A Focused Sequent Calculus Framework for Proof Search in Pure Type Systems

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Abstract

Basic proof search tactics in logic and type theory can be seen as the root-first applications of rules in an appropriate sequent calculus, preferably without the redundancies generated by permutation of rules. This paper addresses the issues of defining such sequent calculi for *Pure Type Systems* (PTS, which are based on natural deduction) and then organizing their rules for effective proof search. First, we introduce the idea of a *Pure Type Sequent Calculus* (PTSC) by enriching a permutation-free sequent calculus for propositional logic due to Herbelin, which is strongly related to natural deduction and already well adapted to proof-search. Such a PTSC admits a normalisation procedure, adapted from Herbelin's and defined by a system of local rewrite rules as in cut-elimination, using explicit substitutions. This system satisfies the *Subject Reduction* property and is confluent. Each PTSC is logically equivalent to its corresponding PTS, and the former is strongly normalising iff the latter is. Second, we make the logical rules of PTSC syntax-directed for proof search, by incorporating the conversion rules as in syntax-directed presentations of the PTS rules for type checking. Third, we consider an extension PTSCα of PTSC with explicitly scoped meta-variables, representing partial proof-terms, and use it to analyse interactive proof construction. This sets up a framework in which we are able to study proof-search strategies, type inhabitant enumeration and unification.

keywords: Type theory, PTS, sequent calculus, strong normalisation, proof-search, interactive proof construction

Contents

Introduction

Barendregt's *Pure Type Systems (PTS)* [Bar92] form a convenient framework for representing a range of different extensions of the simply-typed λ-calculus, such as those comprising the Barendregt Cube. *System F*, *System* F_ω [Gir72], *System* $\lambda\Pi$ [Daa80, HHP87], and the *Calculus of Constructions (CoC)* [CH88] are examples of such systems, on which several major proof assistants are based (e.g. Coq [Coq], Lego [LP92], and the *Edinburgh Logical Framework* [HHP87]; *Higher-Order Logic* can also be presented as a PTS, but this is *not* the basis of its principal implementation [HOL]).

With typed λ-calculus as their basis, such systems are traditionally presented in natural deduction style, with rules introducing and eliminating logical constants (aka type constructors). Dowek [Dow93] and Muñoz [Muñ01] show how to perform proof search in this style, enumerating type inhabitants in such systems.

This however misses out on the advantages of sequent calculus [Gen35] for proof search. As suggested by Plotkin [Plo87], a Gentzen-style sequent calculus (with left and right introduction rules) can be used as a basis for proof search in the case of $\lambda\Pi$ [PW91, Pym95] (later extended to any PTS [GR03]). However, the permutations of inference steps available in a Gentzen-style calculus (such as G3 [Kle52]) introduce some extra non-determinism in proof search.

Herbelin [Her94, Her95] introduced a *permutation-free* calculus LJT for intuitionistic logic, exploiting the focusing ideas of Andreoli [And92], Danos et al [DJS95] and (ultimately) ideas from Girard's linear logic [Gir87]. Such calculi have been considered as a basis for proof search in Intuitionistic logic [DP99b], generalising the uniform proof approach to logic programming (see [MNPS91] for hereditary Harrop logic). A version with cut rules and proof-terms forms an explicit substitution calculus $\overline{\lambda}$ [Her94, DU03] with a strong connection to (call-by-name) β-reduction and abstract machines such as that of Krivine [Kri].

This forms a sequent calculus counterpart to the Curry-Howard correspondence, on the basis of which type theory can be reformulated with a view to formalising proof search. This paper (detailing and extending [LDM06]; see also [Len06]) completes this programme, reformulating PTSs as *Pure Type Sequent Calculi (PTSC)*. It follows the earlier work in [PD98] that relates $\overline{\lambda}$ to proof search in the $\Lambda\Pi$ calculus [PW91, Pym95].

This gives a secure but simple theoretical basis for the implementation of PTS-based systems such as Coq [Coq] and Lego [LP92]; these proof assistants feature interactive proof construction methods using proof search tactics. As noticed by [McK97], the primitive tactics are in an inexact correspondence with the elimination rules of the underlying natural deduction formalism: while the tactic intro *does* correspond to the right-introduction rule for Π-types (whether in natural deduction or in sequent calculus), the tactics apply in Coq and Refine in Lego, however, are much closer (in spirit) to the left-introduction rule for Π-types in the focused sequent calculus LJT (below) than to the Π-elimination rule in natural deduction. The rule

$$\frac{\Gamma \vdash M : A \quad \Gamma; \langle M/x \rangle B \vdash l : C}{\Gamma; \Pi x^A.B \vdash M \cdot l : C}\,\Pi\mathsf{L}$$

types the construct $M \cdot l$ of $\overline{\lambda}$, representing a list of terms with head M and tail l.

However the aforementioned tactics are also able to postpone the investigation of the first premiss and start investigating the second, which leads to incomplete proof-terms and unification constraints to be solved. This paper integrates these features to PTSC using explicitly scoped meta-variables (whereas [McK97] investigated the use of Lego's local definition mechanism [LP92]). This sets up a framework, called PTSCα, for the analysis and definition of interactive proof construction tactics (as in Coq and Lego), as well as type inhabitant enumeration (see [Dow93, Muñ01]).

Of course, formalising proof-search mechanisms has been investigated, if only to design tactic languages like Delahaye's $\mathcal{L}_{tac}$ and $\mathcal{L}_{pdt}$ [Del01]. Also noteworthy here are McBride's and Jojgov's PhD theses [McB00, GJ02], which consider extensions of type theory to admit partial proof objects. Using meta-variables similar to ours, Jojgov shows how to manage explicitly their progressive instantiation via a definitional mechanism and compares this with Delahaye's $\mathcal{L}_{tac}$ and $\mathcal{L}_{pdt}$.

While formalising the connections with this line of research remains as future work, the novelty of our approach here is to use the sequent calculus to bridge the usual gap (particularly wide for PTS and their implementations) between the rules defining a logic and the rules describing proof search steps. A by-product of this bridge is ensuring correctness of proof-search, whose output thus need not be type-checked (which it currently is, in most proof assistants).

One reason why this is possible in our framework is that it can decompose (and thus account for) some mechanisms that are usually externalised and whose outputs usually need to be type-checked, like unification (including higher-order [Hue76]). Indeed, it integrates the fact, first exposed in [Dow93], that proof-search and unification generalise in type theory to a single process.

While the rules of our framework may not be deterministic enough to be considered as specifying an algorithm, they are atomic enough to provide an operational semantics in which algorithms such as above can be specified. They thus provide a semantics not only for type inhabitation algorithms, but also more generally for tactic languages, and more originally to unification algorithms.

It is convenient to summarise the relationship between our work and that of our predecessors as follows:

	Type Theory	Inference rules	Proof-terms	Meta-variables
[Pym95]	$\lambda\Pi$	G3	λ	System U
[Dow93]	CoC	NJ	λ	Existential variables
[PD98]	$\lambda\Pi(\Sigma)$	LJT	$\overline{\lambda}$	NO
[GR03]	PTS	G3	λ	NO
[GJ02]	λHOL	NJ	λ	Higher order
This paper	PTS	LJT	$\overline{\lambda}$	System PE

As an example, we consider the commutativity of conjunction expressed in (the PTSCα corresponding to) System F, presented in [LDM06] with no meta-variables. We show here how meta-variables improve the formalisation of proof-search.

The paper's structure is as follows: Section 1 presents the syntax of PTSCα and gives the rewrite rules for normalisation. Section 2 connects the syntax with that of λ-calculus and derives from this connection the confluence of the PTSCα-calculus. Section 3 presents a parametric typing system PTSC for ground terms (i.e. terms with no meta-variables), and states and proves properties such as *Subject Reduction*. Section 4 establishes the correspondence between a PTSC and the PTS with the same parameters; we show type preservation and the strong normalisation result. Section 5 discusses proof-search in a PTSC. Section 6 introduces the inference system PTSCα, i.e. with meta-variables, as a way to formalise incomplete proofs and operationalise proof-search. Section 7 shows the aforementioned example. These are followed by a conclusion and discussion of directions for further work.

Some ideas and results of this paper (namely Sections 2, 3 and 4, which were already in [LDM06]) have been formalised and proved in Coq [Sil09]. This was done using a De Bruijn index representation, as provided by e.g. [Len06].

1 Syntax and operational semantics of PTSCα

1.1 Syntax

We consider an extension (with type annotations) of the proof-term syntax $\overline{\lambda}$ of Herbelin's focused sequent calculus LJT [Her95]. As in $\overline{\lambda}$, Pure Type Sequent Calculi feature two syntactic categories: that of *terms* and that of *lists*.

The syntax of PTSCα depends on a given set $\mathcal{S}$ of *sorts*, written $s, s', \ldots$, a denumerable set $\mathcal{X}$ of *variables*, written $x, y, z, \ldots$, and two denumerable sets of *meta-variables*: those for terms, written $\alpha, \alpha', \ldots$, and those for lists, written $\beta, \beta', \ldots$. These meta-variables come with an intrinsec notion of *arity*.

Definition 1 (Terms) The set $\mathcal{T}$ of *terms* (denoted $M, N, P, \ldots, A, B, \ldots$) and the set $\mathcal{L}$ of *lists* (denoted $l, l', \ldots$) are inductively defined as

$$\begin{array}{rcl} M, N, P, A, B & ::= & \Pi x^A.B \mid \lambda x^A.M \mid s \mid x\ l \mid M\ l \mid \langle M/x\rangle N \mid \alpha(M_1, \ldots, M_n) \\ l, l' & ::= & [] \mid M{\cdot}l \mid l@l' \mid \langle M/x\rangle l \mid \beta(M_1, \ldots, M_n) \end{array}$$

where n is the arity of α and β.

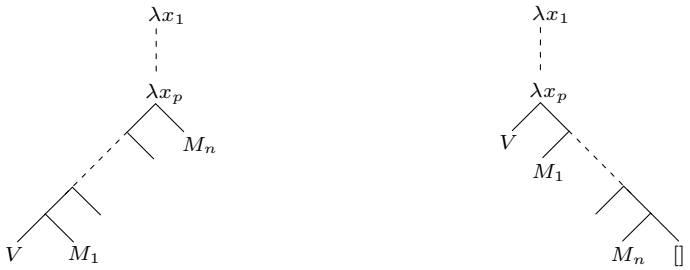
$\Pi x^A.M$, $\lambda x^A.M$, and $\langle N/x\rangle M$ bind x in M, and $\langle M/x\rangle l$ binds x in l, thus defining the free variables of a term M (resp. a list l), denoted $\mathsf{FV}(M)$ (resp. $\mathsf{FV}(l)$), as well as α-conversion, issues of which are treated in the usual way. Note that $\mathsf{FV}(\alpha(M_1, \ldots, M_n)) = \mathsf{FV}(\beta(M_1, \ldots, M_n)) = \bigcup_{i=1}^{n} \mathsf{FV}(M_n)$, see the discussion on meta-variables below.

Let $A{\to}B$ denote $\Pi x^A.B$ when $x \notin \mathsf{FV}(B)$.

Terms and lists without meta-variables are called *ground terms* and *ground lists*, respectively. (Previously, these were just called terms and lists in [LDM06]).

