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## Further Variations on the Six Exponentials Theorem

Michel Waldschmidt

**Abstract.** Let  $\tilde{\mathcal{L}}$  denote the set of linear combinations, with algebraic coefficients, of 1 and logarithms of algebraic numbers. The Strong Six Exponentials Theorem of D. Roy gives sufficient conditions for a  $2 \times 3$  matrix

$$M = \begin{pmatrix} \Lambda_{11} & \Lambda_{12} & \Lambda_{13} \\ \Lambda_{21} & \Lambda_{22} & \Lambda_{23} \end{pmatrix}$$

whose entries are in  $\tilde{\mathcal{L}}$  to have rank 2.

Here we give sufficient conditions so that one at least of the three  $2 \times 2$  determinants

$$\begin{vmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{21} & \Lambda_{22} \end{vmatrix}, \quad \begin{vmatrix} \Lambda_{12} & \Lambda_{13} \\ \Lambda_{22} & \Lambda_{23} \end{vmatrix}, \quad \begin{vmatrix} \Lambda_{13} & \Lambda_{11} \\ \Lambda_{23} & \Lambda_{21} \end{vmatrix}$$

is not in  $\tilde{\mathcal{L}}$

### 1. Main result

We denote by  $\mathbb{Q}$  the field of rational numbers, by  $\overline{\mathbb{Q}}$  the field of algebraic numbers (algebraic closure of  $\mathbb{Q}$  in  $\mathbb{C}$ ), by  $\mathcal{L}$  the  $\mathbb{Q}$ -vector space of logarithms of algebraic numbers:

$$\mathcal{L} = \{\lambda \in \mathbb{C}; e^\lambda \in \overline{\mathbb{Q}}^\times\} = \{\log \alpha; \alpha \in \overline{\mathbb{Q}}^\times\} = \exp^{-1}(\overline{\mathbb{Q}}^\times)$$

and by  $\tilde{\mathcal{L}}$  the  $\overline{\mathbb{Q}}$ -vector subspace of  $\mathbb{C}$  spanned by  $\{1\} \cup \mathcal{L}$ . Hence  $\tilde{\mathcal{L}}$  is the set of linear combinations of 1 and logarithms of algebraic numbers with algebraic coefficients:

$$\tilde{\mathcal{L}} = \left\{ \beta_0 + \beta_1 \log \alpha_1 + \cdots + \beta_n \log \alpha_n; \right. \\ \left. n \geq 0, (\alpha_1, \dots, \alpha_n) \in (\overline{\mathbb{Q}}^\times)^n, (\beta_0, \beta_1, \dots, \beta_n) \in \overline{\mathbb{Q}}^{n+1} \right\}.$$

Here is the so-called strong six exponentials Theorem of D. Roy ([5] Corollary 2 §4 p. 38; see also [7] Corollary 11.16):

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*Key words and phrases.* Transcendental numbers, logarithms of algebraic numbers, four exponentials Conjecture, six exponentials Theorem, algebraic independence.

*Acknowledgements:* A suggestion by D. Roy in Banff in november 2004 turned out to be a key point in the proof of the main result. Thanks also to him and to Guy Diaz for their comments on previous versions of this text.

THEOREM 1.1. *Let  $M$  be a  $2 \times 3$  matrix with entries in  $\tilde{\mathcal{L}}$ :*

$$M = \begin{pmatrix} \Lambda_{11} & \Lambda_{12} & \Lambda_{13} \\ \Lambda_{21} & \Lambda_{22} & \Lambda_{23} \end{pmatrix}.$$

*Assume that the two rows of  $M$  are linearly independent over  $\overline{\mathbb{Q}}$  and also that the three columns are linearly independent over  $\overline{\mathbb{Q}}$ . Then  $M$  has rank 2.*

Consider the three  $2 \times 2$  determinants

$$\Delta_1 = \Lambda_{12}\Lambda_{23} - \Lambda_{13}\Lambda_{22}, \quad \Delta_2 = \Lambda_{13}\Lambda_{21} - \Lambda_{11}\Lambda_{23}, \quad \Delta_3 = \Lambda_{11}\Lambda_{22} - \Lambda_{12}\Lambda_{21}.$$

From the relation

$$\Delta_1 \begin{pmatrix} \Lambda_{11} \\ \Lambda_{21} \end{pmatrix} + \Delta_2 \begin{pmatrix} \Lambda_{12} \\ \Lambda_{22} \end{pmatrix} + \Delta_3 \begin{pmatrix} \Lambda_{13} \\ \Lambda_{23} \end{pmatrix} = 0,$$

it follows from the assumptions of Theorem 1.1 that one at least of the three numbers  $\Delta_1, \Delta_2, \Delta_3$  is transcendental. We want to prove that one at least of these three numbers is not in  $\tilde{\mathcal{L}}$ .

