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# A "milder" version of Calderón's inverse problem for anisotropic conductivities and partial data

El Maati Ouhabaz

## Abstract

Given a general symmetric elliptic operator

$$L_a := \sum_{k,j=1}^d \partial_k(a_{kj}\partial_j) + \sum_{k=1}^d a_k\partial_k - \partial_k(\overline{a_k \cdot}) + a_0$$

we define the associated Dirichlet-to-Neumann (D-t-N) map with partial data, i.e., data supported in a part of the boundary. We prove positivity,  $L^p$ -estimates and domination properties for the semigroup associated with this D-t-N operator. Given  $L_a$  and  $L_b$  of the previous type with bounded measurable coefficients  $a = \{a_{kj}, a_k, a_0\}$  and  $b = \{b_{kj}, b_k, b_0\}$ , we prove that if their partial D-t-N operators (with  $a_0$  and  $b_0$  replaced by  $a_0 - \lambda$  and  $b_0 - \lambda$ ) coincide for all  $\lambda$ , then the operators  $L_a$  and  $L_b$ , endowed with Dirichlet, mixed or Robin boundary conditions are unitarily equivalent. In the case of the Dirichlet boundary conditions, this result was proved recently by Behrndt and Rohleder [6]. We provide a different proof, based on spectral theory, which works for other boundary conditions.

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## 1 Introduction

Let  $\Omega$  be a bounded Lipschitz domain of  $\mathbb{R}^d$  with boundary  $\partial\Omega$ . Let  $\Gamma_0$  be a closed subset of  $\partial\Omega$  with  $\Gamma_0 \neq \partial\Omega$  and  $\Gamma_1$  its complement in  $\partial\Omega$ . We consider the symmetric elliptic operator on  $L^2(\Omega)$  given by the formal expression:

$$L_a(\lambda) := \sum_{k,j=1}^d \partial_k(a_{kj}\partial_j) + \sum_{k=1}^d a_k\partial_k - \partial_k(\overline{a_k \cdot}) + a_0 - \lambda$$

where  $a_{kj} = \overline{a_{jk}}$ ,  $a_k, a_0 = \overline{a_0} \in L^\infty(\Omega)$  and  $\lambda$  is a constant. We define the associated Dirichlet-to-Neumann (D-t-N) operator,  $\mathcal{N}_{\Gamma_1, a}(\lambda)$ , with partial data as follows:

for  $\varphi \in H^{\frac{1}{2}}(\partial\Omega)$  with  $\varphi = 0$  on  $\Gamma_0$ , one solves the Dirichlet problem

$$L_a(\lambda)u = 0 \text{ weakly in } W^{1,2}(\Omega) \text{ with } u = \varphi \text{ on } \partial\Omega, \quad (1.1)$$

and defines (in the weak sense)

$$\mathcal{N}_{\Gamma_1, a}(\lambda)\varphi := \sum_{j=1}^d \left( \sum_{k=1}^d a_{kj}\partial_k u + \overline{a_j}\varphi \right) \nu_j \text{ on } \Gamma_1. \quad (1.2)$$

Here  $\nu = (\nu_1, \dots, \nu_d)$  is the outer unit normal to the boundary of  $\Omega$ . The operator  $\mathcal{N}_{\Gamma_1, a}(\lambda)$  is interpreted as the conormal derivative on the boundary. It is an operator acting on  $L^2(\partial\Omega)$ . See Section 2 for more details.

Let us consider first the case where  $a_{kj} = \sigma(x)\delta^{kj}$ ,  $a_k = 0, k = 0, 1 \dots d$ , where  $\sigma \in L^\infty(\Omega)$  is bounded from below (by a positive constant). A. Calderón's well known inverse problem asks whether one could determine solely the conductivity  $\sigma(x)$  from boundary measurements, i.e., from  $\mathcal{N}_{\Gamma_1}(0)$ . For the global boundary measurements, i.e.,  $\Gamma_1 = \partial\Omega$ , the first global uniqueness result was proved by Sylvester and Uhlmann [27] for a  $C^2$ -smooth conductivity when  $d \geq 3$ . This results was extended to  $C^{1+\epsilon}$ -smooth conductivity by Greenleaf, Lassas and Uhlmann [12] and then by Haberman and Tataru [13] to  $C^1$  and Lipschitz conductivity close to the identity. Haberman [14] proved the uniqueness for Lipschitz conductivity when  $d = 3, 4$  and this was extended to all  $d \geq 3$  by Caro and Rogers [7]. In the two-dimension case with  $C^2$ -smooth conductivity, the global uniqueness was proved by Nachman [21]. This regularity assumption was completely removed by Astala and Päivärinta [4] dealing with  $\sigma \in L^\infty(\Omega)$ . The inverse problem with partial data consists in proving uniqueness (either for the isotropic conductivity or for the potential) when the measurement is made only on a part of the boundary. This means that the trace of the solution  $u$  in (1.1) is supported on a set  $\Gamma_D$  and the D-t-N operator is known on  $\Gamma_N$  for some parts  $\Gamma_D$  and  $\Gamma_N$  of the boundary. This problem has been studied and there are some geometric conditions on  $\Gamma_D$  and  $\Gamma_N$  under which uniqueness is proved. We refer to Isakov [15], Kenig, Sjöstrand and Uhlmann [18], Dos Santos et al. [10], Imanuvilov, Uhlmann and Yamamoto [16] and the review paper [19] by Kenig and Salo for more references and recent developments.

Now we move to the anisotropic case. This corresponds to the general case where the conductivity is given by a general matrix  $a_{kj}$ . As pointed out by Lee and Uhlmann in [20], it is not difficult to see that a change of variables given by a diffeomorphism of  $\Omega$  which is the identity on  $\partial\Omega$  leads to different coefficients  $b_{kj}$  without changing the D-t-N operator on the boundary. Therefore the single coefficients  $a_{kj}$  are not uniquely determined in general. In [20], Lee and Uhlmann proved that for real-analytic coefficients the uniqueness up to a diffeomorphism holds when the dimension  $d$  is  $\geq 3$ . The same result was proved by Astala, Lassas and Päivärinta [5] for the case  $d = 2$  and  $L^\infty$ -coefficients.

In [6], Behrndt and Rohleder considered general elliptic expressions  $L_a$  and  $L_b$  as above and prove that if the corresponding D-t-N operators  $\mathcal{N}_{\Gamma_1, a}(\lambda)$  and  $\mathcal{N}_{\Gamma_1, b}(\lambda)$  coincide for all  $\lambda$  in a set having an accumulation point in  $\rho(L_a^D) \cap \rho(L_b^D)$  then the operators  $L_a^D$  and  $L_b^D$  are unitarily equivalent. Here  $L_a^D$  is the elliptic operator  $L_a$  with Dirichlet boundary conditions. This can be seen as a milder version of the uniqueness problem discussed above. The proof is based on the theory of extensions of symmetric operators and unique continuation results. It is assumed in [6] that the coefficients are Lipschitz continuous on  $\bar{\Omega}$ . We give a different proof of this result which also works for other boundary conditions. Our main result is the following.

**Theorem 1.1.** *Suppose that  $\Omega$  is a bounded Lipschitz domain of  $\mathbb{R}^d$  with  $d \geq 2$ . Let  $\Gamma_0$  be a closed subset of  $\partial\Omega$ ,  $\Gamma_0 \neq \partial\Omega$  and  $\Gamma_1$  its complement. Let  $a = \{a_{kj}, a_k, a_0\}$  and  $b = \{b_{kj}, b_k, b_0\}$  be bounded functions on  $\Omega$  such that  $a_{kj}$  and  $b_{kj}$  satisfy the usual ellipticity condition. If  $d \geq 3$  we assume in addition that the coefficients  $a_{kj}, b_{kj}, a_k$  and  $b_k$  are Lipschitz continuous on  $\bar{\Omega}$ .*

Suppose that  $\mathcal{N}_{\Gamma_1,a}(\lambda) = \mathcal{N}_{\Gamma_1,b}(\lambda)$  for all  $\lambda$  in a set having an accumulation point in  $\rho(L_a^D) \cap \rho(L_b^D)$ . Then:

- i) The operators  $L_a$  and  $L_b$  endowed with Robin boundary conditions are unitarily equivalent.
- ii) The operators  $L_a$  and  $L_b$  endowed with mixed boundary conditions (Dirichlet on  $\Gamma_0$  and Neumann type on  $\Gamma_1$ ) are unitarily equivalent.
- iii) The operators  $L_a$  and  $L_b$  endowed with Dirichlet boundary conditions are unitarily equivalent.

In addition, for Robin or mixed boundary conditions, the eigenfunctions associated to the same eigenvalue  $\lambda \notin \sigma(L_a^D) = \sigma(L_b^D)$  coincide on the boundary of  $\Omega$ .

Note that unlike [6] we do not assume regularity of the coefficients when  $d = 2$ .

