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**A REMARK ON A THEOREM OF A.E.INGHAM
BY
K.G.BHAT and K.RAMACHANDRA**

ABSTRACT. The theorem of INGHAM [AEI]₁ referred to is:

For all $N \geq N_0$ (an absolute constant) the inequality

$$N^3 \leq p \leq (N + 1)^3$$

is solvable in a prime p . (It may be noted that the corresponding theorem for squares is an open question even if we assume RIEMANN HYPOTHESIS). Actually INGHAM proved more, namely

$$\pi(x + h) - \pi(x) \sim h(\log x)^{-1}$$

where $h = x^c$, where $c(> \frac{5}{8})$ is any constant. The purpose of this note is to point out that even this stronger form can be proved without using the functional equation of $\zeta(s)$.

§1.**INTRODUCTION.** The three main ingredients in the proof of INGHAM'S theorems are

(A) I.M.VINOGRADOV's deep result

$$\zeta(s) \neq 0 \quad (s = \sigma + it), \quad \sigma \geq 1 - K_1(\log t)^{-\frac{2}{3}}(\log \log t)^{-\frac{1}{3}}, \quad (1)$$

$t \geq 100$, where $K_1 > 0$ is an absolute constant.

(B) Explicit formula for

$$\sum_{p \leq x} \log p \quad ([AEI]_2). \quad (2)$$

(C)

$$N(\sigma, T) < T^{(\frac{8}{3})(1-\sigma)} (\log T)^{100} \quad (3)$$

where $\frac{1}{2} \leq \sigma \leq 1$, $T \geq 1000$. The precise power of $\log T$ is unimportant. Any constant in place of 100 will do. $N(\sigma, T)$ denotes the number of zeros $\beta + i\gamma$ of $\zeta(s)$ with $\beta \geq \sigma$ and $|\gamma| \leq T$.

1) The toughest part is (A). It follows from the deep result (due to I.M.VINOGRADOV)

$$|\zeta(\sigma + it)| \leq (t^{(1-\sigma)^{\frac{3}{2}}} \log t)^{K_2} \quad \left(\frac{1}{2} \leq \sigma \leq 1, t \geq 100\right) \quad (4)$$

where $K_2 > 0$ is an absolute constant and (1) follows from this in a relatively simple way by a method due to E.LANDAU[KR]₁. For a proof of (4) without using the functional equation see [KR,AS].

2) Explicit formula uses the functional equation, but an alternative approach is due to [KR]₂ by the introduction of HOOLEY-HUXLEY contour.

3) The proof of (3) uses

$$\zeta\left(\frac{1}{2} + it\right) = O(t^{\frac{1}{6}} \log t), t \geq 100, \quad (5)$$

where the O-constant is absolute. The main work in the present note is to sketch a proof of this without using the functional equation of $\zeta(s)$.

§2. **SOME REMARKS** In fact we write

$$\zeta(s, \alpha) = \sum_{n=0}^{\infty} (n + \alpha)^{-s} \quad (0 < \alpha \leq 1, s = \sigma + it, \sigma > 1), \quad (6)$$

and next if X is any positive integer we have

$$\begin{aligned} \zeta(s, \alpha) = \alpha^{-s} - \sum_{n=1}^X (n + \alpha)^{-s} + \sum_{n>X} ((n + \alpha)^{-s} - \int_n^{n+1} \frac{du}{(u + \alpha)^s}) \\ + \int_{X+1}^{\infty} (u + \alpha)^{-s} du. \end{aligned} \quad (7)$$

Since the last term in (7) is

$$\frac{(X + 1 + \alpha)^{1-s}}{s - 1} \quad (8)$$

and the rest is analytic in $\sigma > 0$, (7) gives the analytic continuation in $\sigma > 0$ of (6). We prove our main theorem which is as follows.

THEOREM. *We have*

$$\zeta\left(\frac{1}{2} + it, \alpha\right) - \alpha^{-\frac{1}{2}-it} = O(t^{\frac{1}{6}} \log t), (t \geq 10) \quad (9)$$

uniformly in the real parameter α . (Note that $\zeta(s, 1) = \zeta(s)$).

§3. **PROOF OF THE THEOREM** We use van-der Corput's theorems (Theorems 5.9 and 5.11 of [ECT]) and after the proof of the theorem we make some comments about the Weyl-Hardy-Littlewood method of proof of (9).

THEOREM 5.9. *If $f(x)$ is real and twice continuously differentiable and*

$$0 < \lambda_2 \leq f''(x) \leq h\lambda_2 \text{ (or } 0 < \lambda_2 \leq -f''(x) \leq h\lambda_2)$$

throughout the interval (a, b) and $b \geq a + 1$, then

$$\sum_{a < n \leq b} e^{2\pi i f(n)} = O(h(b-a)\lambda_2^{\frac{1}{2}}) + O(\lambda_2^{-\frac{1}{2}}). \quad (10)$$

THEOREM 5.11. *If $f(x)$ is real and thrice continuously differentiable and*

$$0 < \lambda_3 \leq f'''(x) \leq h\lambda_3 \text{ (or } 0 < \lambda_3 \leq -f'''(x) \leq h\lambda_3)$$

throughout the interval (a, b) and $b \geq a + 1$, then

$$\sum_{a < n \leq b} e^{2\pi i f(n)} = O(h^{\frac{1}{2}}(b-a)\lambda_3^{\frac{1}{6}}) + O((b-a)^{\frac{1}{2}}\lambda_3^{-\frac{1}{6}}). \quad (11)$$

