathbf{u}^{n+1} \, d\mathbf{x} = \iint_{\Omega^2} \frac{\beta(h^n) p_{test}}{p_*} \Phi \, d\mathbf{x} - \iint_{\Omega^2} \Lambda(h^n, p_*) \nabla h^{n+1} \cdot \left[\frac{\partial \beta}{\partial h}(h^n, p_*) \frac{p_{test}}{p_*} \nabla h^n + \frac{\beta(h^n)}{p_*} \nabla p_{test} \right] \, d\mathbf{x}.$$

Consequently, this would require to increase the regularity of p_{test} and to approximate the function $\frac{\partial \beta}{\partial h}$ which may induce instabilities. Replacing the right hand side in (1a) by the left hand side in (1b) enables to overcome all these issues. See [5] for another numerical strategy.

5.2 Data

Choice of the equation of state First, let us observe that a first comparison between the Stiffened Gas law, the Noble-Abel Stiffened Gas one, and the two different tabulated approaches (NIST-p and NIST-0) has been done in Section 4.3, since the differences can already be observed on the steady states in 1D. Further comparisons in 2D are performed in § 5.4.2.

Physical and numerical parameters Based on the numerical schemes detailed above, we perform some simulations with our tabulated EOS aimed at:

1. verifying our code by means of comparisons with explicit steady solutions (given in dimension 1);
2. highlighting the accuracy of the tabulated approach to describe the physics (dimension 2);
3. showing the low influence of the thermal conduction for physically relevant data (computations in dimension 2);
4. showing the ability of the model and of the 2D-scheme to capture the flow complexity by emphasizing the influence of the gravity orientation (computations in dimension 2).

Parameters are set as follows to simulate a PWR:

- Geometry: $L_x = 1$ m, $L_y = 2$ m.
- Discretisation parameters: 40 nodes on the horizontal boundaries, 80 nodes on the vertical boundaries, which results in an unstructured mesh generated by **FreeFem++** with 3740 vertices and a characteristic length of the mesh around 2 cm. The mesh can be seen on Figure 9. The use of the method of characteristics makes the numerical schemes unconditionally stable. Consequently, there is no constrain upon the time step. But to ensure a good accuracy, we cannot take a too large value for the time step. We set $\Delta t = 0.01$ s.
- Reference values: $p_* = 15.5$ MPa, $\Phi_* = 170 \times 10^6$ W · m⁻³, $\varrho_e = 750$ kg · m⁻³, $D_e = 375$ kg · m⁻² · s⁻¹.
- Initial data: $h^0(x, y) = \varrho^{-1}(\varrho_e)$, $\mathbf{u}^0(x, y) = (u^0(x, y), v^0(x, y)) = (0, D_e/\varrho_e)$.
- The power density Φ (constant or not) is specified in each of the following sections.

In every test case, we check that the Mach number remains small in all phases during the whole computation, in order to ensure the validity of our low Mach number model. The Mach number is computed from the speed of sound, with NIST-p tabulated values for pure phases, and Formula (24) given in Appendix C for the mixture.

5.3 Validation of the 2D numerical scheme

In this test case, we aim at validating our two-dimensional numerical scheme. To this end, we chose the power density $\Phi(x, y)$ to be uniformly constant equal to $2\Phi_*$, which produces a quasi 1D case. Starting from a domain initially filled with liquid, this yields appearance of both mixture and vapour. The computation is performed with the NIST-p data

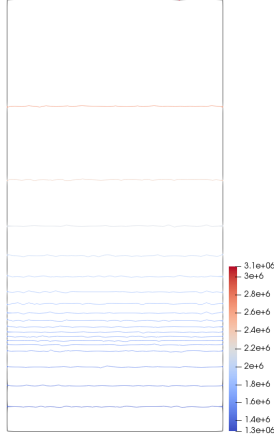


Figure 8: Isolines of enthalpy at time $t = 1.6$ s (asymptotic state) (test case of § 5.3)

given in Tables 2 and 7. In order to compare with the predicted asymptotic solution given in Proposition 4.1, we neglect thermal conductivity for this test case. Thus, the steady solution for the enthalpy in 1D is given by

$$h(y) = \tilde{\varrho}^{-1}(\varrho_e) + \frac{2\Phi_\star}{D_e}y \simeq 1.273 \times 10^6 + 1.36 \times 10^6 y.$$

In our computation, we observe a numerical convergence towards an asymptotic solution. The relative difference between this asymptotic solution and the 1D steady solution is of order 10^{-2} , and reaches a maximum value of 1% at the top of the domain. This shows the good matching between the two solutions.

