Bilaplacian problems with a sign-changing coefficient

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Abstract. We investigate the properties of the operator $\Delta(\sigma\Delta\cdot):\mathrm{H}^2_0(\Omega)\to\mathrm{H}^{-2}(\Omega)$, where σ is a given parameter whose sign can change on the bounded domain Ω. Here, $\mathrm{H}^2_0(\Omega)$ denotes the subspace of $\mathrm{H}^2(\Omega)$ made of the functions v such that $v=\nu\cdot\nabla v=0$ on $\partial\Omega$. The study of this problem arises when one is interested in some configurations of the Interior Transmission Eigenvalue Problem. We prove that $\Delta(\sigma\Delta\cdot):\mathrm{H}^2_0(\Omega)\to\mathrm{H}^{-2}(\Omega)$ is a Fredholm operator of index zero as soon as $\sigma\in\mathrm{L}^\infty(\Omega)$, with $\sigma^{-1}\in\mathrm{L}^\infty(\Omega)$, is such that σ remains uniformly positive (or uniformly negative) in a neighbourhood of $\partial\Omega$. We also study configurations where σ changes sign on $\partial\Omega$ and we prove that Fredholm property can be lost for such situations. In the process, we examine in details the features of a simpler problem where the boundary condition $\nu\cdot\nabla v=0$ is replaced by $\sigma\Delta v=0$ on $\partial\Omega$.

Key words. Sign-changing coefficient, bilaplacian, interior transmission problem, non smooth boundary, singularities.

1 Introduction

1.1 The Interior Transmission Eigenvalue Problem

The motivation for considering bilaplacian operators with a sign-changing coefficient finds its origin in the study of some configurations of the Interior Transmission Eigenvalue Problem (ITEP), a spectral problem introduced in [24, 14] and which appears in inverse scattering theory. In particular, the ITEP arises when one is interested in the reconstruction of the support of a penetrable inclusion embedded in a reference medium from far fields measurements at a given frequency. In this context, it is important to know if for a given frequency, we can find an incident wave for which the field scattered by the inclusion is null. Frequencies for which the answer to this question is positive have an important role: we need to avoid them to implement the Linear Sampling Method, a well-known reconstruction method, and we can also use them to characterize the properties of the inclusion [9, 20]. In the following, we formulate the ITEP. The reader who wants to skip this introductory part may proceed to §1.2.

The reference medium is chosen equal to $\mathbb{R}^d$, $d\ge 1$. The inclusion is a domain $\Omega\subset\mathbb{R}^d$, *i.e.* a bounded and connected open subset of $\mathbb{R}^d$ with Lipschitz boundary $\partial\Omega$. We assume that the propagation of waves in $\mathbb{R}^d$ is governed by the equation $\Delta w+k^2w=0$, where $k\in\mathbb{R}$ is the wave number. On the other hand, we model the inclusion by some physical parameter n so that the total field u (the sum of the incident and scattered fields) satisfies $\Delta u+k^2n^2u=0$. We impose that $n\neq 1$ in Ω and $n=1$ in $\mathbb{R}^d\setminus\Omega$. If we denote $[\cdot]|_{\partial\Omega}$ the jump on $\partial\Omega$ (here, the sign is not important) and ν the unit outward normal vector to $\partial\Omega$ oriented to the exterior of Ω, the total field u satisfies the equations of continuity $[u]|_{\partial\Omega}=[\nu\cdot\nabla u]|_{\partial\Omega}=0$. Now, if w is an incident field which does not scatter, then there holds $u=w$ outside Ω. As a consequence, we must have $u=w$ and $\nu\cdot\nabla u=\nu\cdot\nabla w$ on $\partial\Omega$. To summarize, if w is such that the scattered field is null outside the inclusion, then the pair (u,w)

verifies the problem

$$\left|\begin{array}{lcll} \Delta u + k^2 n^2 u & = & 0 & \text{in } \Omega \\ \Delta w + k^2 w & = & 0 & \text{in } \Omega \\ u - w & = & 0 & \text{on } \partial\Omega \\ \nu \cdot \nabla u - \nu \cdot \nabla w & = & 0 & \text{on } \partial\Omega. \end{array}\right. \tag{1}$$

Let us introduce some basic notations to equip Problem (1) with a functional framework. The space $\mathrm{L}^2(\Omega)$ is endowed with the classical inner product

$$(\varphi, \varphi')_\Omega = \int_\Omega \varphi \, \overline{\varphi'}, \qquad \forall (\varphi, \varphi') \in \mathrm{L}^2(\Omega) \times \mathrm{L}^2(\Omega).$$

For all $\varphi \in \mathrm{L}^2(\Omega)$, we define $\|\varphi\|_\Omega := (\varphi, \varphi)^{1/2}$. We denote $\mathrm{H}^1_0(\Omega)$ (resp. $\mathrm{H}^2_0(\Omega)$) the closure of $\mathscr{C}_0^\infty(\Omega)$ for the H^1-norm (resp. H^2-norm). We endow these spaces with the inner products

$$\begin{array}{rcll} (\varphi, \varphi')_{\mathrm{H}^1_0(\Omega)} & = & (\nabla \varphi, \nabla \varphi')_\Omega, & \qquad \forall (\varphi, \varphi') \in \mathrm{H}^1_0(\Omega) \times \mathrm{H}^1_0(\Omega); \\ (\varphi, \varphi')_{\mathrm{H}^2_0(\Omega)} & = & (\Delta \varphi, \Delta \varphi')_\Omega, & \qquad \forall (\varphi, \varphi') \in \mathrm{H}^2_0(\Omega) \times \mathrm{H}^2_0(\Omega). \end{array}$$

The topological dual space of $\mathrm{H}^1_0(\Omega)$ (resp. $\mathrm{H}^2_0(\Omega)$) is denoted $\mathrm{H}^{-1}(\Omega)$ (resp. $\mathrm{H}^{-2}(\Omega)$).

Definition 1.1. *The elements $k \in \mathbb{C}$ for which there exists a non trivial solution to the problem*

$$\left|\begin{array}{l} \textit{Find } (u, w) \in \mathrm{L}^2(\Omega) \times \mathrm{L}^2(\Omega), \textit{ with } u - w \in \mathrm{H}^2_0(\Omega), \textit{ such that:} \\ \begin{array}{lcll} \Delta u + k^2 n^2 u & = & 0 & \textit{in } \Omega \\ \Delta w + k^2 w & = & 0 & \textit{in } \Omega \end{array} \end{array}\right. \tag{2}$$

are called interior transmission eigenvalues.

Following [39], to avoid having to work with a system of *PDEs*, we rewrite (2) as a fourth order equation. Consider (u, w) a pair satisfying (2). Define $v := u - w$. It verifies the relation

$$\Delta v + k^2 n^2 v = -k^2 (n^2 - 1) w \qquad \text{in } \Omega. \tag{3}$$

Assume that the parameter $n : \Omega \to \mathbb{R}$ is an element of $\mathrm{L}^\infty(\Omega)$ such that $n > 0$ and $n \neq 1$ in Ω. Assume also that $(n^2 - 1)^{-1}$ belongs to $\mathrm{L}^\infty(\Omega)$. Dividing on each side of (3) by $n^2 - 1$ and using the equation $\Delta w + k^2 w = 0$, we obtain, in the sense of distributions,

$$(\Delta + k^2) \left(\frac{1}{n^2 - 1} (\Delta v + k^2 n^2 v) \right) = 0.$$

We deduce that if the pair (u, w) satisfies Problem (2) then $v = u - w$ verifies the problem

$$\left|\begin{array}{l} \text{Find } v \in \mathrm{H}^2_0(\Omega) \text{ such that} \\ \displaystyle\int_\Omega \frac{1}{n^2 - 1} (\Delta v + k^2 n^2 v)(\Delta v' + k^2 v') = 0, \qquad \forall v' \in \mathrm{H}^2_0(\Omega). \end{array}\right. \tag{4}$$

Conversely, one shows (see [39, lemma 3.1] for the details) that if v is a solution of (4) then the pair $(u, w) := ((n^2 - 1)^{-1} (\Delta v + k^2 n^2 v) - k^2 v, (n^2 - 1)^{-1} (\Delta v + k^2 n^2 v))$ satisfies Problem (2). For $k \in \mathbb{C}$, we define the sesquilinear form a_k such that

$$a_k(v, v') = ((n^2 - 1)^{-1} (\Delta v + k^2 n^2 v), (\Delta v' + k^2 v'))_\Omega, \qquad \forall (v, v') \in \mathrm{H}^2_0(\Omega) \times \mathrm{H}^2_0(\Omega).$$

With the Riesz representation theorem, let us introduce the bounded operator $A_k : \mathrm{H}^2_0(\Omega) \to \mathrm{H}^2_0(\Omega)$ associated with a_k such that

$$(A_k v, v')_{\mathrm{H}^2_0(\Omega)} = a_k(v, v'), \qquad \forall (v, v') \in \mathrm{H}^2_0(\Omega) \times \mathrm{H}^2_0(\Omega). \tag{5}$$

For all $k \in \mathbb{C}$, observe that $A_k - A_{k_0}$ is a compact operator on $\mathrm{H}^2_0(\Omega)$. Therefore, according to the analytic Fredholm theorem, we deduce that if there exists $k_0 \in \mathbb{C}$ such that A_{k_0} is an isomorphism of $\mathrm{H}^2_0(\Omega)$, then A_k is an isomorphism of $\mathrm{H}^2_0(\Omega)$ for all $k \in \mathbb{C}\backslash\mathscr{S}$, where $\mathscr{S}$ is a discrete or empty set of $\mathbb{C}$. This is a simple approach to prove that the set of interior transmission eigenvalues is discrete, one of the basic objectives in the theory (another important issue (see [36, 10, 11]) is to establish that interior transmission eigenvalues exist).

When there is a constant $C > 0$ such that $n - 1 \geq C$ in Ω (resp. $-(n-1) \geq C$ in Ω), the form a_0 (resp. $-a_0$) is coercive on $\mathrm{H}^2_0(\Omega) \times \mathrm{H}^2_0(\Omega)$. This allows to prove that A_0 is an isomorphism of $\mathrm{H}^2_0(\Omega)$, and as a consequence, that the set of interior transmission eigenvalues is discrete or empty. This result is known since [39] (see also the review paper [15]). When $n - 1$ changes sign in Ω, the form a_k is no longer coercive nor "coercive+compact" on $\mathrm{H}^2_0(\Omega) \times \mathrm{H}^2_0(\Omega)$. The question of the discreteness of the set of interior transmission eigenvalues in this case has remained completely open for a long time. An important step forward has been made by J. Sylvester in [40]. In this paper, he proves, using an equivalent formulation of (2), that discreteness holds as soon as $n - 1$ is uniformly positive or uniformly negative in a neighbourhood of the boundary $\partial\Omega$. The same assumption is needed in the recent papers [29, 38] where the authors obtain additional results of existence of interior transmission eigenvalues in the case of a smooth coefficient n. However, for the moment, it seems that there is no result when $n - 1$ changes sign on $\partial\Omega$. One of the outcomes of the present article is to prove that Fredholm property (see Definition 1.2 hereafter) can be lost for the operators A_k when $n - 1$ changes sign on $\partial\Omega$. In these situations, the functional framework needs to be modified to apply the analytic Fredholm theorem and to prove that the set of interior transmission eigenvalues is discrete.

1.2 Problem considered in the present paper

To simplify the notations, let us define $\sigma := (n^2 - 1)^{-1} \in \mathrm{L}^\infty(\Omega)$. All along the paper, we shall assume that σ is real valued. Our goal is to investigate the features of the following source term problem

$$(\mathscr{P})\left|\begin{array}{l}\text{Find } v \in \mathrm{H}^2_0(\Omega) \text{ such that:}\\ (\sigma\Delta v, \Delta v')_\Omega = \langle f, v'\rangle_\Omega\,, \qquad \forall v' \in \mathrm{H}^2_0(\Omega)\end{array}\right. \tag{6}$$

corresponding to the principal part (the part which contains the derivatives of higher degree) of the operator A_k. In Problem $(\mathscr{P})$, f is an element of $\mathrm{H}^{-2}(\Omega)$ whereas $\langle\cdot,\cdot\rangle_\Omega$ refers to the duality pairing $\mathrm{H}^{-2}(\Omega) \times \mathrm{H}^2_0(\Omega)$. Let us introduce the sesquilinear form b such that

$$b(v, v') = (\sigma\Delta v, \Delta v')_\Omega, \qquad \forall(v, v') \in \mathrm{H}^2_0(\Omega) \times \mathrm{H}^2_0(\Omega)$$

and the continuous operator $B : \mathrm{H}^2_0(\Omega) \to \mathrm{H}^2_0(\Omega)$ defined by

$$(Bv, v')_{\mathrm{H}^2_0(\Omega)} = b(v, v'), \qquad \forall(v, v') \in \mathrm{H}^2_0(\Omega) \times \mathrm{H}^2_0(\Omega). \tag{7}$$

In recent years, much work (see [16, 6, 2, 4, 35]) has been devoted to the study of the operator $\mathrm{div}(\sigma\nabla\cdot) : \mathrm{H}^1_0(\Omega) \to \mathrm{H}^{-1}(\Omega)$ when σ is a parameter whose sign changes on the domain Ω. This operator arises in the modelling of electromagnetic phenomena in time harmonic regime in media involving usual positive material and metals at optical frequencies or negative metamaterials. In this context, the medium is divided into two regions: one corresponding to the positive material ($\sigma = \sigma_+ > 0$), another corresponding to the negative material ($\sigma = \sigma_- < 0$). Let us present the main results, assuming to simplify that σ_+ and σ_- are some constants. If the interface between the two materials is smooth, the operator $\mathrm{div}(\sigma\nabla\cdot) : \mathrm{H}^1_0(\Omega) \to \mathrm{H}^{-1}(\Omega)$ is Fredholm of index zero if and only the contrast σ_+/σ_- satisfies $\sigma_+/\sigma_- \neq -1$ [16, 2]. When the interface between the two materials has corners, strong singularities can appear. In such configurations, the operator $\mathrm{div}(\sigma\nabla\cdot) : \mathrm{H}^1_0(\Omega) \to \mathrm{H}^{-1}(\Omega)$ is Fredholm of index zero if and only σ_+/σ_- lies outside some interval $I \subset (-\infty; 0)$ [17, 7, 5, 13]. The latter interval always contains the value -1 and depends only on the smallest aperture of the corners of the interface. The goal of the present article is also to compare the features of $\mathrm{div}(\sigma\nabla\cdot) : \mathrm{H}^1_0(\Omega) \to \mathrm{H}^{-1}(\Omega)$ and $\Delta(\sigma\Delta\cdot) : \mathrm{H}^2_0(\Omega) \to \mathrm{H}^{-2}(\Omega)$. Has the change of sign of σ the same consequences for both operators?