We now apply these to

$$E \equiv \sum_{a \leq n \leq b(\leq 2a)} (n + \alpha)^{-it} \text{ with } a \geq 10. \quad (12)$$

Here $f(x) = -\frac{t}{2\pi} \log(x + \alpha)$. We have

$$f'(x) = -\frac{t}{2\pi(x + \alpha)}$$

$$f''(x) = \frac{t}{2\pi(x + \alpha)^2}$$

and

$$f'''(x) = -\frac{2t}{2\pi(x + \alpha)^3}.$$

Thus

$$C_1 \leq f''(x)a^2t^{-1} \leq C_2$$

and $C_3 \leq f'''(x)a^3t^{-1} \leq C_4$

(13)

where C_1, C_2, C_3 and C_4 are absolute positive constants. Thus we have

$$\sum_{a < n \leq b(\leq 2a)} (n + \alpha)^{-it} = O(t^{\frac{1}{2}}) + O(at^{-\frac{1}{2}}) \quad (14)$$

$$\sum_{a < n \leq b(\leq 2a)} (n + \alpha)^{-it} = O(t^{\frac{1}{6}}a^{\frac{1}{2}}) + O(t^{-\frac{1}{6}}a). \quad (15)$$

Hence by partial summation we have

$$\sum_{a < n \leq b(\leq 2a)} (n + \alpha)^{-\frac{1}{2}-it} = O\left(\left(\frac{t}{a}\right)^{\frac{1}{2}}\right) + O\left(\left(\frac{a}{t}\right)^{\frac{1}{2}}\right) \quad (16)$$

and

$$t \left(\sum_{a < n \leq b(\leq 2a)} (n + \alpha)^{-\frac{3}{2}-it} \right) = O\left(\left(\frac{t}{a}\right)^{\frac{3}{2}}\right) + O\left(\left(\frac{t}{a}\right)^{\frac{1}{2}}\right). \quad (17)$$

Also we need

$$\sum_{a < n \leq b(\leq 2a)} (n + \alpha)^{-\frac{1}{2}-it} = O(t^{\frac{1}{6}}) + O(t^{-\frac{1}{6}}a^{\frac{1}{2}}) \quad (18)$$

which follows from (15). From (18) there follows

$$\sum_{1 \leq n \leq t^{\frac{2}{3}}} (n + \alpha)^{-\frac{1}{2}-it} = O(t^{\frac{1}{6}} \log t). \quad (19)$$

From (16) there follows

$$\sum_{t^{\frac{2}{3}} \leq n \leq t^{\frac{4}{3}}} (n + \alpha)^{-\frac{1}{2} - it} = O(t^{\frac{1}{6}} \log t) \quad (20)$$

Thus

$$\sum_{1 \leq n \leq t^{\frac{4}{3}}} (n + \alpha)^{-\frac{1}{2} - it} = O(t^{\frac{1}{6}} \log t). \quad (21)$$

We now fix $X = [t^{\frac{4}{3}}]$. The term (8) contributes $O(t^{-\frac{1}{3}})$. We note that (with $s = \frac{1}{2} + it$)

$$\begin{aligned} & \sum_{n > X} \left((n + \alpha)^{-s} - \int_n^{n+1} \frac{du}{(u + \alpha)^s} \right) \\ &= \sum_{n > X} \int_n^{n+1} ((n + \alpha)^{-s} - (u + \alpha)^{-s}) du \\ &= \sum_{n > X} s \int_0^1 \left(\int_0^u (n + v + \alpha)^{-s-1} dv \right) du \end{aligned}$$

and so its absolute value is

$$O\left(\sum_{a > t^{\frac{4}{3}}} \left(\left(\frac{t}{a}\right)^{\frac{3}{2}} + \left(\frac{t}{a}\right)^{\frac{1}{2}} \right)\right) = O(t^{-\frac{1}{6}}).$$

This proves our main theorem.

REMARK 1 Let X be an arbitrary positive integer $\geq 20(|t| + 20)(K + 1)$. Then by iteration of the method by which we continued $\zeta(s, \alpha)$ in $\sigma > 0$ (incidentally the method is due to E.LANDAU (Handbuch der primzahlen) we can get the analytic continuation in $|\sigma| \leq (K + 1)$ (K being arbitrary constant) and also the inequality

$$\zeta(s, \alpha) = \alpha^{-s} + \sum_{n \leq X} (n + \alpha)^{-s} + \frac{X^{1-s}}{s-1} + O(X^{-\sigma})$$

where $s = \sigma + it$ (σ arbitrary). (O constant depends on K). For this see [KR]₁.

REMARK 2 A remark on Weyl-Hardy-Littlewood method is necessary here. The proof of Theorem 5.5 of [ECT] goes through to prove

$\sum_{1 \leq n \leq t^{\frac{2}{3}}} (n + \alpha)^{-\frac{1}{2} - it} = O(t^{\frac{1}{6}} L)$ except for trivial complications arising from the presence of the real parameter α . This uses the integer parameter k to be 2. However if we use the case $k = 1$ simple computations show that

$$\sum_{t^{\frac{2}{3}} \leq n \leq Ct} (n + \alpha)^{-\frac{1}{2} - it} = O(t^{\frac{1}{6}} L)$$

whatever the constant $C \geq 10$ be. Here L is some fixed power of $\log t$. These considerations prove the main theorem in view of Remark 1 above. We stress once again that functional equations for $\zeta(s)$ or $\zeta(s, \alpha)$ are not necessary in the proof of INGHAM's theorems. L can be any fixed power of $\log t$ and this is enough to prove INGHAM's asymptotic formula mentioned in the abstract.

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