We shall restate this theorem in a more precise way after introducing some necessary material and notation. The proof is given in Section 4. It is based on spectral theory and differs from the one in [6]. Our strategy is to use a relationship between eigenvalues of the D-t-N operator  $\mathcal{N}_{\Gamma_1,a}(\lambda)$  and eigenvalues of the elliptic operator with Robin boundary conditions  $L_a^\mu$  on  $\Omega$  where  $\mu$  is a parameter. One of the main ingredients in the proof is that each eigenvalue of the latter operator is a strictly decreasing map with respect to the parameter  $\mu$ . Next, the equality of  $\mathcal{N}_{\Gamma_1,a}(\lambda)$  and  $\mathcal{N}_{\Gamma_1,b}(\lambda)$  allows us to prove that the spectra of  $L_a^\mu$  and  $L_b^\mu$  are the same and the eigenvalues have the same multiplicity. The similarity of the two elliptic operators with Dirichlet boundary conditions is obtained from the similarity of  $L_a^\mu$  and  $L_b^\mu$  by letting the parameter  $\mu$  tend to  $-\infty$ . During the proof we use some ideas from the papers of Arendt and Mazzeo [2] and [3] which deal with a different subject, namely the Friendlander inequality for the eigenvalues of the Dirichlet and Neumann Laplacian on a Lipschitz domain. The ideas which we borrow from [2] and [3] are then adapted and extended to our general case of D-t-N operators with variable coefficients and partial data.

In Section 2 we define the D-t-N operator with partial data using the method of sesquilinear forms. In particular, for symmetric coefficients it is a self-adjoint operator on  $L^2(\Gamma_1)$ . It can be seen as an operator on  $L^2(\partial\Omega)$  with a non-dense domain and which we extend by 0 to  $L^2(\Gamma_0)$ . Therefore one can associate with this D-t-N operator a semigroup  $(T_t^{\Gamma_1})_{t \geq 0}$  acting on  $L^2(\partial\Omega)$ . In Section 3 we prove positivity, sub-Markovian and domination properties for such semigroups. In particular,  $(T_t^{\Gamma_1})_{t \geq 0}$  extends to a contraction semigroup on  $L^p(\partial\Omega)$  for all  $p \in [1, \infty)$ . Hence, for  $\varphi_0 \in L^p(\Gamma_1)$ , one obtains existence and uniqueness of the solution in  $L^p(\partial\Omega)$  to the evolution problem

$$\partial_t \varphi + \mathcal{N}_{\Gamma_1,a}(\lambda) \varphi = 0, \quad \varphi(0) = \varphi_0.$$

The results of Section 3 are of independent interest and are not used in the proof of the theorem stated above.

## 2 The partial D-t-N operator

Let  $\Omega$  be a bounded open set of  $\mathbb{R}^d$  with Lipschitz boundary  $\partial\Omega$ . The boundary is endowed with the  $(d-1)$ -dimensional Hausdorff measure  $d\sigma$ . Let

$$a_{kj}, a_k, \tilde{a}_k, a_0 : \Omega \rightarrow \mathbb{C}$$

be bounded measurable for  $1 \leq k, j \leq d$  and such that there exists a constant  $\eta > 0$  for which

$$\operatorname{Re} \sum_{k,j=1}^d a_{kj}(x) \xi_k \bar{\xi}_j \geq \eta |\xi|^2 \quad (2.1)$$

for all  $\xi = (\xi_1, \dots, \xi_d) \in \mathbb{C}^d$  and a.e.  $x \in \Omega$ .

Let  $\Gamma_0$  be a closed subset of  $\partial\Omega$  and  $\Gamma_1$  its complement in  $\partial\Omega$ .

### Elliptic operators on $\Omega$ .

We consider the space

$$V = \{u \in W^{1,2}(\Omega), \operatorname{Tr}(u) = 0 \text{ on } \Gamma_0 = 0\}, \quad (2.2)$$

where  $\operatorname{Tr}$  denotes the trace operator. We define the sesquilinear form

$$\mathbf{a} : V \times V \rightarrow \mathbb{C}$$

by the expression

$$\mathbf{a}(u, v) = \sum_{k,j=1}^d \int_{\Omega} a_{kj} \partial_k u \bar{\partial}_j v \, dx + \sum_{k=1}^d \int_{\Omega} a_k \partial_k u \bar{v} + \tilde{a}_k u \bar{\partial}_k v \, dx + a_0 u \bar{v} \, dx \quad (2.3)$$

for all  $u, v \in V$ . Here we use the notation  $\partial_j$  for the partial derivative  $\frac{\partial}{\partial x_j}$ .

It follows easily from the ellipticity assumption (2.1) that the form  $\mathbf{a}$  is quasi-accretive, i.e., there exists a constant  $w$  such that

$$\operatorname{Re} \mathbf{a}(u, u) + w \|u\|_2^2 \geq 0 \quad \forall u \in V.$$

In addition, since  $V$  is a closed subspace of  $W^{1,2}(\Omega)$  the form  $\mathbf{a}$  is closed. Therefore there exists an operator  $L_a$  associated with  $\mathbf{a}$ . It is defined by

$$D(L_a) = \{u \in V, \exists v \in L^2(\Omega) : \mathbf{a}(u, \phi) = \int_{\Omega} v \bar{\phi} \, dx \quad \forall \phi \in V\},$$

$$L_a u := v.$$

Formally,  $L_a$  is given by the expression

$$L_a u = - \sum_{k,j=1}^d \partial_k (a_{kj} \partial_j u) + \sum_{k=1}^d a_k \partial_k u - \partial_k (\tilde{a}_k u) + a_0 u. \quad (2.4)$$

In addition, following [2] or [3] we define the conormal derivative  $\frac{\partial}{\partial \nu}$  in the weak sense (i.e. in  $H^{-1/2}(\partial\Omega)$  the dual space of  $H^{1/2}(\partial\Omega) = \operatorname{Tr}(W^{1,2}(\Omega))$ ), then  $L_a$  is subject to the boundary conditions

$$\begin{cases} \operatorname{Tr}(u) = 0 & \text{on } \Gamma_0 \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \Gamma_1. \end{cases} \quad (2.5)$$

The conormal derivative in our case is usually interpreted as

$$\sum_{j=1}^d \left( \sum_{k=1}^d a_{kj} \partial_k u + \tilde{a}_j u \right) \nu_j,$$

where  $\nu = (\nu_1, \dots, \nu_d)$  is the outer unit normal to the boundary of  $\Omega$ . For all this see [23], Chapter 4.

The condition (2.5) is a mixed boundary condition which consists in taking Dirichlet on  $\Gamma_0$  and Neumann type boundary condition on  $\Gamma_1$ . For this reason we denote this operator by  $L_a^M$ . The subscript  $a$  refers to the fact that the coefficients of the operator are given by  $a = \{a_{kj}, a_k, \tilde{a}_k, a_0\}$  and  $M$  refers to mixed boundary conditions.

We also define the elliptic operator with Dirichlet boundary condition  $\text{Tr}(u) = 0$  on  $\partial\Omega$ . It is the operator associated with the form given by the expression (2.3) with domain  $D(\mathbf{a}) = W_0^{1,2}(\Omega)$ . It is a quasi-accretive and closed form and its associated operator  $L_a^D$  has the same expression as in (2.4) and subject to the Dirichlet boundary condition  $\text{Tr}(u) = 0$  on  $\partial\Omega$ .

Similarly, we define  $L_a^N$  to be the elliptic operator with Neumann type boundary conditions

$$\frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega.$$

It is the operator associated with the form given by the expression (2.3) with domain  $D(\mathbf{a}) = W^{1,2}(\Omega)$ .

Note that  $L_a^D$  coincides with  $L_a^M$  if  $\Gamma_0 = \partial\Omega$  and  $L_a^N$  coincides with  $L_a^M$  if  $\Gamma_0 = \emptyset$ .

Finally we define elliptic operators with Robin boundary conditions. Let  $\mu \in \mathbb{R}$  be a constant and define

$$\begin{aligned} \mathbf{a}^\mu(u, v) = & \sum_{k,j=1}^d \int_{\Omega} a_{kj} \partial_k u \overline{\partial_j v} \, dx + \sum_{k=1}^d \int_{\Omega} a_k \partial_k u \bar{v} + \tilde{a}_k u \overline{\partial_k v} \, dx + a_0 u \bar{v} \, dx \\ & - \mu \int_{\partial\Omega} \text{Tr}(u) \overline{\text{Tr}(v)} \, d\sigma \end{aligned} \quad (2.6)$$

for all  $u, v \in D(\mathbf{a}^\mu) := V$ . Again,  $\text{Tr}$  denotes the trace operator. Using the standard inequality (see [2] or [3]),

$$\int_{\partial\Omega} |\text{Tr}(u)|^2 \leq \varepsilon \|u\|_{W^{1,2}(\Omega)}^2 + c_\varepsilon \int_{\Omega} |u|^2$$

which is valid for all  $\varepsilon > 0$  ( $c_\varepsilon$  is a constant depending on  $\varepsilon$ ) one obtains that for some positive constants  $w$  and  $\delta$

$$\text{Re } \mathbf{a}^\mu(u, u) + w \int_{\Omega} |u|^2 \geq \delta \|u\|_{W^{1,2}(\Omega)}^2.$$

From this it follows that  $\mathbf{a}^\mu$  is a quasi-accretive and closed sesquilinear form. One can associate with  $\mathbf{a}^\mu$  an operator  $L_a^\mu$ . This operator has the same expression (2.4) and it is subject to the Robin boundary conditions

$$\begin{cases} \text{Tr}(u) = 0 & \text{on } \Gamma_0 \\ \frac{\partial u}{\partial \nu} = \mu \text{Tr}(u) & \text{on } \Gamma_1. \end{cases} \quad (2.7)$$

Actually, the boundary conditions (2.7) are mixed Robin boundary conditions in the sense that we have the Dirichlet condition on  $\Gamma_0$  and the Robin one on  $\Gamma_1$ . For simplicity we ignore the word "mixed" and refer to (2.7) as the Robin boundary conditions.

