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# ELLIPTIC EQUATIONS WITH ABSORPTION IN A HALF-SPACE

J. GARCÍA-MELIÁN, A. QUAAS AND B. SIRAKOV

ABSTRACT. We give a necessary and sufficient condition, in the spirit of the classical works by Keller and Osserman, for the elliptic equation  $\Delta u = f(u)$  to have a solution in a half-space of  $\mathbb{R}^N$ . The function  $f$  is supposed to be nondecreasing and nonnegative, and we are interested in solutions whose range is where  $f > 0$ .

## 1. INTRODUCTION AND RESULTS

The aim of this work is to analyze the existence and nonexistence of (sub-)solutions of the problem

$$(P) \quad \Delta u = f(u) \quad \text{in } \mathbb{R}_+^N,$$

where the function  $f$  is continuous, nonnegative and nondecreasing on  $\mathbb{R}$ , and  $\mathbb{R}_+^N$  denotes the half-space  $\mathbb{R}_+^N = \{x \in \mathbb{R}^N : x_N > 0\}$ .

The classical papers by Osserman [7] and Keller [5] have generated a tremendous amount of work on this type of equations, most of which has been devoted to the study of entire solutions or of explosive solutions in a bounded domain. As already explained in [7] (see p. 1644 in that paper) it is meaningful to search for solutions whose range is in the set where  $f$  is strictly positive, since without this assumption the solution set can indeed be very large and variable. Thus, in the following we will consider such solutions only. Note that since  $f$  is nondecreasing and nonnegative, we have  $\{t : f(t) > 0\} = (t_0, \infty)$  for some  $t_0 \geq -\infty$ . Up to a simple transformation we can assume that  $t_0 = 0$  or  $t_0 = -\infty$ , in other words, either  $f(t) > 0$  for  $t > 0$  and  $u > 0$ , or  $f > 0$  on  $\mathbb{R}$  and there is no sign restriction on  $u$ .

In particular, it is known from [7] that the equation  $\Delta u = f(u)$  *does not* have solutions in the whole space  $\mathbb{R}^N$  if and only if

$$(KO) \quad \int_1^\infty \frac{ds}{\sqrt{F(s)}} < \infty,$$

where  $F(s) = \int_0^s f(t)dt$  is a primitive of  $f$  (this condition is nowadays usually referred to as the Keller-Osserman condition). On the other hand, (explosive) solutions in a bounded domain exist if and only if (KO) holds. It is outside the scope of this short paper to attempt a reasonably full bibliography for the many developments that followed [7] and [5], we refer for instance to the surveys [3] and [9], where a large set of references can be found.

However, to the best of our knowledge and maybe somewhat surprisingly, a study of the problem is not available for proper unbounded subdomains of  $\mathbb{R}^N$  such as a half-space (except for a comment on nonexistence of radial solutions in Remark 1.6 of [6]). Elliptic equations on a half-space are both intrinsically important and appear

in the study of other problems, such as the study of existence of a priori bounds for solutions through the Gidas-Spruck blow-up method. Half-spaces and cones also naturally appear in the study of boundary behaviour of solutions of elliptic equations in Lipschitz domains.

In this note we provide a necessary and sufficient condition for (P) to have a solution. This is the content of the following theorem. Throughout the paper, an existence statement will mean that (P) has a strong (or even classical, if  $f$  is supposed to be Hölder continuous) solution, whereas a non-existence statement will mean (P) does not even have solutions in any weak (such as weak-Sobolev or viscosity) sense which leaves valid the comparison principle.

Note that since the existence of a solution in  $\mathbb{R}^N$  implies existence in any subdomain of  $\mathbb{R}^N$ , we only need to study the problem under the hypothesis (KO).

**Theorem 1.** *Suppose (KO) holds. Then equation (P) has a solution if and only if*

$$(HS) \quad \begin{cases} \int_0^1 \frac{ds}{\sqrt{F(s)}} = \infty & \text{if } t_0 = 0, \text{ or} \\ \int_{-\infty}^{-1} f(s) ds < \infty & \text{if } t_0 = -\infty \end{cases}$$

*If this hypothesis is not satisfied, even subsolutions do not exist.*

We stress that this theorem completely characterizes the existence of solutions and subsolutions of (P), and thus represents a counterpart for a half-space of the results of Keller and Osserman for bounded domains or the whole space.

