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**A REMARK ON A THEOREM OF A.E.INGHAM  
BY  
K.G.BHAT and K.RAMACHANDRA**

**ABSTRACT.** The theorem of INGHAM [AEI]<sub>1</sub> referred to is:

For all  $N \geq N_0$  (an absolute constant) the inequality

$$N^3 \leq p \leq (N + 1)^3$$

is solvable in a prime  $p$ . (It may be noted that the corresponding theorem for squares is an open question even if we assume RIEMANN HYPOTHESIS). Actually INGHAM proved more, namely

$$\pi(x + h) - \pi(x) \sim h(\log x)^{-1}$$

where  $h = x^c$ , where  $c(> \frac{5}{8})$  is any constant. The purpose of this note is to point out that even this stronger form can be proved without using the functional equation of  $\zeta(s)$ .

§1.**INTRODUCTION.** The three main ingredients in the proof of INGHAM'S theorems are

(A) I.M.VINOGRADOV's deep result

$$\zeta(s) \neq 0 \quad (s = \sigma + it), \quad \sigma \geq 1 - K_1(\log t)^{-\frac{2}{3}}(\log \log t)^{-\frac{1}{3}}, \quad (1)$$

$t \geq 100$ , where  $K_1 > 0$  is an absolute constant.

(B) Explicit formula for

$$\sum_{p \leq x} \log p \quad ([AEI]_2). \quad (2)$$

(C)

$$N(\sigma, T) < T^{(\frac{8}{3})(1-\sigma)} (\log T)^{100} \quad (3)$$

where  $\frac{1}{2} \leq \sigma \leq 1$ ,  $T \geq 1000$ . The precise power of  $\log T$  is unimportant. Any constant in place of 100 will do.  $N(\sigma, T)$  denotes the number of zeros  $\beta + i\gamma$  of  $\zeta(s)$  with  $\beta \geq \sigma$  and  $|\gamma| \leq T$ .

1) The toughest part is (A). It follows from the deep result (due to I.M.VINOGRADOV)

$$|\zeta(\sigma + it)| \leq (t^{(1-\sigma)^{\frac{3}{2}}} \log t)^{K_2} \quad \left(\frac{1}{2} \leq \sigma \leq 1, t \geq 100\right) \quad (4)$$

where  $K_2 > 0$  is an absolute constant and (1) follows from this in a relatively simple way by a method due to E.LANDAU[KR]<sub>1</sub>. For a proof of (4) without using the functional equation see [KR,AS].

2) Explicit formula uses the functional equation, but an alternative approach is due to [KR]<sub>2</sub> by the introduction of HOOLEY-HUXLEY contour.

3) The proof of (3) uses

$$\zeta\left(\frac{1}{2} + it\right) = O(t^{\frac{1}{6}} \log t), t \geq 100, \quad (5)$$

where the O-constant is absolute. The main work in the present note is to sketch a proof of this without using the functional equation of  $\zeta(s)$ .

§2. **SOME REMARKS** In fact we write

$$\zeta(s, \alpha) = \sum_{n=0}^{\infty} (n + \alpha)^{-s} \quad (0 < \alpha \leq 1, s = \sigma + it, \sigma > 1), \quad (6)$$

and next if  $X$  is any positive integer we have

$$\begin{aligned} \zeta(s, \alpha) = \alpha^{-s} - \sum_{n=1}^X (n + \alpha)^{-s} + \sum_{n>X} ((n + \alpha)^{-s} - \int_n^{n+1} \frac{du}{(u + \alpha)^s}) \\ + \int_{X+1}^{\infty} (u + \alpha)^{-s} du. \end{aligned} \quad (7)$$

Since the last term in (7) is

$$\frac{(X + 1 + \alpha)^{1-s}}{s - 1} \quad (8)$$

and the rest is analytic in  $\sigma > 0$ , (7) gives the analytic continuation in  $\sigma > 0$  of (6). We prove our main theorem which is as follows.