The outline of the paper is the following. In Section 2, we examine the properties of the operator $\tilde{B} : \mathrm{H}^1_0(\Omega) \cap \mathrm{H}^2(\Omega) \to \mathrm{H}^1_0(\Omega) \cap \mathrm{H}^2(\Omega)$ such that $(\tilde{B}v, v')_{\mathrm{H}^2_0(\Omega)} = (\sigma\Delta v, \Delta v')_\Omega$, for all v, v' in

$\mathrm{H}^1_0(\Omega) \cap \mathrm{H}^2(\Omega)$. The latter functional framework corresponds to mixed boundary conditions. We prove that when the domain Ω is smooth or convex, $\tilde{B}$ is an isomorphism without assumption on the sign of σ. The investigation of this simpler problem provides us the way to study the original operator B (with Dirichlet boundary conditions). This is the subject of the first part of Section 3 where we show that B is Fredholm of index zero as soon as σ remains uniformly positive or uniformly negative in a neighbourhood of the boundary $\partial\Omega$. In the second part of Section 3, we are interested in what happens when the sign of σ changes on $\partial\Omega$. In particular, we exhibit situations where Fredholmness in $\mathrm{H}^2_0(\Omega)$ is lost. Then, in Section 4, we complement the analysis of the features of the operator $\tilde{B}$. Adapting a technique used to consider the case $\sigma = 1$ (see [33] and the monograph [18]), we establish that for polygons with reentrant corners (which are non convex and non smooth domains), a kernel and a cokernel can appear for $\tilde{B}$ when σ changes sign. Finally, in Section 5, we introduce the operator $B^\sharp : \mathrm{H}^1_0(\Delta) \to \mathrm{H}^1_0(\Delta)$, such that $(B^\sharp v, v')_{\mathrm{H}^2_0(\Omega)} = (\sigma\Delta v, \Delta v')_\Omega$, for all $v, v' \in \mathrm{H}^1_0(\Delta)$, where $\mathrm{H}^1_0(\Delta) := \{\varphi \in \mathrm{H}^1_0(\Omega) \,|\, \Delta\varphi \in \mathrm{L}^2(\Omega)\}$. We demonstrate that $B^\sharp$ is always an isomorphism and we compare the characteristics of the inverses of $B^\sharp$ and $\tilde{B}$ (when $\tilde{B}$ is invertible). Generally speaking, in this article, we must say that the Interior Transmission Eigenvalue Problem serves as a pretext to make a review of the properties of several bilaplacian operators with a sign-changing coefficient.

In the sequel, on several occasions, we shall rely on Fredholm theory using the following definition.

Definition 1.2. *Let* X *and* Y *be two Banach spaces, and let* $L : \mathrm{X} \to \mathrm{Y}$ *be a continuous linear map. The operator* L *is said to be a Fredholm operator if and only if the following two conditions are fulfilled*

i) $\dim(\ker L) < \infty$ *and* $\mathrm{range}\, L$ *is closed;*

ii) $\dim(\mathrm{coker}\, L) < \infty$ *where* $\mathrm{coker}\, L := (\mathrm{Y}/\mathrm{range}\, L)$.

Besides, the index of a Fredholm operator L *is defined by* $\mathrm{ind}\, L = \dim(\ker L) - \dim(\mathrm{coker}\, L)$.

2 Bilaplacian with mixed boundary conditions: smooth and convex domains

Before investigating the properties of the operator B, let us study the problem obtained replacing in $(\mathscr{P})$ the boundary condition "$\nu \cdot \nabla v = 0$ on $\partial\Omega$" by the condition "$\sigma\Delta v = 0$ on $\partial\Omega$". We will see that in this case the analysis is quite simple. For $f \in (\mathrm{H}^1_0(\Omega) \cap \mathrm{H}^2(\Omega))^*$, the topological dual space of $\mathrm{H}^1_0(\Omega) \cap \mathrm{H}^2(\Omega)$, let us consider the problem

$$\left|\begin{array}{lcll} \multicolumn{4}{l}{\text{Find } v \in \mathrm{H}^1_0(\Omega) \cap \mathrm{H}^2(\Omega) \text{ such that:}} \\ \Delta(\sigma\Delta v) & = & f & \text{in } \Omega \\ \sigma\Delta v & = & 0 & \text{on } \partial\Omega. \end{array}\right. \tag{8}$$

Here, we impose mixed boundary conditions: the condition $v = 0$ on $\partial\Omega$ is said to be *essential*, its appears in the functional space, whereas the condition $\sigma\Delta v = 0$ on $\partial\Omega$ is said to be *natural*. The trace $\sigma\Delta v = 0$ is defined in a weak sense. We shall say that the function $\varphi \in \mathrm{L}^2(\Omega)$ such that $\Delta\varphi \in (\mathrm{H}^1_0(\Omega) \cap \mathrm{H}^2(\Omega))^*$ satisfies $\varphi = 0$ on $\partial\Omega$ if and only if there holds

$$\langle \Delta\varphi, \varphi' \rangle_\Omega = (\varphi, \Delta\varphi')_\Omega, \qquad \forall \varphi' \in \mathrm{H}^1_0(\Omega) \cap \mathrm{H}^2(\Omega), \tag{9}$$

where $\langle \cdot, \cdot \rangle_\Omega$ denotes the duality pairing $(\mathrm{H}^1_0(\Omega) \cap \mathrm{H}^2(\Omega))^* \times \mathrm{H}^1_0(\Omega) \cap \mathrm{H}^2(\Omega)$. Therefore, Problem (8) is equivalent to the following problem

$$(\tilde{\mathscr{P}}) \left|\begin{array}{l} \text{Find } v \in \mathrm{H}^1_0(\Omega) \cap \mathrm{H}^2(\Omega) \text{ such that:} \\ (\sigma\Delta v, \Delta v')_\Omega = \langle f, v' \rangle_\Omega, \qquad \forall v' \in \mathrm{H}^1_0(\Omega) \cap \mathrm{H}^2(\Omega). \end{array}\right. \tag{10}$$

With these mixed boundary conditions, one can solve $(\tilde{\mathscr{P}})$ in two steps. Let f be a source term of $\mathrm{H}^{-1}(\Omega) \subset (\mathrm{H}^1_0(\Omega) \cap \mathrm{H}^2(\Omega))^*$. There exists a unique $p \in \mathrm{H}^1_0(\Omega)$ verifying $-(\nabla p, \nabla p')_\Omega = \langle f, p' \rangle_\Omega$ for all $p' \in \mathrm{H}^1_0(\Omega)$. Let us denote v the unique function satisfying $v \in \mathrm{H}^1_0(\Omega)$ and $\Delta v = \sigma^{-1} p \in \mathrm{L}^2(\Omega)$.

If the domain Ω is of class $\mathscr{C}^2$ ([19, theorem 8.12]) or convex ([21, theorem 3.2.1.2]), we know that v belongs to $\mathrm{H}^2(\Omega)$. Moreover, there holds, for all $v' \in \mathrm{H}^1_0(\Omega) \cap \mathrm{H}^2(\Omega)$,

$$(\sigma\Delta v, \Delta v')_\Omega = (p, \Delta v')_\Omega = -(\nabla p, \nabla v')_\Omega = \langle f, v' \rangle_\Omega .$$

We deduce that v is a solution of $(\tilde{\mathscr{P}})$. Notice that, to obtain this result, the only assumptions for σ are $\sigma \in \mathrm{L}^\infty(\Omega)$ and $\sigma^{-1} \in \mathrm{L}^\infty(\Omega)$. Thus, σ can change sign. Let us complement this first study of Problem $(\tilde{\mathscr{P}})$.

With the Lax-Milgram theorem, we can show that the sesquilinear form $(v, v') \mapsto (v, v)_{\mathrm{H}^2_0(\Omega)} = (\Delta v, \Delta v')_\Omega$ is an inner product on $\mathrm{H}^1_0(\Omega) \cap \mathrm{H}^2(\Omega)$. Moreover, if Ω is of class $\mathscr{C}^2$ or convex, on $\mathrm{H}^1_0(\Omega) \cap \mathrm{H}^2(\Omega)$, the map $v \mapsto \|\Delta v\|_\Omega$ defines a norm which is equivalent to the H^2-norm. Therefore, $\mathrm{H}^1_0(\Omega) \cap \mathrm{H}^2(\Omega)$ endowed with the inner product $(\cdot, \cdot)_{\mathrm{H}^2_0(\Omega)}$ is a Hilbert space. We introduce the sesquilinear form $\tilde{b}$ such that

$$\tilde{b}(v, v') = (\sigma\Delta v, \Delta v')_\Omega, \qquad \forall (v, v') \in \mathrm{H}^1_0(\Omega) \cap \mathrm{H}^2(\Omega) \times \mathrm{H}^1_0(\Omega) \cap \mathrm{H}^2(\Omega), \tag{11}$$

and the continuous operator $\tilde{B} : \mathrm{H}^1_0(\Omega) \cap \mathrm{H}^2(\Omega) \to \mathrm{H}^1_0(\Omega) \cap \mathrm{H}^2(\Omega)$, defined by

$$(\tilde{B}v, v')_{\mathrm{H}^2_0(\Omega)} = \tilde{b}(v, v'), \qquad \forall (v, v') \in \mathrm{H}^1_0(\Omega) \cap \mathrm{H}^2(\Omega) \times \mathrm{H}^1_0(\Omega) \cap \mathrm{H}^2(\Omega). \tag{12}$$

Theorem 2.1. *Assume that the domain $\Omega \subset \mathbb{R}^d$, $d \geq 1$, is of class $\mathscr{C}^2$ or convex. For all $\sigma \in \mathrm{L}^\infty(\Omega)$ such that $\sigma^{-1} \in \mathrm{L}^\infty(\Omega)$, the operator $\tilde{B} : \mathrm{H}^1_0(\Omega) \cap \mathrm{H}^2(\Omega) \to \mathrm{H}^1_0(\Omega) \cap \mathrm{H}^2(\Omega)$ defined in (12) is an isomorphism.*

Proof. Let us introduce the operator $\mathtt{T} : \mathrm{H}^1_0(\Omega) \cap \mathrm{H}^2(\Omega) \to \mathrm{H}^1_0(\Omega) \cap \mathrm{H}^2(\Omega)$ such that, for all $v \in \mathrm{H}^1_0(\Omega) \cap \mathrm{H}^2(\Omega)$, $\mathtt{T}v$ is defined as the unique solution of the problem "find $\mathtt{T}v \in \mathrm{H}^1_0(\Omega)$ satisfying $\Delta(\mathtt{T}v) = \sigma^{-1}\Delta v$". Notice that since Ω is assumed to be of class $\mathscr{C}^2$ ([19, theorem 8.12]) or convex ([21, theorem 3.2.1.2]), $\mathtt{T}v$ is indeed an element of $\mathrm{H}^2(\Omega)$. For all $v, v' \in \mathrm{H}^1_0(\Omega) \cap \mathrm{H}^2(\Omega)$, we can write

$$(\tilde{B}(\mathtt{T}v), v')_{\mathrm{H}^2_0(\Omega)} = (\sigma\Delta(\mathtt{T}v), \Delta v')_\Omega = (\Delta v, \Delta v')_\Omega.$$

Therefore, the operator $\tilde{B} \circ \mathtt{T}$ is equal to the identity of $\mathrm{H}^1_0(\Omega) \cap \mathrm{H}^2(\Omega)$. Since $\tilde{B}$ is selfadjoint, we deduce that $\tilde{B}$ is an isomorphism with $\tilde{B}^{-1} = \mathtt{T}$. □

Let us give another proof of this result, slightly different, using the resolution of $(\tilde{\mathscr{P}})$ in two steps. The resolution of $(\tilde{\mathscr{P}})$ in two steps is interesting for numerical considerations because it is easier to solve two second order problems than a fourth order one. Moreover, it will give us an idea of how to proceed to study configurations where Ω does not satisfy the assumptions of Theorem 2.1 (see Section 4). When Ω is smooth or convex, $\Delta : \mathrm{H}^1_0(\Omega) \cap \mathrm{H}^2(\Omega) \to \mathrm{L}^2(\Omega)$ is an isomorphism. Since the adjoint operator of an isomorphism is also an isomorphism, one has the following classical result.

Proposition 2.1. *Assume that the domain $\Omega \subset \mathbb{R}^d$, $d \geq 1$, is of class $\mathscr{C}^2$ or convex. Then for all $f \in (\mathrm{H}^1_0(\Omega) \cap \mathrm{H}^2(\Omega))^*$, there exists a unique solution to the problem*

$$\left| \begin{array}{l} \textit{Find } p \in \mathrm{L}^2(\Omega) \textit{ such that:} \\ (p, \Delta v')_\Omega = \langle f, v' \rangle_\Omega , \qquad \forall v' \in \mathrm{H}^1_0(\Omega) \cap \mathrm{H}^2(\Omega). \end{array} \right. \tag{13}$$

Proof of Theorem 2.1 (bis): Let $v \in \mathrm{H}^1_0(\Omega) \cap \mathrm{H}^2(\Omega)$ such that

$$(\sigma\Delta v, \Delta v')_\Omega = 0, \qquad \forall v' \in \mathrm{H}^1_0(\Omega) \cap \mathrm{H}^2(\Omega).$$

According to Proposition 2.1, we have necessarily $\sigma\Delta v = 0$, hence $\Delta v = 0$. This implies $v = 0$ since $v \in \mathrm{H}^1_0(\Omega) \cap \mathrm{H}^2(\Omega)$ and proves that $(\tilde{\mathscr{P}})$ admits at most one solution. Then, we consider a source term f in $(\mathrm{H}^1_0(\Omega) \cap \mathrm{H}^2(\Omega))^*$. By virtue of Proposition 2.1, there exists a unique $p \in \mathrm{L}^2(\Omega)$ such that

$$(p, \Delta v')_\Omega = \langle f, v' \rangle_\Omega , \qquad \forall v' \in \mathrm{H}^1_0(\Omega) \cap \mathrm{H}^2(\Omega).$$

Denote v the unique solution to the problem "find $v \in \mathrm{H}^1_0(\Omega) \cap \mathrm{H}^2(\Omega)$ such that $\Delta v = \sigma^{-1}p$". This function v is a solution to $(\tilde{\mathscr{P}})$. ■

The operator $\tilde{B}$ defined in (12) is an isomorphism of $\mathrm{H}_0^1(\Omega)\cap\mathrm{H}^2(\Omega)$ when $\sigma\in\mathrm{L}^\infty(\Omega)$ verifies $\sigma^{-1}\in\mathrm{L}^\infty(\Omega)$ and when Ω is such that $\Delta:\mathrm{H}_0^1(\Omega)\cap\mathrm{H}^2(\Omega)\to\mathrm{L}^2(\Omega)$ constitutes an isomorphism. But what if this last assumption on Ω is not met? What happens for example if Ω is a 2D domain with reentrant corners? The hungry reader who is eager to know the answer to this question can directly jump to Section 4 where we prove that in this case, $\tilde{B}$ is not always an isomorphism. According to the values of the parameter σ, a kernel and a cokernel, both of finite dimension, can appear.

