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A "milder" version of Calderón's inverse problem for anisotropic conductivities and partial data

El Maati Ouhabaz

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Abstract

Given a general symmetric elliptic operator

$$L_a := \sum_{k,j=1}^d \partial_k(a_{kj}\partial_j) + \sum_{k=1}^d a_k\partial_k - \partial_k(\overline{a_k \cdot}) + a_0$$

we define the associated Dirichlet-to-Neumann (D-t-N) operator with partial data, i.e., data supported in a part of the boundary. We prove positivity, L^p -estimates and domination properties for the semigroup associated with this D-t-N operator. Given L_a and L_b of the previous type with bounded measurable coefficients $a = \{a_{kj}, a_k, a_0\}$ and $b = \{b_{kj}, b_k, b_0\}$, we prove that if their partial D-t-N operators (with a_0 and b_0 replaced by $a_0 - \lambda$ and $b_0 - \lambda$) coincide for all λ , then the operators L_a and L_b , endowed with Dirichlet, mixed or Robin boundary conditions are unitary equivalent. In the case of the Dirichlet boundary conditions, this result was proved recently by Behrndt and Rohleder [6] for Lipschitz continuous coefficients. We provide a different proof which works for bounded measurable coefficients and other boundary conditions.

1 Introduction

Let Ω be a bounded Lipschitz domain of \mathbb{R}^d with boundary $\partial\Omega$. Let Γ_0 be an open subset of Ω with $\Gamma_0 \neq \partial\Omega$ and Γ_1 its complement in $\partial\Omega$. We consider the symmetric elliptic operator on $L^2(\Omega)$ given by the formal expression:

$$L_a(\lambda) := \sum_{k,j=1}^d \partial_k(a_{kj}\partial_j) + \sum_{k=1}^d a_k\partial_k - \partial_k(\overline{a_k \cdot}) + a_0 - \lambda$$

where $a_{kj} = \overline{a_{jk}}$, $a_k, a_0 \in L^\infty(\Omega)$ and λ is a constant. We define the associated Dirichlet-to-Neumann (D-t-N), $\mathcal{N}_{\Gamma_1, a}(\lambda)$, with partial data as follows: for $\varphi \in H^{\frac{1}{2}}(\partial\Omega)$ with $\varphi = 0$ on Γ_0 , one solves the Dirichlet problem

$$L_a(\lambda)u = 0 \text{ weakly in } W^{1,2}(\Omega) \text{ with } u = \varphi \text{ on } \partial\Omega, \quad (1.1)$$

and set

$$\mathcal{N}_{\Gamma_1, a}(\lambda)\varphi := \sum_{j=1}^d \left(\sum_{k=1}^d a_{kj}\partial_k u + \overline{a_j} \varphi \right) \nu_j \text{ on } \Gamma_1. \quad (1.2)$$

Here $\nu = (\nu_1, \dots, \nu_d)$ is the outer unit normal to the boundary of Ω . The operator $\mathcal{N}_{\Gamma_1, a}(\lambda)$ is interpreted as the conormal derivative on the boundary. It is an operator acting on $L^2(\partial\Omega)$. See Section 2 for more details.

Let us consider first the case where $a_{kj} = \sigma(x)\delta^{kj}$, $a_k = 0, k = 0, 1 \dots d$, where $\sigma \in L^\infty(\Omega)$ is bounded from below (by a positive constant). A. Calderón's well known inverse problem asks whether one could determine solely the conductivity $\sigma(x)$ from boundary measurements, i.e., from $\mathcal{N}_{\Gamma_1}(0)$. For the global boundary measurements, i.e., $\Gamma_1 = \partial\Omega$, the first global uniqueness result was proved by Sylvester and Uhlmann [23] for a C^2 -smooth conductivity when the dimension $d \geq 3$. This results was extended to $C^{1+\epsilon}$ -smooth conductivity by Greenleaf, Lassas and Uhlman [11] and then by Haberman and Tataru [12] to C^1 and even Lipschitz conductivities in some cases. In the two-dimension case with C^2 -smooth conductivity, the global uniqueness was proved by Nachman [19]. This regularity assumption was completely removed by Astala and Päivärinta [4] dealing with $\sigma \in L^\infty(\Omega)$.

The inverse problem with partial data consists in proving uniqueness (either for the isotropic conductivity or for the potential) when the measurement is made only on a part of the boundary. This means that the trace of the solution u in (1.1) is supported on a set Γ_D and the D-t-N operator is known on Γ_N for some parts Γ_D and Γ_N of the boundary. This problem has been studied and there are some geometric conditions on Γ_D and Γ_N under which uniqueness is proved. We refer to Isakov [13], Kenig, Sjöstrand and Uhlmann [16], Dos Santos et al. [9], Isakov, Uhlmann and Yamamoto [14] and the review paper [17] by Kenig and Salo for more references and recent developments.

Now we move to the anisotropic case. This corresponds to the general case where the conductivity is given by a general matrix a_{kj} . As pointed out by Lee and Uhlmann in [18], it is not difficult to see that a change of variables given by a diffeomorphism of Ω which is the identity on $\partial\Omega$ leads to different coefficients b_{kj} without changing the D-t-N operator on the boundary. Therefore the single coefficients a_{kj} are not uniquely determined in general. In [18], Lee and Uhlmann proved that for real-analytic coefficients the uniqueness up to a diffeomorphism holds when the dimension d is ≥ 3 . The same result was proved by Astala, Lassas and Päivärinta [5] for the case $d = 2$ and L^∞ -coefficients.

In [6], Behrndt and Rohleder considered general elliptic expressions L_a and L_b as above and prove that if the corresponding D-t-N operators $\mathcal{N}_{\Gamma_1, a}(\lambda)$ and $\mathcal{N}_{\Gamma_1, b}(\lambda)$ coincide for all λ in a set having an accumulation point in $\rho(L_a^D) \cap \rho(L_b^D)$ then the operators L_a^D and L_b^D are unitary equivalent. Here L_a^D is the elliptic operator L_a with Dirichlet boundary conditions. This can be seen as a milder version of the uniqueness problem discussed above. The proof is based on the theory of extensions of symmetric operators and unique continuation results. It is assumed in [6] that the coefficients are Lipschitz continuous on $\bar{\Omega}$. We extend this result in the sense that we deal with L^∞ coefficients and also with other boundary conditions. Our main result is the following.