A term M is *closed* if $\mathsf{FV}(M) = \emptyset$.

Lists are used to represent sequences of arguments of a function, with the term $x\ l$ (resp. $M\ l$) representing the application of x (resp. M) to the list of arguments l. Note that a variable alone is not a term; it has to be applied to a list, possibly the empty list, denoted $[]$. The list with head M and tail l is denoted $M \cdot l$, with a typing rule corresponding to the left-introduction of Π-types (cf. Section 3). The following figure shows the generic structure of a λ-term and its representation in $\overline{\lambda}$:



Successive applications give rise to the concatenation of lists, denoted $l@l'$, and $\langle M/x\rangle N$ and $\langle M/x\rangle l$ are explicit substitutions, on terms and lists, respectively. They will be used in two ways: first, to instantiate a universally quantified variable, and second, to describe explicitly the interaction between the constructors in the normalisation process (given in section 1.2).

Among the features that we add to the syntax of $\overline{\lambda}$, our meta-variables can be seen as *higher-order variables*. This is quite similar to CRS [Klo80], in that unknown terms are represented with (meta/higher-order) variables *applied* to the series of (term-)variables that could occur freely in those terms, e.g. $\alpha(x, y)$ (or more formally $\alpha(x\ [], y\ [])$) to represent an unknown term M in which x and y could occur free. These arguments can later be subject to substitutions, so that $\alpha(N, P)$ will represent $\left\{{}^{N,P}/_{x,y}\right\}M$. In other words, a meta-variable on its own stands for something closed, e.g. $x.y.M$ with $\mathsf{FV}(M) \subseteq \{x, y\}$. This allows us to consider a simple notion of α-conversion, with $\lambda x^s.\alpha(x\ [], y\ []) = \lambda z^s.\alpha(z\ [], y\ [])$. [1]

This kind of meta-variable differs from that in [Muñ01], which is rather in the style of ERS [Kha90] where the variables that could occur freely in the unknown term are not specified explicitly. The drawback of our approach is that we have to *know* in advance the free variables that might occur free in the unknown term, but in a typed setting such as proof search these are actually the variables declared in the typing environment. Moreover, although specifying explicitly the variables that could occur free in an unknown term might seem heavy, it actually avoids the usual (non-)confluence problems when terms contain meta-variables in the style of ERS.[2] The solution in [Muñ01] has the drawback of not simulating β-reduction (although the reductions reach the expected normal forms).

1.2 Operational semantics

The operational semantics of PTSCα is given by a reduction system presented in Fig 1, extending (with rules A4, Cα, Dβ) that in [LDM06], and comprising subsystems B, x′, xsubst′ and combinations thereof. Side-conditions to avoid variable capture can be inferred from the reduction rules. Confluence of the system is proved in section 2.

We denote by $\longrightarrow_G$ the contextual closure of the reduction relation defined by any system G of rewrite rules (such as B, xsubst′, x′). The transitive closure

[1] Also, there is no binding mechanism for meta-variables in the syntax of PTSCα, but at the meta-level there is a natural notion of instantiation, which we present only in section 6. We thus emphasise the fact that instantiation of meta-variables never occurs during computation; in that respect, meta-variables really behave like constants or term constructors.

[2] See the discussion at the end of section 2.

$$
\begin{array}{lll}
\mathsf{B} & (\lambda x^A.M)\ (N{\cdot}l) & \longrightarrow\ (\langle N/x\rangle M)\ l \\
\mathsf{x'}\ \{\ \mathsf{B1} & M\ [] & \longrightarrow\ M \\
\mathsf{B2} & (x\ l)\ l' & \longrightarrow\ x\ (l@l') \\
\mathsf{B3} & (M\ l)\ l' & \longrightarrow\ M\ (l@l') \\
\mathsf{A1} & (M{\cdot}l')@l & \longrightarrow\ M{\cdot}(l'@l) \\
\mathsf{A2} & []@l & \longrightarrow\ l \\
\mathsf{A3} & (l@l')@l'' & \longrightarrow\ l@(l'@l'') \\
\mathsf{A4} & l@[] & \longrightarrow\ l \\
\mathsf{xsubst'}:\ \{\ \mathsf{C1} & \langle P/y\rangle\lambda x^A.M & \longrightarrow\ \lambda x^{\langle P/y\rangle A}.\langle P/y\rangle M \\
\mathsf{C2} & \langle P/y\rangle(y\ l) & \longrightarrow\ P\ \langle P/y\rangle l \\
\mathsf{C3} & \langle P/y\rangle(x\ l) & \longrightarrow\ x\ \langle P/y\rangle l \qquad \text{if } x \neq y \\
\mathsf{C4} & \langle P/y\rangle(M\ l) & \longrightarrow\ \langle P/y\rangle M\ \langle P/y\rangle l \\
\mathsf{C5} & \langle P/y\rangle\Pi x^A.B & \longrightarrow\ \Pi x^{\langle P/y\rangle A}.\langle P/y\rangle B \\
\mathsf{C6} & \langle P/y\rangle s & \longrightarrow\ s \\
\mathsf{C}\alpha & \langle P/y\rangle\alpha(M_1,\ldots,M_n) & \longrightarrow\ \alpha(\langle P/y\rangle M_1,\ldots,\langle P/y\rangle M_n) \\
\mathsf{D1} & \langle P/y\rangle[] & \longrightarrow\ [] \\
\mathsf{D2} & \langle P/y\rangle(M{\cdot}l) & \longrightarrow\ (\langle P/y\rangle M){\cdot}(\langle P/y\rangle l) \\
\mathsf{D3} & \langle P/y\rangle(l@l') & \longrightarrow\ (\langle P/y\rangle l)@(\langle P/y\rangle l') \\
\mathsf{D}\beta & \langle P/y\rangle\beta(M_1,\ldots,M_n) & \longrightarrow\ \beta(\langle P/y\rangle M_1,\ldots,\langle P/y\rangle M_n)
\end{array}
$$

Figure 1: Reduction Rules

of $\longrightarrow_G$ is denoted by $\longrightarrow^+_G$, its reflexive and transitive closure is denoted by $\longrightarrow^*_G$, and its symmetric reflexive and transitive closure is denoted by $\longleftrightarrow^*_G$. The set of strongly normalising elements (those from which no infinite $\longrightarrow_G$-reduction sequence starts) is SN^G. When not specified, G is assumed to be the system B,x' from Fig. 1.

We now show that system x' is terminating. If we add rule B, then the system fails to be terminating unless we only consider terms that are typed in a normalising typing system.

We can encode terms and lists into a first-order syntax given by the following signature:

$$\{\star/0, \mathsf{i}/1, \mathsf{ii}/2, \mathsf{cut}/2, \mathsf{sub}/2\} \cup \{\mathsf{tuple}^n/n \mid n \in \mathbb{N}\}$$

We may then equip this signature with the well-founded precedence relation defined by

$$\star \prec \mathsf{i} \prec \mathsf{ii} \prec \mathsf{tuple}^0 \prec \ldots \prec \mathsf{tuple}^n \prec \mathsf{tuple}^{n+1} \prec \ldots \prec \mathsf{cut} \prec \mathsf{sub}$$

The *lexicographic path ordering* (lpo) induced on the first-order terms is also well-founded (definitions and results can be found in [KL80]). The aforementioned encoding is given in Fig 2.

Theorem 1

- *If* $M \longrightarrow_{\mathsf{x}'} M'$ *then* $\mathcal{S}(M) >_{\mathsf{lpo}} \mathcal{S}(M')$.
- *If* $l \longrightarrow_{\mathsf{x}'} l'$ *then* $\mathcal{S}(l) >_{\mathsf{lpo}} \mathcal{S}(l')$.

Proof: By simultaneous induction on M, l. □

$\mathcal{S}(s)$	$= \star$
$\mathcal{S}(\lambda x^A.M)$	$= \mathsf{ii}(\mathcal{S}(A), \mathcal{S}(M))$
$\mathcal{S}(\Pi x^A.M)$	$= \mathsf{ii}(\mathcal{S}(A), \mathcal{S}(M))$
$\mathcal{S}(x\ l)$	$= \mathsf{i}(\mathcal{S}(l))$
$\mathcal{S}(M\ l)$	$= \mathsf{cut}(\mathcal{S}(M), \mathcal{S}(l))$
$\mathcal{S}(\langle M/x\rangle N)$	$= \mathsf{sub}(\mathcal{S}(M), \mathcal{S}(N))$
$\mathcal{S}(\alpha(M_1, \ldots, M_n))$	$= \mathsf{tuple}^n(\mathcal{S}(M_1), \ldots, \mathcal{S}(M_n))$
$\mathcal{S}([])$	$= \star$
$\mathcal{S}(M \cdot l)$	$= \mathsf{ii}(\mathcal{S}(M), \mathcal{S}(l))$
$\mathcal{S}(l @ l')$	$= \mathsf{ii}(\mathcal{S}(l), \mathcal{S}(l'))$
$\mathcal{S}(\langle M/x\rangle l)$	$= \mathsf{sub}(\mathcal{S}(M), \mathcal{S}(l))$
$\mathcal{S}(\beta(M_1, \ldots, M_n))$	$= \mathsf{tuple}^n(\mathcal{S}(M_1), \ldots, \mathcal{S}(M_n))$

Figure 2: First-order encoding

Corollary 2 *System x' is terminating (on all terms and lists).*

2 λ-terms and Confluence

In this section we define encodings between the syntax of PTSCα and that of *Pure Type Systems* (PTS), i.e. a variant of λ-terms. Since, in the latter, the only reduction rule (namely, β) is confluent, we infer from the encodings the confluence of PTSCα. The difficulty lies in the encoding of meta-variables.

We briefly recall the syntax and operational semantics of PTS. The terms have the following syntax:

$$t, u, v, T, U, V, \ldots ::= x \mid\ s \mid\ \Pi x^T.t \mid\ \lambda x^T.t \mid\ t\ u$$

which is equipped with the β-reduction rule $(\lambda x^v.t)\ u \longrightarrow_\beta \ \{^u/_x\}t$, in which the substitution is implicit, i.e. is a meta-operation.

In order to interpret the meta-variables of PTSCα, we need to reserve some of the traditional variables of PTS to the specific purpose of encoding meta-variables. More specifically, for each meta-variable α (resp. β) of arity k, we reserve a λ-calculus variable which we write α^k (resp. β^k).