If the five rows of the matrix  $\begin{pmatrix} M \\ I_3 \end{pmatrix}$  (where  $I_3$  is the  $3 \times 3$  identity matrix) are linearly dependent over  $\overline{\mathbb{Q}}$ , which means that there exists  $(\gamma_1, \gamma_2) \in \overline{\mathbb{Q}}^2 \setminus \{0\}$  such that the three numbers

$$\delta_j = \gamma_1 \Lambda_{1j} + \gamma_2 \Lambda_{2j} \quad (j = 1, 2, 3)$$

are algebraic, then the three numbers  $\Delta_1, \Delta_2, \Delta_3$  are in  $\tilde{\mathcal{L}}$ . Indeed, if  $(j, h, k)$  denotes any of the triples  $(1, 2, 3), (2, 3, 1), (3, 1, 2)$ , then

$$\gamma_1 \Delta_j = \delta_h \Lambda_{2k} - \delta_k \Lambda_{2h} \quad \text{and} \quad \gamma_2 \Delta_j = \delta_k \Lambda_{1h} - \delta_h \Lambda_{1k}.$$

Here is the main result of this paper.

THEOREM 1.2. *Let  $M$  be a  $2 \times 3$  matrix with entries in  $\tilde{\mathcal{L}}$ :*

$$M = \begin{pmatrix} \Lambda_{11} & \Lambda_{12} & \Lambda_{13} \\ \Lambda_{21} & \Lambda_{22} & \Lambda_{23} \end{pmatrix}.$$

*Assume that the five rows of the matrix*

$$\begin{pmatrix} M \\ I_3 \end{pmatrix} = \begin{pmatrix} \Lambda_{11} & \Lambda_{12} & \Lambda_{13} \\ \Lambda_{21} & \Lambda_{22} & \Lambda_{23} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

are linearly independent over  $\overline{\mathbb{Q}}$  and that the five columns of the matrix

$$(I_2, M) = \begin{pmatrix} 1 & 0 & \Lambda_{11} & \Lambda_{12} & \Lambda_{13} \\ 0 & 1 & \Lambda_{21} & \Lambda_{22} & \Lambda_{23} \end{pmatrix}$$

are linearly independent over  $\overline{\mathbb{Q}}$ . Then one at least of the three numbers

$$\Delta_1 = \begin{vmatrix} \Lambda_{12} & \Lambda_{13} \\ \Lambda_{22} & \Lambda_{23} \end{vmatrix}, \quad \Delta_2 = \begin{vmatrix} \Lambda_{13} & \Lambda_{11} \\ \Lambda_{23} & \Lambda_{21} \end{vmatrix}, \quad \Delta_3 = \begin{vmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{21} & \Lambda_{22} \end{vmatrix}$$

is not in  $\tilde{\mathcal{L}}$ .

If  $M$  is  $d \times \ell$  matrix of rank 1, with  $d \geq 2$  and  $\ell \geq 2$ , whose columns are  $\overline{\mathbb{Q}}$ -linearly independent, then the  $d + \ell$  columns of the matrix  $(I_d \ M)$  are also  $\overline{\mathbb{Q}}$ -linearly independent. Hence on the one hand Theorem 1.2 generalizes Theorem 1.1. On the other hand, as noticed by G. Diaz, when one of the six numbers  $\Lambda_{ij}$  is algebraic, Theorem 1.2 reduces to the next consequence of Theorem 1.1 (further related results are given in [1] and [8]).

**COROLLARY 1.3.** *Let  $\Lambda_1, \Lambda_2, \Lambda_3$  be three elements of  $\tilde{\mathcal{L}}$ . Assume that  $\Lambda_1$  is transcendental and that the three numbers  $1, \Lambda_2, \Lambda_3$  are  $\overline{\mathbb{Q}}$ -linearly independent. Then one at least of the two numbers  $\Lambda_1\Lambda_2, \Lambda_1\Lambda_3$  is not in  $\tilde{\mathcal{L}}$ .*

The simple example

$$M = \begin{pmatrix} 0 & \Lambda_2 & \Lambda_3 \\ \Lambda_1 & 0 & 0 \end{pmatrix}.$$

shows that the assumptions of Theorem 1.2 are not sufficient to ensure that none of the three determinants is in  $\tilde{\mathcal{L}}$ .