Theorem 1.1. *Suppose that Ω is a bounded Lipschitz domain of \mathbb{R}^d with $d \geq 2$. Let Γ_0 be an open subset of $\partial\Omega$, $\Gamma_0 \neq \partial\Omega$ and Γ_1 its complement. Let $a = \{a_{kj}, a_k, a_0\}$ and $b = \{b_{kj}, b_k, b_0\}$ be bounded functions on Ω such that a_{kj} and b_{kj} satisfy the usual ellipticity condition. Suppose that $\mathcal{N}_{\Gamma_1, a}(\lambda) = \mathcal{N}_{\Gamma_1, b}(\lambda)$ for all λ in a set having an accumulation point in $\rho(L_a^D) \cap \rho(L_b^D)$. Then:*

i) The operators L_a and L_b endowed with Robin boundary conditions are unitary equivalent.

ii) The operators L_a and L_b endowed with mixed boundary conditions (Dirichlet on Γ_0 and Neumann type on Γ_1) are unitary equivalent.

iii) The operators L_a and L_b endowed with Dirichlet boundary conditions are unitary equivalent.

In addition, for Robin or mixed boundary conditions, the eigenfunctions associated to the same eigenvalue $\lambda \notin \sigma(L_a^D) \cup \sigma(L_b^D)$ coincide on the boundary of Ω .

This theorem is proved in Section 4. Our proof is different from that in [6]. Our strategy is to use a relationship between eigenvalues of the D-t-N operator $\mathcal{N}_{\Gamma_1,a}(\lambda)$ and eigenvalues of the elliptic operator with Robin boundary conditions L_a^μ on Ω where μ is a parameter. The equality of $\mathcal{N}_{\Gamma_1,a}(\lambda)$ and $\mathcal{N}_{\Gamma_1,b}(\lambda)$ allows us then to prove that the spectra of L_a^μ and L_b^μ are the same. The similarity of the two elliptic operators with Dirichlet boundary conditions is obtained from the similarity of L_a^μ and L_b^μ by letting the parameter μ tend to $-\infty$. During the proof we use some ideas from the papers of Arendt and Mazzeo [2] and [3] which deal with a different subject, namely the Friendlander inequality for the eigenvalues of the Dirichlet and Neumann Laplacian on a Lipschitz domain. The ideas which we borrow from [2] and [3] are then adapted and extended to our general case of D-t-N operators with variable coefficients and partial data. In Section 2 we define the D-t-N operator with partial data using the method of sesquilinear forms. In particular, for symmetric coefficients it is a self-adjoint operator on $L^2(\Gamma_1)$. It can be seen as an operator on $L^2(\partial\Omega)$ with a non-dense domain and which we extend by 0 to $L^2(\Gamma_0)$. Therefore one can associate with this D-t-N operator a semigroup $(T_t^{\Gamma_1})_{t \geq 0}$ acting on $L^2(\partial\Omega)$. In Section 3 we prove positivity, sub-Markovian and domination properties for such semigroups. In particular, $(T_t^{\Gamma_1})_{t \geq 0}$ acts as a contraction semigroup on $L^p(\partial\Omega)$ for all $p \in [1, \infty)$. In particular, for $\varphi_0 \in L^p(\Gamma_1)$, one obtains existence and uniqueness of the solution in $L^p(\partial\Omega)$ to the evolution problem

$$\partial_t \varphi + \mathcal{N}_{\Gamma_1,a}(\lambda)\varphi = 0, \quad \varphi(0) = \varphi_0.$$

The results of Section 3 are of independent interest and are not used in the proof of the theorem stated above.

2 The partial D-t-N operator

Let Ω be a bounded open set of \mathbb{R}^d with Lipschitz boundary $\partial\Omega$. The boundary is endowed with the $(d-1)$ -dimensional Hausdorff measure $d\sigma$. Let

$$a_{kj}, a_k, \tilde{a}_k, a_0 : \Omega \rightarrow \mathbb{C}$$

be bounded measurable for $1 \leq k, j \leq d$ and such that there exists a constant $\eta > 0$ for which

$$\operatorname{Re} \sum_{k,j=1}^d a_{kj}(x) \xi_k \bar{\xi}_j \geq \eta |\xi|^2 \quad (2.1)$$

for all $\xi = (\xi_1, \dots, \xi_d)$ and a.e. $x \in \Omega$. Let Γ_0 be an open subset of $\partial\Omega$ and Γ_1 its complement in $\partial\Omega$.

Elliptic operators on Ω .

We consider the space

$$V = \{u \in W^{1,2}(\Omega), u|_{\Gamma_0} = 0\}, \quad (2.2)$$

and define the sesquilinear form $\mathbf{a} : V \times V \rightarrow \mathbb{C}$ by the expression

$$\mathbf{a}(u, v) = \sum_{k,j}^d \int_{\Omega} a_{kj} \partial_k u \overline{\partial_j v} \, dx + \sum_{k=1}^d \int_{\Omega} a_k \partial_k u \bar{v} + \tilde{a}_k u \overline{\partial_k v} \, dx + a_0 u \bar{v} \, dx \quad (2.3)$$

for all $u, v \in V$. Here we use the notation ∂_j for the partial derivative $\frac{\partial}{\partial x_j}$.

Since V is a closed subspace of $W^{1,2}(\Omega)$ the form \mathbf{a} is closed. Therefore there exists an operator L_a associated with \mathbf{a} . It is defined by

$$D(L_a) = \{u \in V, \exists v \in L^2(\Omega) : \mathbf{a}(u, \phi) = \int_{\Omega} v \bar{\phi} \, dx \quad \forall \phi \in V\},$$

$$L_a u := v.$$

Formally, L_a is given by the expression

$$L_a u = - \sum_{k,j=1}^d \partial_k (a_{kj} \partial_j u) + \sum_{k=1}^d a_k \partial_k u - \partial_k (\tilde{a}_k u) + a_0 u, \quad (2.4)$$

and subject to the boundary conditions

$$\begin{cases} u = 0 & \text{on } \Gamma_0 \\ \sum_{j=1}^d \left(\sum_{k=1}^d a_{kj} \partial_k u + \tilde{a}_j u \right) \nu_j = 0 & \text{on } \Gamma_1. \end{cases} \quad (2.5)$$

Here $\nu = (\nu_1, \dots, \nu_d)$ is the outer unit normal to the boundary of Ω . This can be interpreted as mixed boundary conditions, that is Dirichlet on Γ_0 and Neumann type boundary conditions on Γ_1 . For this reason we denote this operator by L_a^M . The subscript a refers to the fact that the coefficients of the operator are given by $a = \{a_{kj}, a_k, \tilde{a}_k, a_0\}$ and M refers to mixed boundary conditions.

We also define the elliptic operator with Dirichlet boundary condition $u = 0$ on $\partial\Omega$. It is the operator associated with the form given by the expression (2.3) with domain $D(\mathbf{a}) = W_0^{1,2}(\Omega)$. It is a closed form and its associated operator L_a^D has the same expression as in (2.4) and subject to Dirichlet boundary condition $u = 0$ on $\partial\Omega$.