Moreover, we observe that this 2D computation on unstructured mesh allows to capture a 2D solution independent of the variable x , as shown on Figure 8.

5.4 Phase transition with a compactly supported heating

In all the test cases of this section, the power density is chosen to be compactly supported within a disc in the lower part of the core, which yields a genuine 2D case:

$$\Phi(x, y) = 20\Phi_\star \mathbf{1}_{\{(x-0.5)^2 + (y-0.5)^2 \leq 0.125^2\}}(x, y).$$

We can see on Figure 9(a) the localisation of the power density (where the fluid is heated).

5.4.1 Reference test case

The values given in § 5.2 with $\lambda \equiv 0$ lead to a scenario in which the temperature increases as the flow passes through the heat source and some mixture appears as seen on Figure 10(a). The NIST-p approach described in § 3.1.2 is used for the equation of state. The mixture phase has a lower density than the liquid phase. Hence, as the flow goes upwards and the gravity field is pointing downwards, a Rayleigh-Taylor instability occurs (see Figure 9(b)): the gravity makes the lighter phase speed up through the heavier phase above (Archimedes' principle). The instability is well recovered by the numerical scheme over this mesh.

In addition, we plot the Mach number of the flow at time $t = 0.55$ s on Figure 10(d), which is the time for which the flow is the most mixed up. We notice that it remains small (of order 10^{-2} at the maximum), which guarantees the validity of our low Mach number model for such flows. It is in particular the case in the mixture phase where the speed of sound is yet decreasing. The temperature and the norm of the velocity are also plotted at the same time on Figure 10. The localisation of mixture in the flow is in particular observed on Figure 10(c), where the temperature is constant and equal to $\hat{T}^s = 617.939$ K.

5.4.2 Influence of the EOS

In this section, we observe the discrepancy induced by the use of different EOS in a 2D unsteady test case. In practical terms, we compare the strategies we introduced in Section 3, namely ❶-❷ for the SG/NASG EOS, ❸ for high-order fitting polynomials and ❹ for constant approximations.

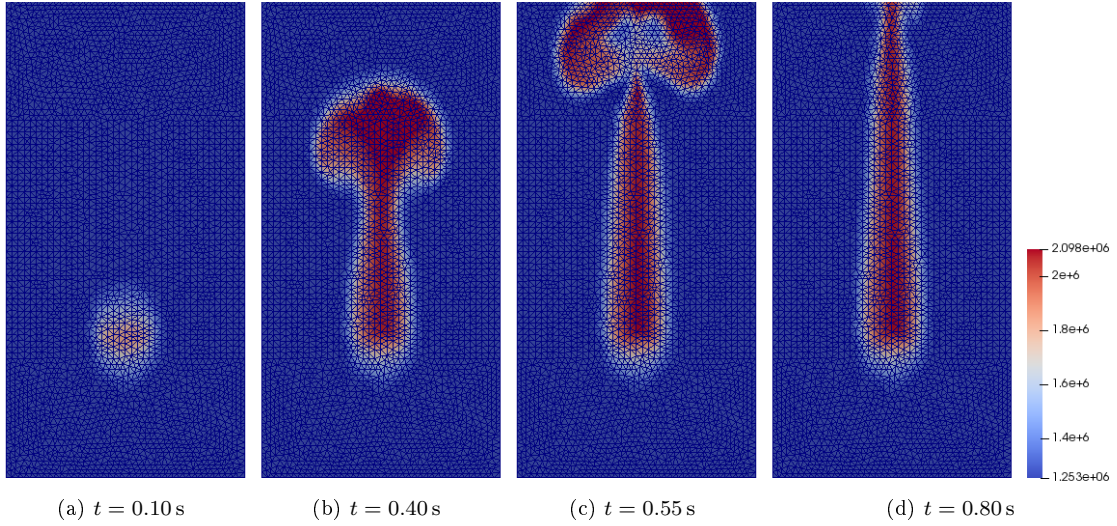


Figure 9: Enthalpy for a 2D flow with phase transition at different times (test case of § 5.4.1, EOS : NIST-p)