$\mathcal{B}(\Pi x^A.B)$	$:=$	$\Pi x^{\mathcal{B}(A)}.\mathcal{B}(B)$	
$\mathcal{B}(\lambda x^A.M)$	$:=$	$\lambda x^{\mathcal{B}(A)}.\mathcal{B}(M)$	
$\mathcal{B}(s)$	$:=$	s	
$\mathcal{B}(x\ l)$	$:=$	$\{^x/_z\}\mathcal{B}^z(l)$	z fresh
$\mathcal{B}(M\ l)$	$:=$	$\{^{\mathcal{B}(M)}/_z\}\mathcal{B}^z(l)$	z fresh
$\mathcal{B}(\langle P/x\rangle M)$	$:=$	$\{^{\mathcal{B}(P)}/_x\}\mathcal{B}(M)$	
$\mathcal{B}(\alpha(M_1, \ldots, M_n))$	$:=$	$\alpha^n\ \mathcal{B}(M_1) \ldots \mathcal{B}(M_n)$	
$\mathcal{B}^y([])$	$:=$	y	
$\mathcal{B}^y(M \cdot l)$	$:=$	$\{^{y\ \mathcal{B}(M)}/_z\}\mathcal{B}^z(l)$	z fresh
$\mathcal{B}^y(l @ l')$	$:=$	$\{^{\mathcal{B}^y(l)}/_z\}\mathcal{B}^z(l')$	z fresh
$\mathcal{B}^y(\langle P/x\rangle l)$	$:=$	$\{^{\mathcal{B}(P)}/_x\}\mathcal{B}^y(l)$	
$\mathcal{B}^y(\beta(M_1, \ldots, M_n))$	$:=$	$\beta^n\ y\ \mathcal{B}(M_1) \ldots \mathcal{B}(M_n)$	

Figure 3: From a PTSCα to a PTS

Fig. 3 shows the encoding of the syntax of PTSCα into that of PTS. Meta-variables for terms are encoded naturally, although note that we make their arity explicit in the encoding. The case of meta-variables for lists is more subtle, since

the translation of lists is parameterised by the future head variable. How can we relate such a variable to a list of terms that is (yet) unknown? We simply give it as an extra argument (the first, in fact) of the encoded meta-variable.

Theorem 3 (Simulation of PTSCα) *$\longrightarrow_\beta$ simulates $\longrightarrow_{\mathsf{Bx'}}$ through $\mathcal{B}$.*

Proof: If $M \longrightarrow_{\mathsf{B}} N$ then $\mathcal{B}(M) \longrightarrow^*_\beta \mathcal{B}(N)$, if $l \longrightarrow_{\mathsf{B}} l'$ then $\mathcal{B}^y(l) \longrightarrow^*_\beta \mathcal{B}^y(l')$, if $M \longrightarrow_{\mathsf{x'}} N$ then $\mathcal{B}(M) = \mathcal{B}(N)$ and if $l \longrightarrow_{\mathsf{x'}} l'$ then $\mathcal{B}^y(l) = \mathcal{B}^y(l')$, which are proved by simultaneous induction on the derivation step and case analysis. □

$\mathcal{A}(s)$	:=	s	
$\mathcal{A}(\Pi x^T.U)$	:=	$\Pi x^{\mathcal{A}(T)}.\mathcal{A}(U)$	
$\mathcal{A}(\lambda x^T.t)$	:=	$\lambda x^{\mathcal{A}(T)}.\mathcal{A}(t)$	
$\mathcal{A}(\alpha^k\ t_1 \dots t_n)$	:=	$\alpha(\mathcal{A}(t_1), \dots, \mathcal{A}(t_n))$	$n \leq k$
$\mathcal{A}(\beta^k\ t\ t_1 \dots t_n)$	:=	$\mathcal{A}_{\beta(\mathcal{A}(t_1),\dots,\mathcal{A}(t_n))}(t)$	$n \leq k$
$\mathcal{A}(t)$	:=	$\mathcal{A}_{[]}(t)$	otherwise
$\mathcal{A}_l(\alpha^k\ t_1 \dots t_n)$	:=	$\alpha(\mathcal{A}(t_1), \dots, \mathcal{A}(t_n))\ l$	$n \leq k$
$\mathcal{A}_l(\beta^k\ t\ t_1 \dots t_n)$	:=	$\mathcal{A}_{\beta(\mathcal{A}(t_1),\dots,\mathcal{A}(t_n))@l}(t)$	$n \leq k$
$\mathcal{A}_l(t\ u)$	:=	$\mathcal{A}_{\mathcal{A}(u)l}(t)$	otherwise
$\mathcal{A}_l(x)$	:=	$x\ l$	
$\mathcal{A}_l(t)$	:=	$\mathcal{A}(t)\ l$	otherwise

Figure 4: From a PTS to a PTSCα

Fig. 4 shows the encoding of the syntax of PTS into that of PTSCα. It is simply the adaptation to the higher-order case of Prawitz's translation from natural deduction to sequent calculus [Pra65]: the encoding of an application relies on a parameterised version of the translation.

Note how we spot the situations which arose from encoded meta-variables, using the explicitly displayed arity to identify the arguments. Note that the case $n < k$ never arises from the encoding of a PTSCα-term, but by only requiring $n \leq k$ (rather than $n = k$) we have a total (rather than partial) translation.

In order to prove confluence, we first need the following results:

Lemma 4

1. *$\mathcal{A}(t)$ is an $\mathsf{x'}$-normal form and, provided that l is $\mathsf{x'}$-normal and if $l = []$ then either $t = x$ or $t = t_1\ t_2$, $\mathcal{A}_l(t)$ is also $\mathsf{x'}$-normal.*
2. *If $l \longrightarrow_{\mathsf{Bx'}} l'$ then $\mathcal{A}_l(t) \longrightarrow_{\mathsf{Bx'}} \mathcal{A}_{l'}(t)$.*
3. *$\mathcal{A}_{l'}(t)\ l \longrightarrow^*_{\mathsf{x'}} \mathcal{A}_{l'@l}(t)$ and $\mathcal{A}(t)\ l \longrightarrow^*_{\mathsf{x'}} \mathcal{A}_l(t)$.*
4. *$\langle \mathcal{A}(u)/x \rangle \mathcal{A}(t) \longrightarrow^*_{\mathsf{x'}} \mathcal{A}(\{^u/_x\}t)$ and $\langle \mathcal{A}(u)/x \rangle \mathcal{A}_l(t) \longrightarrow^*_{\mathsf{x'}} \mathcal{A}_{\langle \mathcal{A}(u)/x \rangle l}(\{^u/_x\}t)$.*

Proof: Each of the above points is obtained by straightforward inductions on t. Note that in order to prove point 4 we need rules A3 and A4. These are not needed (for simulation of β-reduction and for confluence) when only ground terms were concerned. □

Theorem 5 (Simulation of PTS)
$\longrightarrow_{\mathsf{Bx'}}$ (strongly) simulates $\longrightarrow_\beta$ through $\mathcal{A}$.

Proof: If $t \longrightarrow_\beta u$ then $\mathcal{A}(t) \longrightarrow^+_{\mathsf{Bx'}} \mathcal{A}(u)$ and $\mathcal{A}_l(t) \longrightarrow^+_{\mathsf{Bx'}} \mathcal{A}_l(u)$, which are proved by induction on the derivation step, using Lemma 4.4 for the base case and Lemma 4, point 3. □

Now we study the composition of the two encodings:

Lemma 6 *Suppose M and l are x'-normal forms.*

1. *If $l = []$ implies $t = x$ or $t = t_1\ t_2$, then $\mathcal{A}_l(t) = \mathcal{A}(\{\overset{t}{/_x}\}\mathcal{B}^x(l))$ (for any $x \notin \mathsf{FV}(l)$).*
2. *$M = \mathcal{A}(\mathcal{B}(M))$.*

Proof: By simultaneous induction on l and M. Again, rules A3 and A4 (as well as Cα and Dβ) are needed for this lemma to capture the notion of normal form corresponding to the PTS-terms, when meta-variables are present. □

Theorem 7

1. *$\mathcal{B}(\mathcal{A}(t)) = t$*
2. *$M \longrightarrow^*_{\mathsf{x}'} \mathcal{A}(\mathcal{B}(M))$*

Proof:

1. $\mathcal{B}(\mathcal{A}(t)) = t$ and $\mathcal{B}(\mathcal{A}_l(t)) = \{\overset{t}{/_x}\}\mathcal{B}^x(l)$ (with $x \neq \mathsf{FV}(l)$) are obtained by simultaneous induction on t.
2. $M \longrightarrow^*_{\mathsf{x}'} \mathcal{A}(\mathcal{B}(M))$ holds by induction on the longest sequence of x'-reduction from M (x' is terminating): by Lemma 6, point 2, it holds if M is an x'-normal form, and if $M \longrightarrow_{\mathsf{x}'} N$ then we can apply the induction hypothesis on N and by Theorem 3 we have the result.

□

We finally get confluence:

Corollary 8 (Confluence) *$\longrightarrow_{\mathsf{x}'}$ and $\longrightarrow_{\mathsf{B}\mathsf{x}'}$ are confluent.*

Proof: We use the technique of simulation as for instance in [KL05]: consider two reduction sequences starting from a term in a PTSC. They can be simulated through $\mathcal{B}$ by β-reductions, and since a PTS is confluent, we can close the diagram. Now the lower part of the diagram can be simulated through $\mathcal{A}$ back in the PTSC, which closes the diagram there as well, as shown in Fig. 5 for Bx'. Notice that the proof of confluence has nothing to do with typing and does not rely on any result in section 3 (in fact, we use confluence in the proof of Subject Reduction in the Appendix). □

Considering meta-variables in the style of CRS [Klo80] avoids the usual problem of non-confluence coming from the critical pair between B and C4 which generate the two terms $\langle N/x\rangle\langle P/y\rangle M$ and $\langle\langle N/x\rangle P/y\rangle\langle N/x\rangle M$. Indeed, with ERS-style meta-variables these two terms need not reduce to a common term, but with the CRS-approach, they now can (using the rules Cα and Dβ). Again, note how the critical pair between B3 and itself (or B2) needs rule A3 in order to be closed, while it was only there for convenience when all terms were ground.

3 Typing system and properties

Given the set of sorts $\mathcal{S}$, a particular PTSC is specified by a set $\mathcal{A} \subseteq \mathcal{S}^2$ and a set $\mathcal{R} \subseteq \mathcal{S}^3$. We shall see an example in section 4.2. Throughout this section we consider ground terms only.

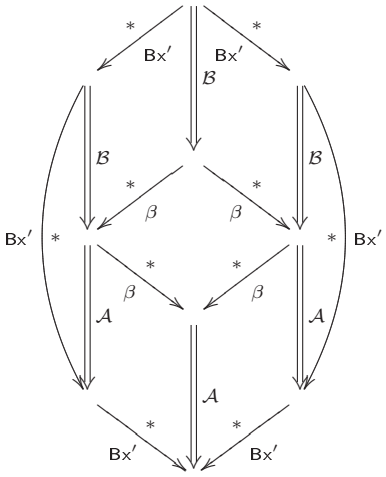


Figure 5: Confluence by simulation

Definition 2 (Environments)

- *Environments* are lists of pairs from $\mathcal{X} \times \mathcal{T}$ denoted $(x : A)$.
- We define the *domain* of an environment and the *application of a substitution to an environment* as follows:

$$\begin{array}{ll} \mathsf{Dom}([]) = [] & \mathsf{Dom}(\Gamma, (x : A)) = \mathsf{Dom}(\Gamma), x \\ \langle P/y\rangle([]) = [] & \langle P/y\rangle(\Gamma, (x : A)) = \langle P/y\rangle\Gamma, (x : \langle P/y\rangle A) \end{array}$$

- It is useful (see Section 6) to define $\mathsf{Dom}(\Gamma)$ as a list, for which the meaning of $x \in \mathsf{Dom}(\Gamma)$ is clear. If $\mathcal{M}$ is a *set* of variables, $\mathcal{M} \subseteq \mathsf{Dom}(\Gamma)$ means for all $x \in \mathcal{M}$, $x \in \mathsf{Dom}(\Gamma)$. Similarly, $\mathsf{Dom}(\Gamma) \cap \mathsf{Dom}(\Delta)$ is the *set* $\{x \in \mathcal{X} \mid x \in \mathsf{Dom}(\Gamma) \wedge x \in \mathsf{Dom}(\Delta)\}$.