At this point a discussion on the possibility of extending the above theorem to more general unbounded domains is in order. The situation turns out to be very different depending on whether  $t_0 = 0$  or  $t_0 = -\infty$ .

First, if  $t_0 = 0$  then the first condition in (HS) actually guarantees the existence of a positive solution in any subdomain of  $\mathbb{R}^N$  whose closure is not the whole space. Note that this condition is what ensures the validity of the strong maximum principle for nonnegative solutions of  $\Delta u = f(u)$ , by the well-known result of Vázquez ([10], see also [8]). On the other hand, if (HS) fails when  $t_0 = 0$  then positive solutions do not exist in any sufficiently "fat" subdomain of  $\mathbb{R}^N$ . Thus, the result in Theorem 1 remains the same if  $t_0 = 0$  and the half-space is replaced by any strictly proper subdomain of  $\mathbb{R}^N$  which contains balls of arbitrary radius – such as, for instance, a nontrivial cone in  $\mathbb{R}^N$  (see Theorem 2 below).

Second, if  $t_0 = -\infty$  then the condition in (HS) appears strongly related to the specific geometry of the half-space. If a different cone in  $\mathbb{R}^N$  is considered, then this condition no longer characterizes the solvability of  $\Delta u = f(u)$ . For instance, the function  $\underline{u} = -x_1^\alpha x_2^\alpha$  is a subsolution of the equation  $\Delta u \geq c|u|^\beta$  in the quarter-space  $\{x_1 > 0, x_2 > 0\}$  for any  $\beta \in (-1, 0)$  (and so  $f(u) = |u|^\beta$  does not satisfy (HS)), provided  $\alpha = \frac{1}{1-\beta}$  and  $c = 2\alpha(1-\alpha)$ . Hence if  $f : \mathbb{R} \rightarrow (0, \infty)$  is increasing and  $f(u) \leq |u|^\beta$  for  $u < 0$  then a solution of  $\Delta u = f(u)$  in the quarterspace exists – since  $\underline{u}$  is a subsolution,  $\bar{u} = 0$  is a supersolution, and they are ordered. Currently we do not know how to obtain a necessary and sufficient condition for existence of

solutions of (P) in a cone different from  $\mathbb{R}^N$  or  $\mathbb{R}_+^N$ , in the case  $t_0 = -\infty$ . We believe this is an interesting open problem.

It is to be stressed that, while the existence and nonexistence of (sub-)solutions of (P) in the whole space depends only on the behavior at infinity of  $f$ , the behavior at  $t_0 = \inf\{t : f(t) > 0\}$  turns out to be important when a proper unbounded subdomain is considered.

## 2. PROOFS

In this section we will prove Theorem 1. The proofs depend on the comparison principle, together with the resolution of some auxiliary problems. We recall that the monotonicity of  $f$  implies the equation is *proper*, thus satisfies the weak comparison principle, in the sense that a subsolution and a supersolution which are ordered on the boundary of any bounded domain are also ordered throughout the domain. Also, Perron's method applies and yields the existence of a solution between any given ordered subsolution and a supersolution (see for instance [2] for classical or weak-Sobolev solutions and [1] for viscosity solutions).

Before giving the proof we briefly recall the construction of an entire (radial) solution of  $\Delta u = f(u)$  when the condition (KO) is not satisfied. Fix a real number  $v_0$  such that  $f(v_0) > 0$  and consider the following ordinary differential equation for a function  $v = v(r)$ ,

$$(2.1) \quad \begin{cases} v'' + \frac{N-1}{r}v' = f(v), & r > 0 \\ v(0) = v_0, \quad v'(0) = 0. \end{cases}$$