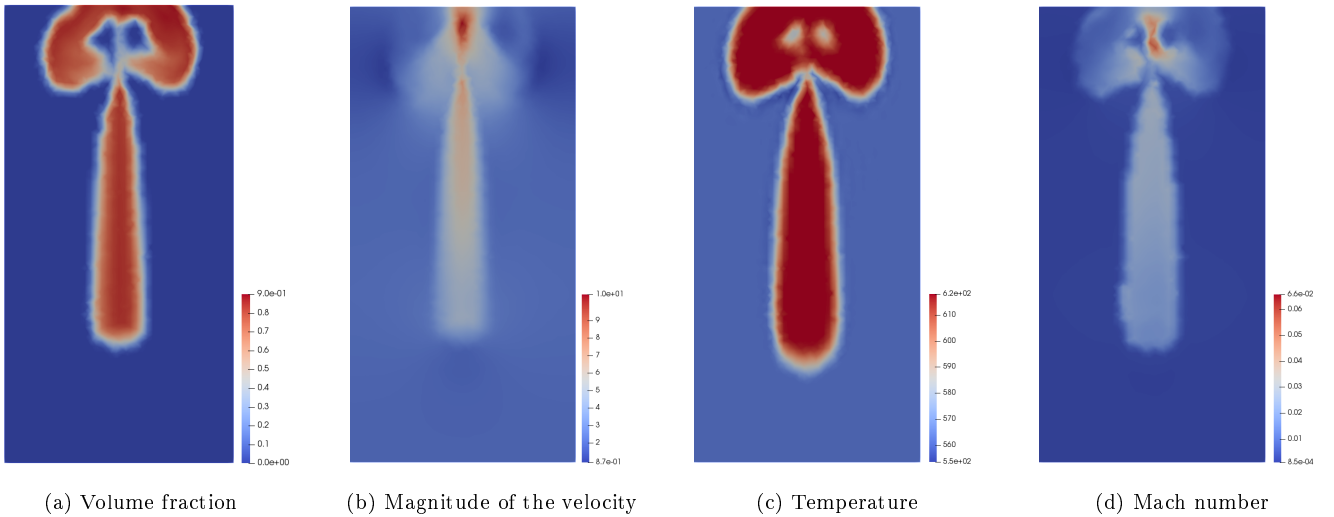


Figure 10: Density, velocity, temperature and Mach number at time $t = 0.55$ s (test case of § 5.4.1, EOS : NIST-p)

We observe on Figure 11 that NIST-p and NIST-0 approaches lead to very similar results, whereas the SG EOS induces a largely over-estimated temperature. The NASG EOS is not presented here, since we already observed on Figure 7(a) that it overestimates the entrance enthalpy h_e computed from ϱ_e , and it thus leads here to very different results, with a much higher temperature. We recover the fact that for such accidental situations, academic EOS like SG/NASG are not appropriate.

5.4.3 Influence of the thermal conductivity

In this test case, the influence of thermal conductivity is investigated. The computations are performed with the NIST-p data given in Tables 2 and 7.

The order of magnitude of the thermal conductivity is given in Table 1 (around 10^{-1}). We observe on Figure 12 the difference between numerical solutions with and without thermal diffusion at time 0.55 s. The error is of order 10^{-4} , which shows that for the physical situations we are interested in, the thermal conductivity does not play a major role.

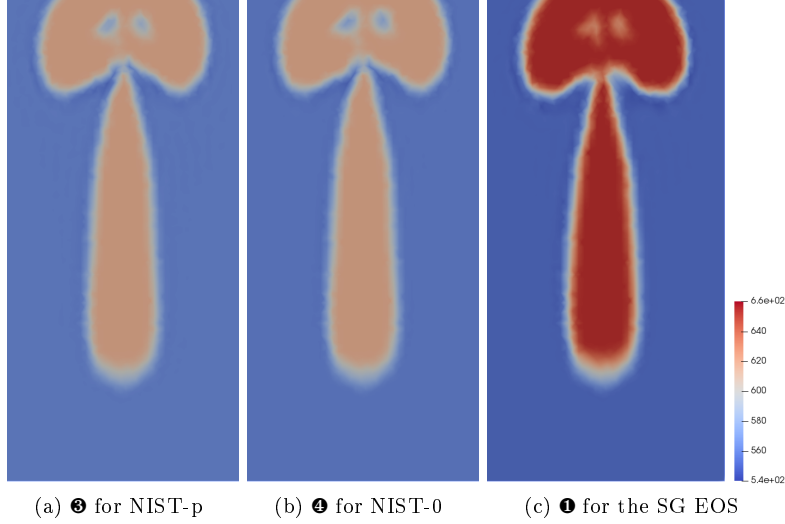


Figure 11: Temperature at time $t = 0.55$ s with different strategies for approximating the EOS (test case of § 5.4.2)