  We define the following *inclusion relation* between environments:
  $\Gamma \sqsubseteq \Delta$ if for all $(x : A) \in \Gamma$, there is $(x : B) \in \Delta$ with $A \longleftrightarrow^* B$

The inference rules in Fig. 6 inductively define the derivability of three kinds of statements:

1. context well-formedness, Γ wf,
2. term typing, $\Gamma \vdash M : A$, and
3. list typing, $\Gamma; B \vdash l : A$; here we say that B is in the *stoup*.

Side-conditions are used, such as $(s_1, s_2, s_3) \in \mathcal{R}$, $x \notin \mathsf{Dom}(\Gamma)$, $A \longleftrightarrow^* B$ or $\Delta \sqsubseteq \Gamma$, and we use the abbreviation $\Delta \sqsubseteq \Gamma$ wf for $\Delta \sqsubseteq \Gamma$ and Γ wf. We freely abuse the notation in the customary way, by not distinguishing between a statement and its derivability according to the rules of Fig. 6.

There are three conversion rules conv_R, conv'_R, and conv_L in order to deal with the two kinds of statements and, for one of them, convert the type in the stoup. Since the substitution of a variable in an environment affects the rest of the environment (which could depend on the variable), the two rules for explicit substitutions (Cut_2 and Cut_4) must have a particular shape that manipulates the environment, if the PTSC is to satisfy basic required properties like those of a PTS. The lemmas of this section are proved by induction on typing derivations:

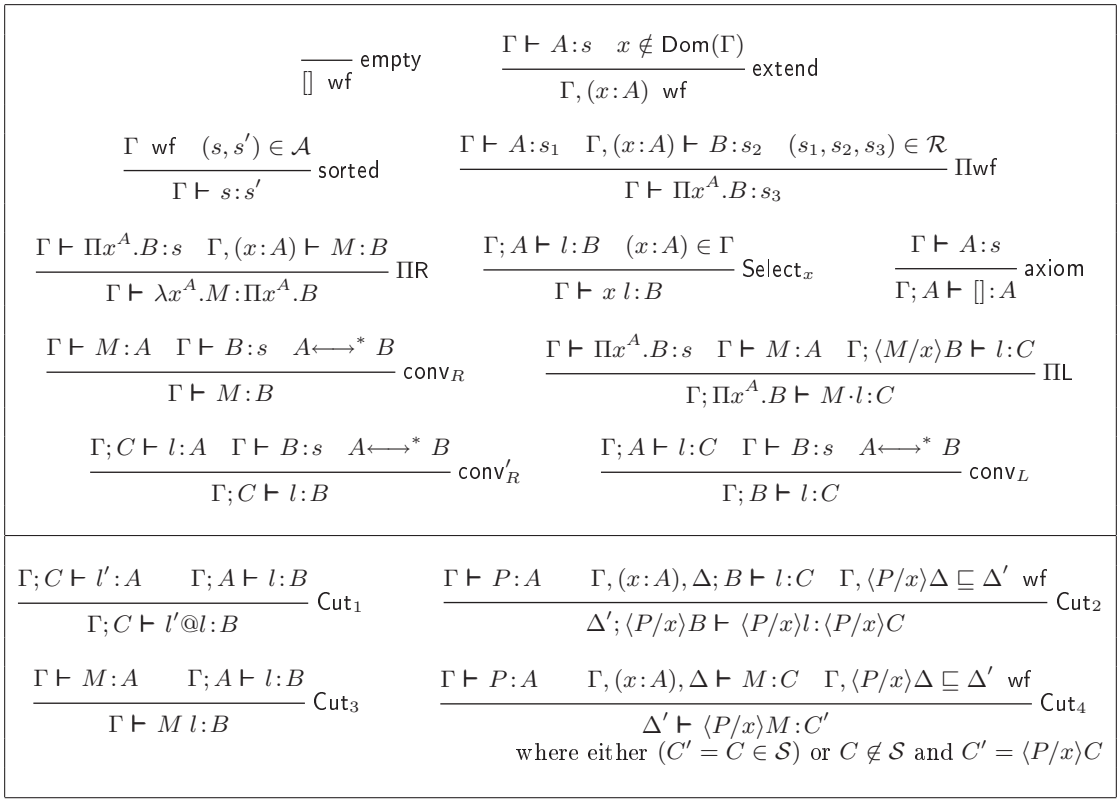


Figure 6: Typing rules of a PTSC

Lemma 9 (Properties of typing statements) *If $\Gamma \vdash M:A$ (resp. $\Gamma;B \vdash l:C$) then $FV(M) \subseteq Dom(\Gamma)$ (resp. $FV(l) \subseteq Dom(\Gamma)$), and the following statements can be derived with strictly smaller typing derivations:*

1. *Γ wf*
2. *$\Gamma \vdash A:s$ for some $s \in \mathcal{S}$, or $A \in \mathcal{S}$*
   (resp. $\Gamma \vdash B:s$ and $\Gamma \vdash C:s'$ for some $s,s' \in \mathcal{S}$)

Corollary 10 (Properties of well-formed environments)

1. *If $\Gamma,x:A,\Delta$ wf then $\Gamma \vdash A:s$ for some $s \in \mathcal{S}$ with $x \notin Dom(\Gamma,\Delta)$ and $FV(A) \subseteq Dom(\Gamma)$ (and in particular $x \notin FV(A)$)*
2. *If Γ,Δ wf then Γ wf.*

Lemma 11 (Weakening) *Suppose Γ,Γ' wf and $Dom(\Gamma') \cap Dom(\Delta) = \emptyset$.*

1. *If $\Gamma,\Delta \vdash M:B$ then $\Gamma,\Gamma',\Delta \vdash M:B$.*
2. *If $\Gamma,\Delta;C \vdash l:B$, then $\Gamma,\Gamma',\Delta;C \vdash l:B$.*
3. *If Γ,Δ wf, then Γ,Γ',Δ wf.*

We can also strengthen the *weakening property* into the *thinning property* by induction on the typing derivation. This allows to weaken the environment, permute it, and convert the types inside, as long as it remains well-formed:

Lemma 12 (Thinning) *Suppose* $\Gamma \sqsubseteq \Gamma'$ *wf and* $Dom(\Gamma') \cap Dom(\Delta) = \emptyset$.

1. *If* $\Gamma, \Delta \vdash M : B$ *then* $\Gamma', \Delta \vdash M : B$.
2. *If* $\Gamma, \Delta; C \vdash l : B$, *then* $\Gamma', \Delta; C \vdash l : B$.
3. *If* Γ, Δ *wf, then* Γ', Δ *wf.*

Using all of the results above, Subject Reduction can be proved (see the Appendix).

Theorem 13 (Subject Reduction in a PTSC)

1. *If* $\Gamma \vdash M : A$ *and* $M \longrightarrow M'$, *then* $\Gamma \vdash M' : A$
2. *If* $\Gamma; A \vdash l : B$ *and* $l \longrightarrow l'$, *then* $\Gamma; A \vdash l' : B$

4 Correspondence with PTS

4.1 Type preservation

There is a logical correspondence between a PTSC given by the sets $\mathcal{S}$, $\mathcal{A}$ and $\mathcal{R}$ and the PTS given by the same sets.

We prove this by showing that (when restricted to ground terms) the encodings preserve typing.

$$\frac{}{[] \text{ wf}} \qquad \frac{\Gamma \vdash_{\mathsf{PTS}} T : s \quad x \notin \mathsf{Dom}(\Gamma)}{\Gamma, (x : T) \text{ wf}} \qquad \frac{\Gamma \text{ wf} \quad (x : T) \in \Gamma}{\Gamma \vdash_{\mathsf{PTS}} x : T}$$

$$\frac{\Gamma \text{ wf} \quad (s, s') \in \mathcal{A}}{\Gamma \vdash_{\mathsf{PTS}} s : s'} \qquad \frac{\Gamma \vdash_{\mathsf{PTS}} U : s_1 \quad \Gamma, (x : U) \vdash_{\mathsf{PTS}} T : s_2 \quad (s_1, s_2, s_3) \in \mathcal{R}}{\Gamma \vdash_{\mathsf{PTS}} \Pi x^U . T : s_3}$$

$$\frac{\Gamma \vdash_{\mathsf{PTS}} \Pi x^U . V : s \quad \Gamma, (x : U) \vdash_{\mathsf{PTS}} t : V}{\Gamma \vdash_{\mathsf{PTS}} \lambda x^U . t : \Pi x^U . V} \qquad \frac{\Gamma \vdash_{\mathsf{PTS}} t : \Pi x^T . U \quad \Gamma \vdash_{\mathsf{PTS}} u : T}{\Gamma \vdash_{\mathsf{PTS}} t\ u : \{u/x\}U}$$

$$\frac{\Gamma \vdash_{\mathsf{PTS}} t : U \quad \Gamma \vdash_{\mathsf{PTS}} V : s \quad U \longleftrightarrow^*_\beta V}{\Gamma \vdash_{\mathsf{PTS}} t : V}$$

Figure 7: Typing rules of a PTS

The terms are typed by the typing rules in Fig. 4.1, which depend on the sets $\mathcal{S}$, $\mathcal{A}$ and $\mathcal{R}$. PTS are confluent and satisfy the following properties (e.g. [Bar92]):

Theorem 14

1. *If* $\Gamma \vdash_{PTS} t : u$ *and* $\Gamma \sqsubseteq \Delta$ *wf then* $\Delta \vdash_{PTS} t : u$ *(where the relation* $\sqsubseteq$ *is defined similarly to that of PTSC, but with* β*-equivalence).*
2. *If* $\Gamma \vdash_{PTS} t : v$ *and* $\Gamma, y : v, \Delta \vdash_{PTS} u : v'$ *then* $\Gamma, \{t/y\}\Delta \vdash_{PTS} \{t/y\}u : \{t/y\}v'$.
3. *If* $\Gamma \vdash_{PTS} t : v$ *and* $t \longrightarrow_\beta u$ *then* $\Gamma \vdash_{PTS} u : v$.