3 Bilaplacian with Dirichlet boundary condition

In this section, we come back to the study of the operator $B:\mathrm{H}_0^2(\Omega)\to\mathrm{H}_0^2(\Omega)$ introduced in (7). First, we provide a sufficient criterion to ensure that B is Fredholm of index zero. Then, we exhibit situations where B is not of Fredholm type.

3.1 Configurations where σ has a constant sign on the boundary

Fredholm property. In this paragraph, Ω is a domain of $\mathbb{R}^d$, with $d\geq 1$. We prove that B is Fredholm of index zero when σ satisfies the following condition.

$(\mathscr{H}_\sigma)$ We assume that $\sigma\in\mathrm{L}^\infty(\Omega)$ is such that $\sigma^{-1}\in\mathrm{L}^\infty(\Omega)$. Moreover, we assume that $\sigma(\boldsymbol{x})\geq C_1>0$ a.e. in $\Omega\backslash\overline{\mathcal{O}}$ or $\sigma(\boldsymbol{x})\leq C_2<0$ a.e. in $\Omega\backslash\overline{\mathcal{O}}$, where C_1, C_2 are two constants and where $\mathcal{O}$ is an open set such that $\overline{\mathcal{O}}\subset\Omega$.

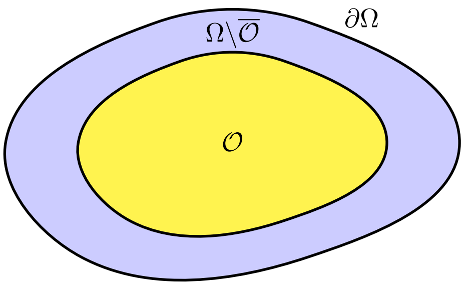

Figure 1: The parameter σ is assumed to be uniformly positive or uniformly negative in the region $\Omega\backslash\overline{\mathcal{O}}$.

In other words, we assume that there exists a neighbourhood of $\partial\Omega$ where $\sigma\geq C_1>0$ or $\sigma\leq C_2<0$. However, outside this neighbourhood, σ can change sign. To prove that B is a Fredholm operator of index zero, we build a right parametrix for B, *i.e.* we build a bounded operator $\mathtt{T}$ such that $B\circ\mathtt{T}=\mathcal{I}+\mathcal{K}$ where $\mathcal{I}:\mathrm{H}_0^2(\Omega)\to\mathrm{H}_0^2(\Omega)$ is an isomorphism and $\mathcal{K}:\mathrm{H}_0^2(\Omega)\to\mathrm{H}_0^2(\Omega)$ is a compact operator.

Theorem 3.1. *Assume that σ satisfies condition $(\mathscr{H}_\sigma)$. Then the operator $B:\mathrm{H}_0^2(\Omega)\to\mathrm{H}_0^2(\Omega)$ verifying $(Bv,v')_{\mathrm{H}_0^2(\Omega)}=(\sigma\Delta v,\Delta v')_\Omega$, for all $(v,v')\in\mathrm{H}_0^2(\Omega)\times\mathrm{H}_0^2(\Omega)$, is Fredholm of index zero.*

Remark 3.1. *Using the approach presented here, we can prove that for all $m\in\mathbb{N}^*:=\{1,2,3,\dots\}$ the operator $\Delta^m(\sigma\Delta^m\cdot):\mathrm{H}_0^{2m}(\Omega)\to\mathrm{H}^{-2m}(\Omega)$ is Fredholm of index zero when σ satisfies condition $(\mathscr{H}_\sigma)$. Here, $\mathrm{H}_0^{2m}(\Omega)$ denotes the closure of $\mathscr{C}_0^\infty(\Omega)$ for the H^{2m}-norm.*

Proof. Let us give the proof in the case where $\sigma\geq C_1>0$ in a neighbourhood of the boundary $\partial\Omega$. The configuration where $\sigma\leq C_2<0$ in a neighbourhood of $\partial\Omega$ can be deduced working with the operator $-B$. Let us introduce $\zeta\in\mathscr{C}_0^\infty(\Omega,[0;1])$ a cut-off function equal to 1 in $\mathcal{O}$. Notice that $1-\zeta$ is an element of $\mathscr{C}^\infty(\overline{\Omega},[0;1])$ which is equal to 1 in a neighbourhood of $\partial\Omega$. Now, let us consider v an element of $\mathrm{H}_0^2(\Omega)$. The function $(1-\zeta)v$ belongs to $\mathrm{H}_0^2(\Omega)$ and, by definition, we have, for all $v'\in\mathrm{H}_0^2(\Omega)$,

$$(\sigma\Delta((1-\zeta)v),\Delta v')_\Omega=b((1-\zeta)v,v')=(B((1-\zeta)v),v')_{\mathrm{H}_0^2(\Omega)}.$$

This allows to write, expanding $\Delta((1-\zeta)v)$,

$$((1-\zeta)\sigma\Delta v, \Delta v')_\Omega = (B((1-\zeta)v), v')_{\mathrm{H}^2_0(\Omega)} + (\sigma(2\nabla v\cdot\nabla\zeta + v\Delta\zeta), \Delta v')_\Omega. \tag{14}$$

On the support of ζ, we must proceed slightly differently because we allow σ to change sign. Let us denote ψ the unique element of $\mathrm{H}^1_0(\Omega)$ such that $\Delta\psi = \sigma^{-1}\Delta v \in \mathrm{L}^2(\Omega)$. The classical results of interior regularity (see for example [22, theorem 2.1.3]) indicate that, for all $\chi \in \mathscr{C}^\infty_0(\Omega)$, $\chi\psi \in \mathrm{H}^2_0(\Omega)$ with the estimate $\|\chi\psi\|_{\mathrm{H}^2_0(\Omega)} \le C\|\sigma^{-1}\Delta v\|_\Omega \le C\|v\|_{\mathrm{H}^2_0(\Omega)}$. In particular, the function $\zeta\psi$ belongs to $\mathrm{H}^2_0(\Omega)$ and depends continuously on v. Since $(\sigma\Delta(\zeta\psi), \Delta v')_\Omega = (B(\zeta\psi), v')_{\mathrm{H}^2_0(\Omega)}$, we obtain, expanding $\Delta(\zeta\psi)$,

$$(\zeta\Delta v, \Delta v')_\Omega = (\sigma\zeta\Delta\psi, \Delta v')_\Omega = (B(\zeta\psi), v')_{\mathrm{H}^2_0(\Omega)} - (\sigma(2\nabla\psi\cdot\nabla\zeta + \psi\Delta\zeta), \Delta v')_\Omega. \tag{15}$$

Let us define the operator $\mathtt{T} : \mathrm{H}^2_0(\Omega) \to \mathrm{H}^2_0(\Omega)$ such that $\mathtt{T}v = \zeta\psi + (1-\zeta)v$ for all $v \in \mathrm{H}^2_0(\Omega)$. With the Riesz representation theorem, we introduce the operators $\mathcal{I} : \mathrm{H}^2_0(\Omega) \to \mathrm{H}^2_0(\Omega)$ and $\mathcal{K} : \mathrm{H}^2_0(\Omega) \to \mathrm{H}^2_0(\Omega)$ such that, for all $(v, v') \in \mathrm{H}^2_0(\Omega) \times \mathrm{H}^2_0(\Omega)$,

$$\begin{array}{rcl}(\mathcal{I}v, v')_{\mathrm{H}^2_0(\Omega)} &=& ((\zeta + (1-\zeta)\sigma)\Delta v, \Delta v')_\Omega \\ (\mathcal{K}v, v')_{\mathrm{H}^2_0(\Omega)} &=& (\sigma(2\nabla(\psi - v)\cdot\nabla\zeta + (\psi - v)\Delta\zeta), \Delta v')_\Omega.\end{array} \tag{16}$$

With these definitions, we have the relation $B \circ \mathtt{T} = \mathcal{I} + \mathcal{K}$. From the Lax-Milgram theorem, we infer that $\mathcal{I} : \mathrm{H}^2_0(\Omega) \to \mathrm{H}^2_0(\Omega)$ is an isomorphism because the sesquilinear form $(v, v') \mapsto ((\zeta + (1-\zeta)\sigma)\Delta v, \Delta v')_\Omega$ on $\mathrm{H}^2_0(\Omega) \times \mathrm{H}^2_0(\Omega)$ is coercive. Lemma 3.1 hereafter indicates that $\mathcal{K} : \mathrm{H}^2_0(\Omega) \to \mathrm{H}^2_0(\Omega)$ is compact. Therefore, the operator $\mathtt{T}$ constitutes a right parametrix for B. Since B is selfadjoint, we deduce that it is a Fredholm operator of index zero. □

Lemma 3.1. *The operator $\mathcal{K} : \mathrm{H}^2_0(\Omega) \to \mathrm{H}^2_0(\Omega)$ defined in (16) is compact.*

Proof. Let us consider $(v'_m)_m$ a bounded sequence of elements of $\mathrm{H}^2_0(\Omega)$. Let us prove that we can extract a subsequence of $(v'_m)_m$, still denoted $(v'_m)_m$, such that $(\mathcal{K}v'_m)_m$ converges in $\mathrm{H}^2_0(\Omega)$. According to the definition (16) of $\mathcal{K}$, we can write

$$\|\mathcal{K}v'_m\|^2_{\mathrm{H}^2_0(\Omega)} \le C\,\|v'_m\|_{\mathrm{H}^2_0(\Omega)}\left(\|v'_m\|_{\mathrm{H}^1(\Omega)} + \|\psi_m\|_{\mathrm{H}^1(\operatorname{supp}\zeta)}\right).$$

In the above equation, "$\operatorname{supp}\zeta$" designates the support of ζ. Let us remind that by virtue of the result of interior regularity, there holds the estimate $\|\psi_m\|_{\mathrm{H}^2(\operatorname{supp}\zeta)} \le C\,\|v'_m\|_{\mathrm{H}^2_0(\Omega)}$. Since the embedding of $\mathrm{H}^2(\Omega)$ (resp. $\mathrm{H}^2(\operatorname{supp}\zeta)$) in $\mathrm{H}^1(\Omega)$ (resp. $\mathrm{H}^1(\operatorname{supp}\zeta)$) is compact, we can extract a subsequence of $(v'_m)_m$, still denoted $(v'_m)_m$, such that $(v'_m)_m$ and $(\psi_m)_m$ converge respectively in $\mathrm{H}^1(\Omega)$ and in $\mathrm{H}^1(\operatorname{supp}\zeta)$ strongly. Let us define $v'_{mp} := v'_m - v'_p$, $\psi_{mp} := \psi_m - \psi_p$ for all $m, p \in \mathbb{N}$. Using the estimate

$$\|\mathcal{K}v'_{mp}\|^2_{\mathrm{H}^2_0(\Omega)} \le C\,\|v'_{mp}\|_{\mathrm{H}^2_0(\Omega)}\left(\|v'_{mp}\|_{\mathrm{H}^1(\Omega)} + \|\psi_{mp}\|_{\mathrm{H}^1(\operatorname{supp}\zeta)}\right),$$

we deduce that $(\mathcal{K}v'_m)_m$ is a Cauchy sequence of $\mathrm{H}^2_0(\Omega)$. As a consequence, $(\mathcal{K}v'_m)_m$ converges. □

Recall that we denoted A_k the operator associated with the original interior transmission problem (see the definition in (5)). For all $k \in \mathbb{C}$, $A_k - B$ is a compact operator. Since the index of an operator is stable under compact perturbations (see for example [41, theorem 12.8]), we have the

Corollary 3.1. *Assume that σ satisfies condition $(\mathscr{H}_\sigma)$. Then, for all $k \in \mathbb{C}$, the operator A_k defined in (5) is Fredholm of index zero.*

Remark 3.2. *In a quite surprising way, this result indicates that Fredholmness for the operator $\Delta(\sigma\Delta\cdot) : \mathrm{H}^2_0(\Omega) \to \mathrm{H}^{-2}(\Omega)$ does not depend on the changes of sign of σ which occur inside the domain Ω. This property is not true for the operator $\operatorname{div}(\sigma\nabla\cdot) : \mathrm{H}^1_0(\Omega) \to \mathrm{H}^{-1}(\Omega)$ (see [17, 7, 2]).*

Study of the injectivity in 1D. In this paragraph, we wish to know whether the result we just obtained is optimal or not. More precisely, we proved that the operator B is Fredholm of index zero when σ remains positive or negative in a neighbourhood of $\partial\Omega$. As for the operator $\tilde{B}$ in convex or smooth domains, we may have a stronger property. Maybe B is an isomorphism of $\mathrm{H}^2_0(\Omega)$ as soon as σ remains positive or negative in a neighbourhood of $\partial\Omega$. We will see on 1D examples for which we can carry explicit computations that this is not true: even for convex domains, B can have a non trivial kernel.