Similarly, we define L_a^N to be the elliptic operator with Neumann type boundary conditions

$$\sum_{j=1}^d \left(\sum_{k=1}^d a_{kj} \partial_k u + \tilde{a}_j u \right) \nu_j = 0 \quad \text{on } \partial\Omega.$$

It is the operator associated with the form given by the expression (2.3) with domain $D(\mathbf{a}) = W^{1,2}(\Omega)$.

Note that L_a^D coincides with L_a^M if $\Gamma_0 = \partial\Omega$ and L_a^N coincides with L_a^M if

$\Gamma_0 = \emptyset$.

Finally we define elliptic operators with Robin boundary conditions. Let $\mu \in \mathbb{R}$ be a constant and define

$$\begin{aligned} \mathbf{a}^\mu(u, v) &= \sum_{k,j}^d \int_{\Omega} a_{kj} \partial_k u \overline{\partial_j v} \, dx + \sum_{k=1}^d \int_{\Omega} a_k \partial_k u \overline{v} + \tilde{a}_k u \overline{\partial_k v} \, dx + a_0 u \overline{v} \, dx \\ &\quad - \mu \int_{\partial\Omega} \text{Tr}(u) \text{Tr}(v) \, d\sigma \end{aligned} \quad (2.6)$$

for all $u, v \in D(\mathbf{a}^\mu) := V$. Here and throughout this paper Tr denotes the trace operator. Using the standard inequality,

$$\int_{\partial\Omega} |\text{Tr}(u)|^2 \leq \varepsilon \|u\|_{W^{1,2}(\Omega)}^2 + c_\varepsilon \int_{\Omega} |u|^2$$

which is valid for all $\varepsilon > 0$ (c_ε is a constant depending on ε) one obtains that for some positive constants w and δ

$$\mathbf{a}^\mu(u, u) + w \int_{\Omega} |u|^2 \geq \delta \|u\|_{W^{1,2}(\Omega)}^2.$$

From this it follows that \mathbf{a}^μ is a closed sesquilinear form. One can associate with \mathbf{a}^μ an operator L_a^μ . This operator has the same expression (2.4) and it is subject to the Robin boundary conditions

$$\begin{cases} \text{Tr}(u) = 0 & \text{on } \Gamma_0 \\ \sum_{j=1}^d \left(\sum_{k=1}^d a_{kj} \partial_k u + \tilde{a}_j u \right) \nu_j = \mu \text{Tr}(u) & \text{on } \Gamma_1. \end{cases} \quad (2.7)$$

According to our previous notation, if $\mu = 0$, then $\mathbf{a}^0 = \mathbf{a}$ and $L_a^0 = L_a^M$. Note that we may choose μ to be a bounded function on the boundary rather than just a constant.

The partial Dirichlet-to-Neumann operator on $\partial\Omega$.

Suppose as before that $a = \{a_{kj}, a_k, \tilde{a}_k, a_0\}$ are bounded measurable and satisfy the ellipticity condition (2.1). Let Γ_0, Γ_1, V be as above and \mathbf{a} is the sesquilinear form defined by (2.3).

We define the space

$$V_H := \{u \in V, \mathbf{a}(u, g) = 0 \text{ for all } g \in W_0^{1,2}(\Omega)\}. \quad (2.8)$$

Then V_H is a closed subspace of V . It is interpreted as the space of harmonic functions for the operator L_a (given by (2.4)) with the additional property that $\text{Tr}(u) = 0$ on Γ_0 .

We start with the following simple lemma.

Lemma 2.1. *Suppose that $0 \notin \sigma(L_a^D)$. Then*

$$V = V_H \oplus W_0^{1,2}(\Omega). \quad (2.9)$$

Proof. . We argue as in [10], Section 2 or [2]. Let us denote by \mathbf{a}^D the form associated with L_a^D , that is, \mathbf{a}^D is given by (2.3) with $D(\mathbf{a}^D) = W_0^{1,2}(\Omega)$. There exists an operator $\mathcal{L}_a^D : W_0^{1,2}(\Omega) \rightarrow W^{-1,2}(\Omega) := W_0^{1,2}(\Omega)'$ (the anti-dual of $W_0^{1,2}(\Omega)$) associated with \mathbf{a}^D in the sense

$$\langle \mathcal{L}_a^D h, g \rangle = \mathbf{a}^D(h, g)$$

for all $h, g \in W_0^{1,2}(\Omega)$. The notation $\langle \cdot, \cdot \rangle$ denotes the duality $W_0^{1,2}(\Omega)' - W_0^{1,2}(\Omega)$. Since $0 \notin \sigma(L_a^D)$, then L_a^D is invertible. Therefore \mathcal{L}_a^D is also invertible on $W_0^{1,2}(\Omega)'$ since the two operators have the same spectrum (see e.g., [ABHN], Proposition 3.10.3). Now we fix $u \in V$ and consider the (anti-)linear functional

$$F : v \mapsto \mathbf{a}(u, v).$$

Clearly, $F \in W_0^{1,2}(\Omega)'$ and hence there exists a unique $u_0 \in W_0^{1,2}(\Omega)$ such that $\mathcal{L}_a^D u_0 = F$. This means that $\mathbf{a}(u - u_0, g) = 0$ for all $g \in W_0^{1,2}(\Omega)$. That is, $u - u_0 \in V_H$ and so $u = u - u_0 + u_0 \in V_H + W_0^{1,2}(\Omega)$. Finally, if $u \in V_H \cap W_0^{1,2}(\Omega)$ then $\mathbf{a}(u, g) = 0$ for all $g \in W_0^{1,2}(\Omega)$. This means that $u \in D(L_a^D)$ with $L_a^D u = 0$. Since L_a^D is invertible we conclude that $u = 0$. \square

As a consequence of Lemma 2.1, the trace operator $\text{Tr} : V_H \rightarrow L^2(\partial\Omega)$ is injective and

$$\text{Tr}(V_H) = \text{Tr}(V). \quad (2.10)$$

In the rest of this section we assume that $0 \notin \sigma(L_a^D)$. We define on $L^2(\partial\Omega, d\sigma)$ the sesquilinear form

$$\mathbf{b}(\varphi, \psi) := \mathbf{a}(u, v) \quad (2.11)$$

where $u, v \in V_H$ are such that $\varphi = \text{Tr}(u)$ and $\psi = \text{Tr}(v)$. This means that $D(\mathbf{b}) = \text{Tr}(V_H)$ and by (2.10)

$$D(\mathbf{b}) = \text{Tr}(V_H) = \text{Tr}(V). \quad (2.12)$$

Lemma 2.2. *There exist positive constants w, δ and M such that*

$$\text{Re } \mathbf{b}(\varphi, \varphi) + w \int_{\partial\Omega} |\varphi|^2 \geq \delta \|u\|_{W^{1,2}(\Omega)}^2 \quad (2.13)$$

and

$$|\mathbf{b}(\varphi, \psi)| \leq M \left[\text{Re } \mathbf{b}(\varphi, \varphi) + w \int_{\partial\Omega} |\varphi|^2 \right]^{1/2} \left[\text{Re } \mathbf{b}(\psi, \psi) + w \int_{\partial\Omega} |\psi|^2 \right]^{1/2} \quad (2.14)$$

for all $\varphi, \psi \in D(\mathbf{b})$. In the first inequality, $u \in V_H$ is such that $\text{Tr}(u) = \varphi$.