According to our previous notation, if  $\mu = 0$ , then  $\mathbf{a}^0 = \mathbf{a}$  and  $L_a^0 = L_a^M$ . Note that we may choose here  $\mu$  to be a bounded measurable function on the boundary rather than just a constant.

The partial Dirichlet-to-Neumann operator on  $\partial\Omega$ .

Suppose as before that  $a = \{a_{kj}, a_k, \tilde{a}_k, a_0\}$  are bounded measurable and satisfy the ellipticity condition (2.1). Let  $\Gamma_0, \Gamma_1, V$  be as above and  $\mathbf{a}$  is the sesquilinear form defined by (2.3).

We define the space

$$V_H := \{u \in V, \mathbf{a}(u, g) = 0 \text{ for all } g \in W_0^{1,2}(\Omega)\}. \quad (2.8)$$

Then  $V_H$  is a closed subspace of  $V$ . It is interpreted as the space of harmonic functions for the operator  $L_a$  (given by (2.4)) with the additional property that  $\text{Tr}(u) = 0$  on  $\Gamma_0$ .

We start with the following simple lemma.

**Lemma 2.1.** *Suppose that  $0 \notin \sigma(L_a^D)$ . Then*

$$V = V_H \oplus W_0^{1,2}(\Omega). \quad (2.9)$$

*Proof.* We argue as in [11], Section 2 or [2]. Let us denote by  $\mathbf{a}^D$  the form associated with  $L_a^D$ , that is,  $\mathbf{a}^D$  is given by (2.3) with  $D(\mathbf{a}^D) = W_0^{1,2}(\Omega)$ . There exists an operator  $\mathcal{L}_a^D : W_0^{1,2}(\Omega) \rightarrow W^{-1,2}(\Omega) := W_0^{1,2}(\Omega)'$  (the anti-dual of  $W_0^{1,2}(\Omega)$ ) associated with  $\mathbf{a}^D$  in the sense

$$\langle \mathcal{L}_a^D h, g \rangle = \mathbf{a}^D(h, g)$$

for all  $h, g \in W_0^{1,2}(\Omega)$ . The notation  $\langle \cdot, \cdot \rangle$  denotes the duality  $W_0^{1,2}(\Omega)' - W_0^{1,2}(\Omega)$ . Since  $0 \notin \sigma(L_a^D)$ , then  $L_a^D$  is invertible. Therefore  $\mathcal{L}_a^D$ , seen as operator on  $W_0^{1,2}(\Omega)'$  with domain  $W_0^{1,2}(\Omega)$ , is also invertible on  $W_0^{1,2}(\Omega)'$  since the two operators  $L_a^D$  and  $\mathcal{L}_a^D$  have the same spectrum (see e.g., [1], Proposition 3.10.3). Now we fix  $u \in V$  and consider the (anti-)linear functional

$$F : v \mapsto \mathbf{a}(u, v).$$

Clearly,  $F \in W_0^{1,2}(\Omega)'$  and hence there exists a unique  $u_0 \in W_0^{1,2}(\Omega)$  such that  $\mathcal{L}_a^D u_0 = F$ , i.e.,  $\langle \mathcal{L}_a^D u_0, g \rangle = F(g)$  for all  $g \in W_0^{1,2}(\Omega)$ . This means that  $\mathbf{a}(u - u_0, g) = 0$  for all  $g \in W_0^{1,2}(\Omega)$  and hence  $u - u_0 \in V_H$ . Thus,  $u = u - u_0 + u_0 \in V_H + W_0^{1,2}(\Omega)$ . Finally, if  $u \in V_H \cap W_0^{1,2}(\Omega)$  then  $\mathbf{a}(u, g) = 0$  for all  $g \in W_0^{1,2}(\Omega)$ . This means that  $u \in D(L_a^D)$  with  $L_a^D u = 0$ . Since  $L_a^D$  is invertible we conclude that  $u = 0$ .  $\square$

As a consequence of Lemma 2.1, the trace operator  $\text{Tr} : V_H \rightarrow L^2(\partial\Omega)$  is injective and

$$\text{Tr}(V_H) = \text{Tr}(V). \quad (2.10)$$

In the rest of this section we assume that  $0 \notin \sigma(L_a^D)$ . We define on  $L^2(\partial\Omega, d\sigma)$  the sesquilinear form

$$\mathfrak{b}(\varphi, \psi) := \mathfrak{a}(u, v) \quad (2.11)$$

where  $u, v \in V_H$  are such that  $\varphi = \text{Tr}(u)$  and  $\psi = \text{Tr}(v)$ . This means that  $D(\mathfrak{b}) = \text{Tr}(V_H)$  and by (2.10)

$$D(\mathfrak{b}) = \text{Tr}(V_H) = \text{Tr}(V). \quad (2.12)$$

**Lemma 2.2.** *There exist positive constants  $w, \delta$  and  $M$  such that*

$$\text{Re } \mathfrak{b}(\varphi, \varphi) + w \int_{\partial\Omega} |\varphi|^2 \geq \delta \|u\|_{W^{1,2}(\Omega)}^2 \quad (2.13)$$

and

$$|\mathfrak{b}(\varphi, \psi)| \leq M \left[ \text{Re } \mathfrak{b}(\varphi, \varphi) + w \int_{\partial\Omega} |\varphi|^2 \right]^{1/2} \left[ \text{Re } \mathfrak{b}(\psi, \psi) + w \int_{\partial\Omega} |\psi|^2 \right]^{1/2} \quad (2.14)$$

for all  $\varphi, \psi \in D(\mathfrak{b})$ . In the first inequality,  $u \in V_H$  is such that  $\text{Tr}(u) = \varphi$ .

*Proof.* It is well known that  $\text{Tr} : W^{1,2}(\Omega) \rightarrow L^2(\partial\Omega)$  is a compact operator and since  $\text{Tr} : V_H \rightarrow L^2(\partial\Omega)$  is injective it follows that for every  $\epsilon > 0$  there exists a constant  $c > 0$  such that

$$\int_{\Omega} |u|^2 \leq \epsilon \|u\|_{W^{1,2}}^2 + c \int_{\partial\Omega} |\text{Tr}(u)|^2 \quad (2.15)$$

for all  $u \in V_H$  (see, e.g., [2]). In particular,

$$\int_{\Omega} |u|^2 \leq \frac{\epsilon}{1-\epsilon} \int_{\Omega} |\nabla u|^2 + \frac{c}{1-\epsilon} \int_{\partial\Omega} |\varphi|^2. \quad (2.16)$$

Now, let  $\varphi \in D(\mathfrak{b}) = \text{Tr}(V_H)$  and  $u \in V_H$  such that  $\varphi = \text{Tr}(u)$ . It follows from the ellipticity assumption (2.1) and the boundedness of the coefficients that for some constant  $c_0 > 0$

$$\text{Re } \mathfrak{a}(u, u) \geq \frac{\eta}{2} \int_{\Omega} |\nabla u|^2 - c_0 \int_{\Omega} |u|^2.$$

Therefore, using (2.16) and the definition of  $\mathfrak{b}$  we obtain

$$\begin{aligned} \text{Re } \mathfrak{b}(\varphi, \varphi) &= \text{Re } \mathfrak{a}(u, u) \\ &\geq \left( \frac{\eta}{2} - \frac{c_0 \epsilon}{1-\epsilon} \right) \int_{\Omega} |\nabla u|^2 - \frac{c c_0}{1-\epsilon} \int_{\partial\Omega} |\varphi|^2. \end{aligned}$$

Taking  $\epsilon > 0$  small enough we obtain (2.13).

In order to prove the second inequality, we use the definition of  $\mathfrak{b}$  and again the boundedness of the coefficients to see that

$$\begin{aligned} |\mathfrak{b}(\varphi, \psi)| &= |\mathfrak{a}(u, v)| \\ &\leq C \|u\|_{W^{1,2}} \|v\|_{W^{1,2}}. \end{aligned}$$

Thus, (2.14) follows from (2.13).  $\square$



**Corollary 2.3.** *The form  $\mathfrak{b}$  is continuous, quasi-accretive and closed.*

*Proof.* Continuity of  $\mathfrak{b}$  is exactly (2.14). Quasi-accretivity means that

$$\operatorname{Re} \mathfrak{b}(\varphi, \varphi) + w \int_{\partial\Omega} |\varphi|^2 \geq 0$$

for some  $w$  and all  $\varphi \in D(\mathfrak{b})$ . This follows from (2.13).