**THEOREM.** *We have*

$$\zeta\left(\frac{1}{2} + it, \alpha\right) - \alpha^{-\frac{1}{2}-it} = O(t^{\frac{1}{6}} \log t), (t \geq 10) \quad (9)$$

*uniformly in the real parameter  $\alpha$ . (Note that  $\zeta(s, 1) = \zeta(s)$ ).*

§3. **PROOF OF THE THEOREM** We use van-der Corput's theorems (Theorems 5.9 and 5.11 of [ECT]) and after the proof of the theorem we make some comments about the Weyl-Hardy-Littlewood method of proof of (9).

**THEOREM 5.9.** *If  $f(x)$  is real and twice continuously differentiable and*

$$0 < \lambda_2 \leq f''(x) \leq h\lambda_2 \text{ (or } 0 < \lambda_2 \leq -f''(x) \leq h\lambda_2)$$

*throughout the interval  $(a, b)$  and  $b \geq a + 1$ , then*

$$\sum_{a < n \leq b} e^{2\pi i f(n)} = O(h(b-a)\lambda_2^{\frac{1}{2}}) + O(\lambda_2^{-\frac{1}{2}}). \quad (10)$$

**THEOREM 5.11.** *If  $f(x)$  is real and thrice continuously differentiable and*

$$0 < \lambda_3 \leq f'''(x) \leq h\lambda_3 \text{ ( or } 0 < \lambda_3 \leq -f'''(x) \leq h\lambda_3)$$

*throughout the interval  $(a, b)$  and  $b \geq a + 1$ , then*

$$\sum_{a < n \leq b} e^{2\pi i f(n)} = O(h^{\frac{1}{2}}(b-a)\lambda_3^{\frac{1}{6}}) + O((b-a)^{\frac{1}{2}}\lambda_3^{-\frac{1}{6}}). \quad (11)$$

We now apply these to

$$E \equiv \sum_{a \leq n \leq b(\leq 2a)} (n + \alpha)^{-it} \text{ with } a \geq 10. \quad (12)$$

Here  $f(x) = -\frac{t}{2\pi} \log(x + \alpha)$ . We have

$$f'(x) = -\frac{t}{2\pi(x + \alpha)}$$

$$f''(x) = \frac{t}{2\pi(x + \alpha)^2}$$

and

$$f'''(x) = -\frac{2t}{2\pi(x + \alpha)^3}.$$

Thus

$$C_1 \leq f''(x)a^2t^{-1} \leq C_2$$

and  $C_3 \leq f'''(x)a^3t^{-1} \leq C_4$

(13)

where  $C_1, C_2, C_3$  and  $C_4$  are absolute positive constants. Thus we have

$$\sum_{a < n \leq b(\leq 2a)} (n + \alpha)^{-it} = O(t^{\frac{1}{2}}) + O(at^{-\frac{1}{2}}) \quad (14)$$

$$\sum_{a < n \leq b(\leq 2a)} (n + \alpha)^{-it} = O(t^{\frac{1}{6}}a^{\frac{1}{2}}) + O(t^{-\frac{1}{6}}a). \quad (15)$$

Hence by partial summation we have

$$\sum_{a < n \leq b(\leq 2a)} (n + \alpha)^{-\frac{1}{2}-it} = O\left(\left(\frac{t}{a}\right)^{\frac{1}{2}}\right) + O\left(\left(\frac{a}{t}\right)^{\frac{1}{2}}\right) \quad (16)$$

and

$$t \left( \sum_{a < n \leq b(\leq 2a)} (n + \alpha)^{-\frac{3}{2}-it} \right) = O\left(\left(\frac{t}{a}\right)^{\frac{3}{2}}\right) + O\left(\left(\frac{t}{a}\right)^{\frac{1}{2}}\right). \quad (17)$$