Proof. It is well known that $\text{Tr} : W^{1,2}(\Omega) \rightarrow L^2(\partial\Omega)$ is a compact operator and since $\text{Tr} : V_H \rightarrow L^2(\partial\Omega)$ is injective it follows that for every $\epsilon > 0$ there exists a constant $c > 0$ such that

$$\int_{\Omega} |u|^2 \leq \epsilon \|u\|_{W^{1,2}}^2 + c \int_{\partial\Omega} |\text{Tr}(u)|^2 \quad (2.15)$$

for all $u \in V_H$ (see, e.g., [2]). Now, let $\varphi \in D(\mathfrak{b}) = \text{Tr}(V_H)$ and $u \in V_H$ such that $\varphi = \text{Tr}(u)$. It follows from the ellipticity assumption (2.1) and the boundedness of the coefficients that for some constant c_0

$$\begin{aligned} \text{Re } \mathfrak{b}(\varphi, \varphi) &= \text{Re } \mathfrak{a}(u, u) \\ &\geq \frac{\eta}{2} \int_{\Omega} |\nabla u|^2 - c_0 \int_{\Omega} |u|^2. \end{aligned}$$

Therefore, (2.13) follows from (2.15).

In order to prove the second inequality, we use the definition of \mathfrak{b} and again the boundedness of the coefficients to see that

$$\begin{aligned} |\mathfrak{b}(\varphi, \psi)| &= |\mathfrak{a}(u, v)| \\ &\leq C \|u\|_{W^{1,2}} \|v\|_{W^{1,2}}. \end{aligned}$$

Thus, (2.14) follows from (2.13). \square

Corollary 2.3. *The form \mathfrak{b} is continuous, quasi-accretive and closed.*

Proof. Continuity of \mathfrak{b} is exactly (2.14). Quasi-accretivity means that

$$\text{Re } \mathfrak{b}(\varphi, \varphi) + w \int_{\partial\Omega} |\varphi|^2 \geq 0$$

for some w and all $\varphi \in D(\mathfrak{b})$. This follows from (2.13).

Now we prove that \mathfrak{b} is closed which means that $D(\mathfrak{b})$ is complete for the norm

$$\|\varphi\|_{\mathfrak{b}} := \left(\text{Re } \mathfrak{b}(\varphi, \varphi) + w \int_{\partial\Omega} |\varphi|^2 \right)^{1/2}$$

in which w is as in (2.13). If (φ_n) is a Cauchy sequence for $\|\cdot\|_{\mathfrak{b}}$ then by (2.13) the corresponding $(u_n) \in V_H$ with $\text{Tr}(u_n) = \varphi_n$ is a Cauchy sequence in V_H . Since V_H is a closed subspace of V it follows that u_n is convergent to some u in V_H . Set $\varphi := \text{Tr}(u)$. We have $\varphi \in D(\mathfrak{b})$ and the definition of \mathfrak{b} together with continuity of Tr as an operator from $W^{1,2}(\Omega)$ to $L^2(\partial\Omega)$ show that φ_n converges to φ for the norm $\|\cdot\|_{\mathfrak{b}}$. This means that \mathfrak{b} is a closed form. \square

Note that the domain $\text{Tr}(V_H)$ of \mathfrak{b} may not be dense in $L^2(\partial\Omega)$ since functions in this domain vanish on Γ_0 . Indeed,

$$H := \overline{D(\mathfrak{b})}^{L^2(\partial\Omega)} = L^2(\Gamma_1) \oplus \{0\}.$$

The direct inclusion follows from the fact that if $\varphi_n \in D(\mathfrak{b})$ converges in $L^2(\partial\Omega)$ then after extracting a subsequence we have a.e. convergence. Since $\varphi_n = 0$ on Γ_0 we obtain that the limit $\varphi = 0$ on Γ_0 . The reverse inclusion follows for example from the fact that $C_c^\infty(\Gamma_1) \subseteq D(\mathfrak{b})$.

We associate with \mathfrak{b} an operator \mathcal{N}_{Γ_1} . It is defined by

$$D(\mathcal{N}_{\Gamma_1}) := \{\varphi \in D(\mathfrak{b}), \exists \psi \in H : \mathfrak{b}(\varphi, \xi) = \int_{\Gamma_1} \psi \bar{\xi} \ \forall \xi \in D(\mathfrak{b})\}, \quad \mathcal{N}_{\Gamma_1} \varphi = \psi.$$

The operator \mathcal{N}_{Γ_1} can be interpreted as an operator on $L^2(\partial\Omega)$ defined as follows: if $\varphi \in D(\mathcal{N}_{\Gamma_1})$ then there exists a unique $u \in V_H$ such that $\varphi = \text{Tr}(u)$ and

$$\varphi|_{\Gamma_0} = 0, \quad \mathcal{N}_{\Gamma_1}(\varphi) = \sum_{j=1}^d \left(\sum_{k=1}^d a_{kj} \partial_k u + \tilde{a}_j \varphi \right) \nu_j \text{ on } \Gamma_1. \quad (2.16)$$

In the particular case where $a_{kj} = \delta_{kj}$ and $a_1 = \dots = a_d = 0$ the right hand side is seen as the normal derivative $\frac{\partial u}{\partial \nu}$. Thus (2.16) reads as

$$\varphi|_{\Gamma_0} = 0, \quad \mathcal{N}_{\Gamma_1}(\varphi) = \frac{\partial u}{\partial \nu} \text{ on } \Gamma_1. \quad (2.17)$$

We call \mathcal{N}_{Γ_1} the *partial Dirichlet-to-Neumann operator* on $L^2(\partial\Omega)$ or the *Dirichlet-to-Neumann operator with partial data*. The term *partial* refers to the fact that \mathcal{N}_{Γ_1} is known only on the part Γ_1 of the boundary $\partial\Omega$.