Since  $f$  is continuous, problem (2.1) has at least one  $C^2$  solution  $v$  defined in a maximal interval of existence  $[0, R)$  for some  $R > 0$ . Since  $(r^{N-1}v')' = r^{N-1}f(v) \geq 0$  we have  $r^{N-1}v' \geq 0$  so that  $v$  is increasing and therefore  $v'' \leq f(v)$ , by (2.1). We can multiply this inequality by  $v'$ , write the result as  $((v')^2)' \leq (2F(v))'$ , and integrate to get

$$\int_{v_0}^{v(r)} \frac{dt}{\sqrt{2F(t)}} \leq r$$

for every  $r \in (0, R)$ , which implies, since (KO) does not hold, that  $R$  is infinite. Thus for any  $x_0 \in \mathbb{R}^N$  the function  $u(x) = v(|x - x_0|)$  is a positive solution of  $\Delta u = f(u)$  in  $\mathbb{R}^N$ . The minimum of  $u$  is attained at  $x_0$  and its value is  $v_0$ .

On the other hand, when (KO) is satisfied, the solution of (2.1) blows up at a finite  $R$ ; indeed, since the equation in (2.1) can also be written as

$$(2.2) \quad v'(r) = \int_0^r \left(\frac{s}{r}\right)^{N-1} f(v(s)) ds,$$

from the monotonicity of  $f$  and  $v$  we have  $v'(r) \leq \frac{r}{N}f(v(r))$ , so that

$$v'' \geq \frac{1}{N}f(v).$$

Multiplying by  $v'$  as above we get

$$\int_{v_0}^{v(r)} \frac{dt}{\sqrt{F(t) - F(v_0)}} \geq \frac{1}{\sqrt{N}} r$$

for every  $r \in (0, R)$ , which implies that  $R$  is finite, by (KO). Thus  $u(x) = v(|x - x_0|)$  solves  $\Delta u = f(u)$  in  $B_R(x_0)$  with  $v = +\infty$  on  $\partial B_R(x_0)$  (throughout the paper,  $B_r(x_0)$  stands for the ball with radius  $r$  centered at  $x_0$ ; we write  $B_r$  when  $x_0 = 0$ ).

**Proof of Theorem 1, case  $t_0 = 0$ .**

*Existence.* Assume that (HS) holds, that is, the Vázquez condition for the strong maximum principle  $\int_0 F^{-1/2} = \infty$  is verified. Take any  $x_0 \in \mathbb{R}^N$  and  $r > 0$  such that  $B_r(x_0) \subset \{x_N < 0\}$ . Since  $f(0) = 0$  and  $f(\delta) > 0$  for some small  $\delta > 0$ , the functions  $\underline{u} = 0$  and  $\bar{u} = \delta$  are respectively a subsolution and a supersolution for the problem

$$(2.3) \quad \begin{cases} \Delta u = f(u) & \text{in } B_R(x_0) \setminus B_r(x_0) \\ u = \delta & \text{on } \partial B_r(x_0) \\ u = 0 & \text{on } \partial B_R(x_0). \end{cases}$$

Hence there exists a solution  $u_R$  of this problem such that  $0 \leq u_R \leq \delta$ . By standard elliptic estimates  $\|u_R\|_{W^{2,p}(K)}$  is bounded independently of  $R$  in any fixed compact set  $K$  of  $\mathbb{R}^N \setminus B_r(x_0)$ , for any  $p < \infty$ . Then by a diagonal argument we get a sequence  $R_n \rightarrow \infty$  such that  $u_{R_n} \rightarrow u$  in  $W_{\text{loc}}^{2,p}(\mathbb{R}^N \setminus B_r(x_0))$ . Passing to the limit we see that  $u$  is a nonnegative solution to the equation  $\Delta u = f(u)$  in  $\mathbb{R}^N \setminus B_r(x_0)$ . This solution does not vanish identically by the boundary condition  $u = \delta$  on  $\partial B_r(x_0)$ . Since the strong maximum principle holds (see [10]) we obtain that  $u$  is strictly positive in  $\mathbb{R}^N \setminus B_r(x_0) \supset \mathbb{R}_+^N$ .

*Nonexistence.* As we recalled above, the Keller-Osserman condition (KO) implies that for each  $v_0 = \varepsilon > 0$  there is a finite number  $R_\varepsilon$  such that the problem  $\Delta w = f(w)$  has a solution in each ball with radius  $R_\varepsilon$ , which blows up at the boundary of the ball, and has value  $\varepsilon$  at the center of the ball.