$\diamond$ EXAMPLE 1. Let us define the domains $\Omega = (a; b)$, $\Omega_1 = (a; 0)$, $\Omega_2 = (0; b)$, with $a < 0$ and $b > 0$. We introduce the function σ such that $\sigma = \sigma_1$ in Ω_1, $\sigma = \sigma_2$ in Ω_2. Here, $\sigma_1 > 0$ and $\sigma_2 < 0$ are some constants. Using the proof of Theorem 3.1, one shows that in this configuration, the operator B is Fredholm of index zero. Therefore, to know whether or not B is an isomorphism, it is sufficient to study $\ker B$. In the sequel, if v is an element of $\mathrm{H}^2_0(\Omega)$, we denote $v_i := v|_{\Omega_i}$ for $i = 1, 2$. Classically, one proves that if $v \in \mathrm{H}^2_0(\Omega)$ satisfies $(\mathscr{P})$ with $f = 0$, then $(v_1, v_2) \in \mathrm{H}^2(\Omega_1) \times \mathrm{H}^2(\Omega_2)$ verifies the transmission problem

$$\left| \begin{array}{lcl} v_1^{(4)} & = & 0 \quad \text{in } \Omega_1 \\ v_2^{(4)} & = & 0 \quad \text{in } \Omega_2 \\ \multicolumn{3}{l}{v_1(a) = v_2(b) = v_1^{(1)}(a) = v_2^{(1)}(b) = 0} \\ \multicolumn{3}{l}{v_1(0) - v_2(0) = v_1^{(1)}(0) - v_2^{(1)}(0) = 0} \\ \multicolumn{3}{l}{\sigma_1 v_1^{(2)}(0) - \sigma_2 v_2^{(2)}(0) = \sigma_1 v_1^{(3)}(0) - \sigma_2 v_2^{(3)}(0) = 0.} \end{array} \right. \tag{17}$$

In the above problem, $v^{(k)}(x)$ designates the kth derivative of v at point x. Of course, if $(v_1, v_2) \in \mathrm{H}^2(\Omega_1) \times \mathrm{H}^2(\Omega_2)$ satisfies (17), then the function v such that $v|_{\Omega_i} = v_i$, for $i = 1, 2$, is an element of $\ker B$. Using the first three lines of (17), we can write

$$v_1(x) = A_1(x-a)^3 + B_1(x-a)^2 \text{ for } x \in \Omega_1 \qquad \text{and} \qquad v_2(x) = A_2(x-b)^3 + B_2(x-b)^2 \text{ for } x \in \Omega_2.$$

The last two lines of (17) impose:

$$\begin{array}{rclcrcl} -a^3 A_1 + a^2 B_1 & = & -b^3 A_2 + b^2 B_2\ ; & \qquad & 3a^2 A_1 - 2aB_1 & = & 3b^2 A_2 - 2b^2 B_2\ ; \\ \sigma_1(-6aA_1 + 2B_1) & = & \sigma_2(-6bA_2 + 2B_2)\ ; & & 6\sigma_1 A_1 & = & 6\sigma_2 A_2. \end{array}$$

We deduce that $\ker B$ is non trivial if and only if the contrast $\kappa_\sigma := \sigma_2/\sigma_1$ satisfies

$$\kappa_\sigma^2 + \Big(-4(b/a) + 6(b/a)^2 - 4(b/a)^3 \Big) \kappa_\sigma + (b/a)^4 = 0.$$

We can check that the discriminant of this polynomial remains positive for $(b/a) \in \mathbb{R}^*_- := (-\infty; 0)$. Thus, for all $(b/a) \in \mathbb{R}^*_-$, there exist two values of the contrast

$$\kappa_\sigma = \Big(2 - 3(b/a) + 2(b/a)^2 \pm 2|(b/a) - 1| \sqrt{((b/a)^2 - (b/a) + 1)} \Big)\, (b/a)$$

for which there exists a non trivial solution to (17). By a straightforward computation, one proves that these two roots are strictly negative for $(b/a) \in \mathbb{R}^*_-$. This is rather reassuring because $(v, v') \mapsto (\sigma\Delta v, \Delta v')_\Omega$ is coercive on $\mathrm{H}^2_0(\Omega) \times \mathrm{H}^2_0(\Omega)$ when $\kappa_\sigma > 0$. In the case where the domain is symmetric with respect to $x = 0$, that is, in the case where $b = -a$, B is not injective for $\kappa_\sigma = -4 \pm \sqrt{3}$. Thus, these computations show that, according to the values of σ, the Fredholm operator $B : \mathrm{H}^2_0(\Omega) \to \mathrm{H}^2_0(\Omega)$ is *not always injective*.

$\diamond$ EXAMPLE 2. Let us look at a configuration where σ has a constant sign in a neighbourhood of $\partial\Omega$. Define the open sets $\Omega = (-1; 1)$, $\Omega_1 = (-1; -\delta) \cup (\delta; 1)$, $\Omega_2 = (-\delta; \delta)$, with $0 < \delta < 1$. We introduce the function σ such that $\sigma = \sigma_1$ in Ω_1, $\sigma = \sigma_2$ in Ω_2. Again, $\sigma_1 > 0$ and $\sigma_2 < 0$ are some constants. According to Theorem 3.1, for all contrast $\kappa_\sigma \in \mathbb{R}^*_-$, the operator $B : \mathrm{H}^2_0(\Omega) \to \mathrm{H}^2_0(\Omega)$ is Fredholm of index zero. Proceeding as for Example 1, we find that it is an isomorphism if and only if

$$\kappa_\sigma \notin \{\delta^3/(\delta^3-1), \delta/(\delta-1)\}.$$

Again, this proves that the result of the previous paragraph is not under-optimal in the sense that B is not always an isomorphism of $\mathrm{H}^2_0(\Omega)$.

To apply the analytic Fredholm theorem to prove that the set of transmission eigenvalues is discrete, we need to find some $k \in \mathbb{C}$ such that the operator $A_k : \mathrm{H}^2_0(\Omega) \to \mathrm{H}^2_0(\Omega)$ introduced in (5) is an isomorphism (we recall that $B = A_0$). The technique we proposed in this paragraph does not allow to obtain this result. We refer the reader to [40] for a proof based on an equivalent formulation of (2). Let us underline again that in this paper, the author also requires the assumption that σ is uniformly positive or uniformly negative in a neighbourhood of the boundary $\partial\Omega$.

3.2 Fredholm property can be lost when σ changes sign on the boundary

Now, our goal is to understand what happens if this assumption on σ is not satisfied. More precisely, what are the properties of B when σ changes sign "on" the boundary $\partial\Omega$? Working with techniques which were developed to study elliptic partial differential equations in non smooth domains, we present occurrences where $B : \mathrm{H}^2_0(\Omega) \to \mathrm{H}^2_0(\Omega)$ is not of Fredholm type.

Let us first introduce the notations. The domain $\Omega \subset \mathbb{R}^2$, is partitioned into two subdomains Ω_1, Ω_2 such that $\overline{\Omega} = \overline{\Omega_1} \cup \overline{\Omega_2}$, $\Omega_1 \cap \Omega_2 = \emptyset$. We assume that $\sigma = \sigma_1$ in Ω_1 and $\sigma = \sigma_2$ in Ω_2, where $\sigma_1 > 0$ and $\sigma_2 < 0$ are two constants. The interface $\Sigma := \overline{\Omega_1} \cap \overline{\Omega_2}$ meets the boundary at exactly two point O, O' like in Figure 2. At O, $\partial\Omega$ and Σ are locally straight lines. Therefore, at this point, the domain Ω, Ω_1 and Ω_2 coincide locally with the unbounded sectors

$$\begin{aligned}
\Xi &:= \{(r\cos\theta, r\sin\theta)\,|\,0 < r < \infty;\ \theta \in (0;\pi)\}\\
\Xi_1 &:= \{(r\cos\theta, r\sin\theta)\,|\,0 < r < \infty;\ \theta \in (0;\alpha)\}\\
\Xi_2 &:= \{(r\cos\theta, r\sin\theta)\,|\,0 < r < \infty;\ \theta \in (\alpha;\pi)\},
\end{aligned}$$

for some $\alpha \in (0;\pi)$.

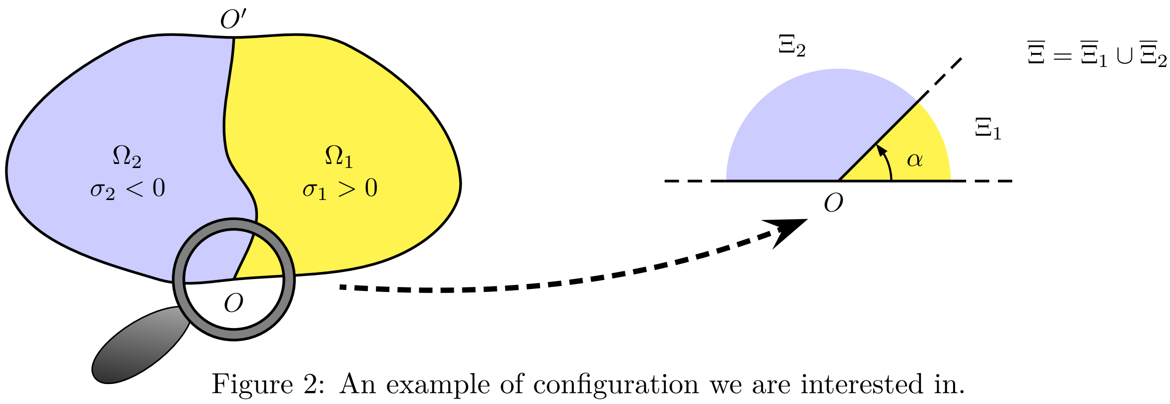

Figure 2: An example of configuration we are interested in.

When one is interested in studying the regularity of the solutions of problem ($\mathscr{P}$), according to [26], we know that a correct start is to compute the *singularities*, that is the non trivial functions of the form

$$s(\boldsymbol{x}) = r^\lambda \varphi(\theta) \tag{18}$$

which satisfy the problem

$$\Delta(\sigma\Delta s) = 0 \text{ a.e. in } \Xi \qquad \text{and} \qquad s = \partial_\nu s = 0 \text{ a.e. on } \partial\Xi.$$

If s satisfies the above equations, then $\lambda \in \mathbb{C}$ is called a *singular exponent*. We denote $\Lambda_{\alpha,\kappa_\sigma}$ the set of singular exponents.

Computation of the singularities. Actually, we will focus our attention only on the computation of some particular singularities: the ones for which the singular exponent verifies $\lambda = 1 + i\eta$, for some $\eta \in \mathbb{R}^* := \mathbb{R} \setminus \{0\}$. The reason is that if such a singularity exists then, as we will prove in §3.2, the operator $B : \mathrm{H}^2_0(\Omega) \to \mathrm{H}^2_0(\Omega)$ is not of Fredholm type. We denote $s_1 = s|_{\Xi_1}$, $s_2 = s|_{\Xi_2}$, $\varphi_1 := \varphi|_{(0;\alpha)}$ and $\varphi_2 := \varphi|_{(\alpha;\pi)}$. If s is a singularity, then (s_1, s_2) satisfies the following transmission problem

$$\left|\begin{array}{rcll} \Delta\Delta s_1 &=& 0 & \text{in } \Xi_1 \\ \Delta\Delta s_2 &=& 0 & \text{in } \Xi_2 \\ s_1 = \nu\cdot\nabla s_1 &=& 0 & \text{on } \partial\Xi\cap\partial\Xi_1\setminus\{0\} \\ s_2 = \nu\cdot\nabla s_2 &=& 0 & \text{on } \partial\Xi\cap\partial\Xi_2\setminus\{0\} \\ s_1 - s_2 = \nu_\Sigma\cdot\nabla(s_1-s_2) &=& 0 & \text{on } \partial\Xi_1\cap\partial\Xi_2\setminus\{0\} \\ \sigma_1\Delta s_1 - \sigma_2\Delta s_2 = \nu_\Sigma\cdot\nabla(\sigma_1\Delta s_1 - \sigma_2\Delta s_2) &=& 0 & \text{on } \partial\Xi_1\cap\partial\Xi_2\setminus\{0\}. \end{array}\right.$$

In these equations, ν (resp. ν_Σ) denotes the unit outward normal vector to $\partial\Xi$ (resp. Σ) oriented to the exterior of Ξ (resp. Ξ_1). In polar coordinates, the bilaplacian operator takes the form

$$\Delta^2 = r^{-4}(\partial_\theta^2 + (r\partial_r - 2)^2)(\partial_\theta^2 + (r\partial_r)^2).$$

Imposing the boundary conditions on $\partial\Xi\cap\partial\Xi_1\setminus\{0\}$ and $\partial\Xi\cap\partial\Xi_2\setminus\{0\}$, we prove (see [28, §7.1.2]) that for $\lambda \in \mathbb{C}\setminus\{0,1,2\}$, the functions φ_1 and φ_2 admit the following expressions

$$\begin{aligned} \varphi_1(\theta) &= A\,(\cos(\lambda\theta) - \cos((\lambda-2)\theta)) + B\,((\lambda-2)\sin(\lambda\theta) - \lambda\sin((\lambda-2)\theta))\,, \\ \varphi_2(\theta) &= C\,(\cos(\lambda(\theta-\pi)) - \cos((\lambda-2)(\theta-\pi))) \\ &\quad + D\,((\lambda-2)\sin(\lambda(\theta-\pi)) - \lambda\sin((\lambda-2)(\theta-\pi)))\,, \end{aligned}$$

where A, B, C and D are some constants. Writing the transmission conditions at $\theta = \alpha$, we obtain a system of four equations with four unknowns. Computing the determinant, we find that $\lambda = 1 + i\eta$, with $\eta \in \mathbb{R}^*$ is a singular exponent if and only there holds

$$\begin{array}{rcl} 0 &=& -1 + \kappa_\sigma - \kappa_\sigma^2 + \eta^2(1-\kappa_\sigma)^2(\cos(2\alpha)-1) \\ && +\kappa_\sigma\cosh(2\pi\eta) + \kappa_\sigma(\kappa_\sigma-1)\cosh(2\alpha\eta) - (\kappa_\sigma-1)\cosh(2\eta(\pi-\alpha)). \end{array} \tag{19}$$

In the above equation, the angle α and the contrast $\kappa_\sigma = \sigma_2/\sigma_1$ are two parameters. The question we wish to solve is the following: for a given problem, *i.e.* for given $\alpha \in (0;\pi)$ and $\kappa_\sigma \in (-\infty;0)$, can we find $\eta \in \mathbb{R}^*$ such that (19) is verified. For $\alpha \in (0;\pi)$ and $\kappa_\sigma \in (-\infty;0)$, we define the function h_{α,κ_σ} such that for all $\eta \in \mathbb{R}$

$$\begin{array}{rcl} h_{\alpha,\kappa_\sigma}(\eta) &=& -1 + \kappa_\sigma - \kappa_\sigma^2 + \eta^2(1-\kappa_\sigma)^2(\cos(2\alpha)-1) \\ && +\kappa_\sigma\cosh(2\pi\eta) + \kappa_\sigma(\kappa_\sigma-1)\cosh(2\alpha\eta) - (\kappa_\sigma-1)\cosh(2\eta(\pi-\alpha)). \end{array} \tag{20}$$

