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## ON AN EXPONENTIAL SUM INVOLVING THE MÖBIUS FUNCTION

H. MAIER<sup>(1)</sup> AND A. SANKARANARAYANAN<sup>(2)</sup>

ABSTRACT. In this paper we study the upper bound for the absolute value of the exponential sum related to the Möbius function unconditionally and present some interesting applications also.

1 2

### 1. Introduction

Let  $\mathbf{e}(\alpha)$  denote  $e^{2\pi i\alpha}$ . We consider the exponential sum

$$(1.1) \quad S(x, \theta) = \sum_{n \leq x} \mu(n) \mathbf{e}(n\theta).$$

R.C. Baker and G. Harman (see [1]) studied this sum under the hypothesis that *for every Dirichlet character  $\chi$ ,  $L(s, \chi)$  has no zeros in the half-plane  $\sigma > a$* . They proved

**THEOREM A.** *Suppose that the above hypothesis holds for every Dirichlet character  $\chi$ . Then*

$$\max_{\theta} |S(x, \theta)| \ll x^{b+\epsilon}$$

with

$$b = \begin{cases} a + \frac{1}{4} & \text{if } \frac{1}{2} \leq a < \frac{11}{20}, \\ \frac{4}{5} & \text{if } \frac{11}{20} \leq a < \frac{3}{5}, \\ \frac{1}{2}(a + 1) & \text{if } \frac{3}{5} \leq a < 1. \end{cases}$$

For other interesting related results we refer to [2], [4], [11] and [13]. Throughout the paper, the integer  $d$  is a divisor of  $q$ ,  $\chi^*$  denotes a Dirichlet character modulo  $q$ ,  $\chi_d$  denotes the principal character modulo  $d$  and  $\chi$  denotes a character modulo  $\left(\frac{q}{d}\right)$ . The constant  $A$  is a generic absolute constant which need not be the same at each occurrence.  $\epsilon$  and  $\delta$  denote arbitrarily small positive constants. We use  $\|\theta\|$  to mean that

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$$||\theta|| = \min_{m \in \mathbb{Z}} |\theta - m|.$$

Throughout the paper, we assume that  $x \geq x_0$  (where  $x_0$  is sufficiently large) and the modulus  $q \geq q_0$  ( $\geq 10$ ).

Now, the main goal of this paper is to give an *unconditional upper bound in terms of zero density functions of Dirichlet L-functions* for  $|S(x, \theta)|$ . Let

$$(1.2) \quad N(\alpha, T, \chi) = \# \{ \rho = \beta + i\gamma : L(\rho, \chi) = 0, \beta \geq \alpha > 0, |\gamma| \leq T \}.$$

We prove

**THEOREM 1.** *Let  $q$  be a positive integer and  $\theta$  be a real number. Then for any rational number  $\frac{r}{q}$  with  $(r, q) = 1$  and for any fixed  $\alpha$  satisfying  $\frac{1}{2} + \delta \leq \alpha \leq 1 - \delta$  (with  $\delta$  being any small positive constant), we have*

$$(1.3) \quad \begin{aligned} S =: S(x, \theta) &\ll q^{-\frac{1}{2}} x^{1+10\epsilon} + \sum_{d|q} \left(\frac{q}{d}\right)^{\frac{1}{2}} \cdot \frac{1}{\phi\left(\frac{q}{d}\right)} \left(1 + x \left|\theta - \frac{r}{q}\right|\right) \times \\ &\times \left( \max_{1 \leq T \leq \left(\frac{x}{d}\right)^3} T^{-1} (qT)^\epsilon \int_{\frac{1}{2}}^1 \left( T \phi\left(\frac{q}{d}\right) \left(\frac{x}{d}\right)^{\alpha+\epsilon} + \left(\frac{x}{d}\right)^{\sigma+\epsilon} \sum_{\chi \pmod{\left(\frac{q}{d}\right)}} N(\sigma, T, \chi \chi_d) \right) d\sigma \right). \end{aligned}$$

Here  $\chi_d$  denotes the principal character modulo  $d$  and the functions  $\chi \chi_d$  are characters modulo  $q$ .

To strengthen our Theorem 1 above, we present here a few applications as Theorems 2, 3 and 4 in the following. Note that Theorem 2 is unconditional whereas Theorem 3 depends on the *Average Zero-Density Hypothesis*. Since they depend on the type of  $\theta$ , let us recall this notion (see page 121 of [7] for more details). Let  $\psi$  be a non-decreasing positive function that is defined at least for all positive integers. The irrational number  $\theta$  is said to be of type  $< \psi$  if  $q||q\theta|| \geq \frac{1}{\psi(q)}$  holds for all positive integers  $q$ . If  $\psi$  is a constant function, then an irrational  $\theta$  of type  $< \psi$  is also called of constant type. Let  $\eta_1$  be a positive real number or infinity. The irrational number  $\theta$  is said to be of type  $\eta_1$  if  $\eta_1$  is the supremum of all  $\delta_1$  for which

$$\liminf_{q \rightarrow \infty} q^{\delta_1} ||q\theta|| = 0$$

where  $q$  runs through the positive integers. The relationship between these two definitions is that an irrational number  $\theta$  is of type  $\eta_1$  if and only if for every  $\tau > \eta_1$  there is a constant  $c = c(\tau, \theta)$  such that  $\theta$  is of type  $< \psi$  where  $\psi(q) = cq^{\tau-1}$ . It is

well-known that almost all numbers are of type 1. From Roth's theorem, we note that all algebraic irrationalities  $\theta$  satisfy

$$\left| \theta - \frac{r}{q} \right| \gg \frac{1}{q^{2+\epsilon}}.$$

Therefore all algebraic irrationalities are of type 1.

Just by using the known average zero-density estimates of  $L(s, \chi)$  over all characters of a fixed modulus, we prove

**THEOREM 2.** *Let  $\theta$  be of type 1. Then for every arbitrarily small  $\epsilon > 0$ , unconditionally we have*

$$S(x, \theta) \ll_{\epsilon} x^{c^* + \epsilon},$$

where  $c^* = \frac{4}{5}$ .

We formulate the zero-density hypothesis for all the characters  $\chi^*$  modulo  $(q)$  as :

**AVERAGE ZERO-DENSITY HYPOTHESIS.** *For  $\sigma \geq \frac{1}{2}$ , we have,*

$$(1.4) \quad \sum_{\chi^*} N(\sigma, T, \chi^*) \ll (qT)^{2(1-\sigma)} (\log(qT))^A,$$

where  $A$  is an absolute constant.

Now, we prove

**THEOREM 3.** *We assume the Average Zero-Density Hypothesis. Let  $\theta$  be of type 1. Then for  $\epsilon > 0$  arbitrarily small, we have*

$$S(x, \theta) \ll x^{\frac{3}{4} + \epsilon}.$$

*Remark.* As we stated earlier, Theorem 1 gives the upper bound for  $|S(x, \theta)|$  which depends on the zero density functions. We observe that Theorem A yields the exponent  $\frac{3}{4} + \epsilon$  (for all real  $\theta$ ) *only on the assumption of the Generalised Riemann Hypothesis*. The author<sup>(2)</sup> jointly with M. Ram Murty studied such exponential sums in [13] using a more general form of Vaughan's method. It should be mentioned here that they obtained for example (for  $\theta$  of type 1)

$$S(x, \theta) \ll x^{\frac{4}{5} + \epsilon},$$

in [13]. The approach of the present paper is entirely different and the results here depend on the zero density functions. As far as Theorem 3 is concerned, we have assumed a weaker hypothesis and obtain a better result when  $\theta$  is of type 1 and it seems to be new.