Now we prove that  $\mathfrak{b}$  is closed which means that  $D(\mathfrak{b})$  is complete for the norm

$$\|\varphi\|_{\mathfrak{b}} := \left( \operatorname{Re} \mathfrak{b}(\varphi, \varphi) + w \int_{\partial\Omega} |\varphi|^2 \right)^{1/2}$$

in which  $w$  is as in (2.13). If  $(\varphi_n)$  is a Cauchy sequence for  $\|\cdot\|_{\mathfrak{b}}$  then by (2.13) the corresponding  $(u_n) \in V_H$  with  $\operatorname{Tr}(u_n) = \varphi_n$  is a Cauchy sequence in  $V_H$ . Since  $V_H$  is a closed subspace of  $V$  it follows that  $u_n$  is convergent to some  $u$  in  $V_H$ . Set  $\varphi := \operatorname{Tr}(u)$ . We have  $\varphi \in D(\mathfrak{b})$  and the definition of  $\mathfrak{b}$  together with continuity of  $\operatorname{Tr}$  as an operator from  $W^{1,2}(\Omega)$  to  $L^2(\partial\Omega)$  show that  $\varphi_n$  converges to  $\varphi$  for the norm  $\|\cdot\|_{\mathfrak{b}}$ . This means that  $\mathfrak{b}$  is a closed form.  $\square$

Note that the domain  $\operatorname{Tr}(V_H)$  of  $\mathfrak{b}$  may not be dense in  $L^2(\partial\Omega)$  since functions in this domain vanish on  $\Gamma_0$ . Indeed,

$$H := \overline{D(\mathfrak{b})}^{L^2(\partial\Omega)} = L^2(\Gamma_1) \oplus \{0\}. \quad (2.17)$$

The direct inclusion follows from the fact that if  $\varphi_n \in D(\mathfrak{b})$  converges in  $L^2(\partial\Omega)$  then after extracting a subsequence we have a.e. convergence. Since  $\varphi_n = 0$  on  $\Gamma_0$  we obtain that the limit  $\varphi = 0$  on  $\Gamma_0$ . The reverse inclusion can be proved as follows. Let  $\Gamma_2$  be a closed subset of  $\mathbb{R}^d$  with  $\Gamma_2 \subset \Gamma_1$  and consider the space  $E = \{u|_{\Gamma_2} : u \in W^{1,\infty}(\mathbb{R}^d), u|_{\Gamma_0} = 0\}$ . Then  $E \subset C(\Gamma_2)$  and an easy application of the Stone-Weierstrass theorem shows that  $E$  is dense in  $C(\Gamma_2)$ . Now given  $\varphi \in C_c(\Gamma_1)$  and  $\epsilon > 0$  we find  $\Gamma_2$  such that  $\|\mathbb{1}_{\Gamma_1 \setminus \Gamma_2}\|_2 < \epsilon$  and  $u|_{\Gamma_2} \in E$  such that  $\|u|_{\Gamma_2} - \varphi\|_{C(\Gamma_2)} < \epsilon$ . Finally we take  $\chi \in C_c^\infty(\mathbb{R}^d)$  such that  $\chi = 1$  on  $\Gamma_2$ . Then  $(u\chi)|_{\Omega} \in V$  and

$$\begin{aligned} \|u\chi - \varphi\|_{L^2(\Gamma_1)} &\leq \|u - \varphi\|_{L^2(\Gamma_2)} + \|\chi\|_{L^2(\Gamma_1 \setminus \Gamma_2)} \\ &\leq \epsilon|\Gamma_2| + \|\chi\|_{\infty}\epsilon. \end{aligned}$$

Here  $|\Gamma_2|$  denotes the measure of  $\Gamma_2$ . These inequalities together with the fact that  $C_c(\Gamma_1)$  is dense in  $L^2(\Gamma_1)$  imply (2.17).

We return to the form  $\mathfrak{b}$  defined above. We associate with  $\mathfrak{b}$  an operator  $\mathcal{N}_{\Gamma_1}$ . It is defined by

$$D(\mathcal{N}_{\Gamma_1}) := \{\varphi \in D(\mathfrak{b}), \exists \psi \in H : \mathfrak{b}(\varphi, \xi) = \int_{\Gamma_1} \psi \bar{\xi} \ \forall \xi \in D(\mathfrak{b})\}, \quad \mathcal{N}_{\Gamma_1} \varphi = \psi.$$

The operator  $\mathcal{N}_{\Gamma_1}$  can be interpreted as an operator on  $L^2(\partial\Omega)$  defined as follows: if  $\varphi \in D(\mathcal{N}_{\Gamma_1})$  then there exists a unique  $u \in V_H$  such that  $\varphi = \operatorname{Tr}(u)$  and

$$\varphi|_{\Gamma_0} = 0, \quad \mathcal{N}_{\Gamma_1}(\varphi) = \frac{\partial u}{\partial \nu} \text{ on } \Gamma_1. \quad (2.18)$$

Again  $\frac{\partial u}{\partial \nu}$  is interpreted in the weak sense as the conormal derivative that is  $\sum_{j=1}^d \left( \sum_{k=1}^d a_{kj} \partial_k u + \tilde{a}_j \varphi \right) \nu_j$ . In the particular case where  $a_{kj} = \delta_{kj}$  and  $a_1 = \dots = a_d = 0$  the right hand side of (2.18) is seen as the normal derivative on the boundary. All this can be made precise by applying the Green formula if the boundary and the coefficients are smooth enough.

We call  $\mathcal{N}_{\Gamma_1}$  the *partial Dirichlet-to-Neumann operator* on  $L^2(\partial\Omega)$  or the *Dirichlet-to-Neumann operator with partial data*. The term *partial* refers to the fact that  $\mathcal{N}_{\Gamma_1}$  is known only on the part  $\Gamma_1$  of the boundary  $\partial\Omega$ .

It follows from the general theory of forms that  $-\mathcal{N}_{\Gamma_1}$  generates a holomorphic semigroup  $e^{-t\mathcal{N}_{\Gamma_1}}$  on  $H$ . We define  $T_t^{\Gamma_1}$  on  $L^2(\partial\Omega)$  by

$$T_t^{\Gamma_1} \varphi = e^{-t\mathcal{N}_{\Gamma_1}} (\varphi \mathbb{1}_{\Gamma_1}) \oplus 0.$$

We shall refer to  $(T_t^{\Gamma_1})_{t \geq 0}$  as the "semigroup" generated by  $-\mathcal{N}_{\Gamma_1}$  on  $L^2(\partial\Omega)$ . It is clear that

$$\|T_t^{\Gamma_1}\|_{\mathcal{L}(L^2(\partial\Omega))} \leq e^{-w_0 t}, \quad t \geq 0, \quad (2.19)$$

for some constant  $w_0$ . Note that if the form  $\mathfrak{a}$  is symmetric, then  $\mathfrak{b}$  is also symmetric and hence  $\mathcal{N}_{\Gamma_1}$  is self-adjoint. In this case, (2.19) holds with  $w_0 = \inf \sigma(\mathcal{N}_{\Gamma_1})$  which also coincides with the first eigenvalue of  $\mathcal{N}_{\Gamma_1}$ . For all this, see e.g. [23], Chapter 1.

### 3 Positivity and domination

In this section we study some properties of the semigroup  $(T_t^{\Gamma_1})_{t \geq 0}$ . We assume throughout this section that

$$a_{jk} = a_{kj}, \quad \tilde{a}_k = a_k, \quad a_0 \in L^\infty(\Omega, \mathbb{R}). \quad (3.1)$$

We recall that  $L_a^D$  is the elliptic operator with Dirichlet boundary conditions defined in the previous section. Its associated symmetric form  $\mathfrak{a}^D$  is given by (2.3) and has domain  $W_0^{1,2}(\Omega)$ . We shall need the accretivity assumption of  $\mathfrak{a}^D$  (or equivalently the self-adjoint operator  $L_a^D$  is non-negative) which means that

$$\mathfrak{a}^D(u, u) \geq 0 \text{ for all } u \in W_0^{1,2}(\Omega). \quad (3.2)$$

**Theorem 3.1.** *Suppose that  $0 \notin \sigma(L_a^D)$ , (3.1) and that  $L_a^D$  is accretive.*

- a) *The semigroup  $(T_t^{\Gamma_1})_{t \geq 0}$  is positive (i.e., it maps non-negative functions of  $L^2(\partial\Omega)$  into non-negative functions).*
- b) *Suppose in addition that  $a_0 \geq 0$  and  $a_k = 0$  for all  $k \in \{1, \dots, d\}$ . Then  $(T_t^{\Gamma_1})_{t \geq 0}$  is a sub-Markovian semigroup.*

Recall that the sub-Markovian property means that for  $\varphi \in L^2(\partial\Omega)$  and  $t \geq 0$

$$0 \leq \varphi \leq 1 \Rightarrow 0 \leq T_t^{\Gamma_1} \varphi \leq 1.$$

This property implies in particular that  $(T_t^{\Gamma_1})_{t \geq 0}$  extends from  $L^2(\partial\Omega)$  to  $L^p(\partial\Omega)$  for all  $p \in [2, \infty[$ . Since  $\mathfrak{a}$  is symmetric then so is  $\mathfrak{b}$  and one obtains by duality that  $(T_t^{\Gamma_1})_{t \geq 0}$  extends also to  $L^p(\partial\Omega)$  for  $p \in [1, 2]$ .