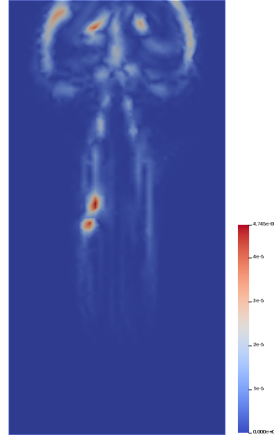


Figure 12: Relative difference between the enthalpy with $\lambda = 0$ and $\lambda \neq 0$ at time $t = 0.55$ s (test case of § 5.4.3, EOS : NIST-p)

5.4.4 Influence of the gravity field

Since our choice of 2D power density generates hydrodynamic instabilities apparently driven by gravity effects, we focus on the impact of the gravity field direction. Again, the computations are performed with the NIST-p data given in Tables 2 and 7.

Figure 13 depicts the same test case as in § 5.4.3 except for the gravity field whose orientation varies (downwards/null/upwards/to the right).

In the classical situation where the gravity is oriented downwards (Figure 13(a)), the flow driven by the material velocity is sped up by the gravity effects which makes the mixture phase (which is lighter than the liquid phase) go upwards (see § 5.4.3 for more details about the appearance of the Rayleigh-Taylor instability). Without gravity (Figure 13(b)), there is a unique physical phenomenon (forced convection) that governs the motion of fluid which explains that at the same time of evolution, the mixture cloud is lower than in the previous case (but hotter as the fluid remained longer within the core) and no Rayleigh-Taylor instability occurs. If the gravity field is upward (Figure 13(c)), there is a balance between the two aforementioned phenomena. The flow velocity makes the mixture go to the top while the gravity makes it go to the bottom as it is lighter compared to the pure liquid phase. The motion is even slower than without gravity. As for the horizontal case (Figure 13(d)), we observe that the mixture tends to the left which is the opposite direction to the gravity field as expected. Hence, we see that our numerical scheme enables to catch various types of behaviour depending on the data.

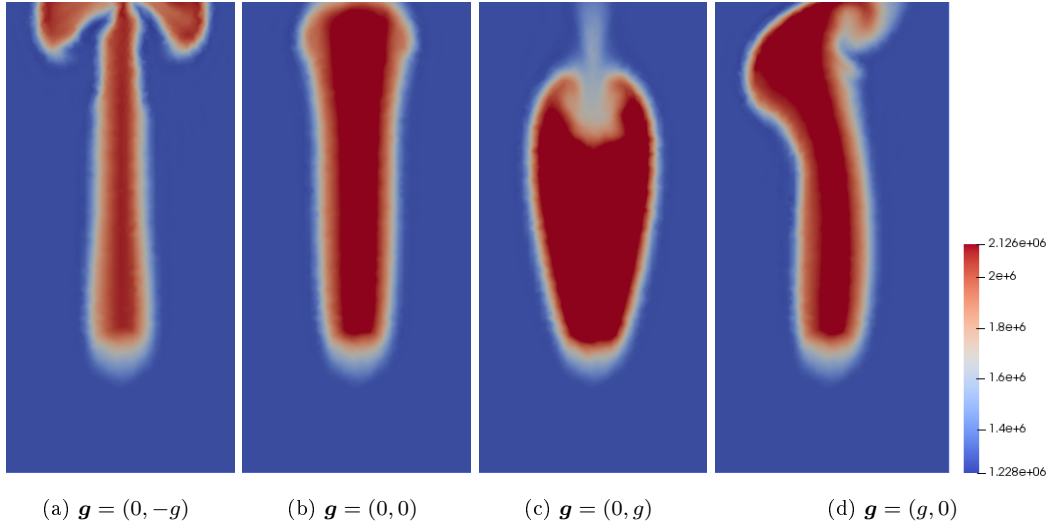


Figure 13: Enthalpy at time 0.55 s for several orientations of the gravity field (test case of § 5.4.4)