As another interesting application of our Theorem 1, we prove

**THEOREM 4.** *Let  $\chi$  be a character modulo  $m$ . For every integer  $m \geq 1$ , we suppose that there exists a constant  $g (\geq 2)$  such that*

$$(1.5) \quad \sum_{\chi(\pmod{m})} N(\sigma, T, \chi) \ll (mT)^{g(1-\sigma)} (\log(mT))^A,$$

*uniformly for  $\frac{1}{2} \leq \sigma \leq 1$ . Here  $A$  is an absolute constant. If  $\theta$  is of type 1, then we have*

$$(1.6) \quad S(x, \theta) \ll x^{\nu+\epsilon}$$

*where*

$$(1.7) \quad \nu = \frac{13}{4} - \frac{3}{g} - \frac{g}{2}.$$

*Remark.* From the known average zero-density results (see the fundamental work of H.L. Montgomery [8], [9], precisely for example see Theorem 12.1 of [10], see (2.23) and a few subsequent lines of [6]), we find that we can take for example  $g = \frac{12}{5}$  in (1.4). We also note that the result of Theorem 4 will be having a better exponent  $\nu$  (which is less than  $\frac{4}{5}$ ) unconditionally for all  $\theta$  of type 1 when  $g < \frac{12}{5}$  in (1.4).

## 2. Some Lemmas

**LEMMA 2.1.** *Let  $\tilde{\beta} = \theta - \frac{x}{q}$  and  $q \leq x^2$ . We have*

$$S(x, \theta) = \sum_{d|q} \frac{\mu(d)}{\phi(q/d)} \sum_{\chi(\pmod{q/d})} \chi(r)\tau(\bar{\chi})S\left(\frac{x}{d}, \chi\chi_d, \tilde{\beta}d\right).$$

*Here  $\chi_d$  denotes the principal character  $(\pmod{d})$  and*

$$S(y, \chi^*, \eta) = \sum_{n \leq y} \mu(n)\chi^*(n)\mathbf{e}(n\eta)$$

*and*

$$\tau(\bar{\chi}) = \sum_{m=1}^{q/d} \bar{\chi}(m)\mathbf{e}\left(\frac{md}{q}\right).$$

*Proof.* See Lemma 1 of [1]. □

*Remark.* The functions  $\chi\chi_d$  appearing in Lemma 2.1 are characters  $(\pmod{q})$ . For the Gauss sum  $\tau(\bar{\chi})$ , we need the well-known bound

$$\tau(\bar{\chi}) \ll \left(\frac{q}{d}\right)^{\frac{1}{2}}.$$

K. Ramachandra and A. Sankaranarayanan studied (see Theorems 1 and 2 of [12]) certain upper bound estimates “*locally*” for the function  $|\log F(s)|$  (where  $F(s)$  is any Dirichlet series satisfying certain general conditions) under the assumption that  $F(s)$  is zero-free in a rectangle of the type  $\{\sigma \geq \frac{1}{2} + \delta, T - H \leq t \leq T + H\}$  of  $t$ -width  $2H$ . Here the parameter  $H$  could be chosen as small as  $H = C \log \log \log T$ . We again emphasize here that the Theorems 1 and 2 of [12] (see the appendix of the paper [12]) are more general and they assume neither the Euler product nor any functional equation of the Dirichlet series  $F(s)$  concerned. We just record a special case of these general theorems as Lemma 2.2 in the sequel.

LEMMA 2.2. *Let  $\frac{1}{2} \leq \alpha^* \leq 1 - \delta$ . Let  $H = C \log \log(qT)$  and  $\chi^*$  be any Dirichlet character modulo  $q$ . Suppose the Dirichlet  $L$ -function  $L(s, \chi^*) \neq 0$  in*

$$\{\sigma > \alpha^*, T - H \leq t \leq T + H\}.$$

*Then for  $(\alpha^* < \sigma \leq 1 - \frac{C_1}{\log \log(qT)}, T - \frac{H}{2} \leq t \leq T + \frac{H}{2})$ , we have :*

$$(2.2.1) \quad |\log L(\sigma + it, \chi^*)| \leq (C_2(\log(qT))(\log \log(qT))^{-1}).$$

*Proof.* It is a special case of the Theorems 1 and 2 of [12]. □

LEMMA 2.3. *Let  $m$  be an integer and  $1 \leq m \leq x$ . We have*

$$\begin{aligned} S(m, \chi^*, 0) &= \frac{1}{2\pi i} \int_{2-iT}^{2+iT} \frac{1}{L(w, \chi^*)} \frac{m^w}{w} dw + O\left(\frac{m^2}{T}\right) \\ &= \frac{1}{2\pi i} \int_{2-iT}^{2+iT} \frac{1}{L(w, \chi^*)} \frac{m^w}{w} dw + O\left(\frac{1}{m}\right), \end{aligned}$$

*with  $T = m^3$ .*

*Proof.* We apply Lemma 3.12 of [14] with  $a_n = \mu(n)\chi^*(n)$ . □

LEMMA 2.4. *Let  $\theta$  be of type 1 and  $\epsilon > 0$  be arbitrarily small. Then for  $x$  sufficiently large, there exist integers  $r, q$  with  $(r, q) = 1$  satisfying :*

$$(2.4.1) \quad x^{\frac{1}{2(1+\epsilon)}} \leq q \leq x^{\frac{1}{2}}$$

*and*

$$(2.4.2) \quad \left| \theta - \frac{r}{q} \right| < x^{-(1-\epsilon)}.$$

*Proof.* In the sequel, let  $\epsilon_j > 0$  denote fixed but arbitrarily small constants. From the definition of the type 1-property, it follows that for all  $\epsilon_1 > 0$ , there is a  $q_0(\epsilon_1)$  such that

$$(2.4.3) \quad \|q\theta\| \geq q^{-(1+\epsilon_1)} \quad \text{for } q \geq q_0(\epsilon_1).$$

We choose  $x$  such that

$$(2.4.4) \quad x^{-\frac{1}{2}} < \min_{q < q_0(\epsilon_1)} \|q\theta\|.$$

By the Dirichlet approximation theorem, there is  $\frac{r}{q} \in \mathbb{Q}$  with  $(r, q) = 1$ ,  $1 \leq q < x^{\frac{1}{2}}$ , such that

$$(2.4.5) \quad \left| \theta - \frac{r}{q} \right| < \frac{1}{qx^{\frac{1}{2}}} \text{ which implies that } \|q\theta\| < \frac{1}{x^{\frac{1}{2}}}.$$

By (2.4.4), we have

$$(2.4.6) \quad q \geq q_0(\epsilon_1).$$

From (2.4.3) and (2.4.5), we have

$$(2.4.7) \quad q^{-(1+\epsilon_1)} \leq \|q\theta\| < x^{-\frac{1}{2}} \text{ which implies that } x^{\frac{1}{2}} \leq q^{(1+\epsilon_1)}.$$