It follows from the general theory of forms that $-\mathcal{N}_{\Gamma_1}$ generates a holomorphic semigroup $e^{-t\mathcal{N}_{\Gamma_1}}$ on H . We define $T_t^{\Gamma_1}$ on $L^2(\partial\Omega)$ by

$$T_t^{\Gamma_1} \varphi = e^{-t\mathcal{N}_{\Gamma_1}}(\varphi \cdot 1_{\Gamma_1}) \oplus 0.$$

We shall refer to $(T_t^{\Gamma_1})_{t \geq 0}$ as the "semigroup" generated by $-\mathcal{N}_{\Gamma_1}$ on $L^2(\partial\Omega)$. It is clear that

$$\|T_t^{\Gamma_1}\|_{\mathcal{L}(L^2(\partial\Omega))} \leq e^{-w_0 t} \quad t \geq 0, \quad (2.18)$$

for some constant w_0 . Note that if the form \mathfrak{a} is symmetric, then \mathfrak{b} is also symmetric and hence \mathcal{N}_{Γ_1} is self-adjoint. In this case, (2.18) holds with $w_0 = \inf \sigma(\mathcal{N}_{\Gamma_1})$ which also coincides with the first eigenvalue of \mathcal{N}_{Γ_1} . For all this, see e.g. [21], Chapter 1.

3 Positivity and domination

In this section we study some properties of the semigroup $(T_t^{\Gamma_1})_{t \geq 0}$. We assume here that

$$a_{jk} = a_{kj}, \quad \tilde{a}_k = a_k, \quad a_0 \in L^\infty(\Omega, \mathbb{R}). \quad (3.1)$$

We recall that L_a^D is the elliptic operator with Dirichlet boundary conditions defined in the previous section. Its associated symmetric form \mathfrak{a}^D is given by (2.3) and has domain $W_0^{1,2}(\Omega)$. We shall need the accretivity assumption of \mathfrak{a}^D (or equivalently the self-adjoint operator L_a^D is non-negative) which means that

$$\text{Re } \mathfrak{a}^D(u, u) \geq 0 \text{ for all } u \in W_0^{1,2}(\Omega). \quad (3.2)$$

Theorem 3.1. *Suppose that $0 \notin \sigma(L_a^D)$, (3.1) and that L_a^D is accretive.*

a) The semigroup $(T_t^{\Gamma_1})_{t \geq 0}$ is positive (i.e., it maps non-negative functions of $L^2(\partial\Omega)$ into non-negative functions).

b) Suppose in addition that $a_0 \geq 0$ and $a_k = 0$ for all $k \in \{1, \dots, d\}$. Then $(T_t^{\Gamma_1})_{t \geq 0}$ is a sub-Markovian semigroup.

Recall that the sub-Markovian property means that for $\varphi \in L^2(\partial\Omega)$ and $t \geq 0$

$$0 \leq \varphi \leq 1 \Rightarrow 0 \leq T_t^{\Gamma_1} \varphi \leq 1.$$

This property implies in particular that $(T_t^{\Gamma_1})_{t \geq 0}$ extends from $L^2(\partial\Omega)$ to $L^p(\partial\Omega)$ for all $p \in [2, \infty[$. Since \mathfrak{a} is symmetric then so is \mathfrak{b} and one obtains by duality that $(T_t^{\Gamma_1})_{t \geq 0}$ extends also to $L^p(\partial\Omega)$ for $p \in [1, 2]$.

Proof. The proof follows exactly the same lines as for Theorem 2.3 in [10].

a) By the well known Beurling–Deny criteria (see [8], Section 1.3 or [21], Theorem 2.6), it suffices to prove that $\varphi^+ \in D(\mathfrak{b})$ and $\mathfrak{b}(\varphi^+, \varphi^-) \leq 0$ for all real-valued $\varphi \in D(\mathfrak{b})$. Note that the fact that $D(\mathfrak{b})$ is not densely defined does not affect the the statements of the Beurling-Deny criteria.

Let $\varphi \in D(\mathfrak{b})$ be real-valued. There exists a real-valued $u \in H_V$ such that $\varphi = \text{Tr}(u)$. Then $\varphi^+ = \text{Tr}(u^+) \in \text{Tr}(V) = \text{Tr}H_V = D(\mathfrak{b})$. This follows from the fact that $v^+ \in V$ for all $v \in V$ (see [21], Section 4.2).

By Lemma 2.1 we can write $u^+ = u_0 + u_1$ and $u^- = v_0 + v_1$ with $u_0, v_0 \in W_0^{1,2}(\Omega)$ and $u_1, v_1 \in H_V$. Hence, $u = u^+ - u^- = (u_0 - v_0) + (u_1 - v_1)$. Since $u, u_1 - v_1 \in H_V$ it follows that $u_0 = v_0$. Therefore,

$$\begin{aligned} \mathfrak{b}(\varphi^+, \varphi^-) &= \mathfrak{a}(u_1, v_1) = \mathfrak{a}(u_1, v_0 + v_1) = \mathfrak{a}(u_0 + u_1, v_0 + v_1) - \mathfrak{a}(u_0, v_0 + v_1) \\ &= \mathfrak{a}(u^+, u^-) - \mathfrak{a}(u_0, v_0) = -\mathfrak{a}(u_0, v_0) \\ &= -\mathfrak{a}(u_0, u_0) = -\mathfrak{a}^D(u_0, u_0). \end{aligned}$$

Here we use the fact that

$$\begin{aligned} \mathfrak{a}(u^+, u^-) &= \sum_{k,j=1}^d \int_{\Omega} a_{kj} \partial_k(u^+) \partial_j(u^-) + \sum_{k=1}^d \int_{\Omega} a_k \partial_k u^+ u^- + a_k u^+ \partial_k u^- \\ &\quad + \int_{\Omega} a_0 u^+ u^- = 0. \end{aligned}$$

By assumption (3.2) we have $\mathfrak{a}^D(u_0, u_0) \geq 0$ and we obtain $\mathfrak{b}(\varphi^+, \varphi^-) \leq 0$. This proves the positivity of $(T_t^{\Gamma_1})_{t \geq 0}$ on $L^2(\partial\Omega)$.

b) By [20] or [21], Corollary 2.17 it suffices to prove that $1 \wedge \varphi := \inf(1, \varphi) \in D(\mathfrak{b})$ and $\mathfrak{b}(1 \wedge \varphi, (\varphi - 1)^+) \geq 0$ for all $\varphi \in D(\mathfrak{b})$ with $\varphi \geq 0$. Let $\varphi \in D(\mathfrak{b})$ and suppose that $\varphi \geq 0$. Let $u \in H_V$ be real-valued such that $\varphi = \text{Tr}(u)$. Note that $1 \wedge u \in V$ (see [21], Section 4.3). We decompose $1 \wedge u = u_0 + u_1 \in W_0^{1,2}(\Omega) \oplus H_V$. Then

$$(u - 1)^+ = u - 1 \wedge u = (-u_0) + (u - u_1) \in W_0^{1,2}(\Omega) \oplus H_V.$$