Conclusion & Perspectives

The ultimate strategy to simulate a fluid flow in a nuclear core would rely on ① a compressible model coupled to a tabulated equation of state satisfying thermodynamic laws and account for phase transition and on ② an accurate and robust numerical scheme able to deal with any Mach regime with a “reasonable” computational cost. In our opinion, this seems currently out of reach. That is why we propose in this paper a compromise between accuracy and efficiency by focusing on a specific regime – namely the low Mach number regime – by means of a reduced complexity model which turns out to be physically relevant after all [45]. This low Mach model (LMNC) is supplemented with the original construction of an incomplete equation of state dedicated to constant thermodynamic pressure states. It consists of a fitting polynomial procedure whose degree is left to the user. This leads to a natural numerical scheme which is easy to implement, which satisfies basic thermodynamic requirements and which leads to a much smaller computational cost than it would be with a compressible model coupled to a tabulated law like IAPWS [11, 51, 52]. The overall method is assessed by means of analytical and stiff test cases involving phase transition where classic analytical EOS fail to provide relevant values.

In particular, we investigated two cases for the degree of fitting polynomials: NIST-p (high-order) and NIST-0 (low-order). Theoretical (§ 4.3) and numerical (§ 5.4.2) results tend to show that the choice of the degree does not have a significant impact on the accuracy. On the contrary, comparisons between NIST-0 and the *Stiffened Gas* law (for which the density has a similar expression) prove that the crucial point is to exactly satisfy saturation points. The discrepancy between h_{κ}^s deduced from the SG law and \hat{h}_{κ}^s used in our strategy induces quite different results as presented in § 5.4.2. We mention that the procedure we described in this paper can be adapted by end-users (numerical method to fit polynomials, range of temperatures, pressure value, ...) depending on the applications.

Our previous papers on the LMNC model were based on the SG law to develop and assess the numerical strategy. The present results henceforth enable to deal with more realistic applications and to provide reliable forecasts in incidental situations. Future works will deal with more general equilibria implying a hierarchy of n -equation models.

A Stiffened Gas (SG) and Noble-Abel – Stiffened Gas (NASG) equations of state

Let us give some details about the complete equation of state (6). Parameters \bar{c}_v , $\bar{\gamma}$, $\bar{\pi}$, \bar{b} , \bar{Q} and $\bar{\mathcal{S}}_0$ are constants describing thermodynamic properties of the fluid: $\bar{c}_v > 0$ is the isochoric heat capacity, $\bar{\gamma} = \frac{\bar{c}_p}{\bar{c}_v} > 1$ the ratio of isobaric and isochoric heat capacities¹⁰, $\bar{\pi} \geq 0$, $\bar{b} > 0$ is the co-volume, \bar{Q} the binding energy and $\bar{\mathcal{S}}_0$ a reference

¹⁰Adopting the same convention as in [3, 40], the polytropic coefficient or adiabatic exponent γ is defined by $\gamma(\tau, \mathcal{S}) \stackrel{\text{def}}{=} -\frac{\tau}{P} \frac{\partial P}{\partial \tau} \Big|_{\mathcal{S}}$. Except in the case of polytropic gases, this coefficient differs from the widely used ratio of heat capacities. For the NASG EOS, γ is related to the ratio of heat capacities by the relation $\gamma = (1 + \frac{\pi}{P})\bar{\gamma}$.

Phase	\bar{c}_v [J · K ⁻¹]	$\bar{\gamma}$	$\bar{\pi}$ [Pa]	\bar{Q} [J · kg ⁻¹]	\bar{b} [m ³ · kg ⁻¹]	\bar{Q}' [J · kg ⁻¹]
Liquid NASG	3202	1.39	8899×10^5	-1244.191×10^3	4.78×10^{-4}	0
Steam NASG	462	1.95	0	2287.484×10^3	0	6417
Liquid SG	1816.2	2.35	10^9	-1167.056×10^3	0	0
Steam SG	1040.14	1.43	0	2030.255×10^3	0	-23 310

Table 8: Water and steam. Values for NASG laws from [36]. Values for SG laws from [35].

entropy. The latter parameter may often be replaced by the following one

$$\bar{Q}' \stackrel{\text{def}}{=} \bar{S}_0 + \bar{c}_v [\bar{\gamma} \ln(\bar{c}_v) + (\bar{\gamma} - 1) \ln(\bar{\gamma} - 1)].$$

The domain of definition of \mathcal{S} is the set $\tau > \bar{b}$ and $\varepsilon - \bar{Q} - \bar{\pi}(\tau - \bar{b}) > 0$. The parameters used in this law describing a pure phase are determined by using a reference curve, usually in the (τ, p) -plane. For example, values for water and steam computed in [35] for SG and in [36] for NASG are given in Table 8. These values yield reasonable approximations over a range of temperature from 300 K to 500 K.