Here is a simple result which follows from Theorem 1.2: *Let  $\Lambda_1, \Lambda_2, \Lambda_3$  be three elements in  $\tilde{\mathcal{L}}$  such that  $1, \Lambda_1, \Lambda_2, \Lambda_3$  are linearly independent over  $\overline{\mathbb{Q}}$ . Then one at least of the three numbers*

$$\Lambda_1^2 - \Lambda_2\Lambda_3, \quad \Lambda_2^2 - \Lambda_3\Lambda_1, \quad \Lambda_3^2 - \Lambda_1\Lambda_2$$

is not in  $\tilde{\mathcal{L}}$ .

In §3 we shall deduce from Theorem 1.2 the following corollary.

**COROLLARY 1.4.** *Let  $M$  be a  $2 \times 3$  matrix with entries in  $\mathcal{L}$ :*

$$M = \begin{pmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} \\ \lambda_{21} & \lambda_{22} & \lambda_{23} \end{pmatrix}.$$

*Assume that the two rows of  $M$  are linearly independent over  $\mathbb{Q}$  and also that the three columns of  $M$  are linearly independent over  $\mathbb{Q}$ . Then one at*

least of the three numbers

$$(1.5) \quad \lambda_{11}\lambda_{22} - \lambda_{12}\lambda_{21}, \quad \lambda_{12}\lambda_{23} - \lambda_{13}\lambda_{22}, \quad \lambda_{13}\lambda_{21} - \lambda_{11}\lambda_{23}$$

is not in  $\tilde{\mathcal{L}}$ .

The six exponentials Theorem of S. Lang ([3], Chap. II § 1) and K. Ramachandra ([4] II § 4) states that, under the assumptions of Corollary 1.4, one at least of the three numbers (1.5) is not zero.

It is expected that a result similar to Theorem 1.2 holds when  $M$  is replaced by a  $2 \times 2$  matrix:

CONJECTURE 1.6. *Let  $M$  be a  $2 \times 2$  matrix with entries in  $\tilde{\mathcal{L}}$ :*

$$M = \begin{pmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{21} & \Lambda_{22} \end{pmatrix}.$$

*Assume that the four rows of the matrix*

$$\begin{pmatrix} M \\ I_2 \end{pmatrix} = \begin{pmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{21} & \Lambda_{22} \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

*are linearly independent over  $\overline{\mathbb{Q}}$  and that the four columns of the matrix*

$$(I_2, M) = \begin{pmatrix} 1 & 0 & \Lambda_{11} & \Lambda_{12} \\ 0 & 1 & \Lambda_{21} & \Lambda_{22} \end{pmatrix}$$

*are linearly independent over  $\overline{\mathbb{Q}}$ . Then the number*

$$\Delta = \begin{vmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{21} & \Lambda_{22} \end{vmatrix}$$

*is not in  $\tilde{\mathcal{L}}$ .*

Conjecture 1.6 follows from the conjecture (see for instance [3], Historical Note of Chapter III, [2], Chap. 6 p. 259 and [7], Conjecture 1.15 and [8] Conjecture 1.1) that  $\mathbb{Q}$ -linearly independent logarithms of algebraic numbers are algebraically independent.

## 2. A consequence of the Linear Subgroup Theorem

Let  $n$  be a positive integer and  $Y$  a  $\overline{\mathbb{Q}}$ -vector subspace of  $\mathbb{C}^n$ . We define

$$\mu(Y, \mathbb{C}^n) = \min_{V \subset \mathbb{C}^n} \frac{\dim_{\overline{\mathbb{Q}}}(Y/Y \cap V)}{\dim_{\mathbb{C}}(\mathbb{C}^n/V)},$$

where  $V$  runs over the set of  $\mathbb{C}$ -vector subspaces of  $\mathbb{C}^n$  with  $V \neq \mathbb{C}^n$ .

For  $x = (x_1, \dots, x_n) \in \mathbb{C}^n$  and  $y = (y_1, \dots, y_n) \in \mathbb{C}^n$  we denote by  $x \cdot y$  the scalar product

$$x \cdot y = x_1 y_1 + \dots + x_n y_n.$$

For  $X$  and  $Y$  two subsets of  $\mathbb{C}^n$ , we denote by  $X \cdot Y$  the set of scalar products  $x \cdot y$  where  $x$  ranges over the set  $X$  and  $y$  over  $Y$ .