We now extend the encodings to environments:

$$\mathcal{A}([]) = [] \qquad \mathcal{B}([]) = []$$
$$\mathcal{A}(\Gamma, (x : v)) = \mathcal{A}(\Gamma), (x : \mathcal{A}(v)) \qquad \mathcal{B}(\Gamma, (x : A)) = \mathcal{B}(\Gamma), (x : \mathcal{B}(A))$$

Now note that the simulations in section 2 imply:

Corollary 15 (Equational theories)
$t \longleftrightarrow^*_\beta u$ if and only if $\mathcal{A}(t) \longleftrightarrow^* \mathcal{A}(u)$
$M \longleftrightarrow^* N$ if and only if $\mathcal{B}(M) \longleftrightarrow^*_\beta \mathcal{B}(N)$

Preservation of typing is proved by induction on the typing derivations:

Theorem 16 (Preservation of typing 1)

1. *If $\Gamma \vdash_{PTS} t : T$ then $\mathcal{A}(\Gamma) \vdash \mathcal{A}(t) : \mathcal{A}(T)$*
2. *If $(\Gamma \vdash_{PTS} t_i : \{^{t_{i-}}/_{x_{i-1}}\} \cdots \{^{t_1}/_{x_1}\} T_i)_{i=1\ldots n}$ and $\mathcal{A}(\Gamma) \vdash \mathcal{A}(\Pi x_1^{T_1} \ldots \Pi x_n^{T_n} . T) : s$ then $\mathcal{A}(\Gamma); \mathcal{A}(\Pi x_1^{T_1} \ldots \Pi x_n^{T_n} . T) \vdash \mathcal{A}(t_1 \ldots t_n) : \mathcal{A}(\{^{t_n}/_{x_n}\} \cdots \{^{t_1}/_{x_1}\} T)$*
3. *If Γ wf then $\mathcal{A}(\Gamma)$ wf*

Theorem 17 (Preservation of typing 2)

1. *If $\Gamma \vdash M : A$ then $\mathcal{B}(\Gamma) \vdash_{PTS} \mathcal{B}(M) : \mathcal{B}(A)$*
2. *If $\Gamma; B \vdash l : A$ then $\mathcal{B}(\Gamma), y : \mathcal{B}(B) \vdash_{PTS} \mathcal{B}^y(l) : \mathcal{B}(A)$ for any fresh y*
3. *If Γ wf then $\mathcal{B}(\Gamma)$ wf*

4.2 Equivalence of Strong Normalisation

Theorem 18 *A PTSC given by the sets $\mathcal{S}$, $\mathcal{A}$, and $\mathcal{R}$ is strongly normalising if and only if the PTS given by the same sets is.*

Proof: Assume that the PTSC is strongly normalising, and let us consider a well-typed t of the corresponding PTS, i.e. $\Gamma \vdash_{\mathsf{PTS}} t : T$ for some Γ, T. By Theorem 16, $\mathcal{A}(\Gamma) \vdash \mathcal{A}(t) : \mathcal{A}(T)$ so $\mathcal{A}(t) \in \mathsf{SN}$. Now by Theorem 5, any reduction sequence starting from t maps to a reduction sequence of at least the same length starting from $\mathcal{A}(t)$, but those are finite.

Now assume that the PTS is strongly normalising and that $\Gamma \vdash M : A$ in the corresponding PTSC. By subject reduction, any N such that $M \longrightarrow^* N$ satisfies $\Gamma \vdash N : A$ and any sub-term P (resp. sub-list l) of any such N is also typable. By Theorem 17, for any such P (resp. l), $\mathcal{B}(P)$ (resp. $\mathcal{B}^y(l)$) is typable in the PTS, so it is strongly normalising by assumption.

We now refine the first-order encoding of any such P and l (as defined in section 1), emulating the technique of Bloo and Geuvers [BG99].

Accordingly, we refine the first-order signature from section 1 by *labelling* the symbols $\mathsf{cut}^t(\_,\_)$ and $\mathsf{sub}^t(\_,\_)$ with all strongly normalising terms t of a PTS, thus generating an infinite signature. The precedence relation is refined as follows

$$\star \prec \mathsf{i}(\_) \prec \mathsf{ii}(\_,\_) \prec \mathsf{cut}^t(\_,\_) \prec \mathsf{sub}^t(\_,\_)$$

but we also set $\mathsf{sub}^t(\_,\_) \prec \mathsf{cut}^{t'}(\_,\_)$ whenever $t' \longrightarrow^+_\beta t$. The precedence is still well-founded, so the induced (lpo) is also still well-founded (definitions and results can be found in [KL80]). The refinement of the encoding is given in Fig 8. An induction on terms shows that reductions decrease the lpo. □

$$
\begin{array}{lll}
\mathcal{T}(s) & & = \star \\
\mathcal{T}(\lambda x^A.M) & = \mathcal{T}(\Pi x^A.M) & = \mathsf{ii}(\mathcal{T}(A), \mathcal{T}(M)) \\
\mathcal{T}(x\ l) & & = \mathsf{i}(\mathcal{T}(l)) \\
\mathcal{T}(M\ l) & & = \mathsf{cut}^{\mathcal{B}(M\ l)}(\mathcal{T}(M), \mathcal{T}(l)) \\
\mathcal{T}(\langle M/x\rangle N) & & = \mathsf{sub}^{\mathcal{B}(\langle M/x\rangle N)}(\mathcal{T}(M), \mathcal{T}(N)) \\
\mathcal{T}([]) & & = \star \\
\mathcal{T}(M{\cdot}l) & & = \mathsf{ii}(\mathcal{T}(M), \mathcal{T}(l)) \\
\mathcal{T}(l@l') & & = \mathsf{ii}(\mathcal{T}(l), \mathcal{T}(l')) \\
\mathcal{T}(\langle M/x\rangle N) & & = \mathsf{sub}^{\mathcal{B}(\langle M/x\rangle l)}(\mathcal{T}(M), \mathcal{T}(l))
\end{array}
$$

Figure 8: First-order encoding

Examples of strongly normalising PTS are the *Calculus of Constructions* [CH88], on which the proof-assistant Coq is based [Coq] (but it also uses inductive types and local definitions), as well as all the other systems of Barendregt's Cube, for all of which we now have a corresponding PTSC that can be used for proof-search.

5 Proof-search

Proof-search considers as inputs an environment Γ and a type A, and the output, if successful, will be a term M such that $\Gamma \vdash M : A$, moreover one in normal form. When we search for a list $\Gamma; B \vdash l : A$, the type B in the stoup is also an input. Henceforth, A will be called simply a *goal*.

The inference rules now need to be *syntax-directed*, that is determined by the shape of the goal (or of the type in the stoup), and the proof-search system (PS, for short) is then obtained by optimising appeals to the conversion rules, yielding the presentation given in Fig. 9. The incorporation of the conversion rules into the other rules is similar to that of the constructive engine in natural deduction [Hue89, vBJMP94]; however the latter algorithm was designed for type synthesis, for which the inputs and outputs are not the same as in proof-search, as mentioned in the introduction.

$$
\frac{A \longleftrightarrow^* A'}{\Gamma; A \vdash_{\mathsf{PS}} [] : A'}\ \mathsf{axiom}
\qquad
\frac{D \longrightarrow^* \Pi x^A.B \quad \Gamma \vdash_{\mathsf{PS}} M : A \quad \Gamma; \langle M/x\rangle B \vdash_{\mathsf{PS}} l : C}{\Gamma; D \vdash_{\mathsf{PS}} M{\cdot}l : C}\ \Pi\mathsf{L}
$$

$$
\frac{C \longrightarrow^* s_3 \quad (s_1, s_2, s_3) \in R \quad \Gamma \vdash_{\mathsf{PS}} A : s_1 \quad \Gamma, (x : A) \vdash_{\mathsf{PS}} B : s_2}{\Gamma \vdash_{\mathsf{PS}} \Pi x^A.B : C}\ \Pi\mathsf{wf}
$$

$$
\frac{C \longrightarrow^* s' \quad (s, s') \in \mathcal{A}}{\Gamma \vdash_{\mathsf{PS}} s : C}\ \mathsf{sorted}
\qquad
\frac{(x : A) \in \Gamma \quad \Gamma; A \vdash_{\mathsf{PS}} l : B}{\Gamma \vdash_{\mathsf{PS}} x\ l : B}\ \mathsf{Select}_x
$$

$$
\frac{C \longrightarrow^* \Pi x^A.B \quad \Gamma, (x : A) \vdash_{\mathsf{PS}} M : B}{\Gamma \vdash_{\mathsf{PS}} \lambda x^A.M : C}\ \Pi\mathsf{R}
$$

Figure 9: Rules for Proof-search

Note one small difference with [LDM06]: we do not, in rule ΠR, require that A be

a normal form. As in [LDM06], soundness and completeness hold, but because of this difference, we get *quasi-normal forms* rather than normal forms.

Definition 3 A term (or a list) is a *quasi-normal form* if all its redexes are within type annotations of λ-abstractions, e.g. A in $\lambda x^A.M$.

Notice that, as we are searching for (quasi-)normal forms, there are *no* cut-rules in PS. However, in PTSC even terms in normal form may need instances of the cut-rule in their typing derivation. This is because, in contrast to logics where well-formedness of formulae is pre-supposed (such as first-order logic, where cut is admissible), PTSC checks well-formedness of types. For instance in rule $\Pi\mathsf{L}$ of PTSC a type which is not normalised ($\langle M/x\rangle B$) occurs in the stoup of the third premiss, so cuts might be needed to type it inside the derivation.

We conjecture that if we modify rule $\Pi\mathsf{L}$ by now requiring in the stoup of its third premiss *a normal form* to which $\langle M/x\rangle B$ reduces, then any typable normal form can be typed with a cut-free derivation. However, this would make rule $\Pi\mathsf{L}$ more complicated and, more importantly, we do not need such a conjecture to hold in order to perform proof-search.

In contrast, system PS avoids this problem by obviating such type-checking constraints altogether, because types are the input of proof-search, and should therefore be checked before starting search. This is the spirit of the type-checking proviso in the following soundness theorem.

PS is sound and complete in the following sense:

Theorem 19

1. *(Soundness) Provided* $\Gamma \vdash A : s$, *if* $\Gamma \vdash_{\mathsf{PS}} M : A$ *then* $\Gamma \vdash M : A$ *and* M *is a quasi-normal form.*
2. *(Completeness) If* $\Gamma \vdash M : A$ *and* M *is a quasi-normal form, then* $\Gamma \vdash_{\mathsf{PS}} M : A$.

Proof: Both proofs are done by induction on typing derivations, with similar statements for lists. For Soundness, the type-checking proviso is verified every time we need the induction hypothesis. For Completeness, the following lemma is required (and also proved inductively): assuming $A \longleftrightarrow^* A'$ and $B \longleftrightarrow^* B'$, if $\Gamma \vdash_{\mathsf{PS}} M : A$ then $\Gamma \vdash_{\mathsf{PS}} M : A'$, and if $\Gamma; B \vdash_{\mathsf{PS}} l : A$ then $\Gamma; B' \vdash_{\mathsf{PS}} l : A'$. □

Note that neither part of the theorem relies on the unsolved problem of *expansion postponement* [vBJMP94, Pol98]. Indeed, as indicated above PS *does not* check types. When recovering a full derivation tree from a PS one by the soundness theorem, expansions and cuts might be introduced at any point, arising from the derivation of the type-checking proviso.

Basic proof-search can be done in PS simply by

- reducing the goal, or the type in the stoup;
- depending on its shape, trying to apply one of the inference rules bottom-up; and
- recursively calling the process on the new goals (called *sub-goals*) corresponding to each premise.

However, some degree of non-determinism is expected in proof search. Such non-determinism is already present in natural deduction, but the sequent calculus version conveniently identifies where it occurs exactly.