Using the fundamental thermodynamic relation $Td\mathcal{S} = d\varepsilon + pd\tau$ and introducing classical potentials, we have

$$\begin{aligned} (\tau, \varepsilon) \mapsto T(\tau, \varepsilon) &= \frac{1}{\left. \frac{\partial \mathcal{S}}{\partial \varepsilon} \right|_{\tau}} = \frac{\varepsilon - \bar{Q} - \bar{\pi}(\tau - \bar{b})}{\bar{c}_v}, & \text{[temperature]} \\ (\tau, \varepsilon) \mapsto p(\tau, \varepsilon) &= T \left. \frac{\partial \mathcal{S}}{\partial \tau} \right|_{\varepsilon} = (\bar{\gamma} - 1) \frac{\varepsilon - \bar{Q}}{\tau - \bar{b}} - \bar{\gamma} \bar{\pi}, & \text{[pressure]} \\ (\tau, \varepsilon) \mapsto h(\tau, \varepsilon) &= \varepsilon + p\tau = \varepsilon + \tau \left[(\bar{\gamma} - 1) \frac{\varepsilon - \bar{Q}}{\tau - \bar{b}} - \bar{\gamma} \bar{\pi} \right], & \text{[enthalpy]} \\ (\tau, \varepsilon) \mapsto c^2(\tau, \varepsilon) &\stackrel{\text{def}}{=} \left. \frac{\partial p}{\partial \varrho} \right|_{\mathcal{S}} = -\frac{\tau^2}{T} \left(\left. \frac{\partial^2 \mathcal{S}}{\partial \tau^2} \right|_{\varepsilon} - 2p \left. \frac{\partial^2 \mathcal{S}}{\partial \tau \partial \varepsilon} \right|_{\varepsilon} + p^2 \left. \frac{\partial^2 \mathcal{S}}{\partial \varepsilon^2} \right|_{\tau} \right) \\ &= \bar{\gamma}(\bar{\gamma} - 1) (\varepsilon - \bar{Q} - \bar{\pi}(\tau - \bar{b})) \frac{\tau^2}{(\tau - \bar{b})^2}. & \text{[speed of sound]} \end{aligned}$$

Thus, by inverting the function $(\tau, \varepsilon) \mapsto (h, p)$

$$\begin{aligned} (h, p) \mapsto \tau(h, p) &= \frac{\bar{\gamma} - 1}{\bar{\gamma}} \frac{h - \bar{Q}}{p + \bar{\pi}} + \bar{b}, & \text{[volume]} \\ (h, p) \mapsto \beta(\mathcal{K}, p) &= p \left. \frac{\partial \tau}{\partial h} \right|_p = \frac{\bar{\gamma} - 1}{\bar{\gamma}} \frac{p}{p + \bar{\pi}}, & \text{[compressibility coefficient]} \\ (h, p) \mapsto q(\mathcal{K}, p) &\stackrel{\text{def}}{=} \bar{Q} - \frac{p}{\beta(\mathcal{K}, p)} \bar{b}, \\ (h, p) \mapsto \varrho(h, p) &= \frac{1}{\tau(h, p)} = \frac{p/\beta(\mathcal{K}, p)}{h - q(\mathcal{K}, p)}, & \text{[density]} \\ (h, p) \mapsto T(h, p) &= \frac{h - \bar{Q} - \bar{b}p}{\bar{\gamma} \bar{c}_v} = \frac{h - \widehat{Q}(p)}{\bar{\gamma} \bar{c}_v}. & \text{[temperature]} \end{aligned}$$

where $\widehat{Q}(p) \stackrel{\text{def}}{=} \bar{Q} + \bar{b}p$. The case of a stiffened gas law is recovered by setting $\bar{b} = 0$ (so that $q = \widehat{Q} = \bar{Q}$). The case of a perfect gas law is recovered by setting $\bar{b} = \bar{Q} = \bar{\pi} = 0$.