*Proof.* The proof follows exactly the same lines as for Theorem 2.3 in [11].

a) By the well known Beurling–Deny criteria (see [9], Section 1.3 or [23], Theorem 2.6), it suffices to prove that  $\varphi^+ \in D(\mathfrak{b})$  and  $\mathfrak{b}(\varphi^+, \varphi^-) \leq 0$  for all real-valued  $\varphi \in D(\mathfrak{b})$ . Note that the fact that  $D(\mathfrak{b})$  is not densely defined does not affect the the statements of the Beurling-Deny criteria.

Let  $\varphi \in D(\mathfrak{b})$  be real-valued. There exists a real-valued  $u \in H_V$  such that  $\varphi = \text{Tr}(u)$ . Then  $\varphi^+ = \text{Tr}(u^+) \in \text{Tr}(V) = \text{Tr}H_V = D(\mathfrak{b})$ . This follows from the fact that  $v^+ \in V$  for all  $v \in V$  (see [23], Section 4.2).

By Lemma 2.1 we can write  $u^+ = u_0 + u_1$  and  $u^- = v_0 + v_1$  with  $u_0, v_0 \in W_0^{1,2}(\Omega)$  and  $u_1, v_1 \in H_V$ . Hence,  $u = u^+ - u^- = (u_0 - v_0) + (u_1 - v_1)$ . Since  $u, u_1 - v_1 \in H_V$  it follows that  $u_0 = v_0$ . Therefore,

$$\begin{aligned} \mathfrak{b}(\varphi^+, \varphi^-) &= \mathfrak{a}(u_1, v_1) = \mathfrak{a}(u_1, v_0 + v_1) = \mathfrak{a}(u_0 + u_1, v_0 + v_1) - \mathfrak{a}(u_0, v_0 + v_1) \\ &= \mathfrak{a}(u^+, u^-) - \mathfrak{a}(u_0, v_0) = -\mathfrak{a}(u_0, v_0) \\ &= -\mathfrak{a}(u_0, u_0) = -\mathfrak{a}^D(u_0, u_0). \end{aligned}$$

Here we use the fact that

$$\begin{aligned} \mathfrak{a}(u^+, u^-) &= \sum_{k,j=1}^d \int_{\Omega} a_{kj} \partial_k(u^+) \partial_j(u^-) + \sum_{k=1}^d \int_{\Omega} a_k \partial_k u^+ u^- + a_k u^+ \partial_k u^- \\ &\quad + \int_{\Omega} a_0 u^+ u^- = 0. \end{aligned}$$

By assumption (3.2) we have  $\mathfrak{a}^D(u_0, u_0) \geq 0$  and we obtain  $\mathfrak{b}(\varphi^+, \varphi^-) \leq 0$ . This proves the positivity of  $(T_t^{\Gamma_1})_{t \geq 0}$  on  $L^2(\partial\Omega)$ .

b) By [22] or [23], Corollary 2.17 it suffices to prove that  $\mathbb{1} \wedge \varphi := \inf(\mathbb{1}, \varphi) \in D(\mathfrak{b})$  and  $\mathfrak{b}(\mathbb{1} \wedge \varphi, (\varphi - \mathbb{1})^+) \geq 0$  for all  $\varphi \in D(\mathfrak{b})$  with  $\varphi \geq 0$ . Let  $\varphi \in D(\mathfrak{b})$  and suppose that  $\varphi \geq 0$ . Let  $u \in H_V$  be real-valued such that  $\varphi = \text{Tr}(u)$ . Note that  $\mathbb{1} \wedge u \in V$  (see [23], Section 4.3). We decompose  $\mathbb{1} \wedge u = u_0 + u_1 \in W_0^{1,2}(\Omega) \oplus H_V$ . Then

$$(u - \mathbb{1})^+ = u - \mathbb{1} \wedge u = (-u_0) + (u - u_1) \in W_0^{1,2}(\Omega) \oplus H_V.$$

Therefore,

$$\begin{aligned} \mathfrak{b}(\mathbb{1} \wedge \varphi, (\varphi - \mathbb{1})^+) &= \mathfrak{a}(u_1, u - u_1) = \mathfrak{a}(u_0 + u_1, u - u_1) \\ &= \mathfrak{a}(u_0 + u_1, -u_0 + u - u_1) + \mathfrak{a}(u_0 + u_1, u_0) \\ &= \mathfrak{a}(u_0 + u_1, -u_0 + u - u_1) + \mathfrak{a}(u_0, u_0) \\ &= \sum_{k,j=1}^d \int_{\Omega} a_{kj} \partial_k(\mathbb{1} \wedge u) \partial_j((u - \mathbb{1})^+) + \\ &\quad \int_{\Omega} a_0(\mathbb{1} \wedge u)(u - \mathbb{1})^+ + \mathfrak{a}^D(u_0, u_0) \\ &= \int_{\Omega} a_0 (u - \mathbb{1})^+ + \mathfrak{a}^D(u_0, u_0) \geq 0. \end{aligned}$$

This proves that  $\mathfrak{b}(\mathbb{1} \wedge \varphi, (\varphi - \mathbb{1})^+) \geq 0$ . □

Next we have the following domination property.

**Theorem 3.2.** *Suppose that  $a_{kj}$ ,  $a_k$ ,  $\tilde{a}_k$  and  $a_0$  satisfy (3.1). Suppose also that  $L_a^D$  is accretive with  $0 \notin \sigma(L_a^D)$ . Let  $\Gamma_0$  and  $\tilde{\Gamma}_0$  be two closed subsets of the boundary such that  $\Gamma_0 \subseteq \tilde{\Gamma}_0$ . Then for every  $0 \leq \varphi \in L^2(\partial\Omega)$*

$$0 \leq T_t^{\tilde{\Gamma}_1} \varphi \leq T_t^{\Gamma_1} \varphi.$$

*Proof.* Let  $\tilde{\Gamma}_1$  be the complement of  $\tilde{\Gamma}_0$  in  $\partial\Omega$ . Denote by  $\mathbf{b}$  and  $\tilde{\mathbf{b}}$  the sesquilinear forms associated with  $\mathcal{N}_{\Gamma_1}$  and  $\mathcal{N}_{\tilde{\Gamma}_1}$ , respectively. Clearly,  $\tilde{\mathbf{b}}$  is a restriction of  $\mathbf{b}$  and hence it is enough to prove that  $D(\tilde{\mathbf{b}})$  is an ideal of  $D(\mathbf{b})$  and apply [22] or [23], Theorem 2.24. For this, let  $0 \leq \varphi \leq \psi$  with  $\varphi \in D(\mathbf{b})$  and  $\psi \in D(\tilde{\mathbf{b}})$ . This means that  $\varphi$  and  $\psi$  are respectively the traces on  $\partial\Omega$  of  $u, v \in W^{1,2}(\Omega)$  such that

$$\varphi = \text{Tr}(u) = 0 \text{ on } \Gamma_0 \quad \text{and} \quad \psi = \text{Tr}(v) = 0 \text{ on } \tilde{\Gamma}_0.$$

Since  $0 \leq \varphi \leq \psi$  we have  $\varphi = 0$  on  $\tilde{\Gamma}_0$ . This equality gives  $\varphi \in D(\tilde{\mathbf{b}})$  and this shows that  $D(\tilde{\mathbf{b}})$  is an ideal of  $D(\mathbf{b})$ .  $\square$

The next result shows monotonicity with respect to the potential  $a_0$ . This was already proved in [11] Theorem 2.4, in the case where  $L_a^D = -\Delta + a_0$ . The proof given there works also in the general framework of the present paper.

As above let  $a_{kj}$ ,  $a_k$  and  $a_0$  be real-valued and let  $(T_t^{\Gamma_1, a_0})_{t \geq 0}$  denote the semigroup  $(T_t^{\Gamma_1})_{t \geq 0}$  defined above. Suppose that  $b_0$  is a real-valued function and denote by  $(T_t^{\Gamma_1, b_0})_{t \geq 0}$  be the semigroup of  $\mathcal{N}_{\Gamma_1}$  with coefficients  $a_{kj}$ ,  $a_k$  and  $b_0$  (i.e.  $a_0$  is replaced by  $b_0$ ). Then we have

**Theorem 3.3.** *Suppose that  $a_{kj}$ ,  $a_k$ ,  $\tilde{a}_k$  and  $a_0$  satisfy (3.1). Suppose again that  $0 \notin \sigma(L_a^D)$  and  $L_a^D$  is accretive. If  $a_0 \leq b_0$  then*

$$0 \leq T_t^{\Gamma_1, b_0} \varphi \leq T_t^{\Gamma_1, a_0} \varphi$$

for all  $0 \leq \varphi \in L^2(\partial\Omega)$  and  $t \geq 0$ .

## 4 Proof of the main result

In this section we prove Theorem 1.1. We recall briefly the operators introduced in Section 2.