Therefore,

$$\begin{aligned} \mathfrak{b}(1 \wedge \varphi, (\varphi - 1)^+) &= \mathfrak{a}(u_1, u - u_1) = \mathfrak{a}(u_0 + u_1, u - u_1) \\ &= \mathfrak{a}(u_0 + u_1, -u_0 + u - u_1) + \mathfrak{a}(u_0 + u_1, u_0) \\ &= \mathfrak{a}(u_0 + u_1, -u_0 + u - u_1) + \mathfrak{a}(u_0, u_0) \\ &= \sum_{k,j=1}^d \int_{\Omega} a_{kj} \partial_k(1 \wedge u) \partial_j((u - 1)^+) + \\ &\quad \int_{\Omega} a_0(1 \wedge u)(u - 1)^+ + \mathfrak{a}^D(u_0, u_0) \\ &= \int_{\Omega} a_0(u - 1)^+ + \mathfrak{a}^D(u_0, u_0) \geq 0. \end{aligned}$$

This proves that $\mathfrak{b}(1 \wedge \varphi, (\varphi - 1)^+) \geq 0$. \square

Next we have the following domination property.

Theorem 3.2. *Suppose that a_{kj} , a_k , \tilde{a}_k and a_0 are all real-valued and L_a^D is accretive with $0 \notin \sigma(L_a^D)$. Let Γ_0 and $\tilde{\Gamma}_0$ be two open subsets of the boundary such that $\Gamma_0 \subseteq \tilde{\Gamma}_0$. Then for every $0 \leq \varphi \in L^2(\partial\Omega)$*

$$0 \leq T_t^{\tilde{\Gamma}_1} \varphi \leq T_t^{\Gamma_1} \varphi.$$

Proof. Let $\tilde{\Gamma}_1$ be the complement of $\tilde{\Gamma}_0$ in $\partial\Omega$. Denote by \mathfrak{b} and $\tilde{\mathfrak{b}}$ the sesquilinear forms associated with \mathcal{N}_{Γ_1} and $\mathcal{N}_{\tilde{\Gamma}_1}$, respectively. Clearly, $\tilde{\mathfrak{b}}$ is a restriction of \mathfrak{b} and hence it is enough to prove that $D(\tilde{\mathfrak{b}})$ is an ideal of $D(\mathfrak{b})$ and apply [20] or [21], Theorem 2.24. For this, let $0 \leq \varphi \leq \psi$ with $\varphi \in D(\mathfrak{b})$ and $\psi \in D(\tilde{\mathfrak{b}})$. This means that φ and ψ are respectively the traces on $\partial\Omega$ of $u, v \in W^{1,2}(\Omega)$ such that

$$\varphi = \text{Tr}(u) = 0 \text{ on } \Gamma_0 \quad \text{and} \quad \psi = \text{Tr}(v) = 0 \text{ on } \tilde{\Gamma}_0.$$

Since $0 \leq \varphi \leq \psi$ we have $\varphi = 0$ on $\tilde{\Gamma}_0$. This equality gives $\varphi \in D(\tilde{\mathfrak{b}})$ and this shows that $D(\tilde{\mathfrak{b}})$ is an ideal of $D(\mathfrak{b})$. \square

The next result shows monotonicity with respect to the potential a_0 . This was already proved in [10] Theorem 2.4, in the case where $L_a^D = -\Delta + a_0$. The proof given there works also in the general framework of the present paper.

As above let a_{kj} , a_k and a_0 be real-valued and let $(T_t^{\Gamma_1, a_0})_{t \geq 0}$ denote the semigroup $(T_t^{\Gamma_1})_{t \geq 0}$ defined above. Suppose that b_0 is a real-valued function and denote by $(T_t^{\tilde{\Gamma}_1, b_0})_{t \geq 0}$ be the semigroup of $\mathcal{N}_{\tilde{\Gamma}_1}$ with coefficients a_{kj} , a_k and b_0 (i.e. a_0 is replaced by b_0). Then we have

Theorem 3.3. *Suppose again that $0 \notin \sigma(L_a^D)$ and L_a^D is non-negative. If $a_0 \leq b_0$ then*

$$0 \leq T_t^{\tilde{\Gamma}_1, b_0} \varphi \leq T_t^{\Gamma_1, a_0} \varphi$$

for all $0 \leq \varphi \in L^2(\partial\Omega)$ and $t \geq 0$.

4 Proof of the main result

In this section we prove Theorem 1.1. We first recall some notation.

Let $\mu \in \mathbb{R}$ and \mathfrak{a}^μ be the form given by (2.6) with domain $D(\mathfrak{a}^\mu) := V$ and V is again given by (2.2). The operator associated with \mathfrak{a}^μ is L_a^μ . It is given by the formal expression (2.4) and it is subject to mixed and Robin boundary conditions (2.7).

Fix $\lambda \notin \sigma(L_a^D)$. We denote by $\mathcal{N}_{\Gamma_1, a}(\lambda)$ the partial D-t-N operator with the coefficients $\{a_{kj}, a_k, a_0 - \lambda\}$. It is the operator associated with the form

$$\mathfrak{b}(\varphi, \psi) := \sum_{k,j}^d \int_{\Omega} a_{kj} \partial_k u \overline{\partial_j v} \, dx + \sum_{k=1}^d \int_{\Omega} a_k \partial_k u \overline{v} + \overline{a_k} u \overline{\partial_k v} \, dx + (a_0 - \lambda) u \overline{v} \, dx$$

where $u, v \in V_H(\lambda)$ with $\text{Tr}(u) = \varphi$, $\text{Tr}(v) = \psi$ and

$$V_H(\lambda) := \{u \in V, \mathfrak{a}(u, g) = \lambda \int_{\Omega} u \overline{g} \text{ for all } g \in W_0^{1,2}(\Omega)\}, \quad (4.1)$$

This space is the same as in (2.8) but now with a_0 replaced by $a_0 - \lambda$.

We start with the following result which was proved in [2] and [3] in the case where $a_{kj} = \delta_{kj}$, $a_k = 0$, a_0 is a constant and $\Gamma_1 = \partial\Omega$.

Theorem 4.1. *Let $\mu, \lambda \in \mathbb{R}$ and $\lambda \notin \sigma(L_a^D)$. Then:*

- 1) $\mu \in \sigma(\mathcal{N}_{\Gamma_1, a}(\lambda)) \Leftrightarrow \lambda \in \sigma(L_a^\mu)$. In addition, if $u \in \text{Ker}(\lambda - L_a^\mu)$, $u \neq 0$ then $\varphi := \text{Tr}(u) \in \text{Ker}(\mu - \mathcal{N}_{\Gamma_1, a}(\lambda))$ and $\varphi \neq 0$. Conversely, if $\varphi \in \text{Ker}(\mu - \mathcal{N}_{\Gamma_1, a}(\lambda))$, $\varphi \neq 0$, then there exists $u \in \text{Ker}(\lambda - L_a^\mu)$, $u \neq 0$ such that $\varphi = \text{Tr}(u)$.
- 2) $\dim \text{Ker}(\mu - \mathcal{N}_{\Gamma_1, a}(\lambda)) = \dim \text{Ker}(\lambda - L_a^\mu)$.