This means that

$$(2.4.8) \quad q \geq x^{\frac{1}{2(1+\epsilon_2)}} \text{ and hence } \left| \theta - \frac{r}{q} \right| < x^{-(1-\epsilon_3)}.$$

This proves the lemma.  $\square$

LEMMA 2.5. *Let  $q$  be a positive integer and  $d$  be any divisor of  $q$ . Let  $\chi$  denote a character modulo  $\left(\frac{q}{d}\right)$  and  $\chi_d$  denote the principal character modulo  $d$ . Then for  $\sigma \geq \frac{1}{2}$ , we have*

$$N(\sigma, T, \chi\chi_d) = N(\sigma, T, \chi).$$

*Proof.* We have (for  $\Re s > 1$ )

$$L(s, \chi) = \prod_p (1 - \chi(p)p^{-s})^{-1}$$

and

$$L(s, \chi\chi_d) = \prod_p (1 - \chi(p)\chi_d(p)p^{-s})^{-1}.$$

We note that

$$\chi_d(p) = \begin{cases} 0 & \text{if } p|d, \\ 1 & \text{if } p \nmid d. \end{cases}$$

Therefore, we have

$$\begin{aligned} L(s, \chi\chi_d) &= \prod_{p|d} (1 - \chi(p)p^{-s})^{-1} \prod_{p|d} (1 - \chi(p)\chi_d(p)p^{-s})^{-1} \\ &= L(s, \chi) \prod_{p|d} (1 - \chi(p)p^{-s}). \end{aligned}$$

We also note that

$$\prod_{p|d} (1 - \chi(p)p^{-s}) \neq 0 \text{ for } \Re s > 0.$$

Hence we obtain

$$N(\sigma, T, \chi\chi_d) = N(\sigma, T, \chi).$$

This proves the lemma.  $\square$

### 3. The construction of the contour $\mathcal{C}$ of integration

In the following,  $C^*$  is a generic absolute constant which need not be the same at each occurrence.

**Definition:** A zero  $\rho =: \beta + i\gamma$  of the Dirichlet  $L$ -function  $L(s, \chi^*)$  is said to be *good* if  $\beta < 1 - \frac{C^*}{(\log \log(q(|t|+2)))}$  and  $\rho$  is said to be *exceptional* otherwise.

Let  $T_0 \leq T (\leq x^3)$  and  $x \geq x_0$  where  $x_0$  is sufficiently large. Let  $\mathcal{G}$  and  $\mathcal{E}$  denote the set of all *good* and *exceptional zeros* respectively of all the Dirichlet  $L$ -functions  $L(s, \chi^*)$  with  $|\gamma| \leq T$ . We denote  $|\mathcal{G}|$  and  $|\mathcal{E}|$  to mean the cardinality of the sets  $\mathcal{G}$  and  $\mathcal{E}$  respectively.

Let  $\alpha$  be any fixed constant satisfying  $\frac{1}{2} + \delta \leq \alpha \leq 1 - \delta$  with  $\delta$  being any arbitrarily small fixed positive constant. The contour is symmetric with respect to the real axis. So, it suffices to describe it in the upper half-plane. We assume that  $|\mathcal{E}| = 0$ . Therefore,  $L(s, \chi^*) \neq 0$  in the region  $\left\{ \sigma > 1 - \frac{C^*}{(\log \log 2q(U+10))}, U \leq t \leq 2U \right\}$  where  $C^*$  is a suitable absolute positive constant and we construct the contour accordingly. We let  $T = 2^{l_0}$ . We choose  $c$  with  $\frac{1}{2} \leq c \leq 1$  such that  $H_0 = c \log \log(qT) = 2^L$  with a positive integer  $L$ . For  $l \geq L$ , put  $U = U^{(l)} = 2^l$ . We define the contour for  $U < t \leq 2U$ . We let  $H = H(U^{(l)}) = c_l \log \log(qU^{(l)})$  and choose  $c_l$  satisfying  $\frac{1}{2} \leq c_l \leq 1$  such that  $\frac{U}{2H}$  is a positive integer. Then we partition the interval  $[U, 2U]$  into  $\frac{U}{2H}$  disjoint abutting small intervals  $I_j = I_j^{(l)}$  of equal length  $2H$  for  $1 \leq j \leq \frac{U}{2H}$ . Let  $I_j = [U_j - H, U_j + H]$  and let

$$\beta_j = \sup \{ \beta : \rho = \beta + i\gamma, L(\rho, \chi^*) = 0, \beta \geq \alpha, \gamma \in [U_j - 2H, U_j + 2H] \}$$

and



$$\beta_j^* = \beta_j + \frac{C^*}{(\log \log 2q(U+10))}.$$

We also define (with  $H'_0 = H_0 + 2(\log(qH_0))^2$ )

$$\beta_0 = \sup \{ \beta : \rho = \beta + i\gamma, L(\rho, \chi^*) = 0, \beta \geq \alpha, \gamma \in [0, 2H'_0] \}$$

and

$$\beta_0^* = \beta_0 + \frac{C^*}{(\log \log qH_0)}.$$

If there is no zero of  $L(s, \chi^*)$  in the rectangle

$$\{ \sigma \geq \alpha, U_j - 2H \leq t \leq U_j + 2H \},$$

then we define  $\beta_j^* = \alpha$ . Similar notion applies to  $\beta_0^*$  also.

Thus, the contour  $\mathcal{C}$  consists of *vertical pieces*

$$V_j =: \begin{cases} [\beta_j^* + i(U_j - H + \epsilon), \beta_j^* + i(U_j + H - \epsilon)] & \text{if } \beta_j^* < \min(\beta_{j-1}^*, \beta_{j+1}^*), \\ [\beta_j^* + i(U_j - H - \epsilon), \beta_j^* + i(U_j + H + \epsilon)] & \text{if } \beta_j^* > \max(\beta_{j-1}^*, \beta_{j+1}^*), \\ [\beta_j^* + i(U_j - H - \epsilon), \beta_j^* + i(U_j + H - \epsilon)] & \text{if } \min(\beta_{j-1}^*, \beta_{j+1}^*) < \beta_j^* < \max(\beta_{j-1}^*, \beta_{j+1}^*), \end{cases}$$

the vertical piece

$$V_0 =: \begin{cases} [\beta_0^*, \beta_0^* + i(H_0 - \epsilon)] & \text{if } \beta_1^* > \beta_0^*, \\ [\beta_0^*, \beta_0^* + i(H_0 + \epsilon)] & \text{if } \beta_1^* < \beta_0^*, \end{cases}$$

the horizontal pieces  $h_j$  with

$$h_j(\text{top}) = [\beta_j^* + i(U_j + H - \epsilon), \beta_{j+1}^* + i(U_j + H - \epsilon)]$$

and

$$h_j(\text{bottom}) = [\beta_j^* + i(U_j - H + \epsilon), \beta_{j-1}^* + i(U_j - H + \epsilon)]$$

provided  $\beta_j^* < \min(\beta_{j-1}^*, \beta_{j+1}^*)$ ,

$$h_j(\text{top}) = [\beta_{j+1}^* + i(U_j + H + \epsilon), \beta_j^* + i(U_j + H + \epsilon)]$$