First, we notice that h_{α,κ_σ} is even: if $1 + i\eta$, with $\eta \in \mathbb{R}^*$ is a singular exponent, then $1 - i\eta$ is also a singular exponent. Therefore, is it sufficient to study $\eta \mapsto h_{\alpha,\kappa_\sigma}(\eta)$ on $(0;+\infty)$. Then, we observe that there holds $h_{\alpha,\kappa_\sigma}(\eta) = 0$ if and only there holds $h_{\pi-\alpha,1/\kappa_\sigma}(\eta) = 0$. This is reassuring since the singularities for the problem with an angle of aperture $\pi - \alpha$ and a contrast equal to $1/\kappa_\sigma$ are the same as the singularities for the problem with an angle of aperture α and a contrast equal to κ_σ. For all $(\alpha,\kappa_\sigma) \in (0;\pi)\times(-\infty;0)$, we have $h_{\alpha,\kappa_\sigma}(\eta) \to -\infty$ when $\eta \to +\infty$. Moreover, there holds $h_{\alpha,\kappa_\sigma}(0) = 0$. A Taylor expansion of h_{α,κ_σ} at $\eta = 0$ gives

$$h_{\alpha,\kappa_\sigma}(\eta) = g_\alpha(\kappa_\sigma)\,\eta^2 + O(\eta^4) \tag{21}$$

with $g_\alpha(\kappa_\sigma) = 2(\alpha^2 - \sin^2(\alpha))\,\kappa_\sigma^2 - 4(\alpha^2 - \sin^2(\alpha) - \alpha\pi)\,\kappa_\sigma + 2(\alpha^2 - \sin^2(\alpha) + \pi^2 - 2\alpha\pi)$. For a given $\alpha \in (0;\pi)$, we find that $g_\alpha(\kappa_\sigma)$ is strictly positive when

$$\kappa_\sigma \in I(\alpha) := (-\infty;\ell_-(\alpha)) \cup (\ell_+(\alpha);0) \tag{22}$$

where $\quad \ell_-(\alpha) := -\dfrac{\pi-\alpha+\sin(\pi-\alpha)}{\alpha-\sin\alpha} \quad$ and $\quad \ell_+(\alpha) := -\dfrac{\pi-\alpha-\sin(\pi-\alpha)}{\alpha+\sin\alpha}$.

Let us define the region (see Figure 3)

$$\mathscr{R} := \{(\alpha',\kappa'_\sigma) \in (0;\pi)\times(-\infty;0)\,|\,\kappa'_\sigma \in I(\alpha')\}. \tag{23}$$

When (α,κ_σ) belongs to $\mathscr{R}$, the continuous function h_{α,κ_σ} satisfies $h_{\alpha,\kappa_\sigma}(0)=0$, $h_{\alpha,\kappa_\sigma}(\eta)=g_\alpha(\kappa_\sigma)\,\eta^2+O(\eta^4)$ at $\eta=0$, with $g_\alpha(\kappa_\sigma)>0$, and $\lim_{\eta\to+\infty}h_{\alpha,\kappa_\sigma}(\eta)=-\infty$. This allows to deduce that h_{α,κ_σ} vanishes at least once on $(0;+\infty)$. Thus, if $(\alpha,\kappa_\sigma)\in\mathscr{R}$, then there exist singularities of the form $s(r,\theta)=r^{1+i\eta}\varphi(\theta)$ with $\eta\in\mathbb{R}^*$. In [12, proposition 5.1], it is proven that if $(\alpha,\kappa_\sigma)\in\mathscr{R}$, then the set of singular exponents $\Lambda_{\alpha,\kappa_\sigma}$ is such that $\Lambda_{\alpha,\kappa_\sigma}\cap\{\lambda\in\mathbb{C}\setminus\{1\}\,|\,\Re e\,\lambda=1\}=\{1\pm i\eta_0\}$, for some $\eta_0>0$. On the other hand, [12, proposition 5.2] shows that if (α,κ_σ) is located in $((0;\pi)\times(-\infty;0))\setminus\overline{\mathscr{R}}$, then $\Lambda_{\alpha,\kappa_\sigma}\cap\{\lambda\in\mathbb{C}\setminus\{1\}\,|\,\Re e\,\lambda=1\}=\emptyset$.

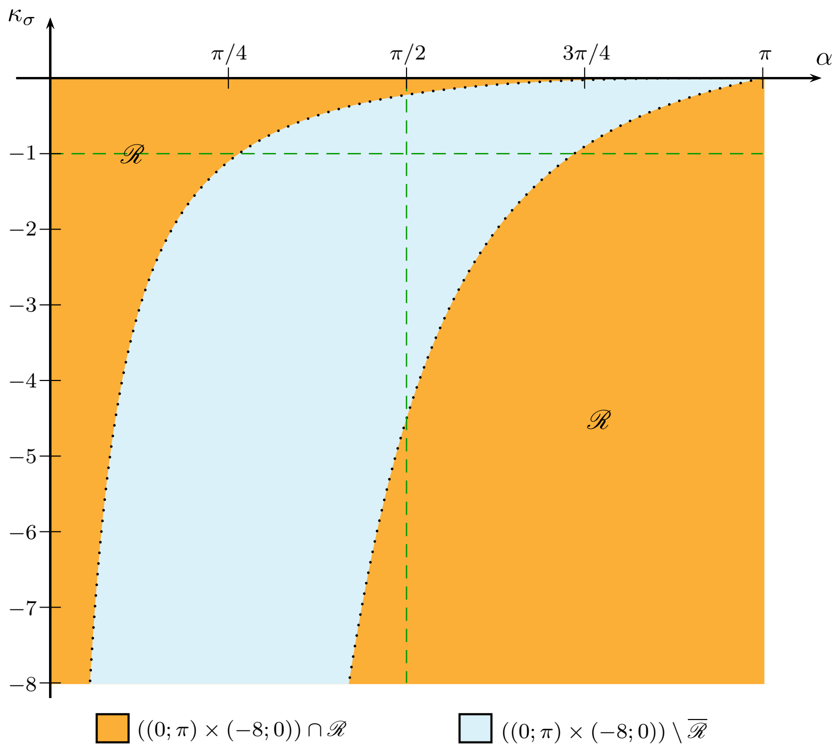

Figure 3: When (α,κ_σ) belongs to the orange shaded region $\mathscr{R}$, there exist singularities of the form $s(r,\theta)=r^{1+i\eta}\varphi(\theta)$ with $\eta\in\mathbb{R}^*$. Therefore, in these situations the operator $B:\mathrm{H}^2_0(\Omega)\to\mathrm{H}^2_0(\Omega)$ defined in (7) is not of Fredholm type.

Ill-posedness. The following lemma, known as the Peetre's lemma [37] (see also lemma 5.1 in [30, Chap. 2], or lemma 3.4.1 in [27]), provides a necessary and sufficient condition to ensure that a bounded selfadjoint operator is of Fredholm type.

Lemma 3.2. *Let* X, Y *and* Z *be three Banach spaces such that* X *is compactly embedded into* Z. *Let* $L:\mathrm{X}\to\mathrm{Y}$ *be a continuous linear map. Then the assertions below are equivalent:*

$i)$ $\dim(\ker L)<\infty$ *and* $\operatorname{range}L$ *is closed in* Y*;*
$ii)$ *there exists* $C>0$ *such that* $\|v\|_{\mathrm{X}}\le C\,(\,\|Lv\|_{\mathrm{Y}}+\|v\|_{\mathrm{Z}}\,)$, $\qquad\forall v\in\mathrm{X}$.

Now, we state and prove the main result of §3.2.

Proposition 3.1. *Assume that* (σ,κ_σ) *belongs to the region* $\mathscr{R}$ *defined in (23). Then, the operator* $B:\mathrm{H}^2_0(\Omega)\to\mathrm{H}^2_0(\Omega)$ *defined in (7) is not of Fredholm type.*

Proof. When (σ,κ_σ) belongs to the region $\mathscr{R}$, we know that there exists a singularity $s(\boldsymbol{x})=r^{1+i\eta}\varphi(\theta)$ with $\eta\in\mathbb{R}^*$ such that $\Delta(\sigma\Delta s)=0$ a.e. in Ξ and $s=\partial_\nu s=0$ a.e. on $\partial\Xi$. Let $\zeta\in\mathscr{C}^\infty_0([0;+\infty),[0;1])$ be a cut-off function equal to 1 in a neighbourhood of O. For $m\in\mathbb{N}^*=\{1,2,3,\dots\}$, we define

$$s_m(\boldsymbol{x}):=r^{1+i\eta+1/m}\varphi(\theta)\qquad\text{and}\qquad v_m(\boldsymbol{x}):=\zeta(r)s_m(\boldsymbol{x}).$$

We assume that the support of ζ is such that $v_m|_{\partial\Omega} = 0$. Since $\Re e\,(1 + i\eta + 1/m) > 1$, one can check by a direct computation that the function v_m belongs to $\mathrm{H}^2_0(\Omega)$ for all $m \in \mathbb{N}^*$. Our goal is to establish the following properties

$$\lim_{m\to+\infty} \|v_m\|_{\mathrm{H}^2_0(\Omega)} = +\infty \qquad \text{and} \qquad \|Bv_m\|_{\mathrm{H}^{-2}(\Omega)} + \|v_m\|_{\mathrm{H}^1_0(\Omega)} \le C, \qquad \forall m \in \mathbb{N}^*.$$

Together with Lemma 3.2, this will prove that B is not of Fredholm type when $(\sigma, \kappa_\sigma) \in \mathscr{R}$.

$\star$ Behaviour of $(\|v_m\|_{\mathrm{H}^2_0(\Omega)})_m$. By definition, we have $\|v_m\|_{\mathrm{H}^2_0(\Omega)} = \|\Delta v_m\|_\Omega \ge \|\Delta v_m\|_{\widetilde{\Omega}}$, where $\widetilde{\Omega} := \{\boldsymbol{x} \in \Omega \,|\, \zeta(r) = 1\}$. On $\widetilde{\Omega}$, we find

$$\Delta v_m = r^{-1+i\eta+1/m}((1 + i\eta + 1/m)^2 \varphi(\theta) + \varphi''(\theta)).$$

Let us define the function ς_m such that $\varsigma_m(\theta) = (1 + i\eta + 1/m)^2\varphi(\theta) + \varphi''(\theta)$. We have

$$\begin{array}{rcl} \|\Delta v_m\|^2_{\widetilde{\Omega}} & \ge & \displaystyle\int_0^\delta \int_0^\pi r^{-2+2/m} |\varsigma_m(\theta)|^2 \, r d\theta dr \\ & \ge & \displaystyle\|\varsigma_m\|^2_{(0;\pi)} \int_0^\delta r^{-1+2/m} dr = \|\varsigma_m\|^2_{(0;\pi)} \, \frac{m}{2} \, \delta^{2/m}. \end{array} \tag{24}$$

Above $\delta > 0$ is a fixed small number such that $\zeta(r) = 1$ on $[0; \delta]$. Since $(1+i\eta)^2\varphi + \varphi'' \not\equiv 0$ (otherwise, we should have $\Delta s = 0$, which is impossible due to the boundary conditions), there holds $\|\varsigma_m\|^2_{(0;\pi)} \ne 0$ for m large enough. From (24), we deduce that $\|v_m\|_{\mathrm{H}^2_0(\Omega)} \underset{m\to+\infty}{\to} +\infty$.

$\star$ Behaviour of $(\|v_m\|_{\mathrm{H}^1_0(\Omega)})_m$. By a direct computation, we can check that there exists a constant $C > 0$ such that for all $m \in \mathbb{N}^*$, we have $\|v_m\|_{\mathrm{H}^1_0(\Omega)} \le C$.

$\star$ Behaviour of $(\|Bv_m\|_{\mathrm{H}^{-2}(\Omega)})_m$. Now, let us prove that the sequence $(\|\Delta(\sigma\Delta v_m)\|_{\mathrm{H}^{-2}(\Omega)})_m$ remains bounded. By definition, we have

$$\|\Delta(\sigma\Delta v_m)\|_{\mathrm{H}^{-2}(\Omega)} := \sup_{v\in\mathrm{H}^2_0(\Omega)\,|\,\|\Delta v\|_\Omega=1} |(\sigma\Delta v_m, \Delta v)_\Omega| = \sup_{v\in\mathscr{C}^\infty_0(\Omega)\,|\,\|\Delta v\|_\Omega=1} |(\sigma\Delta v_m, \Delta v)_\Omega|\,.$$

We compute, for all $v \in \mathscr{C}^\infty_0(\Omega)$, $\Delta v_m = \Delta(\zeta s_m) = \zeta\Delta s_m + 2\nabla s_m\nabla\zeta + s_m\Delta\zeta$ and $\Delta(\zeta v) = \zeta\Delta v + 2\nabla v\nabla\zeta + v\Delta\zeta$. This allows to write

$$\begin{array}{rcl} (\sigma\Delta v_m, \Delta v)_\Omega & = & (\sigma(\zeta\Delta s_m + 2\nabla v_m\nabla\zeta + v_m\Delta\zeta), \Delta v)_\Omega \\ & = & (\sigma\Delta s_m, \Delta(\zeta v) - 2\nabla v\nabla\zeta - v\Delta\zeta)_\Omega + (\sigma(2\nabla s_m\nabla\zeta + s_m\Delta\zeta), \Delta v)_\Omega. \end{array}$$

We deduce

$$\begin{array}{rcl} |(\sigma\Delta v_m, \Delta v)_\Omega| & \le & \underset{❶}{|(\sigma\Delta s_m, \Delta(\zeta v))_\Omega|} + 2\underset{❷}{|(\sigma\Delta s_m, \nabla v\nabla\zeta)_\Omega|} + \underset{❸}{|(\sigma\Delta s_m, v\Delta\zeta)_\Omega|} \\ & & +2\underset{❹}{|(\sigma\nabla s_m\nabla\zeta, \Delta v)_\Omega|} + \underset{❺}{|(\sigma s_m\Delta\zeta, \Delta v)_\Omega|}. \end{array}$$