Proof. We follow a similar idea as in [2] and [3]. It is enough to prove that the mapping

$$S : \text{Ker}(\lambda - L_a^\mu) \rightarrow \text{Ker}(\mu - \mathcal{N}_{\Gamma_1, a}(\lambda)), \quad u \mapsto \text{Tr}(u)$$

is an isomorphism. First, we prove that S is well defined. Let $u \in \text{Ker}(\lambda - L_a^\mu)$. Then $u \in D(L_a^\mu)$ and $L_a^\mu u = \lambda u$. By the definition of L_a^μ we have $u \in V$ and for all $v \in V$

$$\begin{aligned} & \sum_{k,j=1}^d \int_{\Omega} a_{kj} \partial_k u \overline{\partial_j v} + \sum_{k=1}^d \int_{\Omega} a_k \partial_k u \overline{v} + \overline{a_k} u \overline{\partial_k v} \\ & + \int_{\Omega} a_0 u \overline{v} - \lambda \int_{\Omega} u \overline{v} = \mu \int_{\partial\Omega} \text{Tr}(u) \overline{\text{Tr}(v)}. \end{aligned} \quad (4.2)$$

Taking $v \in W_0^{1,2}(\Omega)$ yields $u \in V_H(\lambda)$. For $v \in V_H(\lambda)$ we have

$$\begin{aligned} & \sum_{k,j=1}^d \int_{\Omega} a_{kj} \partial_k u \overline{\partial_j v} + \sum_{k=1}^d \int_{\Omega} a_k \partial_k u \overline{v} + \overline{a_k} u \overline{\partial_k v} \\ & + \int_{\Omega} a_0 u \overline{v} - \lambda \int_{\Omega} u \overline{v} = \mu \int_{\partial\Omega} \text{Tr}(u) \overline{\text{Tr}(v)}. \end{aligned}$$

It follows from the definition of $\mathcal{N}_{\Gamma_1, a}(\lambda)$ that

$$\varphi := \text{Tr}(u) \in D(\mathcal{N}_{\Gamma_1, a}(\lambda)) \text{ and } \mathcal{N}_{\Gamma_1, a}(\lambda) \varphi = \mu \varphi.$$

This means that $S(u) \in \text{Ker}(\mu - \mathcal{N}_{\Gamma_1, a}(\lambda))$.

Suppose now that $u \in \text{Ker}(\lambda - L_a^\mu)$ with $u \neq 0$. If $S(u) = 0$ then $u \in W_0^{1,2}(\Omega)$. Therefore, it follows from (4.2) that for all $v \in V$

$$\sum_{k,j=1}^d \int_{\Omega} a_{kj} \partial_k u \overline{\partial_j v} + \sum_{k=1}^d \int_{\Omega} a_k \partial_k u \overline{v} + \overline{a_k} u \overline{\partial_k v} + \int_{\Omega} (a_0 - \lambda) u \overline{v} = 0. \quad (4.3)$$

This implies that $u \in V_H(\lambda)$. We conclude by Lemma 2.1 that $u = 0$. Thus S is injective.

We prove that S is surjective. Let $\varphi \in \text{Ker}(\mu - \mathcal{N}_{\Gamma_1, a}(\lambda))$. Then by the definition of $\mathcal{N}_{\Gamma_1, a}(\lambda)$, there exists $u \in V_H(\lambda)$ such that $\varphi = \text{Tr}(u)$ and u satisfies (4.2) for all $v \in V_H(\lambda)$. If $v \in V$ we write $v = v_0 + v_1 \in W_0^{1,2}(\Omega) \oplus V_H(\lambda)$ and see that (4.2) holds for u and v . This means that $u \in D(L_a^\mu)$ and $L_a^\mu u = \lambda u$. \square

Lemma 4.2. *For $\lambda \in \mathbb{R}$ large enough, $(\lambda + L_a^\mu)^{-1}$ converges in $\mathcal{L}(L^2(\Omega))$ to $(\lambda + L_a^D)^{-1}$ as $\mu \rightarrow -\infty$.*

This is Proposition 2.6 in [2] when $a_{kj} = \delta_{kj}$, $a_k = a_0 = 0$. The proof given in [2] remains valid in our setting. Note that the idea of proving the uniform

convergence here is based on a criterion from [7] (see Appendix B) which states that it is enough to check that for all $(f_n), f \in L^2(\Omega)$

$$f_n \rightharpoonup f \Rightarrow (\lambda + L_a^{\mu_n})^{-1} f_n \rightarrow (\lambda + L_a^D)^{-1} f, \quad (4.4)$$

for every sequence $\mu_n \rightarrow -\infty$. The first convergence is in the weak sense in $L^2(\Omega)$ and the second one is the strong convergence. It is not difficult to check (4.4).

An immediate consequence of the previous lemma is that the spectrum of L_a^μ converges to the spectrum of L_a^D . More precisely, let $\sigma(L_a^\mu) = \{\lambda_{a,1}^\mu, \lambda_{a,2}^\mu, \dots\}$ and $\sigma(L_a^D) = \{\lambda_{a,1}^D, \lambda_{a,2}^D, \dots\}$ be the corresponding eigenvalues. Then for all k ,

$$\lambda_{a,k}^\mu \rightarrow \lambda_{a,k}^D \text{ as } \mu \rightarrow -\infty. \quad (4.5)$$

In addition we have (compare with [3], Proposition 3).