and

$$h_j(\text{bottom}) = [\beta_{j-1}^* + i(U_j - H - \epsilon), \beta_j^* + i(U_j - H - \epsilon)]$$

provided  $\beta_j^* > \max(\beta_{j-1}^*, \beta_{j+1}^*)$ ,

$$h_j(\text{top}) = [\beta_j^* + i(U_j + H - \epsilon), \beta_{j+1}^* + i(U_j + H - \epsilon)]$$

and

$h_j(\text{bottom}) = [\beta_{j-1}^* + i(U_j - H - \epsilon), \beta_j^* + i(U_j - H - \epsilon)]$   
 provided  $\beta_{j-1}^* < \beta_j^* < \beta_{j+1}^*$ ,

$$h_j(\text{top}) = [\beta_{j+1}^* + i(U_j + H - \epsilon), \beta_j^* + i(U_j + H - \epsilon)]$$

and

$$h_j(\text{bottom}) = [\beta_j^* + i(U_j - H - \epsilon), \beta_{j-1}^* + i(U_j - H - \epsilon)]$$

provided  $\beta_{j+1}^* < \beta_j^* < \beta_{j-1}^*$ , and *the similar horizontal pieces*  $h_{0,l}$  (this is just a notation only) that link the top (respectively the bottom) vertical pieces of the ranges

$$U^{(l-1)} \leq t \leq 2U^{(l-1)} \quad (\text{respectively}) \quad U^{(l)} \leq t \leq 2U^{(l)}.$$

#### 4. The treatment of $S(y, \chi^*, \eta)$

We keep in mind that the modulus  $q$  satisfies the inequality  $q \geq q_0$  and there is no exceptional zero in the following.

We have

$$(4.1) \quad S(y, \chi^*, \eta) = \sum_{m=1}^{y-1} S(m, \chi^*, 0) \mathbf{e}(\eta m) (1 - \mathbf{e}(\eta)) + S(y, \chi^*, 0) \mathbf{e}(\eta y).$$

By Lemma 2.3, we have (with  $T = m^3 (\leq y^3 \leq (\frac{x}{d})^3 \leq x^3)$ )

$$(4.2) \quad \begin{aligned} \sum_{m=1}^{y-1} S(m, \chi^*, 0) \mathbf{e}(\eta m) &= \frac{1}{2\pi i} \int_{2-iT}^{2+iT} \frac{1}{L(w, \chi^*) w} \left( \sum_{m=1}^{y-1} m^w \mathbf{e}(\eta m) \right) dw + O(\log y) \\ &= I^* + O(\log y) \quad (\text{say}). \end{aligned}$$

We now replace the path  $[2 - iT, 2 + iT]$  by the contour  $\mathcal{C}$  constructed in section 3.

By Lemma 2.2 (for large  $|t| (\geq H_0)$ ), we have

$$(4.3) \quad |L(\sigma + it, \chi^*)|^{-1} \ll (qU)^\epsilon$$

for  $\sigma + it \in V_j$  and  $\sigma + it \in h_j$ . The horizontal slab with  $|t| \in [0, H_0]$  is treated as follows. We redefine

$$\beta_0^* = \begin{cases} \beta_{0,1}^* & \text{if } |t| \in [0, 2^7], \\ \beta_{0,2}^* & \text{if } |t| \in [2^3, H_0']. \end{cases}$$

For the portion  $|t| \in [2^5, H_0]$ , we first observe that the region

$$\left\{ 10 \geq \sigma \geq \beta_{0,2}^* \geq \beta_0 + \frac{C^*}{(\log \log q H_0)}, \quad 2^3 \leq |t| \leq H_0' \right\}$$

is zero-free for  $L(s, \chi^*)$  and hence applying Borel-Caratheodary theorem, we get (for  $10 \geq \sigma \geq \beta_0 + \frac{C^*}{(\log \log q H_0)}$ )

$$(4.4) \quad |\log L(\sigma + it, \chi^*)| \ll (\log(qH_0))^{1-\epsilon} \ll (\log(qT))^{1-\epsilon},$$

and so the above estimate holds when  $\beta_{0,2}^* + it \in V_0$  with  $t \in [2^5, H_0]$  provided  $T \geq T_0$  (where  $T_0$  is sufficiently large).

The portion  $|t| \in [0, 2^5]$  is dealt as follows. Note that by our assumption,  $L(s, \chi^*)$  does not have any exceptional zero. We first observe that the region

$$\left\{ 10 \geq \sigma \geq 1 - \frac{C^{**}}{2(\log \log q)} \geq \beta_{0,1}^*, \quad |t| \leq 2^7 \right\}$$

is zero-free for  $L(s, \chi^*)$  with a suitable absolute constant  $C^{**}$ . If  $\sigma \geq 1 - \frac{C^{**}}{2(\log \log q)}$ , then we have

$$(4.5) \quad |L(s, \chi^*)| \ll (\log q)^{1-\epsilon}.$$

If  $|s-1| \leq \frac{C^{**}}{10(\log \log q)}$ , then we observe that

$$(4.6) \quad \begin{aligned} |L(s, \chi^*)| &\leq |L(s, \chi^*) - L(1, \chi^*)| + |L(1, \chi^*)| \\ &\ll |s-1|(\log q)^{1-\epsilon} + \log q \\ &\ll (\log q). \end{aligned}$$

Now, we apply the Borel-Caratheodary Theorem suitably to the function  $\log((s-1)L(s, \chi^*))$  with concentric circles having centre at  $2+it$  and radii  $\frac{1}{2}$ ,  $1 + \frac{C^{**}}{2(\log \log q)}$  and conclude that

$$(4.7) \quad |\log L(\beta_{0,1}^* + it, \chi^*)| \ll \left| \log \left( \frac{1}{1 - \beta_{0,1}^*} \right) \right| + (\log q)^{1-\epsilon} \ll (\log(qT))^{1-\epsilon},$$

for  $T \geq T_0$ . This implies that

$$(4.8) \quad |L(\beta_{0,1}^* + it, \chi^*)|^{-1} \ll (qT)^\epsilon$$

whenever  $|t| \in [0, 2^5]$ .

By partial summation, we obtain

$$(4.9) \quad \sum_{m=1}^{y-1} m^{\beta_j^* + it} \mathbf{e}(\eta m) = (y-1)\beta_j^* \sum_{m \leq y-1} m^{it} \mathbf{e}(\eta m) - \beta_j^* \int_1^{y-1} \left( \sum_{m \leq u} m^{it} \mathbf{e}(\eta m) \right) u^{\beta_j^* - 1} du.$$