Let us study the term ❶. Integrating by parts, we find $(\sigma\Delta s_m, \Delta(\zeta v))_\Omega = (\Delta(\sigma\Delta s_m), \zeta v)_\Omega$. A direct computation allows to establish that $\Delta(\sigma\Delta s_m)(\boldsymbol{x}) = r^{-3+i\eta+1/m}\tilde{\varphi}_m(\theta)$ where $\tilde{\varphi}_m \in \mathrm{L}^2(0;\pi)$ is such that $\|\tilde{\varphi}_m\|_{(0;\pi)} \le C/m$ for some constant C independent of m. Therefore, we can write

$$\begin{array}{rcl} |(\sigma\Delta s_m, \Delta(\zeta v))_\Omega| & = & |(r^{-3+i\eta+1/m}\tilde{\varphi}_m, \zeta v)_\Omega| \\ & = & C\,|(r^{-1+i\eta+1/m}\tilde{\varphi}_m, r^{-2}(r\partial_r)^2(\zeta v))_\Omega| \\ & \le & C\,\|r^{-1+i\eta+1/m}\tilde{\varphi}_m\|_\Omega \|v\|_{\mathrm{H}^2_0(\Omega)} \;\le\; C^{2/m}\|v\|_{\mathrm{H}^2_0(\Omega)}. \end{array}$$

In the above estimate, C is a constant independent of m which can change from one line to another. The second line is obtained proceeding to an integration by part with respect to the r variable. This proves that ❶ is bounded as $m \to +\infty$. To deal with the terms ❷ and ❸, we use that $\nabla\zeta$ and $\Delta\zeta$ vanish in a neighbourhood of O to obtain

$$\begin{aligned} &|(\sigma\Delta s_m, \nabla v\nabla\zeta)_\Omega| = |(\sigma\Delta s_m, \nabla v\nabla\zeta)_{\Omega\setminus B(O,\delta)}| \le C\,\|\Delta s_m\|_{\Omega\setminus B(O,\delta)}\|v\|_{\mathrm{H}^2_0(\Omega)} \\ \text{and}\quad &|(\sigma\Delta s_m, v\Delta\zeta)_\Omega| = |(\sigma\Delta s_m, v\Delta\zeta)_{\Omega\setminus B(O,\delta)}| \le C\,\|\Delta s_m\|_{\Omega\setminus B(O,\delta)}\|v\|_{\mathrm{H}^2_0(\Omega)}. \end{aligned}$$

Since the sequence $(\|\Delta v_m\|_{\Omega\setminus B(O,\delta)})_m$ is bounded, we deduce that the terms ❷ and ❸ remain bounded. Finally, to deal with the terms ❹ and ❺, we write

$$|(\sigma\nabla s_m\nabla\zeta, \Delta v)_\Omega| + |(\sigma s_m\Delta\zeta, \Delta v)_\Omega| \le C\,\|s_m\|_{\mathrm{H}^1_0(\Omega)}\|v\|_{\mathrm{H}^2_0(\Omega)}.$$

Since the sequence $(\|s_m\|_{\mathrm{H}^1_0(\Omega)})_m$ is bounded, we deduce that the terms ❹ and ❺ are bounded. All these intermediate results allow to conclude that the sequence $(\|\Delta(\sigma\Delta v_m)\|_{\mathrm{H}^{-2}(\Omega)})_m$ remains bounded as $m \to +\infty$. □

Remark 3.3. *For the the anisotropic version of the ITEP [8, 25, 11], set in* H^1, *analogous strong singularities can also appear as proven in [1]. In this article, it is shown how to take them into account and how to modify the functional framework to recover a problem well-posed in the Fredholm sense. One can probably adapt the approach to handle situations where* $B : \mathrm{H}^2_0(\Omega) \to \mathrm{H}^2_0(\Omega)$ *is not of Fredholm type. However, to be in position to apply the analytic Fredholm theorem to establish that the set of interior transmission eigenvalues is discrete, it is necessary to demonstrate that there exists one frequency such that the ITEP in this new framework is injective. This is still an open question.*

4 Bilaplacian with mixed boundary conditions: polygonal domains with reentrant corners

The operator $\tilde{B} : \mathrm{H}^1_0(\Omega) \cap \mathrm{H}^2(\Omega) \to \mathrm{H}^1_0(\Omega) \cap \mathrm{H}^2(\Omega)$ defined in (12) is an isomorphism when Ω is convex or smooth. In this section, we analyse the properties of $\tilde{B}$ when $\Omega \subset \mathbb{R}^2$ is an open set with a polygonal boundary $\partial\Omega$ presenting reentrant corners. For such domains, using an integration by parts, one proves the *a priori* estimate (see [22, theorem 2.2.3] or [23, 31]):

$$\|v\|_{\mathrm{H}^2(\Omega)} \le C\,\|\Delta v\|_\Omega, \qquad \forall v \in \mathrm{H}^1_0(\Omega) \cap \mathrm{H}^2(\Omega),$$

where C is a constant which depends only on Ω. This estimate provides lot of information. It proves that the operator $\Delta : \mathrm{H}^1_0(\Omega) \cap \mathrm{H}^2(\Omega) \to \mathrm{L}^2(\Omega)$ is injective and that its range is closed (it is a monomorphism). Then we can try to characterize the orthogonal complement of the range of $\Delta : \mathrm{H}^1_0(\Omega) \cap \mathrm{H}^2(\Omega) \to \mathrm{L}^2(\Omega)$. Theorem 2.3.7 of [22] indicates that this orthogonal complement is of finite dimension N, where N is equal to the number of corners of $\partial\Omega$ whose aperture is strictly larger than π. Thus, $\Delta : \mathrm{H}^1_0(\Omega)\cap\mathrm{H}^2(\Omega) \to \mathrm{L}^2(\Omega)$ is an injective Fredholm operator of index $-N$. When there is no reentrant corner in $\partial\Omega$, *i.e.* when Ω is convex, we find back that $\Delta : \mathrm{H}^1_0(\Omega) \cap \mathrm{H}^2(\Omega) \to \mathrm{L}^2(\Omega)$ is an isomorphism.

4.1 Saponjyan paradox in the case of a positive σ

We assume in this paragraph that there exists a constant C such that $\sigma \ge C > 0$ a.e. in Ω. In this case, the form $\tilde{b}$ defined in (11) is coercive on $\mathrm{H}^1_0(\Omega)\cap\mathrm{H}^2(\Omega) \times \mathrm{H}^1_0(\Omega)\cap\mathrm{H}^2(\Omega)$ whether the domain Ω is convex or not. According to the Lax-Milgram theorem, ($\tilde{\mathscr{P}}$) has a unique solution $v \in \mathrm{H}^1_0(\Omega)\cap\mathrm{H}^2(\Omega)$ and $\tilde{B}$ is an isomorphism of $\mathrm{H}^1_0(\Omega) \cap \mathrm{H}^2(\Omega)$.

Now, let us try to solve ($\tilde{\mathscr{P}}$) in two steps. To simplify the explanation, we assume here that f belongs to $\mathrm{H}^{-1}(\Omega)$. Let us denote $p_0 \in \mathrm{H}^1_0(\Omega)$ the function such that $-(\nabla p_0, \nabla p')_\Omega = \langle f, p'\rangle_\Omega$ for all $p' \in \mathrm{H}^1_0(\Omega)$. Then, let us introduce v_0 the element of $\mathrm{H}^1_0(\Omega)$ satisfying $\Delta v_0 = \sigma^{-1}p_0 \in \mathrm{L}^2(\Omega)$. When Ω is convex, the function v_0 is an element of $\mathrm{H}^1_0(\Omega)\cap\mathrm{H}^2(\Omega)$. In this case, v_0 verifies ($\tilde{\mathscr{P}}$). Since this problem is well-posed, we deduce $v_0 = v$. When $\partial\Omega$ has one or several reentrant corners, it can happen that $v_0 \notin \mathrm{H}^1_0(\Omega)\cap\mathrm{H}^2(\Omega)$. In this situation, we have $(\sigma\Delta v_0, \Delta v')_\Omega = \langle f, v'\rangle_\Omega$ for all $v' \in \mathrm{H}^1_0(\Omega)\cap\mathrm{H}^2(\Omega)$ but

v_0 is not equal to the solution v of $(\tilde{\mathscr{P}})$. This is what S.A. Nazarov and G.H. Sweers call, in the very interesting papers [32, 33, 34], the *Saponjyan's paradox*. Saponjyan was an Armenian scientist from the XXth century. Thanks to conformal mapping techniques, he obtained a solution which was not of finite mechanical energy. Since he had no explanation for that, he called this phenomenon a paradox.

Nevertheless, and this is the topic of articles [32, 33, 34], even when Ω has a reentrant corner, there exists a mean of solving $(\tilde{\mathscr{P}})$ in two steps while obtaining the solution of finite energy, that is the one in $\mathrm{H}^1_0(\Omega)\cap\mathrm{H}^2(\Omega)$. Let us describe the process. To simplify, we assume that Ω has only one reentrant corner O of aperture equal to $\alpha \in (\pi; 2\pi)$ (cf. Figure 4). The method consists in solving in a clever way the two Laplace problems with homogeneous Dirichlet boundary condition which appears in $(\tilde{\mathscr{P}})$. Before proceeding further, we need to recall some classical results (see for example [21, 22]) of the theory of singularities for the Laplace operator in non convex polygons.

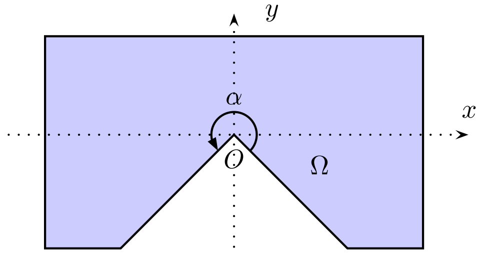

Figure 4: An example of a non convex polygonal boundary with one reentrant corner.

Let us introduce ζ such that
$$\zeta(\boldsymbol{x}) = r^{-\pi/\alpha}\sin(\pi\theta/\alpha) + \tilde{\zeta}(\boldsymbol{x}). \tag{25}$$
where $\tilde{\zeta}$ is the unique function of $\mathrm{H}^1(\Omega)$ satisfying $\Delta\tilde{\zeta} = 0$ a.e. in Ω and $\tilde{\zeta} = -r^{-\pi/\alpha}\sin(\pi\theta/\alpha)$ a.e. on $\partial\Omega$. Here, (r,θ) are the polar coordinates centered at O, such that $\theta = 0$ or $\theta = \alpha$ on $\partial\Omega$ in a neighbourhood of O. We assume that Ω is not convex at O. This imposes $0 < \pi/\alpha < 1$. By a straightforward computation, one proves that $\boldsymbol{x} \mapsto r^{-\pi/\alpha}\sin(\pi\theta/\alpha)$ belongs to $\mathrm{L}^2(\Omega)\backslash\mathrm{H}^1(\Omega)$. Since $\tilde{\zeta} \in \mathrm{H}^1(\Omega)$, we deduce that the function ζ defined in (25) satisfies $\zeta \in \mathrm{L}^2(\Omega)\backslash\mathrm{H}^1(\Omega)$. By definition of $\tilde{\zeta}$, there hold $\Delta\zeta = 0$ a.e. in Ω and $\zeta = 0$ a.e. on $\partial\Omega$. According to (9), this is equivalent to the following integral identity: $(\zeta, \Delta v')_\Omega = 0$ for all $v' \in \mathrm{H}^1_0(\Omega)\cap\mathrm{H}^2(\Omega)$. Actually, $\{\zeta\}$ constitutes a basis of the set of L^2-functions satisfying such property (see [22, lemma 2.3.6]). Let us include this result in the following proposition.

Proposition 4.1. *Assume that the boundary $\partial\Omega$ is a polygon with one reentrant corner. Then for all $f \in (\mathrm{H}^1_0(\Omega)\cap\mathrm{H}^2(\Omega))^*$, there exists a solution to the problem*

$$\left|\begin{array}{l} \text{Find } p \in \mathrm{L}^2(\Omega) \text{ such that:} \\ (p, \Delta v')_\Omega = \langle f, v'\rangle_\Omega\,, \qquad \forall v' \in \mathrm{H}^1_0(\Omega)\cap\mathrm{H}^2(\Omega). \end{array}\right. \tag{26}$$

Moreover, if p_1, p_2 are two solutions of Problem (26), then there exists $a \in \mathbb{C}$ such that $p_2 = p_1 + a\zeta$, where ζ is introduced in (25).

Proof. The map $\Delta : \mathrm{H}^1_0(\Omega)\cap\mathrm{H}^2(\Omega) \to \mathrm{L}^2(\Omega)$ is a monomorphism. As a consequence, the adjoint operator is onto and for all $f \in (\mathrm{H}^1_0(\Omega)\cap\mathrm{H}^2(\Omega))^*$, Problem (26) has at least one solution. Now, if p_1, p_2 are two solutions of Problem (26), then we have $(p_2 - p_1, \Delta v')_\Omega = 0$ for all $v' \in \mathrm{H}^1_0(\Omega)\cap\mathrm{H}^2(\Omega)$. With [22, lemma 2.3.6], we deduce that there holds $p_2 - p_1 = a\zeta$, where a is a constant. $\square$

Now, we provide a result of decomposition (see for example [22, theorem 2.4.3] or [27, p. 263]) which states that the elements of $\mathrm{H}^1_0(\Omega)$ whose Laplacian belongs to $\mathrm{L}^2(\Omega)$ split as the sum of an explicit singular part and a regular part in $\mathrm{H}^2(\Omega)$.

Proposition 4.2. *Consider $\varphi \in \mathrm{H}^1_0(\Omega)$ such that $\Delta\varphi = g \in \mathrm{L}^2(\Omega)$. Then φ admits the decomposition*

$$\varphi(\boldsymbol{x}) = c\, r^{\pi/\alpha}\sin(\pi\theta/\alpha) + \tilde{\varphi}(\boldsymbol{x}), \tag{27}$$

with $\tilde{\varphi} \in \mathrm{H}^2(\Omega)$. Moreover, the coefficient c in (27) is given by the following expression

$$c = -(\pi)^{-1}(g, \zeta)_\Omega,$$

where ζ is defined in (25).