B Proof of Proposition 3.1

We note $\tau \stackrel{\text{def}}{=} 1/\varrho$, so that $\beta = p \left. \frac{\partial \tau}{\partial h} \right|_p$. We recall the following classical definitions:

$$(c^*)^2(\tau, \mathcal{S}) \stackrel{\text{def}}{=} -\tau^2 \left. \frac{\partial p}{\partial \tau} \right|_{\mathcal{S}}, \quad c_v \stackrel{\text{def}}{=} T \left. \frac{\partial \mathcal{S}}{\partial T} \right|_{\tau}, \quad c_p \stackrel{\text{def}}{=} T \left. \frac{\partial \mathcal{S}}{\partial T} \right|_p, \quad (20)$$

where \mathcal{S} is the specific entropy. We first remark that¹¹

$$\frac{c_p}{c_v} = \frac{T \left. \frac{\partial \mathcal{S}}{\partial T} \right|_p}{T \left. \frac{\partial \mathcal{S}}{\partial T} \right|_\tau} = \frac{- \left. \frac{\partial T}{\partial \tau} \right|_S \left. \frac{\partial \tau}{\partial \mathcal{S}} \right|_T}{- \left. \frac{\partial p}{\partial \mathcal{S}} \right|_T \left. \frac{\partial T}{\partial p} \right|_S} = \frac{\left. \frac{\partial p}{\partial \tau} \right|_S \left. \frac{\partial \tau}{\partial p} \right|_T}{- \frac{(c^*)^2}{\tau^2} \left. \frac{\partial \tau}{\partial p} \right|_T}. \quad (21)$$

Then

$$c_p - c_v = T \left(\left. \frac{\partial \mathcal{S}}{\partial T} \right|_p - \left. \frac{\partial \mathcal{S}}{\partial T} \right|_\tau \right) = T \left(\left. \frac{\partial \mathcal{S}}{\partial T} \right|_\tau + \left. \frac{\partial \mathcal{S}}{\partial \tau} \right|_T \left. \frac{\partial \tau}{\partial T} \right|_p - \left. \frac{\partial \mathcal{S}}{\partial T} \right|_\tau \right) = T \left. \frac{\partial \mathcal{S}}{\partial \tau} \right|_T \left. \frac{\partial \tau}{\partial T} \right|_p = T \left. \frac{\partial p}{\partial T} \right|_\tau \left. \frac{\partial \tau}{\partial T} \right|_p$$

due to the Maxwell relation associated to the free energy \mathcal{F} that satisfies $d\mathcal{F} = -pd\tau - \mathcal{S}dT$. Hence

$$c_p - c_v = -T \frac{\left. \frac{\partial \tau}{\partial T} \right|_p}{\left. \frac{\partial T}{\partial \tau} \right|_p \left. \frac{\partial \tau}{\partial p} \right|_T} \stackrel{(21)}{=} T \frac{c_v (c^*)^2}{c_p \tau^2} \left(\left. \frac{\partial \tau}{\partial T} \right|_p \right)^2. \quad (22)$$

Since the enthalpy satisfies the principle $dh = Td\mathcal{S} + \tau dp$, then $T \left. \frac{\partial \mathcal{S}}{\partial T} \right|_p = \left. \frac{\partial h}{\partial T} \right|_p$ so that

$$\frac{1}{c_p} = \left. \frac{\partial T}{\partial h} \right|_p. \quad (23)$$

Finally

$$\beta = p \left. \frac{\partial \tau}{\partial h} \right|_p = p \left. \frac{\partial \tau}{\partial T} \right|_p \left. \frac{\partial T}{\partial h} \right|_p \stackrel{(23)}{=} \frac{p}{c_p} \left. \frac{\partial \tau}{\partial T} \right|_p \stackrel{(22)}{=} \frac{p\tau}{c^* \sqrt{T}} \sqrt{\frac{1}{c_v} - \frac{1}{c_p}}.$$

C Mach number and speed of sound of the two-phase flow

To evaluate the Mach number in the differential system and verify that it is low we need to compute the speed of sound. It is given by

$$c^*(h, p) \stackrel{\text{def}}{=} \sqrt{\left. \frac{\partial p}{\partial \varrho} \right|_S} = \frac{1}{\sqrt{\frac{1}{\varrho(h, p)} \left. \frac{\partial \varrho}{\partial h} \right|_S + \left. \frac{\partial \varrho}{\partial p} \right|_S}} = \begin{cases} c_l^*(h, p), & \text{if } h \leq h_l^s, \\ c_m^*(h, p), & \text{if } h_l^s < h < h_g^s, \\ c_g^*(h, p), & \text{if } h \geq h_g^s. \end{cases}$$

In pure phases, we can fit the speed of sound with a polynomial based on the NIST table [37] as previously but for the sake of consistency, we choose to match the new expression (8) of β which provides a fitting function (which is no more a polynomial). For instance, with a NIST-p strategy for both β and c_p , the relative error between this construction and the values extracted from the tables are 6.85×10^{-3} in the liquid phase and 5.64×10^{-3} in the vapour phase.