From (4.3) and (4.9), we observe that the contribution of the vertical path  $V_j$  to  $I^*$  is

$$\begin{aligned}
I^*(V_j) &=: \sum_{m=1}^{y-1} \frac{1}{2\pi i} \int_{V_j} \frac{1}{L(w, \chi^*)} \frac{m^w}{w} \mathbf{e}(\eta m) dw \\
(4.10) \quad &\ll HU^{-1} (qU)^\epsilon y^{\beta_j^*} \max_{u \leq y} \sup_{t \in [U, 2U]} \left| \sum_{m \leq u} m^{it} \mathbf{e}(\eta m) \right|,
\end{aligned}$$

an analogous estimate

$$\begin{aligned}
I^*(V_0) &\ll H_0 (qT)^\epsilon y^{\beta_0^*} \times \\
&\times \left\{ \max_{u \leq y} \sup_{t \in [0, 2^5]} \left| \sum_{m \leq u} m^{it} \mathbf{e}(\eta m) \right| + \max_{u \leq y} \sup_{t \in [2^5, H_0]} \left| \sum_{m \leq u} m^{it} \mathbf{e}(\eta m) \right| \right\},
\end{aligned}$$

and the contribution of the horizontal path  $h_j$  to  $I^*$  is

$$\begin{aligned}
I^*(h_j) &=: \sum_{m=1}^{y-1} \frac{1}{2\pi i} \int_{h_j} \frac{1}{L(w, \chi^*)} \frac{m^w}{w} \mathbf{e}(\eta m) dw \\
(4.11) \quad &\ll U^{-1} (qU)^\epsilon y^{\beta_j^*} \max_{u \leq y} \sup_{t \in [U, 2U]} \left| \sum_{m \leq u} m^{it} \mathbf{e}(\eta m) \right|.
\end{aligned}$$

For the horizontal pieces  $h_{0,t}$ , an analogous estimate applies.

## 5. Zero density functions

We presume that  $|\mathcal{E}| = 0$ . As before, we let  $\alpha$  be any fixed constant satisfying  $\frac{1}{2} + \delta \leq \alpha \leq 1 - \delta$  with  $\delta$  being any arbitrarily small fixed positive constant. We now choose a partition of the interval  $[\alpha, 1]$  namely

$$(5.1) \quad \alpha = \alpha_0 < \alpha_1 < \cdots < \alpha_{j-1} < \alpha_j = 1,$$

with

$$(5.2) \quad \alpha_j - \alpha_{j-1} < \epsilon.$$

The number of  $j$ -values for which  $\beta_j^* \in [\alpha_{l-1}, \alpha_l]$  is bounded by  $N(\alpha_{l-1}, 2U, \chi^*)$ . From (4.10) and (4.11), we observe that the total contribution from all the vertical bits and the horizontal bits to the integral  $I^*$  (for  $U \leq t \leq 2U$ ) is

$$\begin{aligned}
C(I^*) &=: \sum_j (I^*(V_j) + I^*(h_j)) \\
&\ll HU^{-1}(qU)^\epsilon \left( \max_{u \leq y} \sup_{t \in [U, 2U]} \left| \sum_{m \leq u} m^{it} \mathbf{e}(\eta m) \right| \right) \left( \sum_j y^{\beta_j^*} \right) \\
&\ll HU^{-1}(qU)^\epsilon \left( \max_{u \leq y} \sup_{t \in [U, 2U]} \left| \sum_{m \leq u} m^{it} \mathbf{e}(\eta m) \right| \right) \left( Uy^{\alpha+\epsilon} + \int_\alpha^1 y^{\sigma+\epsilon} N(\sigma, 2U, \chi^*) d\sigma \right),
\end{aligned}$$

since  $\beta_j^* > \beta_j$ .

Thus we arrive at,

$$\begin{aligned}
S^* &=: \sum_{m=1}^{y-1} S(m, \chi^*, 0) \mathbf{e}(\eta m) \\
&\ll \sum_{\substack{L \leq j \leq l_0, \\ U=2^j}}^{y-1} \left\{ HU^{-1}(qU)^\epsilon \max_{u \leq y} \sup_{t \in [U, 2U]} \left| \sum_{m \leq u} m^{it} \mathbf{e}(\eta m) \right| \left( Uy^{\alpha+\epsilon} + \int_\alpha^1 y^{\sigma+\epsilon} N(\sigma, 2U, \chi^*) d\sigma \right) \right\} \\
&+ H_0(qT)^\epsilon \left\{ \max_{u \leq y} \sup_{t \in [0, 2^5]} \left| \sum_{m \leq u} m^{it} \mathbf{e}(\eta m) \right| + \max_{u \leq y} \sup_{t \in [2^5, H_0]} \left| \sum_{m \leq u} m^{it} \mathbf{e}(\eta m) \right| \right\} \times \\
&\times \left( 2^L y^{\alpha+\epsilon} + 2^L \int_\alpha^1 y^{\sigma+\epsilon} \frac{N(\sigma, 2^L, \chi^*)}{2^L} d\sigma \right) + \log y \\
&\ll H(qT)^{4\epsilon} \max_{u \leq y} \left( \sup_{t \in [0, 2]} \left| \sum_{m \leq u} m^{it} \mathbf{e}(\eta m) \right| + \sum_{2 \leq j \leq l_0} \sup_{t \in [2^{j-1}, 2^j]} \left| \sum_{m \leq u} m^{it} \mathbf{e}(\eta m) \right| \right) \times \\
&\times \left( y^{\alpha+\epsilon} + \int_\alpha^1 y^{\sigma+\epsilon} \frac{N(\sigma, T, \chi^*)}{T} d\sigma \right) + \log y \\
&\ll \max_{1 \leq T \leq y^3} T^{-1} (qT)^{5\epsilon} \max_{u \leq y} \left( \sup_{t \in [0, 2]} \left| \sum_{m \leq u} m^{it} \mathbf{e}(\eta m) \right| + \sum_{2 \leq j \leq l_0} \sup_{t \in [2^{j-1}, 2^j]} \left| \sum_{m \leq u} m^{it} \mathbf{e}(\eta m) \right| \right) \times \\
(5.3) \quad &\times \left( Ty^{\alpha+\epsilon} + \int_\alpha^1 y^{\sigma+\epsilon} N(\sigma, T, \chi^*) d\sigma \right) + \log y.
\end{aligned}$$

We treat  $S(y, \chi^*, 0) \mathbf{e}(\eta y)$  in a similar fashion and obtain

(5.4)

$$S(y, \chi^*, 0) \mathbf{e}(\eta y) \ll \max_{1 \leq T \leq y^3} T^{-1} (qT)^{5\epsilon} \left( T y^{\alpha+\epsilon} + \int_{\alpha}^1 y^{\sigma+\epsilon} N(\sigma, T, \chi^*) d\sigma \right) + \log y.$$

Let  $t^*$  be the value at which the quantity  $\left| \sum_{m \leq u} m^{it} \mathbf{e}(\eta m) \right|$  attains its supremum. We just estimate this sum trivially and get

$$(5.5) \quad \sup_{t \in [0, 2^{t_0}]} \left| \sum_{m \leq u} m^{it} \mathbf{e}(\eta m) \right| \leq u \leq |\eta|^{-1} (1 + |\eta| u)$$

*Remark.* Any non-trivial bound to the sum in the left hand side of (5.5) only in terms of  $u$  will surely sharpen Theorem 1.