**THEOREM 2.1.** *Let  $X$  and  $Y$  be two  $\overline{\mathbb{Q}}$ -vector subspaces of  $\mathbb{C}^n$ . Assume  $X$  has dimension  $d$  with  $d > n$ . Assume further*

$$\mu(Y, \mathbb{C}^n) > \frac{d}{d-n}.$$

*Then the set  $X \cdot Y$  is not contained in  $\tilde{\mathcal{L}}$ .*

**PROOF.** This is essentially Proposition 6.1 of [6], where  $\mathbb{Q}$  is replaced by  $\overline{\mathbb{Q}}$  and the  $\mathbb{Q}$ -vector space  $\mathcal{L}$  by the  $\overline{\mathbb{Q}}$ -vector space  $\tilde{\mathcal{L}}$ . Henceforth the proof runs as follows.

Like in Lemme 5.2 of [6], one checks that if  $X$  and  $Y$  are two vector subspaces of  $\mathbb{C}^n$  over  $\overline{\mathbb{Q}}$ , of dimensions  $d$  and  $\ell$  respectively, then there exist a positive integer  $n' \leq n$  and two vector subspaces  $X'$  and  $Y'$  of  $\mathbb{C}^{n'}$ , of dimensions  $d'$  and  $\ell'$  respectively, such that

$$\mu(X', \mathbb{C}^{n'}) = \frac{d'}{n'} \geq \frac{d}{n}, \quad \mu(Y', \mathbb{C}^{n'}) = \frac{\ell'}{n'} \geq \mu(Y, \mathbb{C}^n)$$

and

$$(2.2) \quad X' \cdot Y' \subset X \cdot Y.$$

This shows that for the proof of Theorem 2.1, there is no loss of generality to assume  $\mu(X, \mathbb{C}^n) = d/n$  and  $\mu(Y, \mathbb{C}^n) = \ell/n$ . The assumption  $\mu(Y, \mathbb{C}^n) > d/(d-n)$  reduces to  $ld > n(\ell + d)$ .

Following the argument of Lemme 5.4 in [6], one proves that if  $X$  and  $Y$  are two vector subspaces of  $\mathbb{C}^n$  over  $\overline{\mathbb{Q}}$ , of dimensions  $d$  and  $\ell$  respectively,  $X_1$  a subspace of  $X$  of dimension  $d_1$  and  $Y_1$  a subspace of  $Y$  of dimension  $\ell_1$  such that  $X_1 \cdot Y_1 = \{0\}$ , then

$$(2.3) \quad (d - d_1)\mu(Y, \mathbb{C}^n) + (\ell - \ell_1)\mu(X, \mathbb{C}^n) \geq n\mu(X, \mathbb{C}^n)\mu(Y, \mathbb{C}^n).$$

In Lemme 5.4 in [6] an extra assumption is required, namely

$$\mu(X, \mathbb{C}^n)\mu(Y, \mathbb{C}^n) \geq \mu(X, \mathbb{C}^n) + \mu(Y, \mathbb{C}^n),$$

but we do not need it here, since our assumption  $X_1 \cdot Y_1 = \{0\}$  is stronger than the assumption in Lemme 5.4 of [6] that  $X_1 \cdot Y_1$  has rank  $\leq 1$ .

Next we introduce the coefficient  $\theta(M)$  attached to a  $d \times \ell$  matrix  $M$  with entries in  $\mathbb{C}$ . It is defined as follows:

$$\theta(M) = \min \frac{\ell'}{d'},$$

where  $(d', \ell')$  ranges over the set of pairs of integers satisfying  $0 \leq \ell' \leq \ell$ ,  $1 \leq d' \leq d$ , such that there exist a  $d \times d$  regular matrix  $P$  and a regular  $\ell \times \ell$  regular matrix  $Q$ , both with entries in  $\overline{\mathbb{Q}}$ , with

$$PMQ = \begin{pmatrix} A & B \\ C & 0 \end{pmatrix} \begin{matrix} \} d^* \\ \} d' \end{matrix}$$