There are three potential sources of such non-determinism:

- The choice of a variable x for applying rule Select_x, knowing only Γ and B (this corresponds in natural deduction to the choice of the head-variable of the proof-term). Not every variable of the environment will work, since the type in the stoup will eventually have to be unified with the goal, so we still need backtracking.

- When the goal reduces to a Π-type, there is an overlap between rules $\Pi\mathsf{R}$ and Select_x; similarly, when the type in the stoup reduces to a Π-type, there is an overlap between rules $\Pi\mathsf{L}$ and axiom. Both overlaps disappear when Select_x is restricted to the case when the goal does not reduce to a Π-type (and sequents with stoups never have a goal reducing to a Π-type). This corresponds to looking only for η-long normal forms in natural deduction. This restriction also brings the derivations in LJT (and in our PTSC) closer to the notion of uniform proofs. Further work includes the addition of η to the notion of conversion in PTSC.

- When the goal reduces to a sort s, three rules can be applied (in contrast to the first two points, this source of non-determinism does not already appear in the propositional case).

Such classification is often called *"don't care"* non-determinism in the case of the choice to apply an invertible rule and *"don't know"* non-determinism when the choice identifies a potential backtracking point.

Don't know non-determinism can be in fact quite constrained by the need to eventually unify the stoup with the goal, as an example in Section 7 below illustrates. Indeed, the dependency created by a Π-type forces the searches for proofs of the two premisses of rule $\Pi\mathsf{L}$ to be sequentialised in a way that might prove inefficient: the proof-term produced for the first premiss, selected among others at random, might well lead to the failure to solve the second premiss, leading to endless backtracking.

Hence, there is much to be gained by postponing the search for a proof of the first premiss and trying to solve the second with incomplete inputs. This might not terminate with success or failure but will send back constraints that may be useful in helping to solve the first premiss with the correct proof-term. "Helping" could just be giving some information to orient and speed-up the search for the right proof-term, but it could well define it completely (saving numerous attempts with proof-terms that will lead to failure). Unsurprisingly, these constraints are produced by the axiom rule as *unification* constraints.

In Coq [Coq], the proof-search tactic `apply x` can be decomposed into the bottom-up application of Select_x followed by a series of bottom-up applications of $\Pi\mathsf{L}$ and finally axiom, but it either postpones the solution of sub-goals or automatically solves them from the unification attempt, often avoiding obvious back-tracking.

In the next section we use the framework with meta-variables we have introduced to capture this behaviour in an extended sequent calculus.

6 Using meta-variables for proof-search

We now use the meta-variables in $\mathsf{PTSC}\alpha$ to delay the solution of sub-goals created by the application of rules such as $\Pi\mathsf{L}$. In this way, the extension from PTSC to $\mathsf{PTSC}\alpha$ supports not only an account of tactics such as `apply x` of Coq, but also the specification of algorithms for type inhabitant enumeration and unification. It provides the search-trees that such algorithms have to explore. Our approach has two main novelties in comparison with similar approaches (in the setting of natural deduction) by Dowek [Dow93] and Muñoz [Muñ01].

The first main novelty is that the search-tree is made of the inference rules of sequent calculus and its exploration is merely the root-first construction of a derivation tree; this greatly simplifies the understanding and the description of what such algorithms do.

The second main novelty is the avoidance of the complex phenomenon known as *r-splitting* that features in traditional inhabitation and unification algorithms (e.g. [Dow93]). In natural deduction, lists of arguments are not first-class objects; hence, when choosing a head variable in the construction of a λ-term, one also has to anticipate how many arguments it will be applied to (with polymorphism, there could be infinitely many choices). This anticipation can require a complex analysis of the sorting relations during a single search step and result in an infinitely branching search-tree whose exploration requires interleaving techniques. This is avoided by the use of meta-variables for lists of unknown length, which allows the choice of a head variable without commitment to the number of its arguments.

In contrast to section 4, where we confined our attention to *ground* terms in a PTSC and their relation to the corresponding PTS, here we consider the full language of *open* terms, representing incomplete proofs and partially solved goals. Correspondingly, *(open) environments* are now lists of pairs, denoted $(x : A)$, where x is a variable and A is a (possibly open) term (while *ground environments* only feature ground terms). Ground terms and environments are the eventual targets of successful proof-search, with all meta-variables instantiated. We further consider a new environment Σ that contains the sub-goals that remain to be proved:

Definition 4 (Goal environment, constraint, solved constraint, substitution)

- A *goal environment* Σ is a list of:
  - Triples of the form $\Gamma \vdash \alpha : A$, declaring the meta-variable α and called *(term-)goals*, where A is an open term and Γ is an open environment.
  - 4-tuples of the form $\Gamma; B \vdash \beta : A$, declaring the meta-variable β and called *(list-)goals*, where A and B are open terms and Γ is an open environment.
  - Triples of the form $A \overset{\Gamma}{=} B$, called *constraints*, where Γ is an open environment and A and B are open terms.

  Goals of a goal environment are required to declare distinct meta-variables.
- A constraint is *solved* if it is of the form $A \overset{\Gamma}{=} B$ where A and B are ground and $A \longleftrightarrow^* B$.
- A goal environment is *solved* if it contains no term or list goals and consists only of solved constraints.
- A *substitution* is a finite function that maps a meta-variable for term (resp. list), of arity n, to a *higher-order* term (resp. list) of arity n, that is to say, a term (resp. list) under a series of n bindings that capture (at least) its free variables (e.g. $x.y.M$ with $\mathsf{FV}(M) \subseteq \{x, y\}$).

  Such a series of bindings can be provided by a typing environment Γ, e.g. $\mathsf{Dom}(\Gamma).M$ (which is a useful notation when e.g. $\Gamma \vdash M : A$).
- The application of a substitution to terms and lists is defined by induction on these. Only the base cases are interesting:

  If $\sigma(\alpha) = x_1 \dots x_n.M$, then $\sigma(\alpha(N_1, \dots, N_n))$ is the x'-normal form[3] of

  $$\langle \sigma(N_1)/x_1 \rangle \dots \langle \sigma(N_n)/x_n \rangle M$$

[3] x' is convergent even on untyped terms, see Theorems 1 and Corollary 8

(with the usual capture-avoiding conditions). Similarly, if $\sigma(\beta) = x_1 \ldots x_n.l$, then $\sigma(\beta(N_1, \ldots, N_n))$ is the x'-normal form of

$$\langle \sigma(N_1)/x_1 \rangle \ldots \langle \sigma(N_n)/x_n \rangle l$$

The application of a substitution to an environment is the straightforward extension of the above.

For instance on the example of section 1.1, for an *actual* term M with $\mathsf{FV}(M) = \{x, y\}$ and $\sigma(\alpha) = x.y.M$, we have that $\sigma(\alpha(N, P))$ is the x'-normal form of $\langle \sigma(N)/x \rangle \langle \sigma(P)/y \rangle M$.

The reason why we x'-normalise the instantiation of meta-variables is that if M is already x'-normal then $(\alpha \mapsto x_1 \ldots x_n.M)(\alpha(y_1\ [], \ldots, y_n\ []))$ really *is* a renaming of M (and also an x'-normal form). This ensures that only normal forms are output by our system for proof-search, which we can more easily relate to PS.

We now introduce this system, called PE for *Proof Enumeration*, which can be seen as an extension of PS to open terms.

Definition 5 (An inference system for proof enumeration)
The inference rules for proof enumeration, presented in Fig. 10, manipulate three kinds of statement:

- The first two are of the form $\Gamma \vdash M : A \mid \Sigma$ and $\Gamma; B \vdash M : A \mid \Sigma$.
- The third kind of statement is of the form $\Sigma \Longrightarrow \sigma$, where
  - Σ is a goal environment.
  - σ is a substitution as defined above.

In the bottom part of the figure we use the notational convention that a substitution denoted σ_Σ has the meta-variables of the goal environment Σ as its domain.

Derivability in PE of the three kinds of statement is denoted $\Gamma \vdash_{\mathsf{PE}} M : A \mid \Sigma$, $\Gamma; B \vdash_{\mathsf{PE}} M : A \mid \Sigma$ and $\Sigma \Longrightarrow_{\mathsf{PE}} \sigma$.

The sequents $\Gamma \vdash M : A \mid \Sigma$ and $\Gamma; B \vdash M : A \mid \Sigma$ have the same intuitive meaning as the corresponding statements in system PS, but note the extra goal environment Σ, which represents the list of sub-goals and constraints that have been produced by proof-search and that remain to be solved. Thus, the inputs of proof synthesis are Γ and A (and B for the second kind of statement) and the outputs are M (or l) and Σ. Statements of PS are in fact particular cases of these statements with Σ being always solved.

In contrast, in a statement of the form $\Sigma \Longrightarrow \sigma$, Σ is the list of goals to solve, together with the constraints that the solutions must satisfy. It is the input of proof synthesis and σ is meant to be its solution, i.e. the output.

Now we prove that PE is *sound*. For that we need the following notion:

Definition 6 (Solution) We define the property *σ is a solution of a goal environment* Σ, by induction on the length of Σ.

- σ is a solution of $[]$.
- If σ is a solution of Σ and

$$x_1 : \sigma(A_1), \ldots, x_n : \sigma(A_n) \vdash_{\mathsf{PS}} (\sigma(\alpha))(x_1\ [], \ldots, x_n\ []) : \sigma(A)$$

then σ is a solution of $\Sigma, (x_1 : A_1, \ldots, x_n : A_n \vdash \alpha : A)$.

$$\frac{\Gamma = x_1 : A_1, \ldots, x_n : A_n}{\Gamma; D \vdash_{\mathsf{PE}} \beta(x_1\ [], \ldots, x_n\ []) : C \mid (\Gamma; D \vdash \beta : C)}$$

$$\frac{}{\Gamma; D \vdash_{\mathsf{PE}} [] : C \mid D \stackrel{\Gamma}{=} C}$$

$$\frac{D \longrightarrow^*_{\mathsf{Bx}} \Pi x^A.B \quad \Gamma \vdash_{\mathsf{PE}} M : A \mid \Sigma_1 \quad \Gamma; \langle M/x \rangle B \vdash_{\mathsf{PE}} l : C \mid \Sigma_2}{\Gamma; D \vdash_{\mathsf{PE}} M \cdot l : C \mid \Sigma_1, \Sigma_2}$$

$$\frac{\Gamma = x_1 : A_1, \ldots, x_n : A_n}{\Gamma \vdash_{\mathsf{PE}} \alpha(x_1\ [], \ldots, x_n\ []) : C \mid (\Gamma \vdash \alpha : C)}$$

$$\frac{C \longrightarrow^*_{\mathsf{Bx}} s \quad (s', s) \in \mathcal{A}}{\Gamma \vdash_{\mathsf{PE}} s' : C \mid []}$$

$$\frac{C \longrightarrow^*_{\mathsf{Bx}} s \quad (s_1, s_2, s) \in \mathcal{R} \quad \Gamma \vdash_{\mathsf{PE}} A : s_1 \mid \Sigma_1 \quad \Gamma, x : A \vdash_{\mathsf{PE}} B : s_2 \mid \Sigma_2}{\Gamma \vdash_{\mathsf{PE}} \Pi x^A.B : C \mid \Sigma_1, \Sigma_2}$$