## 6. Proof of the Theorems

**Proof of Theorem 1.** We take  $\chi^* = \chi \chi_d$  which are characters modulo  $q$  for various  $\chi$  modulo  $\frac{q}{d}$ . Here  $\chi_d$  denotes the principal character modulo  $d$ . We fix our  $\eta = \beta d$ . The proof follows on substituting the estimates obtained in (5.3), (5.4) and (5.5) in Lemma 2.1 appropriately. Thus using the estimates (5.3), (5.4) and (5.5), we first obtain

$$(6.1) \quad \begin{aligned} S(y, \chi^*, \eta) &\ll \max_{1 \leq T \leq y^3} (1 + y|\eta|) T^{-1} (qT)^{10\epsilon} \left( T y^{\alpha+\epsilon} + \int_{\alpha}^1 y^{\sigma+\epsilon} N(\sigma, T, \chi^*) d\sigma \right) \\ &\quad + \log y \\ &\ll \max_{1 \leq T \leq y^3} (1 + y|\eta|) T^{-1} (qT)^{10\epsilon} \left( T y^{\alpha+\epsilon} + \int_{\frac{1}{2}}^1 y^{\sigma+\epsilon} N(\sigma, T, \chi^*) d\sigma \right), \end{aligned}$$

provided  $|\mathcal{E}| = 0$ . Thus, the set  $\mathcal{G}$  contributes to  $\sum_{\chi \pmod{\frac{q}{d}}} S(y, \chi^*, \eta)$  precisely an amount which is in absolute value

$$\ll \sum_{\chi \pmod{\frac{q}{d}}} |S(y, \chi^*, \eta)|$$

and from (6.1) and Lemma 2.1 (with  $y = \frac{q}{d}$ ), we observe that this leads to the second sum term (appearing in the right hand side of (1.3)) over all the divisors  $d$  of  $q$  with maximum over  $T$  of certain integral involving the zero density functions provided the set  $\mathcal{E}$  is empty.

The case  $|\mathcal{E}| \geq 1$  is dealt as follows. From the density estimates (see for example (6.4) and (6.5)), we have (for  $\frac{1}{2} \leq \sigma \leq 1$ )

$$\sum_{\chi^*} N(\sigma, T, \chi^*) \ll (qT)^{\frac{12}{5}(1-\sigma)} (\log(qT))^A$$

with an absolute constant  $A$ . Therefore we have,

$$\begin{aligned} |\mathcal{E}| &\ll \sum_{\chi^*} N\left(1 - \frac{C^*}{(\log \log(qT))}, T, \chi^*\right) \\ &\ll \exp\left(C_1^* \left(\frac{\log(qT)}{\log \log(qT)}\right)\right) \\ (6.2) \quad &\ll (qT)^\epsilon. \end{aligned}$$

For each of these exceptional zero from the set  $\mathcal{E}$ , we estimate  $S(y, \chi^*, \eta)$  trivially and from Lemma 2.1 (with  $y = \frac{x}{d}$ ), we observe that the contribution coming from the set  $\mathcal{E}$  to  $S(x, \theta)$  is in absolute value

$$(6.3) \quad \ll q^{-\frac{1}{2}+\epsilon} (qT)^\epsilon x \ll q^{-\frac{1}{2}} x^{1+10\epsilon},$$

since,  $q \leq x^2$ ,  $T \leq x^3$ . This leads to the first term appearing in the right hand side of (1.3). This proves Theorem 1.

**Proof of Theorem 2.** The fundamental work on zero density estimates of Montgomery (see [8] and [9]) and the methods of [5] and Theorem 1 of [6] (for example see (2.23) and a few subsequent lines of [6]) enable one to deduce the following zero density theorem (which is a variation of Theorem 12.1 of [10]).

**AVERAGE ZERO-DENSITY THEOREM.** *Suppose that  $q \geq 1$  and  $T \geq 2$ . Let  $\chi^*$  denote characters modulo  $q$ . For  $\frac{1}{2} \leq \sigma \leq \frac{3}{4} + \epsilon$ , we have*

$$(6.4) \quad \sum_{\chi^*} N(\sigma, T, \chi^*) \ll (qT)^{\frac{3(1-\sigma)}{2-\sigma}} (\log(qT))^9 \left( \ll (qT)^{\frac{12}{5}(1-\sigma)} (\log(qT))^9 \right),$$

and for  $\frac{3}{4} + \epsilon \leq \sigma \leq 1$ , we have

$$(6.5) \quad \sum_{\chi^*} N(\sigma, T, \chi^*) \ll (qT)^{\frac{12}{5}(1-\sigma)} (\log(qT))^A.$$

Let  $f_1(\sigma) =: \sigma + \frac{3(1-\sigma)}{2(2-\sigma)}$ . Then,  $f_1^{(1)}(\sigma) = 1 - \frac{3}{2(2-\sigma)^2}$ . If  $\sigma_0 = 2 - \left(\frac{3}{2}\right)^{\frac{1}{2}}$ , then  $f_1^{(1)}(\sigma_0) = 0$  and  $f_1^{(2)}(\sigma_0) < 0$ . We notice that the interval  $[\frac{1}{2}, \frac{3}{4} + \epsilon]$  is contained in the interval  $[\frac{1}{2}, 2 - \left(\frac{3}{2}\right)^{\frac{1}{2}}]$ . We find that  $f_1^{(1)}(\frac{1}{2}) > 0$  and hence the function  $f_1(\sigma)$  is increasing in the interval  $[\frac{1}{2}, \frac{3}{4} + \epsilon]$ . Therefore

$$(6.6) \quad \max_{\frac{1}{2} \leq \sigma \leq \frac{3}{4} + \epsilon} f_1(\sigma) = f_1\left(\frac{3}{4} + \epsilon\right) \leq \frac{21}{20} + 5\epsilon.$$

If  $f_2(\sigma) =: \sigma + \frac{6}{5}(1 - \sigma)$ , then we find that (for  $\frac{3}{4} + \epsilon \leq \sigma \leq 1$ )

$$(6.7) \quad f_2(\sigma) \leq \frac{21}{20} + 5\epsilon.$$

We fix our  $\alpha = \frac{1}{2} + \delta$  in Theorem 1 where  $\delta$  is any arbitrarily small positive constant. Hence from Theorem 1, Lemmas 2.4 and 2.5, we have

$$(6.8) \quad \begin{aligned} S =: S(x, \theta) &\ll q^{-\frac{1}{2}} x^{1+10\epsilon} + \sum_{d|q} \left(\frac{q}{d}\right)^{\frac{1}{2}} \cdot \frac{1}{\phi\left(\frac{q}{d}\right)} \left(1 + x \left|\theta - \frac{r}{q}\right|\right) \times \\ &\times \left( \max_{1 \leq T \leq \left(\frac{q}{d}\right)^3} T^{-1} (qT)^{10\epsilon} \int_{\frac{1}{2}}^1 \left( T \phi\left(\frac{q}{d}\right) \left(\frac{x}{d}\right)^{\alpha+\epsilon} + \left(\frac{x}{d}\right)^{\sigma+\epsilon} \left( \sum_{\chi \pmod{\left(\frac{q}{d}\right)}} N(\sigma, T, \chi) \right) \right) d\sigma \right), \end{aligned}$$

with  $q \geq x^{\frac{1}{2(1+\epsilon)}}$  and  $\left|\theta - \frac{r}{q}\right| < x^{-(1-\epsilon)}$ . Now, we split the integral into two integrals, one in which the range of integration varies from  $\frac{1}{2}$  to  $\frac{3}{4} + \epsilon$  whereas the range in the other varies from  $\frac{3}{4} + \epsilon$  to 1.