Also we need

$$\sum_{a < n \leq b(\leq 2a)} (n + \alpha)^{-\frac{1}{2}-it} = O(t^{\frac{1}{6}}) + O(t^{-\frac{1}{6}}a^{\frac{1}{2}}) \quad (18)$$

which follows from (15). From (18) there follows

$$\sum_{1 \leq n \leq t^{\frac{2}{3}}} (n + \alpha)^{-\frac{1}{2}-it} = O(t^{\frac{1}{6}} \log t). \quad (19)$$

From (16) there follows

$$\sum_{t^{\frac{2}{3}} \leq n \leq t^{\frac{4}{3}}} (n + \alpha)^{-\frac{1}{2} - it} = O(t^{\frac{1}{6}} \log t) \quad (20)$$

Thus

$$\sum_{1 \leq n \leq t^{\frac{4}{3}}} (n + \alpha)^{-\frac{1}{2} - it} = O(t^{\frac{1}{6}} \log t). \quad (21)$$

We now fix  $X = [t^{\frac{4}{3}}]$ . The term (8) contributes  $O(t^{-\frac{1}{3}})$ . We note that (with  $s = \frac{1}{2} + it$ )

$$\begin{aligned} & \sum_{n > X} \left( (n + \alpha)^{-s} - \int_n^{n+1} \frac{du}{(u + \alpha)^s} \right) \\ &= \sum_{n > X} \int_n^{n+1} ((n + \alpha)^{-s} - (u + \alpha)^{-s}) du \\ &= \sum_{n > X} s \int_0^1 \left( \int_0^u (n + v + \alpha)^{-s-1} dv \right) du \end{aligned}$$

and so its absolute value is

$$O\left(\sum_{a > t^{\frac{4}{3}}} \left( \left(\frac{t}{a}\right)^{\frac{3}{2}} + \left(\frac{t}{a}\right)^{\frac{1}{2}} \right)\right) = O(t^{-\frac{1}{6}}).$$

This proves our main theorem.

**REMARK 1** Let  $X$  be an arbitrary positive integer  $\geq 20(|t| + 20)(K + 1)$ . Then by iteration of the method by which we continued  $\zeta(s, \alpha)$  in  $\sigma > 0$  (incidentally the method is due to E.LANDAU (Handbuch der primzahlen) we can get the analytic continuation in  $|\sigma| \leq (K + 1)$  ( $K$  being arbitrary constant) and also the inequality

$$\zeta(s, \alpha) = \alpha^{-s} + \sum_{n \leq X} (n + \alpha)^{-s} + \frac{X^{1-s}}{s-1} + O(X^{-\sigma})$$

where  $s = \sigma + it$  ( $\sigma$  arbitrary). ( $O$  constant depends on  $K$ ). For this see [KR]<sub>1</sub>.

**REMARK 2** A remark on Weyl-Hardy-Littlewood method is necessary here. The proof of Theorem 5.5 of [ECT] goes through to prove

$\sum_{1 \leq n \leq t^{\frac{2}{3}}} (n + \alpha)^{-\frac{1}{2} - it} = O(t^{\frac{1}{6}} L)$  except for trivial complications arising from the presence of the real parameter  $\alpha$ . This uses the integer parameter  $k$  to be 2. However if we use the case  $k = 1$  simple computations show that

$$\sum_{t^{\frac{2}{3}} \leq n \leq Ct} (n + \alpha)^{-\frac{1}{2} - it} = O(t^{\frac{1}{6}} L)$$

whatever the constant  $C \geq 10$  be. Here  $L$  is some fixed power of  $\log t$ . These considerations prove the main theorem in view of Remark 1 above. We stress once again that functional equations for  $\zeta(s)$  or  $\zeta(s, \alpha)$  are not necessary in the proof of INGHAM's theorems.  $L$  can be any fixed power of  $\log t$  and this is enough to prove INGHAM's asymptotic formula mentioned in the abstract.

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