$\underbrace{\hspace{1.5cm}}_{\ell'} \quad \underbrace{\hspace{1.5cm}}_{\ell^*}$

From (2.3) with  $d_1 = d'$  and  $\ell_1 = \ell^*$  it follows that if

$$X = \overline{\mathbb{Q}}x_1 + \cdots + \overline{\mathbb{Q}}x_d \quad \text{and} \quad Y = \overline{\mathbb{Q}}y_1 + \cdots + \overline{\mathbb{Q}}y_\ell$$

are again two vector subspaces of  $\mathbb{C}^n$  over  $\overline{\mathbb{Q}}$ , of dimensions  $d$  and  $\ell$  respectively, satisfying  $\mu(X, \mathbb{C}^n) = d/n$ , then the matrix

$$(2.4) \quad M = (x_i \cdot y_j)_{1 \leq i \leq d, 1 \leq j \leq \ell}$$

has

$$\theta(M) \geq \frac{n}{d} \cdot \mu(Y, \mathbb{C}^n).$$

In particular if  $\mu(X, \mathbb{C}^n) = d/n$  and  $\mu(Y, \mathbb{C}^n) = \ell/n$ , then  $\theta(M) = \ell/d$ . Finally Theorem 4 in [5] (which is Proposition 11.19 or Theorem 12.19 in [7]) shows that the rank  $r$  of a  $d \times \ell$  matrix  $M$  with entries in  $\tilde{\mathcal{L}}$  satisfies

$$r \geq \frac{d\theta}{1 + \theta},$$

where  $\theta = \theta(M)$ . Using this result for the matrix  $M$  given by (2.4) whose rank  $r$  is  $\leq n$ , one concludes that if  $\mu(X, \mathbb{C}^n) = d/n$  and  $\mu(Y, \mathbb{C}^n) = \ell/n$  with  $X \cdot Y \subset \tilde{\mathcal{L}}$ , then

$$n \geq \frac{\ell d}{\ell + d}.$$

Theorem 2.1 follows. □

REMARK. Theorem 1.1 is equivalent with the case  $n = 1$  of Theorem 2.1.

### 3. Proof of the main results

In this section we prove Theorem 1.2 and Corollary 1.4.

PROOF OF THEOREM 1.2. Assume that the hypotheses of Theorem 1.2 are satisfied. Define elements  $v_1, \dots, v_5$  in  $\tilde{\mathcal{L}}^2$  by

$$v_1 = e_1, \quad v_2 = e_2, \quad v_{2+j} = (\Lambda_{1j}, \Lambda_{2j}), \quad (j = 1, 2, 3),$$

where  $e_1 = (1, 0)$  and  $e_2 = (0, 1)$ . For  $v = (x, y) \in \mathbb{C}^2$ , set  $v' = (-y, x)$ , so that  $v' \cdot v = 0$ . Consider the  $5 \times 5$  matrix

$$A = (v'_i \cdot v_j)_{1 \leq i, j \leq 5}.$$

From its very definition, it is plain that  $A$  has rank 2. Explicitly one has

$$A = \begin{pmatrix} 0 & 1 & \Lambda_{21} & \Lambda_{22} & \Lambda_{23} \\ -1 & 0 & -\Lambda_{11} & -\Lambda_{12} & -\Lambda_{13} \\ -\Lambda_{21} & \Lambda_{11} & 0 & \Delta_3 & -\Delta_2 \\ -\Lambda_{22} & \Lambda_{12} & -\Delta_3 & 0 & \Delta_1 \\ -\Lambda_{23} & \Lambda_{13} & \Delta_2 & -\Delta_1 & 0 \end{pmatrix}.$$

Let  $X$  be the  $\overline{\mathbb{Q}}$ -vector space spanned by  $v_1, \dots, v_5$  in  $\mathbb{C}^2$  and similarly let  $Y$  be the subspace of  $\mathbb{C}^2$  spanned by  $v'_1, \dots, v'_5$  over  $\overline{\mathbb{Q}}$ . We claim

$$(3.1) \quad \mu(X, \mathbb{C}^2) = \mu(Y, \mathbb{C}^2) \geq 2.$$

The equality  $\mu(X, \mathbb{C}^2) = \mu(Y, \mathbb{C}^2)$  follows from the fact that the map  $(x, y) \mapsto (-y, x)$  is an automorphism of  $\mathbb{C}^2$ .

Since the five columns of  $(I_2 \ M)$  are linearly independent over  $\overline{\mathbb{Q}}$ ,  $\dim_{\overline{\mathbb{Q}}} X = 5$ .