For  $\mu \in \mathbb{R}$  and recall the operator  $L_a^\mu$  associated with the form  $\mathbf{a}^\mu$  given by (2.6) with domain  $D(\mathbf{a}^\mu) := V$  and  $V$  is again given by (2.2). The operator associated with  $\mathbf{a}^\mu$  is  $L_a^\mu$ . It is given by the formal expression (2.4) and it is subject to mixed and Robin boundary conditions (2.7).

We also recall that  $L_a^D$  is the operator subject to the Dirichlet boundary conditions and  $L_a^M$  is subject to mixed boundary conditions.

Fix  $\lambda \notin \sigma(L_a^D)$ . We denote by  $\mathcal{N}_{\Gamma_1, a}(\lambda)$  the partial D-t-N operator with the coefficients  $\{a_{kj}, a_k, a_0 - \lambda\}$ . It is the operator associated with the form

$$\mathbf{b}(\varphi, \psi) := \sum_{k,j}^d \int_{\Omega} a_{kj} \partial_k u \overline{\partial_j v} \, dx + \sum_{k=1}^d \int_{\Omega} a_k \partial_k u \overline{v} + \overline{a_k} u \overline{\partial_k v} \, dx + (a_0 - \lambda) u \overline{v} \, dx$$

where  $u, v \in V_H(\lambda)$  with  $\text{Tr}(u) = \varphi$ ,  $\text{Tr}(v) = \psi$  and

$$V_H(\lambda) := \{u \in V, \mathbf{a}(u, g) = \lambda \int_{\Omega} u \overline{g} \text{ for all } g \in W_0^{1,2}(\Omega)\}, \quad (4.1)$$

This space is the same as in (2.8) but now with  $a_0$  replaced by  $a_0 - \lambda$ .

We restate the main theorem using the notation introduced in Section 2.

**Theorem 4.1.** *Suppose that  $\Omega$  is a bounded Lipschitz domain of  $\mathbb{R}^d$  with  $d \geq 2$ . Let  $\Gamma_0$  be a closed subset of  $\partial\Omega$ ,  $\Gamma_0 \neq \partial\Omega$  and  $\Gamma_1 = \partial\Omega \setminus \Gamma_0$ . Let  $a = \{a_{kj} = \overline{a_{jk}}, a_k = \overline{a_k}, a_0 = \overline{a_0}\}$  and  $b = \{b_{kj} = \overline{b_{jk}}, b_k = \overline{b_k}, b_0 = \overline{b_0}\}$  be bounded measurable functions on  $\Omega$  such that  $a_{kj}$  and  $b_{kj}$  satisfy the ellipticity condition (2.1). If  $d \geq 3$  we assume in addition that the coefficients  $a_{kj}, b_{kj}, a_k$  and  $b_k$  are Lipschitz continuous on  $\overline{\Omega}$ .*

*Suppose that  $\mathcal{N}_{\Gamma_1, a}(\lambda) = \mathcal{N}_{\Gamma_1, b}(\lambda)$  for all  $\lambda$  in a set having an accumulation point in  $\rho(L_a^D) \cap \rho(L_b^D)$ . Then:*

- i) The operators  $L_a^\mu$  and  $L_b^\mu$  are unitarily equivalent for all  $\mu \in \mathbb{R}$ .*
- ii) The operators  $L_a^M$  and  $L_b^M$  are unitarily equivalent.*
- iii) The operators  $L_a^D$  and  $L_b^D$  are unitarily equivalent.*

*Moreover, for every  $\lambda \in \sigma(L_a^\mu) = \sigma(L_b^\mu)$  with  $\lambda \notin \sigma(L_a^D) = \sigma(L_b^D)$ , the sets  $\{\text{Tr}(u), u \in \text{Ker}(\lambda I - L_a^\mu)\}$  and  $\{\text{Tr}(v), v \in \text{Ker}(\lambda I - L_b^\mu)\}$  coincide. The same property holds for the operators  $L_a^M$  and  $L_b^M$ .*

We shall need several preparatory results. We start with the following theorem which was proved in [2] and [3] in the case where  $a_{kj} = \delta_{kj}$ ,  $a_k = 0$ ,  $a_0$  is a constant and  $\Gamma_1 = \partial\Omega$ .

**Theorem 4.2.** *Let  $a = \{a_{kj} = \overline{a_{jk}}, a_k = \overline{a_k}, a_0 = \overline{a_0}\}$  be bounded measurable functions on  $\Omega$  such that  $a_{kj}$  satisfy the ellipticity condition (2.1).*

*Let  $\mu, \lambda \in \mathbb{R}$  and  $\lambda \notin \sigma(L_a^D)$ . Then:*

- 1)  $\mu \in \sigma(\mathcal{N}_{\Gamma_1, a}(\lambda)) \Leftrightarrow \lambda \in \sigma(L_a^\mu)$ . In addition, if  $u \in \text{Ker}(\lambda - L_a^\mu)$ ,  $u \neq 0$  then  $\varphi := \text{Tr}(u) \in \text{Ker}(\mu - \mathcal{N}_{\Gamma_1, a}(\lambda))$  and  $\varphi \neq 0$ . Conversely, if  $\varphi \in \text{Ker}(\mu - \mathcal{N}_{\Gamma_1, a}(\lambda))$ ,  $\varphi \neq 0$ , then there exists  $u \in \text{Ker}(\lambda - L_a^\mu)$ ,  $u \neq 0$  such that  $\varphi = \text{Tr}(u)$ .*
- 2)  $\dim \text{Ker}(\mu - \mathcal{N}_{\Gamma_1, a}(\lambda)) = \dim \text{Ker}(\lambda - L_a^\mu)$ .*

*Proof.* We follow a similar idea as in [2] and [3]. It is enough to prove that the mapping

$$S : \text{Ker}(\lambda - L_a^\mu) \rightarrow \text{Ker}(\mu - \mathcal{N}_{\Gamma_1, a}(\lambda)), \quad u \mapsto \text{Tr}(u)$$

is an isomorphism. First, we prove that  $S$  is well defined. Let  $u \in \text{Ker}(\lambda - L_a^\mu)$ . Then  $u \in D(L_a^\mu)$  and  $L_a^\mu u = \lambda u$ . By the definition of  $L_a^\mu$  we have  $u \in V$  and for all  $v \in V$

$$\begin{aligned} & \sum_{k,j=1}^d \int_{\Omega} a_{kj} \partial_k u \overline{\partial_j v} + \sum_{k=1}^d \int_{\Omega} a_k \partial_k u \overline{v} + \overline{a_k} u \overline{\partial_k v} \\ & + \int_{\Omega} a_0 u \overline{v} - \lambda \int_{\Omega} u \overline{v} = \mu \int_{\partial\Omega} \text{Tr}(u) \overline{\text{Tr}(v)}. \end{aligned} \quad (4.2)$$

Taking  $v \in W_0^{1,2}(\Omega)$  yields  $u \in V_H(\lambda)$ . Note that (4.2) also holds for  $v \in V_H(\lambda)$ . Hence it follows from the definition of  $\mathcal{N}_{\Gamma_1, a}(\lambda)$  that

$$\varphi := \text{Tr}(u) \in D(\mathcal{N}_{\Gamma_1, a}(\lambda)) \text{ and } \mathcal{N}_{\Gamma_1, a}(\lambda) \varphi = \mu \varphi.$$

This means that  $S(u) \in \text{Ker}(\mu - \mathcal{N}_{\Gamma_1, a}(\lambda))$ .

Suppose now that  $u \in \text{Ker}(\lambda - L_a^\mu)$  with  $u \neq 0$ . If  $S(u) = 0$  then  $u \in W_0^{1,2}(\Omega)$ . Therefore, it follows from (4.2) that for all  $v \in V$

$$\sum_{k,j=1}^d \int_{\Omega} a_{kj} \partial_k u \overline{\partial_j v} + \sum_{k=1}^d \int_{\Omega} a_k \partial_k u \overline{v} + \overline{a_k} u \overline{\partial_k v} + \int_{\Omega} (a_0 - \lambda) u \overline{v} = 0. \quad (4.3)$$

This implies that  $u \in V_H(\lambda)$ . We conclude by Lemma 2.1 that  $u = 0$ . Thus  $S$  is injective.

We prove that  $S$  is surjective. Let  $\varphi \in \text{Ker}(\mu - \mathcal{N}_{\Gamma_1, a}(\lambda))$ . Then by the definition of  $\mathcal{N}_{\Gamma_1, a}(\lambda)$ , there exists  $u \in V_H(\lambda)$  such that  $\varphi = \text{Tr}(u)$  and  $u$  satisfies (4.2) for all  $v \in V_H(\lambda)$ . If  $v \in V$  we write  $v = v_0 + v_1 \in W_0^{1,2}(\Omega) \oplus V_H(\lambda)$  and see that (4.2) holds for  $u$  and  $v$ . This means that  $u \in D(L_a^\mu)$  and  $L_a^\mu u = \lambda u$ .  $\square$

**Lemma 4.3.** *For  $\lambda \in \mathbb{R}$  large enough,  $(\lambda + L_a^\mu)^{-1}$  converges in  $\mathcal{L}(L^2(\Omega))$  to  $(\lambda + L_a^D)^{-1}$  as  $\mu \rightarrow -\infty$ .*

This is Proposition 2.6 in [2] when  $a_{kj} = \delta_{kj}$ ,  $a_k = a_0 = 0$ . The proof given in [2] remains valid in our setting. Note that the idea of proving the uniform convergence here is based on a criterion from [8] (see Appendix B) which states that it is enough to check that for all  $(f_n), f \in L^2(\Omega)$

$$f_n \rightharpoonup f \Rightarrow (\lambda + L_a^{\mu_n})^{-1} f_n \rightarrow (\lambda + L_a^D)^{-1} f, \quad (4.4)$$

for every sequence  $\mu_n \rightarrow -\infty$ . The first convergence is in the weak sense in  $L^2(\Omega)$  and the second one is the strong convergence. It is not difficult to check (4.4).