In the mixture, the speed of sound at mechanic and thermodynamic equilibrium is given by (see [4] for details)

$$\left(\frac{1}{c_m^*} \right)^2 (h, p) = \frac{[-\alpha(h, p)(\varrho_g^s)'(p) - (1 - \alpha(h, p))(\varrho_l^s)'(p)]q_m(p) + \alpha(h, p)(\varrho_g^s h_g^s)'(p) + (1 - \alpha(h, p))(\varrho_l^s h_l^s)'(p) - 1}{h - q_m(p)}. \quad (24)$$

where $(\varrho_\kappa^s)'$ and $(\varrho_\kappa^s h_\kappa^s)'$ are the derivatives of $p \mapsto \varrho_\kappa^s(p)$ and $p \mapsto \varrho_\kappa^s(p) h_\kappa^s(p)$. When one uses analytical EOS in pure phases, we can differentiate these relations to obtain those derivatives (as in [4] with the SG law). In the present case, as we use tabulated values in pure phases, $(\varrho_\kappa^s)'$ and $(h_\kappa^s)'$ are approximated by centered finite differences:

$$(\varrho_\kappa^s)'(p_*) = \frac{\varrho_\kappa^s(p_{-1}) - \varrho_\kappa^s(p_1)}{p_{-1} - p_1}, \quad (\varrho_\kappa^s h_\kappa^s)'(p_*) = \frac{\varrho_\kappa^s(p_{-1}) h_\kappa^s(p_{-1}) - \varrho_\kappa^s(p_1) h_\kappa^s(p_1)}{p_{-1} - p_1},$$

where p_i , $\varrho_\kappa^s(p_i)$ and $h_\kappa^s(p_i)$, $i = -1, 0, 1$, are obtained from experimental data (see Table 9 for the pressure of interest $p_* = 15.5$ MPa).

Note that $c_m^*(h_\kappa^s(p), p) < c_\kappa^*(h_\kappa^s(p), p)$, see for example [20, 40].

¹¹We make use of the following formula: for an EOS $f(X_1, X_2, X_3) = 0$, we have $\left. \frac{\partial X_1}{\partial X_2} \right|_{X_3} \left. \frac{\partial X_2}{\partial X_3} \right|_{X_1} \left. \frac{\partial X_3}{\partial X_1} \right|_{X_2} = -1$.

p [MPa]	ϱ_l^s [kg · m ⁻³]	h_l^s [kJ · kg ⁻¹]	ϱ_g^s [kg · m ⁻³]	h_g^s [kJ · kg ⁻¹]
$p_{-1} = 15.499$	594.397161363	1629.84051905	101.919399943	2596.14861770
$p_0 = 15.500$	594.378648626	1629.87998125	101.930084802	2596.11873446
$p_1 = 15.501$	594.360134886	1629.91944396	101.940770824	2596.08884821

Table 9: Data on saturation curve to compute $(\varrho_l^s)'$, $(\varrho_g^s)'$, $(h_l^s)'$ and $(h_g^s)'$ depending on p . Values for water [37]

Remark C.1. In [42, 46], another equivalent expression is used:

$$\left(\frac{1}{c_m^*}\right)^2 (h, p) = \varrho_m(h, p) \left(\frac{\alpha(h, p)}{\varrho_g^s(p)(c_g^s)^2(p)} + \frac{1 - \alpha(h, p)}{\varrho_l^s(p)(c_l^s)^2(p)} + T^s(p) \left(\alpha(h, p) \varrho_g^s(p) \gamma_g c_{v_g} \chi_g^2(p) + (1 - \alpha(h, p)) \varrho_l^s(p) \gamma_l c_{v_l} \chi_l^2(p) \right) \right)$$

where $\chi_\kappa(h, p) = \frac{\beta_\kappa(h, p) - \beta_m(p)}{p}$ and $(c_\kappa^*)^s(p) = c_\kappa^*(h_\kappa^s(p), p)$ is the speed of sound of the pure phase κ at saturation.

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