Assume problem (P) has a positive solution  $u$ . Take a (small) number  $\delta > 0$  and fix any ball  $\hat{B} = B_{R_\varepsilon + \delta}(x_0) \subset \mathbb{R}_+^N$ . We claim that  $u < 2\varepsilon$  in  $B_\delta(x_0)$ . In fact, if we had  $u(\bar{x}) \geq 2\varepsilon$  for some  $\bar{x} \in B_\delta(x_0)$ , we could define  $w(x) = v(|x - \bar{x}|)$ , where  $v$  is the solution of (2.1) with  $v(0) = \varepsilon$ , and use the comparison principle in the ball  $B_{R_\varepsilon}(\bar{x}) \subset R_+^n$  to get  $u \leq w$  in this ball. However, this is a contradiction since  $u(\bar{x}) \geq 2\varepsilon$  while  $w(\bar{x}) = v(0) = \varepsilon$ .

Finally, we claim that the failure of (HS) implies that there exists a supersolution  $z$  of (P) in  $B_\delta(x_0)$  with  $z = 2\varepsilon$  on  $\partial B_\delta(x_0)$  and  $z = 0$  on  $B_{\delta/2}(x_0)$  (a dead core). By using the comparison principle again, this time in  $B_\delta(x_0)$ , we find that  $u \leq 2\varepsilon = z$  on  $\partial B_\delta(x_0)$  implies  $u \leq z = 0$  in  $B_{\delta/2}(x_0)$ , contradicting the fact that  $u$  is positive.

Hence to conclude the proof of the theorem only this last claim remains to be proved. We will find a radial super-solution  $z$  such that  $z(x) = \beta(|x - x_0|)$ , and  $\beta$  is a real function which satisfies the ODE

$$(2.4) \quad \beta'' + \frac{N-1}{r} \beta' \leq f(\beta) \quad \text{in } (0, \delta),$$

and is such that  $\beta = 0$  on  $(0, \delta/2)$  and  $\beta(\delta) = 2\varepsilon$ . For this purpose it suffices to find a solution in  $(\delta/2, \delta)$  such that  $\beta(\delta/2) = 0$ ,  $\beta'(\delta/2) = 0$  and  $\beta(\delta) = 2\varepsilon$ , and then extend this solution as zero in  $(0, \delta/2)$ .

To find  $\beta$  we assume  $N \geq 3$  and perform the change of variables  $s = \frac{1}{N-2}r^{2-N}$ ,  $\tilde{\beta}(s) = \beta(r)$  (a similar argument can be made in the case  $N = 2$  with the change of variables  $s = \log r$ ; we will skip the details) and obtain the one-dimensional problem

$$(2.5) \quad \begin{cases} \tilde{\beta}'' \leq cs^{-\gamma} f(\tilde{\beta}) & \text{in } (s_0, s_1) \\ \tilde{\beta}(s_0) = \varepsilon, \tilde{\beta}(s_1) = \tilde{\beta}'(s_1) = 0, \end{cases}$$

where  $\gamma = \frac{2(N-1)}{N-2}$ ,  $c = (N-2)^{-\gamma}$ ,  $s_0 = \delta^{2-N}/(N-2)$  and  $s_1 = 2^{N-2}s_0$ . Observe that  $\gamma > 2$ .

Since the integral  $\int_0^\varepsilon F^{-1/2}$  is convergent we can find adequately small  $\varepsilon > 0$  and  $\delta > 0$  such that

$$\int_0^\varepsilon \frac{ds}{\sqrt{2F(s)}} = \sqrt{cs_1^{-\gamma}}(s_1 - s_0) = \tilde{c}\tilde{\delta}^{\tilde{\gamma}}.$$

where  $\tilde{c}, \tilde{\gamma}$  are positive constants which depend only on the dimension  $N$ . Thus we can define implicitly the function  $\tilde{\beta}$  through the equality