From now on, we denote by  $(\lambda_{a,n}^\mu)_{n \geq 1}$  the eigenvalues of  $L_a^\mu$ , repeated according to their multiplicities. We have for each  $\mu \in \mathbb{R}$

$$\lambda_{a,1}^\mu \leq \lambda_{a,2}^\mu \leq \dots \rightarrow +\infty.$$

Similarly for the eigenvalues  $(\lambda_{a,n}^D)_{n \geq 1}$  of  $L_a^D$ . These eigenvalues satisfy the standard min-max principle since the operators  $L_a^\mu$  and  $L_a^D$  are self-adjoint by our assumptions.

A well known consequence of the previous lemma is that the spectrum of  $L_a^\mu$  converges to the spectrum of  $L_a^D$ . More precisely, for all  $k$ ,

$$\lambda_{a,k}^\mu \rightarrow \lambda_{a,k}^D \text{ as } \mu \rightarrow -\infty. \quad (4.5)$$

In addition, we have the following lemma which will play a fundamental role.

**Lemma 4.4.** *Let  $a = \{a_{kj} = \overline{a_{jk}}, a_k = \overline{a_k}, a_0 = \overline{a_0}\}$  be bounded measurable functions on  $\Omega$  such that  $a_{kj}$  satisfy the ellipticity condition (2.1). If  $d \geq 3$  we assume in addition that the coefficients  $a_{kj}$  and  $a_k$  are Lipschitz continuous on  $\overline{\Omega}$ . Then for each  $k$ ,  $\mu \mapsto \lambda_{a,k}^\mu$  is strictly decreasing on  $\mathbb{R}$  and  $\lambda_{a,k} \rightarrow -\infty$  as  $\mu \rightarrow +\infty$ .*

*Proof.* Firstly, by the min-max principle  $\lambda_{a,k}^\mu \leq \lambda_{a,k}^D$  and the function  $\mu \mapsto \lambda_{a,k}^\mu$  is non-increasing. Fix  $k \geq 0$  and suppose that  $\mu \mapsto \lambda_{a,k}^\mu$  is constant on  $[\alpha, \beta]$  for some  $\alpha < \beta$ . For each  $\mu$  we take a normalized eigenvector  $u^\mu$  such that  $\text{Tr}(u^{\mu+h}) \rightarrow \text{Tr}(u^\mu)$  in  $L^2(\partial\Omega)$  as  $h \rightarrow 0$  (or as  $h_n \rightarrow 0$  for some sequence  $h_n$ ). Indeed, due to regularity properties  $\mu \mapsto \lambda_{a,k}^\mu$  is continuous (see [17], Chapter VII) and hence  $(\lambda_{a,k}^{\mu+h})_h$  is bounded for small  $h$ . The equality  $\mathbf{a}^{\mu+h}(u^{\mu+h}, u^{\mu+h}) = \lambda_{a,k}^{\mu+h}$  implies that  $\mathbf{a}^{\mu+h}(u^{\mu+h}, u^{\mu+h})$  is bounded w.r.t.  $h$  (for small  $h$ ). This latter property and ellipticity easily imply that  $(u^{\mu+h})_h$  is bounded in  $V$ . After extracting a sequence we may assume that  $(u^{\mu+h})_h$  converges weakly in  $V$  to some  $u$  as  $h \rightarrow 0$ . The compactness embedding of  $V$

in  $L^2(\Omega)$  as well as the compactness of the trace operator show that  $(u^{\mu+h})_h$  converges to  $u$  in  $L^2(\Omega)$  and  $\text{Tr}(u^{\mu+h})$  converges to  $\text{Tr}(u)$  in  $L^2(\partial\Omega)$ . On the other hand for every  $v \in V$ , the equality

$$\mathfrak{a}^{\mu+h}(u^{\mu+h}, v) = \lambda_{a,k}^{\mu+h} \int_{\Omega} u^{\mu+h} v \, dx$$

shows that the limit  $u$  is a normalized eigenvector of  $L_a^\mu$  for the eigenvalue  $\lambda_{a,k}^\mu$ . We take  $u^\mu := u$  and obtain the claim stated above.

Observe that

$$\int_{\Gamma_1} \text{Tr}(u^{\mu+h}) \overline{\text{Tr}(u^\mu)} \, d\sigma = 0 \quad (4.6)$$

for all  $h \neq 0$  and  $\mu, \mu + h \in [\alpha, \beta]$ . Indeed, using the definition of the form  $\mathfrak{a}^\mu$  (see (2.6)) we have

$$\begin{aligned} \lambda \int_{\Omega} u^{\mu+h} \overline{u^\mu} \, dx &= \mathfrak{a}^{\mu+h}(u^{\mu+h}, u^\mu) \\ &= \mathfrak{a}^\mu(u^{\mu+h}, u^\mu) - h \int_{\Gamma_1} \text{Tr}(u^{\mu+h}) \overline{\text{Tr}(u^\mu)} \, d\sigma \\ &= \lambda \int_{\Omega} u^{\mu+h} \overline{u^\mu} \, dx - h \int_{\Gamma_1} \text{Tr}(u^{\mu+h}) \overline{\text{Tr}(u^\mu)} \, d\sigma. \end{aligned}$$

This gives (4.6). Now, letting  $h \rightarrow 0$  we obtain from (4.6) and the fact that  $\text{Tr}(u^{\mu+h})$  converges to  $\text{Tr}(u^\mu)$  as  $h \rightarrow 0$  that  $\text{Tr}(u^\mu) = 0$  on  $\Gamma_1$  for all  $\mu \in [\alpha, \beta]$ . Hence  $\text{Tr}(u^\mu) = 0$  on  $\partial\Omega$  since  $u^\mu \in V$ . Hence,  $L^\mu$  has an eigenfunction  $u^\mu \in W_0^{1,2}(\Omega)$ . Note that if  $d = 2$  or if  $d \geq 3$  and the coefficients  $a_{kj}$  and  $a_k$  are Lipschitz continuous on  $\tilde{\Omega}$ , then the operator  $L_a$  has the unique continuation property (see [26] for the case  $d = 2$  and [28] for  $d \geq 3$ ). If  $d \geq 3$  and hence the coefficients are Lipschitz on  $\tilde{\Omega}$ , we apply Proposition 2.5 in [6] to conclude that  $u^\mu = 0$ , but this is not possible since  $\|u^\mu\|_2 = 1$ . If  $d = 2$  we argue in a similar way. Indeed, let  $\tilde{\Omega}$  be an open subset of  $\mathbb{R}^2$  containing  $\Omega$  and such that  $\Gamma_0 \subset \partial\tilde{\Omega}$  and  $\tilde{\Omega} \setminus \Omega$  contains an open ball. We extend all the coefficients to bounded measurable function  $\tilde{a}_{kj}, \tilde{a}_k$  and  $\tilde{a}_0$  on  $\tilde{\Omega}$ . In addition,  $\tilde{a}_{kj} = \overline{\tilde{a}_{jk}}$  on  $\tilde{\Omega}$  and satisfy the ellipticity condition. We extend  $u^\mu$  to  $\tilde{u}^\mu \in W_0^{1,2}(\tilde{\Omega})$  by 0 outside  $\Omega$ . We define in  $\tilde{\Omega}$  the elliptic operator  $L_{\tilde{a}}$  as previously. For  $v \in C_c^\infty(\tilde{\Omega})$  we note that  $v|_\Omega \in V$  and hence

$$\begin{aligned} \int_{\tilde{\Omega}} L_{\tilde{a}}(\tilde{u}^\mu) \bar{v} \, dx &= \mathfrak{a}^\mu(u^\mu, v|_\Omega) \\ &= \lambda \int_{\Omega} u^\mu \bar{v}|_\Omega = \lambda \int_{\tilde{\Omega}} \tilde{u}^\mu \bar{v}. \end{aligned}$$

The term  $\int_{\tilde{\Omega}} L_{\tilde{a}}(\tilde{u}^\mu) \bar{v}$  is of course interpreted in the sense of the associated sesquilinear form and the first equality uses the fact that  $\tilde{u}^\mu$  is 0 on  $\tilde{\Omega} \setminus \Omega$  and  $u^\mu \in W_0^{1,2}(\Omega)$ . Hence,  $\tilde{u}^\mu$  satisfies

$$(L_{\tilde{a}} - \lambda)(\tilde{u}^\mu) = 0$$

in the weak sense on  $\tilde{\Omega}$ . We conclude by the unique continuation property ([26]) that  $\tilde{u}^\mu = 0$  on  $\tilde{\Omega}$  since it is 0 on an open ball contained in  $\tilde{\Omega} \setminus \Omega$ . We arrive as above to a contradiction. Hence,  $\mu \mapsto \lambda_{a,k}^\mu$  is strictly decreasing on  $\mathbb{R}$ .