Noticing that $\boldsymbol{x} \mapsto r^{\pi/\alpha} \sin(\pi\theta/\alpha)$ belongs to $\mathrm{H}^1(\Omega) \backslash \mathrm{H}^2(\Omega)$ (since $\pi/\alpha < 1$), we deduce the

Corollary 4.1. *Let $\varphi \in \mathrm{H}^1_0(\Omega)$ be such that $\Delta\varphi = g \in \mathrm{L}^2(\Omega)$. Then $\varphi \in \mathrm{H}^2(\Omega)$ if and only if $(g, \zeta)_\Omega = 0$.*

Now, we have all the tools we need to solve $(\tilde{\mathscr{P}})$ in two steps. For any source term $f \in (\mathrm{H}^1_0(\Omega) \cap \mathrm{H}^2(\Omega))^*$, let us introduce p_0 an element of $\mathrm{L}^2(\Omega)$ such that $(p_0, \Delta v')_\Omega = \langle f, v' \rangle_\Omega$ for all $v' \in \mathrm{H}^1_0(\Omega) \cap \mathrm{H}^2(\Omega)$. The existence of such a function p_0 is guaranteed by Proposition 4.1. For $a \in \mathbb{C}$, we define $p = p_0 + a\zeta$, where ζ is given by (25), and we denote v the unique function of $\mathrm{H}^1_0(\Omega)$ such that $\Delta v = \sigma^{-1} p$. We want v to be in $\mathrm{H}^2(\Omega)$. According to Corollary 4.1, we must take a such that

$$\begin{aligned} & 0 = (\sigma^{-1} p, \zeta)_\Omega = (\sigma^{-1} p_0, \zeta)_\Omega + a\, (\sigma^{-1} \zeta, \zeta)_\Omega \\ \Leftrightarrow \quad & a = -(\sigma^{-1} p_0, \zeta)_\Omega / (\sigma^{-1} \zeta, \zeta)_\Omega. \end{aligned} \tag{28}$$

To conclude that v constitutes the solution of $(\tilde{\mathscr{P}})$, it just remains to notice that, for all $v' \in \mathrm{H}^1_0(\Omega) \cap \mathrm{H}^2(\Omega)$, there holds

$$(\sigma \Delta v, \Delta v')_\Omega = (p, \Delta v')_\Omega = (p_0, \Delta v')_\Omega = \langle f, v' \rangle_\Omega .$$

4.2 Study in the case where σ changes sign

When σ changes sign, the sesquilinear form $\tilde{b}$ associated with $(\tilde{\mathscr{P}})$ is not coercive. When the domain Ω is convex or of class $\mathscr{C}^2$, constructing the inverse of $\tilde{B}$, we proved with Theorem 2.1 that $(\tilde{\mathscr{P}})$ is well-posed. Moreover, in the second proof of Theorem 2.1, we established that under one of these two assumptions, $(\tilde{\mathscr{P}})$ can also be solved in two steps. In the previous paragraph, for a positive σ, we presented how to solve $(\tilde{\mathscr{P}})$ in two steps when Ω has one reentrant corner. Our goal is to extend this approach to deal with configurations where σ changes sign. The novelty is that, according to the values of σ, a kernel and a cokernel, whose dimensions are less or equal to the number of reentrant corners of the domain, can appear.

To clarify the presentation, we shall assume that $\partial\Omega$ has only one reentrant corner located at O, of aperture $\alpha \in (\pi; 2\pi)$. The approach to consider domains with several reentrant corners is very similar and requires only some simple additional arguments of linear algebra (we refer the reader to [12, §2.2.2.2] for the details). Working as in §4.1, we see that the resolution in two steps can be used to prove that the operator $\tilde{B}$ associated with $(\tilde{\mathscr{P}})$ is onto as soon as σ satisfies $(\sigma^{-1}\zeta, \zeta)_\Omega \neq 0$. This leads us to consider two cases: either σ is such that $(\sigma^{-1}\zeta, \zeta)_\Omega \neq 0$ or σ is such that $(\sigma^{-1}\zeta, \zeta)_\Omega = 0$.

⋆⋆⋆⋆⋆

Case $(\sigma^{-1}\zeta, \zeta)_\Omega \neq 0$

⋆⋆⋆⋆⋆

Proposition 4.3. *Assume that the boundary $\partial\Omega$ is a polygon with one reentrant corner. Assume that $\sigma \in \mathrm{L}^\infty(\Omega)$ verifies $\sigma^{-1} \in \mathrm{L}^\infty(\Omega)$ and $(\sigma^{-1}\zeta, \zeta)_\Omega \neq 0$, where ζ is defined in (25). Then, the operator $\tilde{B} : \mathrm{H}^1_0(\Omega) \cap \mathrm{H}^2(\Omega) \to \mathrm{H}^1_0(\Omega) \cap \mathrm{H}^2(\Omega)$ defined in (12) is an isomorphism.*

Proof. Let us introduce the operator $\mathtt{T}$ such that, for $v \in \mathrm{H}^1_0(\Omega) \cap \mathrm{H}^2(\Omega)$, the function $\mathtt{T}v \in \mathrm{H}^1_0(\Omega)$ satisfies $\Delta(\mathtt{T}v) = \sigma^{-1}(\Delta v + a\zeta)$ with $a = -(\sigma^{-1}\Delta v, \zeta)_\Omega / (\sigma^{-1}\zeta, \zeta)_\Omega$. Since $(\sigma^{-1}(\Delta v + a\zeta), \zeta)_\Omega = 0$, we know that $\mathtt{T}v$ belongs to $\mathrm{H}^2(\Omega)$ according to Corollary 4.1. Thus, $\mathtt{T}$ is a continuous operator from $\mathrm{H}^1_0(\Omega) \cap \mathrm{H}^2(\Omega)$ to $\mathrm{H}^1_0(\Omega) \cap \mathrm{H}^2(\Omega)$. For all $(v, v') \in \mathrm{H}^1_0(\Omega) \cap \mathrm{H}^2(\Omega) \times \mathrm{H}^1_0(\Omega) \cap \mathrm{H}^2(\Omega)$, we then compute

$$\begin{array}{rclcl} (\tilde{B}(\mathtt{T}v), v')_{\mathrm{H}^2_0(\Omega)} & = & \tilde{b}(\mathtt{T}v, v') & = & (\sigma \Delta(\mathtt{T}v), \Delta v')_\Omega \\ & & & = & (\Delta v + a\zeta, \Delta v')_\Omega \\ & & & = & (\Delta v, \Delta v')_\Omega = (v, v')_{\mathrm{H}^2_0(\Omega)}. \end{array}$$

The last line is obtained noticing that $(\zeta, \Delta v')_\Omega = 0$ because $v' \in \mathrm{H}^1_0(\Omega) \cap \mathrm{H}^2(\Omega)$ (Corollary 4.1). Thus, there holds $\tilde{B} \circ \mathtt{T} = \mathrm{Id}$. Since $\tilde{B}$ is selfadjoint, we deduce that $\tilde{B}$ is an isomorphism with $\tilde{B}^{-1} = \mathtt{T}$. □

We stop here the study of the case $(\sigma^{-1}\zeta,\zeta)_\Omega \neq 0$ and focus now our attention on the configuration $(\sigma^{-1}\zeta,\zeta)_\Omega = 0$.

⋆⋆⋆⋆⋆

Case $(\sigma^{-1}\zeta,\zeta)_\Omega = 0$

⋆⋆⋆⋆⋆

In this case, we can no longer use the precious degree of freedom to construct a solution to $(\tilde{\mathscr{P}})$ in $\mathrm{H}^1_0(\Omega)\cap\mathrm{H}^2(\Omega)$. Let us denote ψ the function of $\mathrm{H}^1_0(\Omega)$ satisfying

$$\Delta\psi = \sigma^{-1}\zeta. \tag{29}$$

Since $(\sigma^{-1}\zeta,\zeta)_\Omega = 0$, Corollary 4.1 indicates that ψ belongs to $\mathrm{H}^2(\Omega)$. Moreover, for all $v'\in\mathrm{H}^1_0(\Omega)\cap\mathrm{H}^2(\Omega)$, there holds $(\sigma\Delta\psi,\Delta v')_\Omega = (\zeta,\Delta v')_\Omega = 0$. Therefore, ψ constitutes an element of $\ker\tilde{B}$.

Proposition 4.4. *Assume that the boundary $\partial\Omega$ is a polygon with one reentrant corner. Assume that $\sigma\in\mathrm{L}^\infty(\Omega)$ verifies $\sigma^{-1}\in\mathrm{L}^\infty(\Omega)$ and $(\sigma^{-1}\zeta,\zeta)_\Omega = 0$. Then,*

- $\dim(\ker\tilde{B}) = 1$ *with* $\ker\tilde{B} = \mathrm{span}(\psi)$*;*
- $\dim(\mathrm{coker}\,\tilde{B}) = 1$ *and for all $f\in(\mathrm{H}^1_0(\Omega)\cap\mathrm{H}^2(\Omega))^*$, $(\tilde{\mathscr{P}})$ has a solution if and only if $\langle f,\psi\rangle_\Omega = 0$.*

In this statement, the functions ζ and ψ are respectively defined in (25) and (29).

Proof. ⋆ Kernel. If v belongs to $\ker\tilde{B}$ then, according to Proposition 4.1, we have $\sigma\Delta v = a\zeta$ where a is a constant. Thus, $\ker\tilde{B}\subset\mathrm{span}(\psi)$. As indicated above, we have $\psi\in\ker\tilde{B}$ and so $\ker\tilde{B} = \mathrm{span}(\psi)$.
⋆ Cokernel. Let us consider $f\in(\mathrm{H}^1_0(\Omega)\cap\mathrm{H}^2(\Omega))^*$ such that $\langle f,\psi\rangle_\Omega = 0$. Let us introduce p_0 an element of $\mathrm{L}^2(\Omega)$ such that $(p_0,\Delta v')_\Omega = \langle f,v'\rangle_\Omega$ for all $v'\in\mathrm{H}^1_0(\Omega)\cap\mathrm{H}^2(\Omega)$. The existence of such a function p_0 is ensured by Proposition 4.1. The function p_0 satisfies the compatibility condition $(\sigma^{-1}p_0,\zeta)_\Omega = 0$. Indeed, since $\Delta\psi = \sigma^{-1}\zeta$, we can write $(\sigma^{-1}p_0,\zeta)_\Omega = (p_0,\sigma^{-1}\zeta)_\Omega = (p_0,\Delta\psi)_\Omega = \langle f,\psi\rangle_\Omega = 0$. As a consequence, by virtue of Corollary 4.1, the function $v\in\mathrm{H}^1_0(\Omega)$ verifying $\Delta v = \sigma^{-1}p_0$ is in $\mathrm{H}^2(\Omega)$. Moreover, for all $v'\in\mathrm{H}^1_0(\Omega)\cap\mathrm{H}^2(\Omega)$, there holds

$$(\sigma\Delta v,\Delta v')_\Omega = (p_0,\Delta v')_\Omega = \langle f,v'\rangle_\Omega\,.$$

Therefore, v constitutes a solution to $(\tilde{\mathscr{P}})$. Now, let us consider $f\in(\mathrm{H}^1_0(\Omega)\cap\mathrm{H}^2(\Omega))^*$ such that $\langle f,\psi\rangle_\Omega \neq 0$. Let us assume that there exists a solution v to $(\tilde{\mathscr{P}})$. Then, by Proposition 4.1, we have $\sigma\Delta v = p_0 + a\zeta$, where $p_0\in\mathrm{L}^2(\Omega)$ satisfies $(p_0,\Delta v')_\Omega = \langle f,v'\rangle_\Omega$ for all $v'\in\mathrm{H}^1_0(\Omega)\cap\mathrm{H}^2(\Omega)$ and where a is a constant. This imposes $(\sigma^{-1}p_0,\zeta)_\Omega = 0$. But there holds $(\sigma^{-1}p_0,\zeta)_\Omega = (p_0,\sigma^{-1}\zeta)_\Omega = (p_0,\Delta\psi)_\Omega = \langle f,\psi\rangle_\Omega$. Thus, we obtain an absurdity. This ends to prove that there exists a solution to $(\tilde{\mathscr{P}})$ if and only if $\langle f,\psi\rangle_\Omega = 0$. □

⋄ Example. Consider the open set Ω of Figure 4 which presents the particularity to be symmetric with respect to the axis (Oy). Notice also that the vertex of the reentrant corner is located on (Oy). For this configuration, we can prove that the function ζ defined in (25) is symmetric with respect to (Oy). Indeed, the function $\hat{\zeta}:(x,y)\mapsto\zeta(-x,y)$ verifies Problem (26) with $f=0$. But Proposition 4.1 indicates that $\{\zeta\}$ is a basis of the set of functions satisfying Problem (26) with $f=0$. Using the behaviour of $\hat{\zeta}$ at O, this allows us to conclude that $\hat{\zeta}=\zeta$. Therefore, when σ is skewsymmetric with respect to the axis (Oy), *i.e.* when $\sigma(x,y) = -\sigma(-x,y)$ a.e. in Ω, there holds $(\sigma^{-1}\zeta,\zeta)_\Omega = 0$. In this situation, $(\tilde{\mathscr{P}})$ has a kernel and a cokernel which are both of dimension one.

4.3 Discussion

Analogies with the Maxwell's problem. The method developed to prove well-posedness of Problem $(\tilde{\mathscr{P}})$ in geometries with reentrant corners presents strong analogies with the one used in [3] to establish well-posedness of Maxwell's equations when the permittivity ε and/or the permeability μ change sign on the domain. Let us clarify this point.