Therefore from (6.4), (6.5) and (6.8), we get

$$(6.9) \quad S(x, \theta) \ll x^{25\epsilon} (Q_0 + Q_1 + Q_2 + Q_3),$$

where

$$(6.10) \quad Q_0 \ll x^{\frac{3}{4}} ; \quad Q_1 =: \max_{d \leq x^{\frac{1}{2}}} \frac{x^{\frac{3}{4} + \delta + \epsilon}}{d^{1+\delta}},$$

$$(6.11) \quad Q_2 =: \max_{d \leq x^{\frac{1}{2}}} \max_{1 \leq T \leq \left(\frac{q}{d}\right)^3} \max_{\frac{1}{2} \leq \sigma \leq \frac{3}{4} + \epsilon} T^{-1} \left(\frac{x^{\frac{1}{2}}}{d}\right)^{-\frac{1}{2}} \left(\frac{x}{d}\right)^{\sigma+\epsilon} \left(\frac{x^{\frac{1}{2}} T}{d}\right)^{\frac{3(1-\sigma)}{2-\sigma}}$$

and

$$(6.12) \quad Q_3 =: \max_{d \leq x^{\frac{1}{2}}} \max_{1 \leq T \leq \left(\frac{q}{d}\right)^3} \max_{\frac{3}{4} + \epsilon \leq \sigma \leq 1} T^{-1} \left(\frac{x^{\frac{1}{2}}}{d}\right)^{-\frac{1}{2}} \left(\frac{x}{d}\right)^{\sigma+\epsilon} \left(\frac{x^{\frac{1}{2}} T}{d}\right)^{\frac{12(1-\sigma)}{5}}.$$

We observe that the expressions on the right hand side of (6.10), (6.11) and (6.12) above (without maximums) are decreasing in the variables  $d$  and  $T$ . Therefore the maximum is attained for  $d = 1$  and  $T = 1$ . Hence, it is easy to see that



$$(6.13) \quad Q_0 \ll x^{\frac{3}{4}} ; \quad Q_1 \ll x^{\frac{3}{4}+10\delta+\epsilon},$$

$$(6.14) \quad Q_2 \ll \max_{\frac{1}{2} \leq \sigma \leq \frac{3}{4} + \epsilon} x^{\sigma + \epsilon + \frac{3(1-\sigma)}{2(2-\sigma)} - \frac{1}{4}} \ll x^{\frac{4}{5} + 5\epsilon}$$

and

$$(6.15) \quad Q_3 \ll \max_{\frac{3}{4} + \epsilon \leq \sigma \leq 1} x^{\sigma + \epsilon + \frac{6(1-\sigma)}{5} - \frac{1}{4}} \ll x^{\frac{4}{5} + 5\epsilon}.$$

We note that

$$c^* = \frac{4}{5}.$$

Thus, from (6.13), (6.14), (6.15) and (6.9), we obtain

$$(6.16) \quad S(x, \theta) \ll x^{c^* + 50\epsilon + 10\delta}.$$

Since  $\epsilon$  and  $\delta$  are arbitrarily small positive constants, the proof of Theorem 2 is complete.

**Proof of Theorem 3.** As in the proof of Theorem 2, we now choose our  $\alpha$  here to be  $\alpha = \frac{1}{2} + \delta$  in Theorem 1 so that (for  $\theta$  of type 1) we have

$$\begin{aligned} S &=: S(x, \theta) \\ &\ll q^{-\frac{1}{2}} x^{1+10\epsilon} + \sum_{d|q} \left(\frac{q}{d}\right)^{\frac{1}{2}} \cdot \frac{1}{\phi\left(\frac{q}{d}\right)} \left(1 + x \left|\theta - \frac{r}{q}\right|\right) \times \\ &\times \left( \max_{1 \leq T \leq \left(\frac{x}{d}\right)^3} T^{-1} (qT)^{10\epsilon} \left\{ \int_{\frac{1}{2}}^1 \left( T \phi\left(\frac{q}{d}\right) \left(\frac{x}{d}\right)^{\alpha+\epsilon} + \left(\frac{x}{d}\right)^{\sigma+\epsilon} \left( \sum_{\chi \pmod{\left(\frac{q}{d}\right)}} N(\sigma, T, \chi) \right) \right) d\sigma \right\} \right) \\ &\ll q^{-\frac{1}{2}} x^{1+10\epsilon} + x^{25\epsilon} \max_{d \leq x^{\frac{1}{2}}} \left(\frac{x^{\frac{1}{2}}}{d}\right)^{\frac{1}{2}} \left(\frac{x^{\frac{1}{2}}}{d}\right)^{-1} (1 + x^\epsilon) \times \\ &\times \left( \left( \max_{1 \leq T \leq \left(\frac{x}{d}\right)^3} \left( \left(\frac{x^{\frac{1}{2}}}{d}\right) \left(\frac{x}{d}\right)^{\frac{1}{2}+\delta+\epsilon} + T^{-1} \max_{\frac{1}{2} \leq \sigma \leq 1} \left(\frac{x}{d}\right)^{\sigma+\epsilon} \left(\frac{x^{\frac{1}{2}}}{d} T\right)^{2(1-\sigma)+\epsilon} \right) \right) \right) \\ &\ll x^{\frac{3}{4}+20\epsilon} \\ (6.17) \quad &+ x^{-\frac{1}{4}+30\epsilon} \left\{ \max_{d \leq x^{\frac{1}{2}}} \left(\frac{x^{1+10\delta+\epsilon}}{d}\right) + \max_{d \leq x^{\frac{1}{2}}} \max_{1 \leq T \leq \left(\frac{x}{d}\right)^3} \max_{\frac{1}{2} \leq \sigma \leq 1} d^{\frac{1}{2}} T^{-1} \left(\frac{x}{d}\right)^{\sigma+\epsilon} \left(\frac{x^{\frac{1}{2}}}{d} T\right)^{2(1-\sigma)+\epsilon} \right\}. \end{aligned}$$

Observing the monotonicity in  $d$  and  $T$ , we see that the maximum is attained for  $d = 1$  and  $T = 1$ . Thus we obtain

$$S(x, \theta) \ll x^{\frac{3}{4} + 50\epsilon + 10\delta}.$$

Since  $\epsilon$  and  $\delta$  are arbitrarily small positive constants, this completes the proof of Theorem 3.