$$\frac{(x : A) \in \Gamma \quad \Gamma; A \vdash_{\mathsf{PE}} l : C \mid \Sigma'}{\Gamma \vdash_{\mathsf{PE}} x\ l : C \mid \Sigma'}$$

$$\frac{C \longrightarrow^*_{\mathsf{Bx}} \Pi x^A.B \quad \Gamma, x : A \vdash_{\mathsf{PE}} M : B \mid \Sigma'}{\Gamma \vdash_{\mathsf{PE}} \lambda x^A.M : C \mid \Sigma'}$$

$$\frac{\Gamma; B \vdash_{\mathsf{PE}} l : A \mid \Sigma'' \qquad \Sigma, \Sigma'', (\beta \mapsto \mathsf{Dom}(\Gamma).l)(\Sigma') \Longrightarrow_{\mathsf{PE}} \sigma_\Sigma, \sigma_{\Sigma''}, \sigma_{\Sigma'}}{\Sigma, (\Gamma; B \vdash \beta : A), \Sigma' \Longrightarrow_{\mathsf{PE}} \sigma_\Sigma, (\beta \mapsto \mathsf{Dom}(\Gamma).\sigma_\Sigma(l)), \sigma_{\Sigma'}}$$

$$\frac{\Gamma \vdash_{\mathsf{PE}} M : A \mid \Sigma'' \qquad \Sigma, \Sigma'', (\alpha \mapsto \mathsf{Dom}(\Gamma).M)(\Sigma') \Longrightarrow_{\mathsf{PE}} \sigma_\Sigma, \sigma_{\Sigma''}, \sigma_{\Sigma'}}{\Sigma, (\Gamma \vdash \alpha : A), \Sigma' \Longrightarrow_{\mathsf{PE}} \sigma_\Sigma, (\alpha \mapsto \mathsf{Dom}(\Gamma).\sigma_\Sigma(M)), \sigma_{\Sigma'}}$$

$$\frac{\Sigma \text{ is solved}}{\Sigma \Longrightarrow_{\mathsf{PE}} \emptyset}$$

Figure 10: Proof-term enumeration $\vdash_{\mathsf{PE}}$

- If σ is a solution of Σ and

  $$x_1 : \sigma(A_1), \ldots, x_n : \sigma(A_n); \sigma(B) \vdash_{\mathsf{PS}} (\sigma(\beta))(x_1\ [], \ldots, x_n\ []) : \sigma(A)$$

  then σ is a solution of $\Sigma, (x_1 : A_1, \ldots, x_n : A_n; B \vdash \beta : A)$.
- If σ is a solution of Σ and

  $$\sigma(M) \longleftrightarrow^* \sigma(N)$$

  then σ is a solution of $\Sigma, M \stackrel{\Gamma}{=} N$.

For soundness we also need the following lemma:

Lemma 20 *Suppose that $\sigma(M)$ and $\sigma(l)$ are ground.*

1. *If $M \longrightarrow_{Bx'} N$ then $\sigma(M) \longrightarrow^*_{Bx} \sigma(N)$.*
2. *If $l \longrightarrow_{Bx'} l'$ then $\sigma(l) \longrightarrow^*_{Bx} \sigma(l')$.*

Proof: By simultaneous induction on the derivation of the reduction step, checking all rules for the base case of root reduction. □

Theorem 21 (Soundness) *Suppose σ is a solution of Σ.*

1. *If $\Gamma \vdash_{PE} M : A \mid \Sigma$ then $\sigma(\Gamma) \vdash_{PS} \sigma(M) : \sigma(A)$.*
2. *If $\Gamma; B \vdash_{PE} M : A \mid \Sigma$ then $\sigma(\Gamma); \sigma(B) \vdash_{PS} \sigma(M) : \sigma(A)$.*

Proof: By induction on derivations. □

Corollary 22 *If $\Sigma \Longrightarrow_{PE} \sigma$ then σ is a solution of Σ.*

Proof: By induction on the derivation, using Theorem 21. □

System PE is *complete* in the following sense:

Theorem 23 (Completeness)

1. *If $\Gamma \vdash_{PS} M : A$ then $\Gamma \vdash_{PE} M : A \mid \Sigma$ for some solved goal environment Σ.*
2. *If $\Gamma; B \vdash_{PS} M : A$ then $\Gamma; B \vdash_{PE} M : A \mid \Sigma$ for some solved Σ.*

Proof: By induction on derivations. The rules of PE generalise those of PS. □

In fact, the completeness of the full system PE is not surprising, since it is quite general. In particular, nothing is said about when the process should decide to abandon the current goal and start working on another one. Hence we should be interested in completeness of particular *strategies* dealing with that question. For instance:

- We can view the system PS as supporting the strategy of eagerly solving sub-goals as soon as they are created, never delaying them with the sub-goal environment.
- The algorithm for proof enumeration in [Dow93] would correspond here to the "lazy" strategy that always abandons the sub-goal generated by rule $\Pi\mathsf{L}_{\mathsf{PS}}$, but this in fact enables unification constraints to guide the solution of this sub-goal later, so in that case laziness is probably more efficient than eagerness. This is probably what should be chosen for automated theorem proving.
- Mixtures of the two strategies can also be considered and could be the basis of interactive theorem proving. Indeed in some cases the user's input might be more efficient than the automated algorithm, and rule $\Pi\mathsf{L}_{\mathsf{PS}}$ would be a good place to ask whether the user has any clue to solve the sub-goal (since it could help solving the rest of the unification). If he or she has none, then by default the algorithm might abandon the sub-goal and leave it for later.

  In Coq, the tactic `apply x` does something similar: it tries to automatically solve the sub-goals that interfere with the unification constraint (leaving the other ones for later, visible to the user), but, if unification fails, it is always possible for the user to use the tactic and give explicitly the proof-term to make it work. However, such an input is not provided in proof synthesis mode and the user really has to give it fully, since the tactic will fail if unification fails. In PE, the unification constraint can remain partially solved.

All these behaviours can be simulated in PE, which is therefore a useful framework for the study of proof synthesis strategies in type theory and for comparison with the work of Jojgov [GJ02], McBride [McB00] and Delahaye [Del01].

7 Example: commutativity of conjunction

We now give an example of proof-search (first introduced in [LDM06] without the use of meta-variables).

We consider the PTSC equivalent to System F, i.e. the one given by the sets:

$$\mathcal{S} = \{\star, \Box\},\ \mathcal{A} = \{(\star, \Box)\},\ \text{and}\ \mathcal{R} = \{(\star,\star),(\Box,\star)\}$$

For brevity we omit the types on λ-abstractions, we abbreviate $x\ []$ as x for any variable x and simplify $\langle N/x\rangle P$ as P when $x \notin \mathsf{FV}(P)$. We also write $A \wedge B$ for $\Pi Q^{\star}.(A{\to}(B{\to}Q)){\to}Q$.

Proof-search in system PS would result in the following derivation:

$$\dfrac{\dfrac{\pi_B}{\Gamma \vdash_{\mathsf{PS}} N_B : B} \qquad \dfrac{\dfrac{\pi_A}{\Gamma \vdash_{\mathsf{PS}} N_A : A} \qquad \dfrac{}{\Gamma; Q \vdash_{\mathsf{PS}} [] : Q}\,\mathsf{axiom}}{\Gamma; A{\to}Q \vdash_{\mathsf{PS}} N_A\cdot[] : Q}\,\Pi\mathsf{L}}{\dfrac{\Gamma; B{\to}(A{\to}Q) \vdash_{\mathsf{PS}} N_B\cdot N_A\cdot[] : Q}{\Gamma \vdash_{\mathsf{PS}} y\ N_B\cdot N_A\cdot[] : Q}\,\mathsf{Select}_y}\,\Pi\mathsf{L}}{A:\star, B:\star \vdash_{\mathsf{PS}} \lambda x.\lambda Q.\lambda y.y\ N_B\cdot N_A\cdot[] : (A\wedge B){\to}(B\wedge A)}\,\Pi\mathsf{R}$$

where $\Gamma = A:\star, B:\star, x : A\wedge B, Q:\star, y : B{\to}(A{\to}Q)$, and π_A is the following derivation ($N_A = x\ A\cdot(\lambda x'.\lambda y'.x')\cdot[]$):

$$\dfrac{\dfrac{\dfrac{}{\Gamma;\star \vdash_{\mathsf{PS}} [] : \star}\,\mathsf{axiom}}{\Gamma \vdash_{\mathsf{PS}} A : \star}\,\mathsf{Select}_A \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma, x' : A, y' : B; A \vdash_{\mathsf{PS}} [] : A}\,\mathsf{axiom}}{\Gamma, x' : A, y' : B \vdash_{\mathsf{PS}} x' : A}\,\mathsf{Select}_{x'}}{\Gamma \vdash_{\mathsf{PS}} \lambda x'.\lambda y'.x' : A{\to}(B{\to}A)}\,\Pi\mathsf{R} \qquad \dfrac{}{\Gamma; A \vdash_{\mathsf{PS}} [] : A}\,\mathsf{axiom}}{\Gamma; (A{\to}(B{\to}A)){\to}A \vdash_{\mathsf{PS}} (\lambda x'.\lambda y'.x')\cdot[] : A}\,\Pi\mathsf{L}}{\dfrac{\Gamma; A\wedge B \vdash_{\mathsf{PS}} A\cdot(\lambda x'.\lambda y'.x')\cdot[] : A}{\Gamma \vdash_{\mathsf{PS}} x\ A\cdot(\lambda x'.\lambda y'.x')\cdot[] : A}\,\mathsf{Select}_x}\,\Pi\mathsf{L}$$

and π_B is the derivation similar to π_A ($N_B = x\ B\cdot(\lambda x'.\lambda y'.y')\cdot[]$) with conclusion $\Gamma \vdash_{\mathsf{PS}} x\ B\cdot(\lambda x'.\lambda y'.y')\cdot[] : B$.

We now reconsider the above example in the light of system PE. It illustrates the need for delaying the search for a proof of the first premiss of rule $\Pi\mathsf{L}$. Let

$$\begin{array}{ll}
\Gamma & = A:\star, B:\star, x : A\wedge B, Q:\star, y : B{\to}A{\to}Q \\
\alpha_A(\Gamma) & = \alpha_A(A,B,x,Q,y) \\
\alpha_B(\Gamma) & = \alpha_B(A,B,x,Q,y) \\
M' & = \lambda x.\lambda Q.\lambda y.y\ \alpha_B(\Gamma)\cdot\alpha_A(\Gamma)\cdot[] \\
\Sigma & = (\Gamma \vdash \alpha_B : B), (\Gamma \vdash \alpha_A : A), (Q \overset{\Gamma}{=} Q)
\end{array}$$

We get the PE-derivation below:

$$\dfrac{\dfrac{\dfrac{}{\Gamma \vdash \alpha_B(\Gamma) : B \mid (\Gamma \vdash \alpha_B : B)} \qquad \dfrac{\dfrac{}{\Gamma \vdash \alpha_A(\Gamma) : A \mid (\Gamma \vdash \alpha_A : A)} \qquad \dfrac{}{\Gamma; Q \vdash [] : Q \mid (Q \overset{\Gamma}{=} Q)}}{\Gamma; A{\to}Q \vdash \alpha_A(\Gamma)\cdot[] : Q \mid (\Gamma \vdash \alpha_A : A), (Q \overset{\Gamma}{=} Q)}}{\dfrac{\Gamma; B{\to}A{\to}Q \vdash \alpha_B(\Gamma)\cdot\alpha_A(\Gamma)\cdot[] : Q \mid \Sigma}{\Gamma \vdash y\ \alpha_B(\Gamma)\cdot\alpha_A(\Gamma)\cdot[] : Q \mid \Sigma}} \qquad \dfrac{\dots}{\Sigma \Longrightarrow \sigma_\Sigma}}{(A:\star, B:\star \vdash \alpha : (A\wedge B){\to}(B\wedge A)) \Longrightarrow (\alpha \mapsto \sigma_\Sigma(M'))}$$

where $\sigma_\Sigma = (\alpha_B \mapsto \mathsf{Dom}(\Gamma).N_B, \alpha_A \mapsto \mathsf{Dom}(\Gamma).N_A)$ is the solution to be obtained from the right premiss.