Let D be a simply connected domain of $\mathbb{R}^3$ with a connected boundary ∂D. We denote ν the unit outward normal vector to ∂D. In the study of the Maxwell's problem, we are led to consider the operator $\tilde{M} : \mathrm{V}_T(D) \to \mathrm{V}_T(D)$ such that

$$(\tilde{M}\boldsymbol{v}, \boldsymbol{v}')_{\mathbf{curl}} = (\sigma\,\mathbf{curl}\,\boldsymbol{v}, \mathbf{curl}\,\boldsymbol{v}')_D, \qquad \forall(\boldsymbol{v}, \boldsymbol{v}') \in \mathrm{V}_T(D) \times \mathrm{V}_T(D), \tag{30}$$

where $\mathrm{V}_T(D) := \{\boldsymbol{\varphi} \in \mathrm{L}^2(D)^3 \,|\, \mathbf{curl}\,\boldsymbol{\varphi} \in \mathrm{L}^2(D)^3,\ \mathrm{div}\,\boldsymbol{\varphi} = 0 \text{ in } D \text{ and } \boldsymbol{\varphi}\cdot\nu = 0 \text{ on } \partial D\}$ and where $(\cdot,\cdot)_{\mathbf{curl}} = (\mathbf{curl}\,\cdot, \mathbf{curl}\,\cdot)_D + (\cdot,\cdot)_D$. In (30), σ corresponds to the inverse of ε. We recall that when D is simply connected, the map $(\boldsymbol{\varphi}, \boldsymbol{\varphi}') \mapsto (\mathbf{curl}\,\boldsymbol{\varphi}, \mathbf{curl}\,\boldsymbol{\varphi}')_D$ defines an inner product on $\mathrm{V}_T(D)$ and the associated norm is equivalent to the canonical norm $\boldsymbol{\varphi} \mapsto (\boldsymbol{\varphi}, \boldsymbol{\varphi})^{1/2}_{\mathbf{curl}}$. This allows to obtain the decomposition $\mathrm{L}^2(D)^3 = \mathbf{curl}\,(\mathrm{V}_T(D)) \overset{\perp}{\oplus} \nabla(\mathrm{H}^1_0(D))$ (which is the analog of $\mathrm{L}^2(\Omega) = \Delta(\mathrm{H}^1_0(\Omega) \cap \mathrm{H}^2(\Omega)) \overset{\perp}{\oplus} \mathrm{span}(\zeta)$, where ζ is defined in (25), for the polygonal domain with one reentrant corner). Assume now that the operator $S : \mathrm{H}^1_0(D) \to \mathrm{H}^1_0(D)$ such that

$$(\nabla(S\varphi), \nabla\varphi')_\Omega = (\sigma^{-1}\nabla\varphi, \nabla\varphi')_\Omega, \qquad \forall(\varphi, \varphi') \in \mathrm{H}^1_0(D) \times \mathrm{H}^1_0(D),$$

is an isomorphism (which is the analog of $(\sigma^{-1}\zeta, \zeta)_\Omega \neq 0$). Then in this case, we can show that $\tilde{M}$ is an isomorphism working as in the proof of Proposition 4.3. More precisely, for $\boldsymbol{v} \in \mathrm{V}_T(D)$, introduce ψ the element of $\mathrm{H}^1_0(D)$ such that $(\sigma^{-1}\nabla\psi, \nabla\psi')_\Omega = (\sigma^{-1}\mathbf{curl}\,\boldsymbol{v}, \nabla\psi')_\Omega$ for all $\psi' \in \mathrm{H}^1_0(D)$. The term $\sigma^{-1}(\mathbf{curl}\,\boldsymbol{v} - \nabla\psi)$ belongs to $\nabla(\mathrm{H}^1_0(D))^\perp$ and there is a unique function $\mathbb{T}\boldsymbol{v} \in \mathrm{V}_T(D)$ such that $\mathbf{curl}\,(\mathbb{T}\boldsymbol{v}) = \sigma^{-1}(\mathbf{curl}\,\boldsymbol{v} - \nabla\psi)$. This defines a continuous operator $\mathbb{T} : \mathrm{V}_T(D) \to \mathrm{V}_T(D)$. Moreover, one finds

$$\begin{aligned}(\tilde{M}(\mathbb{T}\boldsymbol{v}), \boldsymbol{v}')_{\mathbf{curl}} &= (\sigma\,\mathbf{curl}\,(\mathbb{T}\boldsymbol{v}), \mathbf{curl}\,\boldsymbol{v}')_D\\ &= (\mathbf{curl}\,\boldsymbol{v}, \mathbf{curl}\,\boldsymbol{v}')_D - (\nabla\psi, \mathbf{curl}\,\boldsymbol{v}')_D \;=\; (\mathbf{curl}\,\boldsymbol{v}, \mathbf{curl}\,\boldsymbol{v}')_D,\end{aligned}$$

which demonstrates that $\mathbb{T}$ is the inverse of $\tilde{M}$.

Higher dimensions. In dimension $d \geq 3$, the properties of $\Delta : \mathrm{H}^1_0(\Omega) \cap \mathrm{H}^2(\Omega) \to \mathrm{L}^2(\Omega)$ are more varied than in 2D. In [12], we consider in details the case of conical tips in $\mathbb{R}^3$. Some of these conical tips provide simple examples of non smooth and non convex domains for which $\Delta : \mathrm{H}^1_0(\Omega)\cap\mathrm{H}^2(\Omega) \to \mathrm{L}^2(\Omega)$ is an isomorphism. As a consequence, for these geometries, $\tilde{B} : \mathrm{H}^1_0(\Omega) \cap \mathrm{H}^2(\Omega) \to \mathrm{H}^1_0(\Omega) \cap \mathrm{H}^2(\Omega)$ constitutes an isomorphism for all $\sigma \in \mathrm{L}^\infty(\Omega)$ such that $\sigma^{-1} \in \mathrm{L}^\infty(\Omega)$. Working with edges in 3D, one can also find (non convex) domains for which $\Delta : \mathrm{H}^1_0(\Omega) \cap \mathrm{H}^2(\Omega) \to \mathrm{L}^2(\Omega)$ and $\tilde{B}$ are not of Fredholm type.

5 Bilaplacian in $\mathrm{H}^1_0(\Delta)$

In Sections 2 and 4, we have imposed the boundary condition $\sigma\Delta v = 0$ on $\partial\Omega$ in a weak way working with the integral identity "Find $v \in \mathrm{H}^1_0(\Omega) \cap \mathrm{H}^2(\Omega)$ such that $(\sigma\Delta v, \Delta v')_\Omega = \langle f, v'\rangle_\Omega$ for all $v' \in \mathrm{H}^1_0(\Omega) \cap \mathrm{H}^2(\Omega)$". Notice that this variational formulation has also a sense, for a source term smooth enough, when the functions v, v' are chosen in the space $\mathrm{H}^1_0(\Delta) := \{\varphi \in \mathrm{H}^1_0(\Omega) \,|\, \Delta\varphi \in \mathrm{L}^2(\Omega)\}$. Let Ω be a domain (with a Lipschitz boundary) of $\mathbb{R}^d$, $d \geq 1$. In this section, we wish to study the problem,

$$(\mathscr{P}^\sharp)\ \left|\begin{array}{l}\text{Find } v \in \mathrm{H}^1_0(\Delta) \text{ such that:}\\ (\sigma\Delta v, \Delta v')_\Omega = \langle f, v'\rangle_\Omega\,, \qquad \forall v' \in \mathrm{H}^1_0(\Delta).\end{array}\right. \tag{31}$$

Here, f is a given source term of $\mathrm{H}^1_0(\Delta)^*$, the topological dual space of $\mathrm{H}^1_0(\Delta)$. According to the Lax-Milgram theorem, we have $\|\varphi\|_{\mathrm{H}^1_0(\Omega)} \leq C\,\|\Delta\varphi\|_\Omega$ for all $\varphi \in \mathrm{H}^1_0(\Delta)$. Therefore, $(v, v') \mapsto (v, v')_{\mathrm{H}^2_0(\Omega)} = (\Delta v, \Delta v')_\Omega$ defines an inner product on $\mathrm{H}^1_0(\Delta)$ and the associated norm is equivalent to the natural norm $v \mapsto (\|v\|^2_{\mathrm{H}^1_0(\Omega)} + \|\Delta v\|^2_\Omega)^{1/2}$. In order to study the properties of $(\mathscr{P}^\sharp)$, we introduce with the Riesz representation theorem, the bounded operator $B^\sharp : \mathrm{H}^1_0(\Delta) \to \mathrm{H}^1_0(\Delta)$ such that

$$(B^\sharp v, v')_{\mathrm{H}^2_0(\Omega)} = (\sigma\Delta v, \Delta v')_\Omega, \qquad \forall(v, v') \in \mathrm{H}^1_0(\Delta) \times \mathrm{H}^1_0(\Delta). \tag{32}$$

Proposition 5.1. *Assume that $\sigma \in \mathrm{L}^\infty(\Omega)$ is such that $\sigma^{-1} \in \mathrm{L}^\infty(\Omega)$. Then the operator $B^\sharp : \mathrm{H}^1_0(\Delta) \to \mathrm{H}^1_0(\Delta)$ defined in (32) is an isomorphism.*

Proof. Let us introduce the operator $\mathtt{T}$ such that, for $v \in \mathrm{H}^1_0(\Delta)$, $\mathtt{T}v \in \mathrm{H}^1_0(\Delta)$ is the function satisfying $\Delta(\mathtt{T}v) = \sigma^{-1}\Delta v$. It is clear that $B^\sharp$ is a continuous operator. For all $(v, v') \in \mathrm{H}^1_0(\Delta) \times \mathrm{H}^1_0(\Delta)$, we have

$$(B^\sharp(\mathtt{T}v), v')_{\mathrm{H}^2_0(\Omega)} = (\sigma\Delta(\mathtt{T}v), \Delta v')_\Omega = (\Delta v, \Delta v')_\Omega.$$

This proves that $B^\sharp \circ \mathtt{T} = \mathrm{Id}$. Since $B^\sharp$ is selfadjoint, we deduce that $B^\sharp$ is an isomorphism. Its inverse is equal to $\mathtt{T}$. □

Let us study the smoothness of the solution $v^\sharp$ of Problem $(\mathscr{P}^\sharp)$ defined in (31). More precisely, our goal is to compare $v^\sharp$ with $\tilde{v}$, the solution of Problem $(\tilde{\mathscr{P}})$ defined in (10), when the latter is well-posed. Since $(\mathrm{H}^1_0(\Omega) \cap \mathrm{H}^2(\Omega)) \subset \mathrm{H}^1_0(\Delta)$, there holds $\mathrm{H}^1_0(\Delta)^* \subset (\mathrm{H}^1_0(\Omega) \cap \mathrm{H}^2(\Omega))^*$. Therefore, the source term f can be chosen in $\mathrm{H}^1_0(\Delta)^*$.

As soon as the domain Ω is such that the operator $\Delta : \mathrm{H}^1_0(\Omega) \cap \mathrm{H}^2(\Omega) \to \mathrm{L}^2(\Omega)$ is an isomorphism, we have $\mathrm{H}^1_0(\Delta) = \mathrm{H}^1_0(\Omega) \cap \mathrm{H}^2(\Omega)$. In this case, for example when $\Omega \subset \mathbb{R}^d$ is convex or of class $\mathscr{C}^2$, there holds $v^\sharp = \tilde{v}$.

Now, let us study the situation where $\mathrm{H}^1_0(\Delta) \neq \mathrm{H}^1_0(\Omega) \cap \mathrm{H}^2(\Omega)$. Let us work in dimension 2, for a domain Ω which has, to set ideas, one reentrant corner located at O. We reintroduce the function ζ defined in (25) verifying $\zeta \in \mathrm{L}^2(\Omega) \backslash \mathrm{H}^1(\Omega)$, $\Delta\zeta = 0$ a.e. in Ω and $\zeta = 0$ a.e. on $\partial\Omega$. As in (29), we define $\psi \in \mathrm{H}^1_0(\Omega)$ the function such that $\Delta\psi = \sigma^{-1}\zeta$. Since $\zeta \in \mathrm{L}^2(\Omega)$, we have $\psi \in \mathrm{H}^1_0(\Delta)$ and so $(\sigma\Delta v^\sharp, \Delta\psi)_\Omega = \langle f, \psi\rangle_\Omega$. This can also be written $(\Delta v^\sharp, \zeta)_\Omega = \langle f, \psi\rangle_\Omega$. Thus, by virtue of Corollary 4.1, we have $v^\sharp \in \mathrm{H}^1_0(\Omega) \cap \mathrm{H}^2(\Omega)$ if and only if $\langle f, \psi\rangle_\Omega = 0$. Let us distinguish two cases.

- If $(\sigma^{-1}\zeta, \zeta)_\Omega \neq 0$, according to Proposition 4.3, the operator $\tilde{B}$ defined in (12) is an isomorphism. Therefore, the solution $\tilde{v}$ of $(\tilde{\mathscr{P}})$ is defined uniquely.
  - If $\langle f, \psi\rangle_\Omega = 0$, then $v^\sharp$ verifies the same problem as $\tilde{v}$. We deduce that $v^\sharp = \tilde{v}$ in this configuration.
  - If $\langle f, \psi\rangle_\Omega \neq 0$, then $v^\sharp \notin \mathrm{H}^2(\Omega)$ and so $v^\sharp \neq \tilde{v}$. More precisely, since $(\sigma\Delta(v^\sharp - \tilde{v}), \Delta v')_\Omega = 0$ for all $v' \in \mathrm{H}^1_0(\Omega) \cap \mathrm{H}^2(\Omega)$, we deduce that $\Delta(v^\sharp - \tilde{v}) = a\,\sigma^{-1}\zeta$, where a is constant. Multiplying by ζ, integrating by parts on Ω and using that $\tilde{v} \in \mathrm{H}^1_0(\Omega) \cap \mathrm{H}^2(\Omega)$, we deduce that $a = (\Delta v^\sharp, \zeta)_\Omega / (\sigma^{-1}\zeta, \zeta)_\Omega = \langle f, \psi\rangle_\Omega / (\sigma^{-1}\zeta, \zeta)_\Omega$. Thus, in this configuration, we have
  $$v^\sharp - \tilde{v} = \frac{\langle f, \psi\rangle_\Omega}{(\sigma^{-1}\zeta, \zeta)_\Omega}\,\psi.$$
- If $(\sigma^{-1}\zeta, \zeta)_\Omega = 0$, then, according to Proposition 4.4, $(\tilde{\mathscr{P}})$ has a solution if and only if $\langle f, \psi\rangle_\Omega = 0$. Let us assume that this holds true. In this case, according to the above discussion, the solution $v^\sharp \in \mathrm{H}^1_0(\Delta)$ of $(\mathscr{P}^\sharp)$ is actually in $\mathrm{H}^1_0(\Omega) \cap \mathrm{H}^2(\Omega)$. Now, let us explain why ψ belongs to $\ker \tilde{B}$ but not to $\ker B^\sharp$. A function $v \in \mathrm{H}^1_0(\Omega) \cap \mathrm{H}^2(\Omega)$ belongs to $\ker \tilde{B}$ if and only if it satisfies $(\sigma\Delta v, \Delta v')_\Omega = 0$ for all $v' \in \mathrm{H}^1_0(\Omega) \cap \mathrm{H}^2(\Omega)$. Since $(\sigma\Delta\psi, \Delta v')_\Omega = (\zeta, \Delta v')_\Omega$, we have $\psi \in \ker \tilde{B}$. On the other hand, $\psi \in \ker B^\sharp$ if and only if we have $(\sigma\Delta\psi, \Delta v')_\Omega = 0$ for all $v' \in \mathrm{H}^1_0(\Delta) \supset (\mathrm{H}^1_0(\Omega) \cap \mathrm{H}^2(\Omega))$. Testing against v' such that $\Delta v' = \zeta$, we see that ψ is not an element of $\ker B^\sharp$. Of course, such a v' does not belong to $\mathrm{H}^1_0(\Omega) \cap \mathrm{H}^2(\Omega)$ because $(\zeta, \zeta)_\Omega \neq 0$.

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