**Proof of Theorem 4.** From the known density estimates (see Theorem 12.1 of [10], the methods of [5] and Theorem 1 of [6]), we can very well use the bound

$$(6.18) \quad \sum_{\chi \pmod{\frac{g}{2}}} N(\sigma, T, \chi) \ll \begin{cases} \left(\frac{qT}{d}\right)^{\frac{3(1-\sigma)}{2-\sigma}} \left(\log\left(\frac{qT}{d}\right)\right)^A & \text{if } \frac{1}{2} \leq \sigma \leq 2 - \frac{3}{g}, \\ \left(\frac{qT}{d}\right)^{g(1-\sigma)} \left(\log\left(\frac{qT}{d}\right)\right)^A & \text{if } 2 - \frac{3}{g} \leq \sigma \leq 1, \end{cases}$$

for  $2 \leq g \leq \frac{12}{5}$  with a suitable absolute constant  $A$ . We observe that we can split the integral in (6.8) into two integrals, one in which the range of integration runs from  $\frac{1}{2}$  to  $2 - \frac{3}{g}$  and for the other, the range of integration runs from  $2 - \frac{3}{g}$  to 1. We use the first estimate of (6.18) when  $\sigma$  varies in the interval  $\left[\frac{1}{2}, 2 - \frac{3}{g}\right]$  whereas we use the second estimate of (6.18) when  $\sigma$  varies in the interval  $\left[2 - \frac{3}{g}, 1\right]$ . We note that  $f_1(\sigma) =: \sigma + \frac{3(1-\sigma)}{2(2-\sigma)}$  is an increasing function throughout in the interval  $\left[\frac{1}{2}, 2 - \frac{3}{g}\right]$ . Therefore, we have

$$(6.19) \quad -\frac{1}{4} + \left(\max_{\frac{1}{2} \leq \sigma \leq 2 - \frac{3}{g}} f_1(\sigma)\right) \leq \frac{13}{4} - \frac{3}{g} - \frac{g}{2} + \epsilon$$

and

$$(6.20) \quad -\frac{1}{4} + \left(\max_{2 - \frac{3}{g} \leq \sigma \leq 1} \sigma + \frac{g}{2}(1-\sigma)\right) \leq \frac{13}{4} - \frac{3}{g} - \frac{g}{2} + \epsilon.$$

Thus, we obtain (for  $\theta$  of type 1)

$$(6.21) \quad S(x, \theta) \ll x^{c^{**} + \epsilon}$$

where

$$(6.22) \quad c^{**} = \max\left(\frac{3}{4}, \frac{13}{4} - \frac{3}{g} - \frac{g}{2}\right) = \frac{13}{4} - \frac{3}{g} - \frac{g}{2} = \nu,$$

since  $\frac{12}{5} \geq g \geq 2$ . We observe that whenever  $g < \frac{12}{5}$  we obtain a better exponent  $c^{**}$  which is strictly less than  $\frac{4}{5}$  provided  $\theta$  is of type 1.

### 7. Concluding Remarks

If  $\theta$  is of type  $\eta_1$  ( $\geq 1$ ), then we have

$$\|q\theta\| \geq q^{-(\eta_1+\epsilon)} \text{ for } q \geq q_0(\epsilon)$$

and hence we do have (as in Lemma 2.4)

$$q \geq x^{\frac{1}{2\eta_1}-\epsilon} \text{ and so } \left| \theta - \frac{r}{q} \right| < \frac{1}{qx^{\frac{1}{2}}} < x^{-\left(\frac{1}{2}+\frac{1}{2\eta_1}-\epsilon\right)}.$$

Therefore, the proof of Theorem 2 essentially leads to :

CONCLUSION 1. *Let  $\eta_1$  be a real number satisfying  $1 \leq \eta_1 < \frac{1}{2c^*-1}$ . Let  $\theta$  be of type  $\eta_1$ . Then for every arbitrarily small  $\epsilon > 0$ , we have*

$$S(x, \theta) \ll x^{c^*+\frac{1}{2}-\frac{1}{2\eta_1}+\epsilon}$$

where  $c^*$  is the same positive constant as in Theorem 2

and the proof of Theorem 3 leads to

CONCLUSION 2. *Let  $\eta_1$  be a real number satisfying  $1 \leq \eta_1 < 2$ . Let  $\theta$  be of type  $\eta_1$ . If we assume the Average Zero- Density Hypothesis, then for every arbitrarily small  $\epsilon > 0$ , we have*

$$S(x, \theta) \ll x^{\frac{5}{4}-\frac{1}{2\eta_1}+\epsilon}.$$

From a theorem of Walfisz (see [15]), we have unconditionally

$$(7.1) \quad M(x) =: S(x, 0) \ll x \exp\left(-C(\log x)^{\frac{3}{5}}(\log \log x)^{-\frac{1}{5}}\right)$$

and this is the best unconditional result available till today (when  $\theta = 0$ ). It is a well-known fact that *The Riemann Hypothesis implies that*

$$(7.2) \quad M(x) =: S(x, 0) \ll x^{\frac{1}{2}+\epsilon}$$

for any small positive constant  $\epsilon$ .

However, in view of Theorem A (as mentioned earlier in remark 1 of Theorem 2), *Even on the assumption of the Generalised Riemann Hypothesis*, all that we can conclude is that (for all real  $\theta$ )

$$(7.3) \quad S(x, \theta) \ll x^{\frac{3}{4}+\epsilon}.$$

It should be mentioned here that the Theorem 1 is powerful and yields definitely a very nice non-trivial upper bound for  $|S(x, \theta)|$  unconditionally whenever we know the type of  $\theta$  as mentioned in the conclusion 1 (i.e for a certain type of irrational numbers  $\theta$ ) and conditionally as mentioned in the conclusion 2. However, we must also note that the Theorem 1 may yield only the trivial upper bound for  $|S(x, \theta)|$  for example

when  $\theta = 0$  (or  $\theta$  being any rational integer for that matter) and it looks to be a disadvantage of Theorem 1.

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#### REFERENCES

- [1] R.C. Baker and G. Harman, *Exponential sums formed with the Möbius function*, J. London Math. Soc., (2) **43** (1991), 193-198.
- [2] H. Davenport, *On some infinite series involving arithmetical functions-II*, Quart. J. Math., **8** (1937), 313-320.
- [3] H. Davenport, *Multiplicative number theory*, 2nd edition, revised by H.L. Montgomery (Springer, New York, 1980).
- [4] D. Hajela and B. Smith, *On the maximum of an exponential sum of the Möbius function*, Lecture Notes in Mathematics, **1240** (Springer, Berlin, 1987), 145-164.
- [5] M.N. Huxley, *On the difference between consecutive primes*, Invent. Math., **15** (1972), 164-170.
- [6] M.N. Huxley, *Large values of Dirichlet polynomials*, III, Acta Arith., **26** (1975), 435-444.
- [7] L. Kuipers and H. Niederreiter, *Uniform distributions of sequences*, Wiley, (1974).
- [8] H.L. Montgomery, *Mean and large values of Dirichlet polynomials*, Invent. Math., **8** (1969), 334-345.
- [9] H.L. Montgomery, *Zeros of L- functions*, Invent. Math., **8** (1969), 346-354.
- [10] H.L. Montgomery, *Topics in multiplicative number theory*, Lecture Notes in Mathematics, **227** (Springer, Berlin, 1971).
- [11] H.L. Montgomery and R.C. Vaughan, *Exponential sums with multiplicative coefficients*, Invent. Math., **43** (1977), 69-82.
- [12] K. Ramachandra and A. Sankaranarayanan, *On some theorems of Littlewood and Selberg-III*, Ann. Acad. Sci. Fenn. Ser. A I Math., **16** (1991), 139-149.
- [13] M. Ram Murty and A. Sankaranarayanan, *Averages of exponential twists of the Liouville function*, Forum Math., **14** (2002), no: 2, 273-291.
- [14] E.C. Titchmarsh, *The theory of the Riemann zeta-function*, 2nd Edition ( edited by D.R. Heath-Brown ), Clarendon Press ( Oxford ) (1986).
- [15] A. Walfisz, *Weylsche Exponentialsummen in der neueren Zahlentheorie*, Mathematische Forschungsberichte **15**, Berlin (1963).

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