$$\int_{\tilde{\beta}(t)}^\varepsilon \frac{ds}{\sqrt{2F(s)}} = \sqrt{cs_1^{-\gamma}}(t - s_0)$$

for  $t \in (s_0, s_1)$ . Therefore  $\tilde{\beta}$  satisfies  $\tilde{\beta}'' = cs_1^{-\gamma} f(\tilde{\beta})$ ,  $\tilde{\beta}(s_0) = \varepsilon$ ,  $\tilde{\beta}(s_1) = \tilde{\beta}'(s_1) = 0$  and  $\tilde{\beta} \leq \varepsilon$ . Using  $s_1^{-\gamma} \leq s^{-\gamma}$  for  $s \in (s_0, s_1)$  we infer (2.5) and the claim follows. This concludes the proof.  $\square$

Observe that in the above proof we did not actually use the fact that the domain in which the equation  $\Delta u = f(u)$  is set is a half-space. In the existence part of the proof we only used that we can find a ball outside the domain, whereas in the non-existence part we only used that the domain contains a sufficiently large ball. In other words, we have actually proved the following result.

**Theorem 2.** *Suppose (KO) holds, and  $f(0) = 0$ ,  $f(t) > 0$  for  $t > 0$  is monotone increasing. Assume  $\Omega$  is such that  $\bar{\Omega} \neq \mathbb{R}^N$  and for each  $R > 0$  there exists a ball with radius  $R$  included in  $\Omega$ . Then the equation  $\Delta u = f(u)$  has a positive solution in  $\Omega$  if and only if  $\int_0^1 \frac{ds}{\sqrt{F(s)}} = \infty$ . If this hypothesis is not satisfied, even positive subsolutions do not exist in  $\Omega$ .*

**Proof of Theorem 1, case  $t_0 = -\infty$ .**

*Existence.* Assume (HS) holds, that is,  $\int_{-\infty}^0 f$  is convergent. Fix a positive constant  $C^*$  such that  $C^* > 2 \int_{-\infty}^0 f(t)dt = -2F(-\infty)$ . We define implicitly  $v$  as follows:

$$\int_{v(t)}^0 \frac{ds}{\sqrt{2F(s) + C^*}} = t.$$

Observe that  $v$  is well defined by (HS) and the choice of  $C^*$ . Moreover, by differentiating

$$-v'(t) = \sqrt{2F(v(t)) + C^*},$$

and therefore  $v'$  is strictly negative and bounded for  $t \geq 0$ . By differentiating once more we see that  $v$  satisfies  $v'' = f(v)$  in  $(0, +\infty)$  and thus  $u(x) = v(x_N)$  is a (one-dimensional and negative) solution of (P) in the half-space.

*Nonexistence.* Let us assume that (KO) holds,  $\int_{-\infty} f$  is divergent and there exists a sub-solution  $u$  of (P) in  $\mathbb{R}_+^N$ .

First we observe that

$$(2.6) \quad \lim_{x_N \rightarrow +\infty} u(x) = -\infty \quad \text{uniformly for } x' \in \mathbb{R}^{N-1},$$

where  $x = (x', x_N)$ .

In fact, if this were not the case, there would exist  $\gamma < 0$  and a sequence  $x^{(n)}$  with  $x_N^{(n)} \rightarrow \infty$  such that  $u(x^{(n)}) \geq \gamma$  as  $n \rightarrow \infty$ . As we observed before the proof of Theorem 1, we may use the condition (KO) to ensure that there exists a radial solution  $w_n$  of the equation  $\Delta w = f(w)$  in  $B_R(x^{(n)})$ , with  $w_n(x^{(n)}) = 2\gamma$  and  $w_n = +\infty$  on  $\partial B_R(x^{(n)})$ , for some sufficiently large and fixed  $R > 0$ . Observe that  $B_R(x^{(n)}) \subset \mathbb{R}_+^N$  for large  $n$ , therefore we can apply the comparison principle and deduce  $u(x) \leq w_n(x)$  in  $B_R(x^{(n)})$  for large  $n$ . We obtain a contradiction since  $w_n(x^{(n)}) = 2\gamma < \gamma \leq u(x_n)$ .