Let  $V$  is a vector subspace of  $\mathbb{C}^2$  of dimension 1 and let  $t_1 z_1 + t_2 z_2 = 0$  be an equation of  $V$  in  $\mathbb{C}^2$ , with  $(t_1, t_2) \in \mathbb{C}^2 \setminus \{0\}$ . Consider the linear map

$$p : \begin{array}{ccc} \mathbb{C}^2 & \rightarrow & \mathbb{C} \\ (z_1, z_2) & \mapsto & t_1 z_1 + t_2 z_2 \end{array}$$

whose kernel is  $V$ . Since the five rows of  $\begin{pmatrix} M \\ I_3 \end{pmatrix}$  are  $\overline{\mathbb{Q}}$ -linearly independent,

$$\dim_{\overline{\mathbb{Q}}}((X \cap V)/V) = \dim_{\overline{\mathbb{Q}}} p(X) \geq 2.$$

This completes the proof of (3.1).

From (3.1) we deduce that the hypothesis  $\mu(Y, \mathbb{C}^2) > d/(d-n)$  of Theorem 2.1 is satisfied with  $d = 5$  and  $n = 2$ , hence the set  $X \cdot Y$  is not contained in  $\tilde{\mathcal{L}}$ . Consequently one at least of the three numbers  $\Delta_1, \Delta_2, \Delta_3$  is not in  $\tilde{\mathcal{L}}$ .



This completes the proof of the Main Theorem 1.2.  $\square$

REMARK. In (3.1) we may have equality: for instance if  $\Lambda_{22} = \Lambda_{23} = 0$  then  $\mu(X, \mathbb{C}^2) = \mu(Y, \mathbb{C}^2) = 2$ .

However the proof of Theorem 2.1 shows that in the case  $\mu(X, \mathbb{C}^2) = \mu(Y, \mathbb{C}^2) < 5/2$ , Theorem 1.2 should follow from Theorem 1.1. Indeed after a change of variables rational over  $\overline{\mathbb{Q}}$  one needs only to consider a matrix

$$M = \begin{pmatrix} 0 & \Lambda_{12} & \Lambda_{13} \\ \Lambda_{21} & 0 & 0 \end{pmatrix},$$

which is the situation of Corollary 1.3. If  $X$  is the  $\overline{\mathbb{Q}}$ -subspace of  $\mathbb{C}^2$  spanned by

$$v_1 = (1, 0), \quad v_2 = (0, 1), \quad v_3 = (0, \Lambda_{21}), \quad v_4 = (\Lambda_{12}, 0), \quad v_5 = (\Lambda_{13}, 0)$$

and  $Y$  the subspace spanned by

$$v'_1 = (0, 1), \quad v'_2 = (-1, 0), \quad v'_3 = (-\Lambda_{21}, 0), \quad v'_4 = (0, \Lambda_{12}), \quad v'_5 = (0, \Lambda_{13}),$$

then

$$X' = \overline{\mathbb{Q}} + \overline{\mathbb{Q}}\Lambda_{12} + \overline{\mathbb{Q}}\Lambda_{13} \quad \text{and} \quad Y' = \overline{\mathbb{Q}} + \overline{\mathbb{Q}}\Lambda_{21}$$

are  $\overline{\mathbb{Q}}$ -subspaces of  $\mathbb{C}$  satisfying (2.2). Here  $\mu(X', \mathbb{C}) = 3 > d/n = 5/2$  and  $\mu(Y', \mathbb{C}) = 2 = \mu(Y, \mathbb{C}^2)$ .

PROOF OF COROLLARY 1.4. From Baker's Theorem it follows that if  $Y_0$  is a  $\mathbb{Q}$ -vector subspace of  $\mathcal{L}^n$  of dimension  $\ell$ , then the  $\overline{\mathbb{Q}}$ -vector subspace of  $\widetilde{\mathcal{L}}^n$  spanned by  $\overline{\mathbb{Q}}^n \cup Y_0$  has dimension  $\ell + n$  (see Exercise 1.5 (iii) of [7]). Taking firstly  $n = 2$ ,  $\ell = 3$ , and secondly  $n = 3$ ,  $\ell = 2$ , we deduce that the matrix  $M$  of corollary 1.4 satisfies the assumptions of Theorem 1.2. Corollary 1.4 follows.  $\square$

#### 4. Erratum to [8]

We take the opportunity of this paper to point out a mistake in the statement of Corollary 2.12 p. 347 of [8]: the assumption that  $\Lambda_{21}$  is not zero and  $\Lambda_{11}/\Lambda_{21}$  is transcendental should be replaced by the assumption that the three numbers 1,  $\Lambda_{11}$  and  $\Lambda_{21}$  are linearly independent over the field of algebraic numbers. Otherwise a counterexample is obtained for instance with  $\Lambda_{21} = 1$  and  $\Lambda_{2j} = 0$  for  $2 \leq j \leq 5$ .

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