It remains to prove that for any  $k$ ,  $\lambda_{a,k}^\mu \rightarrow -\infty$  as  $\mu \rightarrow +\infty$ . By the min-max principle

$$\lambda_1^\mu \leq \sum_{k,j=1}^d \int_{\Omega} a_{kj} \partial_k u \overline{\partial_j u} + 2 \operatorname{Re} \sum_{k=1}^d \int_{\Omega} a_k \partial_k u \bar{u} + \int_{\Omega} a_0 |u|^2 - \mu \int_{\Gamma_1} |\operatorname{Tr}(u)|^2$$

for every normalized  $u \in V$ . Taking  $u$  such that  $\operatorname{Tr}(u) \neq 0$  shows that  $\lambda_{a,1}^\mu \rightarrow -\infty$  as  $\mu \rightarrow +\infty$ . Suppose now that  $\lambda_{a,k}^\mu > w$  for some  $w \in \mathbb{R}$ ,  $k > 1$  and all  $\mu \in \mathbb{R}$ . Taking the smallest possible  $k$  we have  $\lambda_{a,j}^\mu \rightarrow -\infty$  as  $\mu \rightarrow +\infty$  for  $j = 1, \dots, k-1$ . Of course,  $\lambda_{a,j}^\mu > w$  for all  $j \geq k$  and we may choose  $w \notin \sigma(L_a^D)$ . Remember also that  $\mu \mapsto \lambda_{a,j}^\mu$  is strictly decreasing for  $j = 1, \dots, k-1$ . On the other hand, by Theorem 4.2 we have  $\sigma(\mathcal{N}_{\Gamma_1,a}(w)) \subset \{\mu \in \mathbb{R}, \lambda_{a,j}^\mu = w, j = 1, \dots, k-1\}$ . Using the fact that  $\lambda_{a,j}^\mu \rightarrow -\infty$  as  $\mu \rightarrow +\infty$  and  $\mu \mapsto \lambda_{a,j}^\mu$  is strictly decreasing for  $j = 1, \dots, k-1$  we see that we can choose  $w$  such that the set  $\{\mu \in \mathbb{R}, \lambda_{a,j}^\mu = w, j = 1, \dots, k-1\}$  is finite and hence  $\sigma(\mathcal{N}_{\Gamma_1,a}(w))$  is finite which is not possible since  $L^2(\Gamma_1)$  has infinite dimension.  $\square$

Related results to Lemma 4.4 can be found in [3] (see Proposition 3) and [25]. In both papers the proofs use the unique continuation property.

We shall also need the following lemma.

**Lemma 4.5.** *For every  $\varphi, \psi \in \operatorname{Tr}(V)$ , the mapping*

$$\lambda \mapsto \langle \mathcal{N}_{\Gamma_1,a}(\lambda) \varphi, \psi \rangle$$

*is holomorphic on  $\mathbb{C} \setminus \sigma(L_a^D)$ .*

This result is easy to prove, see Lemma 2.4 in [6].

*Proof of Theorem 1.1.* As above, we denote by  $(\lambda_{b,n}^\mu)_{n \geq 1}$  and  $(\lambda_{b,n}^D)_{n \geq 1}$  the eigenvalues of the self-adjoint operators  $L_b^\mu$  and  $L_b^D$ , respectively.

It follows from Lemma 4.5 and the assumptions that  $\mathcal{N}_{\Gamma_1,a}(\lambda) = \mathcal{N}_{\Gamma_1,b}(\lambda)$  for all  $\lambda \in \mathbb{C} \setminus (\sigma(L_a^D) \cup \sigma(L_b^D))$ .

i) We show that for all  $\mu \in \mathbb{R}$

$$\sigma(L_a^\mu) = \sigma(L_b^\mu), \quad (4.7)$$

and the eigenvalues have the same multiplicity.

Fix  $\mu \in \mathbb{R}$  and suppose that  $\lambda = \lambda_{a,k}^\mu \in \sigma(L_a^\mu) \setminus (\sigma(L_a^D) \cup \sigma(L_b^D))$ . By Theorem 4.2,  $\mu \in \sigma(\mathcal{N}_{\Gamma_1,a}(\lambda)) = \sigma(\mathcal{N}_{\Gamma_1,b}(\lambda))$  and hence  $\lambda \in \sigma(L_b^\mu)$ . Thus,  $\lambda = \lambda_{a,k}^\mu = \lambda_{b,j}^\mu$  for some  $j \geq 1$ . The second assertion of Theorem 4.2 shows that  $\lambda_{a,k}^\mu$  and  $\lambda_{b,j}^\mu$  have the same multiplicity. In addition,  $j = k$ . Indeed, if  $k < j$  then

$$\lambda_{b,1}^\mu \leq \lambda_{b,2}^\mu \leq \dots \leq \lambda_{b,k}^\mu \leq \dots \leq \lambda_{b,j}^\mu = \lambda_{a,k}^\mu.$$

Each  $\lambda_{b,m}^\mu$  coincides with an eigenvalue of  $L_a^\mu$  (with the same multiplicity) and hence  $\lambda_{a,k}^\mu$  is (at least) the  $j$ -th eigenvalue of  $L_a^\mu$  with  $j > k$  which is not possible. The same argument works if  $j < k$ . Using Lemma 4.4 we see that for any  $k$  there exists a discrete set  $J \subset \mathbb{R}$  such that  $\lambda_{a,k}^\mu = \lambda_{b,k}^\mu$  for every  $\mu \in \mathbb{R} \setminus J$ . By continuity of  $\mu \mapsto \lambda_{a,k}^\mu$  and  $\mu \mapsto \lambda_{b,k}^\mu$  these two functions coincide on  $\mathbb{R}$ . This



proves (4.7) and also that the multiplicities of the eigenvalues  $\lambda_{a,k}^\mu$  and  $\lambda_{b,k}^\mu$  are the same.

The similarity property follows by a classical argument. Recall that  $L_a^\mu$  and  $L_b^\mu$  are self-adjoint operators with compact resolvents. It follows that there exist orthonormal bases  $\Phi_n$  and  $\Psi_n$  of  $L^2(\Omega)$  which are eigenfunctions of  $L_a^\mu$  and  $L_b^\mu$ , respectively. Define the mapping

$$\mathcal{U} : L^2(\Omega) \rightarrow L^2(\Omega), \Phi_n \mapsto \Psi_n.$$

Thus for  $f = \sum_n (f, \Phi_n) \Phi_n \in L^2(\Omega)$ ,  $\mathcal{U}(f) = \sum_n (f, \Phi_n) \Psi_n$ . The notation  $(f, \Phi_n)$  is the scalar product in  $L^2(\Omega)$ . Clearly,

$$\|\mathcal{U}(f)\|_2^2 = \sum_n |(f, \Phi_n)|^2 = \|f\|_2^2.$$

The mapping  $\mathcal{U}$  is an isomorphism. In addition, if  $L_a^\mu \Phi_n = \lambda_{a,n}^\mu \Phi_n$  then for  $f \in D(L_b^\mu)$

$$\begin{aligned} \mathcal{U} L_a^\mu \mathcal{U}^{-1}(f) &= \mathcal{U} L_a^\mu \mathcal{U}^{-1} \left( \sum_n (f, \Psi_n) \Psi_n \right) \\ &= \mathcal{U} L_a^\mu \left( \sum_n (f, \Psi_n) \Phi_n \right) \\ &= \mathcal{U} \left( \sum_n (f, \Psi_n) \lambda_{a,n}^\mu \Phi_n \right) \\ &= \sum_n (f, \lambda_{b,n}^\mu \Psi_n) \Psi_n \\ &= L_b^\mu(f). \end{aligned}$$

Thus,  $L_a^\mu$  and  $L_b^\mu$  are unitarily equivalent. This proves assertion *i*).

*ii*) Choose  $\mu = 0$  in the previous assertion.

*iii*) As mentioned above, by Lemma 4.3 we have (4.5). The same property holds for  $L_b^\mu$ , that is,  $\lambda_{b,k}^\mu \rightarrow \lambda_{b,k}^D$  as  $\mu \rightarrow -\infty$ . It follows from assertion *i*) that  $\lambda_{a,k}^D = \lambda_{b,k}^D$  for all  $k \geq 1$  and have the same multiplicity. We conclude as above that  $L_a^D$  and  $L_b^D$  are unitarily equivalent.

Finally, another application of Theorem 4.2 shows that  $\text{Tr}(\text{Ker}(\lambda - L_a^\mu)) = \text{Tr}(\text{Ker}(\lambda - L_b^\mu))$  for  $\lambda \notin \sigma(L_a^D) = \sigma(L_b^D)$ .  $\square$

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