Lemma 4.3. *For each k , $\mu \mapsto \lambda_{a,k}^\mu$ is strictly decreasing on \mathbb{R} .*

Proof. Firstly, by the min-max principle $\lambda_{a,k}^\mu \leq \lambda_{a,k}^D$ and the function $\mu \mapsto \lambda_{a,k}^\mu$ is non-increasing. Suppose that $\lambda_{a,k}^\mu = \lambda$ for all μ in some interval $[a, b]$ with $a < b$. By perturbation theory, the eigenvalues of L_a^μ could be described as holomorphic functions of the parameter μ (see [15], Chapter VII, especially, Theorem 3.9 and Remark 4.22). Therefore one of the eigenvalues of L_a^μ is constant on \mathbb{R} . We show that this is not possible. Indeed, by the min-max principle

$$\lambda_1^\mu \leq \sum_{k,j=1}^d \int_{\Omega} a_{kj} \partial_k u \overline{\partial_j u} + 2 \operatorname{Re} \sum_{k=1}^d \int_{\Omega} a_k \partial_k u \bar{u} + \int_{\Omega} a_0 |u|^2 - \mu \int_{\Gamma_1} |\operatorname{Tr}(u)|^2$$

for every normalized $u \in V$. Taking u such that $\operatorname{Tr}(u) \neq 0$ shows that $\lambda_{a,1}^\mu \rightarrow -\infty$ as $\mu \rightarrow +\infty$. Suppose now that $\lambda_{a,k}^\mu > w$ for some $w \in \mathbb{R}$, $k > 1$ and all $\mu \in \mathbb{R}$. Taking the smallest possible k we have $\lambda_{a,j}^\mu \rightarrow -\infty$ as $\mu \rightarrow +\infty$ for $j = 1, \dots, k-1$. Of course, $\lambda_{a,j}^\mu > w$ for all $j \geq k$ and we may choose $w \notin \sigma(L_a^D)$. By Theorem 4.1 we have $\sigma(\mathcal{N}_{\Gamma_1, a}(w)) \subset \{\mu \in \mathbb{R}, \lambda_{a,j}^\mu = w, j = 1, \dots, k-1\}$. Using the fact that $\lambda_{a,j}^\mu \rightarrow -\infty$ as $\mu \rightarrow +\infty$ for $j = 1, \dots, k-1$ we see that we can choose w such that the latter set is finite and hence $\sigma(\mathcal{N}_{\Gamma_1, a}(w))$ is finite which not possible since $L^2(\Gamma_1)$ has infinite dimension. \square

We shall also need the following lemma

Lemma 4.4. *For every $\varphi, \psi \in \operatorname{Tr}(V)$, the mapping*

$$\lambda \mapsto \langle \mathcal{N}_{\Gamma_1, a}(\lambda) \varphi, \psi \rangle$$

is holomorphic on $\mathbb{C} \setminus \sigma(L_a^D)$.

This result is easy to prove, see Lemma 2.4 in [6].

Proof of Theorem 1.1. It follows from Lemma 4.4 and the assumptions that $\mathcal{N}_{\Gamma_1, a}(\lambda) = \mathcal{N}_{\Gamma_1, b}(\lambda)$ for all $\lambda \in \mathbb{C} \setminus \sigma(L_a^D) \cup \sigma(L_b^D)$.

i) We show that for all $\mu \in \mathbb{R}$

$$\sigma(L_a^\mu) = \sigma(L_b^\mu). \quad (4.6)$$

Fix $\mu \in \mathbb{R}$ and suppose that $\lambda = \lambda_{a,k}^\mu \in \sigma(L_a^\mu) \setminus \sigma(L_a^D) \cup \sigma(L_b^D)$. By Theorem 4.1, $\mu \in \sigma(\mathcal{N}_{\Gamma_1,a}(\lambda)) = \sigma(\mathcal{N}_{\Gamma_1,b}(\lambda))$ and hence $\lambda \in \sigma(L_b^\mu)$. Thus, $\lambda = \lambda_{a,k}^\mu = \lambda_{b,j}^\mu$ for some $j \geq 1$. Using Lemma 4.3 we see that there exists a discrete set $J \subset \mathbb{R}$ such that for each k and $\mu \in \mathbb{R} \setminus J$, there exists $j \geq 1$ such that $\lambda_{a,k}^\mu = \lambda_{b,j}^\mu$. Note that $j = k$. Indeed, if $k < j$ then for $\mu \in \mathbb{R} \setminus J$ we have

$$\lambda_{b,1}^\mu < \lambda_{b,2}^\mu < \dots < \lambda_{b,k}^\mu < \dots < \lambda_{b,j}^\mu = \lambda_{a,k}^\mu$$

and hence there are more than $k - 1$ different eigenvalues of L_a^μ which are less than $\lambda_{a,k}^\mu$ but this is not possible. The same argument works if $j < k$. Now we have $\lambda_{a,k}^\mu = \lambda_{b,k}^\mu$ for all $\mu \in \mathbb{R} \setminus J$ and remember that the functions $\mu \mapsto \lambda_{a,k}^\mu$ and $\mu \mapsto \lambda_{b,k}^\mu$ are continuous. Thus they coincide on \mathbb{R} . This proves (4.6).

The similarity property follows by a classical argument. Recall that L_a^μ and L_b^μ are self-adjoint operators with compact resolvents. It follows that there exist orthonormal bases Φ_n and Ψ_n of $L^2(\Omega)$ which are eigenfunctions of L_a^μ and L_b^μ , respectively. Define the mapping

$$\mathcal{U} : L^2(\Omega) \rightarrow L^2(\Omega), \quad \Phi_n \mapsto \Psi_n.$$

Thus for $f = \sum_n (f, \Phi_n) \Phi_n \in L^2(\Omega)$, $\mathcal{U}(f) = \sum_n (f, \Phi_n) \Psi_n$. The notation (f, Φ_n) is the scalar product in $L^2(\Omega)$. Clearly,

$$\|\mathcal{U}(f)\|_2^2 = \sum_n |(f, \Phi_n)|^2 = \|f\|_2^2.$$

The mapping \mathcal{U} is an isomorphism. In addition, if $L_a^\mu \Phi_n = \lambda_n \Phi_n$ then for $f \in D(L_b^\mu)$ and

$$\begin{aligned} \mathcal{U} L_a^\mu \mathcal{U}^{-1}(f) &= \mathcal{U} L_a^\mu \mathcal{U}^{-1} \left(\sum_n (f, \Psi_n) \Psi_n \right) \\ &= \mathcal{U} L_a^\mu \left(\sum_n (f, \Psi_n) \Phi_n \right) \\ &= \mathcal{U} \left(\sum_n (f, \Psi_n) \lambda_n \Phi_n \right) \\ &= \sum_n (f, \lambda_n \Psi_n) \Psi_n \\ &= L_b^\mu(f). \end{aligned}$$

Thus, L_a^μ and L_b^μ are unitary equivalent. This proves assertion i).

ii) Choose $\mu = 0$ in the previous assertion.

iii) As mentioned above, by Lemma 4.2 we have (4.5). The same property holds for L_b^μ , that is, $\lambda_{b,k}^\mu \rightarrow \lambda_{b,k}^D$ as $\mu \rightarrow -\infty$. It follows from assertion (i) that $\lambda_{a,k}^D = \lambda_{b,k}^D$ for all $k \geq 1$. We conclude as above that L_a^D and L_b^D are unitary

equivalent.

Finally, another application of Theorem 4.1 shows that $\text{Tr}(\text{Ker}(\lambda - L_a^\mu)) = \text{Tr}(\text{Ker}(\lambda - L_b^\mu))$ for $\lambda \notin \sigma(L_a^D) \cup \sigma(L_b^\mu)$. \square

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