By the above argument, and replacing if necessary the half-space  $\{x_N > 0\}$  by the half-space  $\{x_N > A\}$  for some conveniently chosen  $A > 0$ , we can assume that  $u < 0$  in  $\mathbb{R}_+^N$ . Moreover, for a given  $M > 0$  there exists  $R = R(M) > 0$  such that  $u \leq -M$  on  $\mathbb{R}^{N-1} \times \{R\}$  and  $R(M) \rightarrow +\infty$  as  $M \rightarrow +\infty$ .

For any  $R, T > 0$  consider the finite cylinder

$$D_{T,R} := \{x = (x', x_N) \in \mathbb{R}_+^N : |x'| < T, x_N < R\}.$$

Let  $\Omega_n$  be a sequence of smooth bounded domains such that  $D_{n,R} \subset \Omega_n \subset D_{n+1,R}$ , and let  $h_n$  be a continuous function defined on  $\partial\Omega_n$  such that  $-M \leq h_n \leq 0$  on  $\partial\Omega_n$ ,  $h_n = -M$  on  $\partial\Omega_n \cap \{|x'| < n, x_N = R\}$  and  $h_n = 0$  on  $\partial\Omega_n \cap \{x_N < R\}$ . Consider the problem

$$(2.7) \quad \begin{cases} \Delta w = f(w) & \text{in } \Omega_n \\ w = h_n & \text{on } \partial\Omega_n. \end{cases}$$

By the construction of  $h_n$  we see that  $u$  is a subsolution of this problem, while  $\bar{u} = 0$  is a supersolution (recall that  $f(0) > 0$ ), and they are ordered. Therefore there exists a solution  $w_n$  of (2.7) verifying  $u \leq w_n \leq 0$  in  $\Omega_n$ . Again by the usual elliptic estimates the sequence  $w_n$  is bounded in  $W^{2,p}(K)$  for each compact  $K \subset \{0 \leq x_N \leq R\}$ . Therefore we can pass to the limit by using a diagonal argument, to obtain a solution  $w_M$  of the following problem in a strip

$$(2.8) \quad \begin{cases} \Delta w = f(w) & \text{in } \{0 \leq x_N \leq R\} \\ w = 0 & \text{on } x_N = 0 \\ w = -M & \text{on } x_N = R \end{cases}$$

Of course we also have  $u \leq w_M \leq 0$ . We claim that actually  $w_M$  is the maximal solution of (2.8) in the order interval  $[u, 0]$ . To show this, let  $\tilde{w}$  be any solution of (2.8) such that  $u \leq \tilde{w} \leq 0$ . Therefore  $\tilde{w}$  is a subsolution of (2.7), so by comparison  $\tilde{w} \leq w_n$  in  $\Omega_n$  for every  $n$ . At the limit  $\tilde{w} \leq w_M$  and maximality follows.

By maximality,  $w_M$  depends only on the variable  $x^N$  (since any translation of  $w_M$  in a direction parallel to  $\{x_N = 0\}$  is still a solution of (2.8)). This means that  $v_M(x_N) = w_M(x)$  is a classical solution of  $v_M'' = f(v_M)$  in  $(0, R)$  with  $v_M(0) = 0$  and  $v_M(R) = -M$ . We can pass to the limit as  $M \rightarrow +\infty$  as before to obtain that  $v_{M_n} \rightarrow v$  in  $C_{\text{loc}}^2[0, +\infty)$  for some sequence  $M_n \rightarrow +\infty$ , where

$$v'' = f(v) \quad \text{in } (0, +\infty)$$

with  $v(0) = 0$ . Now  $w(x) = v(x_N)$  is itself a solution of (P), and hence, by (2.6), we have  $v(x_N) \rightarrow -\infty$  as  $x_N \rightarrow \infty$ .

However, as we already observed,  $(v')^2 = 2F(v) + v'(0)^2$  and so  $F(v(x_N))$  is bounded from below as  $x_N \rightarrow \infty$ , while on the other hand the failure of (HS) means precisely that  $F(t) = -\int_t^0 f$  tends to  $-\infty$  as  $t \rightarrow -\infty$ . The proof is concluded.  $\square$

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