In the above derivation, we have systematically abandoned the sub-goals and recorded them for later. The only choice we made was that of the head-variable y, because it led to the production of the (solved) unification constraint $(Q \stackrel{\Gamma}{=} Q)$.

We now continue the proof-search with the right premiss, solving the two sub-goals $(\Gamma \vdash \alpha_B : B)$ and $(\Gamma \vdash \alpha_A : A)$ that have been delayed. For instance, we can now decide to solve $(\Gamma \vdash \alpha_A : A)$, which will eventually produce $\alpha_A \mapsto \mathsf{Dom}(\Gamma).N_A$ with $N_A = x\ A{\cdot}(\lambda x'y'.x'){\cdot}[]$:

$$\dfrac{\dfrac{}{\Gamma \vdash \alpha_1(\Gamma) : \star \mid \Sigma_1} \qquad \dfrac{\dfrac{\dfrac{}{\Gamma' \vdash \alpha'_1(\Gamma') : \alpha_1(\Gamma) \mid \Sigma'_1}}{\Gamma \vdash \lambda x'y'.\alpha'_1(\Gamma') : A{\to}B{\to}\alpha_1(\Gamma) \mid \Sigma'_1} \qquad \dfrac{}{\Gamma; \alpha_1(\Gamma) \vdash [] : A \mid \Sigma''_1}}{\Gamma; (A{\to}B{\to}\alpha_1(\Gamma)){\to}\alpha_1(\Gamma) \vdash (\lambda x'y'.\alpha'_1(\Gamma')){\cdot}[] : A \mid \Sigma'_1, \Sigma''_1}}{\dfrac{\Gamma; A \wedge B \vdash \alpha_1(\Gamma){\cdot}(\lambda x'y'.\alpha'_1(\Gamma')){\cdot}[] : A \mid \Sigma_1, \Sigma'_1, \Sigma''_1}{\Gamma \vdash x\ \alpha_1(\Gamma){\cdot}(\lambda x'y'.\alpha'_1(\Gamma')){\cdot}[] : A \mid \Sigma_1, \Sigma'_1, \Sigma''_1} \qquad \dfrac{\ldots}{(\Gamma \vdash \alpha_B : B), \Sigma_1, \Sigma'_1, \Sigma''_1, (Q \stackrel{\Gamma}{=} Q) \Longrightarrow \sigma}}{\Sigma \Longrightarrow (\alpha_B \mapsto \mathsf{Dom}(\Gamma).N_B, \alpha_A \mapsto \mathsf{Dom}(\Gamma).x\ A{\cdot}(\lambda x'y'.x'){\cdot}[])}$$

where

$$\begin{array}{ll}
\alpha_1(\Gamma) & = \alpha_1(A, B, x, Q, y) \\
\Sigma_1 & = (\Gamma \vdash \alpha_1 : \star) \\
\Gamma' & = \Gamma, x' : A, y' : B \\
\alpha'_1(\Gamma) & = \alpha'_1(A, B, x, Q, y, x', y') \\
\Sigma'_1 & = (\Gamma' \vdash \alpha'_1 : \alpha_1(\Gamma)) \\
\Sigma''_1 & = (\alpha_1(\Gamma) \stackrel{\Gamma}{=} A) \\
\sigma & = (\alpha_B \mapsto \mathsf{Dom}(\Gamma).N_B, \quad \alpha_1 \mapsto \mathsf{Dom}(\Gamma).A, \quad \alpha'_1 \mapsto \mathsf{Dom}(\Gamma').x')
\end{array}$$

In the above derivation, we have also abandoned the generated sub-goals. Again we made one committing choice: that of the head-variable x, which led to the unification constraint $\alpha_1(\Gamma) \stackrel{\Gamma}{=} A$. Any other choice of head-variable would have led to a unification constraint with no solution. Here, this fact (and the subsequent choice of x) can be mechanically noticed by a simple syntactic check.

We now continue the proof-search with the right premiss. We can decide to solve $(\Gamma \vdash \alpha_B : B)$, $(\Gamma \vdash \alpha_1 : \star)$, or $(\Gamma' \vdash \alpha'_1 : \alpha_1(\Gamma))$. The order in which we solve $(\Gamma \vdash \alpha_B : B)$ has little importance (the structure is similar to that of the derivation above), but clearly we cannot solve $(\Gamma' \vdash \alpha'_1 : \alpha_1(\Gamma))$ before we know $\alpha_1(\Gamma)$. Hence, we need to solve $(\Gamma \vdash \alpha_1 : \star)$ first, which will produce $\alpha_1 \mapsto \mathsf{Dom}(\Gamma).A$:

$$\dfrac{\dfrac{\dfrac{}{\Gamma; \star \vdash [] : \star \mid \star \stackrel{\Gamma}{=} \star}}{\Gamma \vdash A\ [] : \star \mid \star \stackrel{\Gamma}{=} \star} \qquad \dfrac{\ldots}{(\Gamma \vdash \alpha_B : B), (\star \stackrel{\Gamma}{=} \star), (\Gamma' \vdash \alpha'_1 : A), (A \stackrel{\Gamma}{=} A), (Q \stackrel{\Gamma}{=} Q) \Longrightarrow \sigma'}}{(\Gamma \vdash \alpha_B : B), (\Gamma \vdash \alpha_1 : \star), (\Gamma' \vdash \alpha'_1 : \alpha_1(\Gamma)), (\alpha_1(\Gamma) \stackrel{\Gamma}{=} A), (Q \stackrel{\Gamma}{=} Q) \Longrightarrow \sigma}$$

where $\sigma' = (\alpha_B \mapsto \mathsf{Dom}(\Gamma).N_B, \quad \alpha'_1 \mapsto \mathsf{Dom}(\Gamma').x')$.

In this derivation we had to inhabit $\star$. This is a fundamental step of the proof, even when expressed with ground terms (in system PS) as above. Here, having delayed the solution of sub-goals, we are now able to infer the correct inhabitation, directly from the unification constraint $(\alpha_1(\Gamma) \stackrel{\Gamma}{=} A)$ which we have generated previously. Our delaying mechanism thus avoids many situations in which the correct choice for inhabiting a type has to be guessed in advance, anticipating the implicit constraints that such a choice will have to satisfy at some point. This is hardly mechanisable and thus leads to numerous backtrackings.

Finally we proceed to the right premiss by solving $(\Gamma' \vdash \alpha'_1 : A)$:

$$\dfrac{\dfrac{}{\Gamma'; A \vdash [] : A \mid A \overset{\Gamma'}{=} A}}{\Gamma' \vdash x'\ [] : A \mid A \overset{\Gamma'}{=} A} \qquad \dfrac{\qquad \ldots \qquad}{(\Gamma \vdash \alpha_B : B), (\star \overset{\Gamma}{=} \star), (A \overset{\Gamma'}{=} A), (A \overset{\Gamma}{=} A), (Q \overset{\Gamma}{=} Q) \Longrightarrow (\alpha_B(\Gamma) \mapsto N_B)}$$

$$(\Gamma \vdash \alpha_B : B), (\star \overset{\Gamma}{=} \star), (\Gamma' \vdash \alpha'_1 : A), (A \overset{\Gamma}{=} A), (Q \overset{\Gamma}{=} Q) \Longrightarrow \sigma'$$

In this derivation we had to inhabit A. Again we made one committing choice: that of the head-variable x', which led to the unification constraint $A \overset{\Gamma'}{=} A$. Again, any other choice of head-variable would have led to obvious failure, a fact which can be mechanically noticed by a simple syntactic check.

We can then proceed with $(\Gamma \vdash \alpha_B : B)$, in a way very similar to that of $(\Gamma \vdash \alpha_A : A)$. We get eventually $N_B = x\ B \cdot (\lambda x' y'.y') \cdot []$.

Putting everything together, system PE has produced a proof for commutativity of conjunction:

$$A : \star, B : \star \vdash \lambda x Q y.y\ (x\ B \cdot (\lambda x' y'.y') \cdot []) \cdot (x\ A \cdot (\lambda x' y'.x') \cdot []) \cdot [] : (A \wedge B) \rightarrow (B \wedge A)$$

The system has mechanically inferred the relevant choices of the head-variables structuring the proof-term, by finite checks and using the unification constraints generated by delaying the solution of sub-goals.

Conclusion and Further Work

In this paper we have developed a framework that serves as a good theoretical basis for proof-search in type theory.

Proof-search tactics in natural deduction depart from the simple bottom-up application of the typing rules, so that their readability and usage become more complex, as illustrated in proof-assistants such as Coq. Just as in propositional logic [DP99a], permutation-free sequent calculi can be a useful theoretical approach to study and design those tactics, in the hope of improving semi-automated reasoning.

Following these ideas, we have defined a parameterised formalism that gives a sequent calculus for each PTS. It comprises a syntax, a rewrite system and typing rules. In contrast to previous work, the syntax of both types and proof-terms of PTSC is in a sequent-calculus style, thus avoiding the use of implicit or explicit conversions to natural deduction [GR03, PD98].

A strong correspondence with natural deduction has been established (regarding both logic and strong normalisation), and we have derived from it the confluence of each PTSC. These results and their proofs were formalised in Coq [Sil09]. We can give as examples the corners of Barendregt's Cube, for which we now have an elegant theoretical framework for proof-search: We have shown how to deal with conversion rules so that basic proof-search tactics are simply the root-first application of the typing rules.

These ideas have then been extended, in the calculi PTSCα, by the use of meta-variables to formalise the notion of incomplete proofs, and their theory has been studied. The approach differs from [Muñ01] both in that we use sequent calculus rules, which match proof-search tactics, and in that our system can simulate β-reduction.

We have shown that, in particular, the explicit use of meta-variables avoids the phenomenon of *r-splitting* and allows for more flexibility in proof-search, where sub-goals can be tackled in the order that is most suitable for each situation. Such a flexibility avoids some of the need for "guess-work" in proof-search, and formalises

some mechanisms of proof-search tactics in proof assistants. This approach has been illustrated by the example of commutativity of conjunction.

Our system does not commit to specific proof-search strategies *a priori*, so that it can be used as a general framework to investigate these strategies, as discussed at the end of Section 6. This could reflect various degrees of user interaction in proof-search.

Ongoing work includes the incorporation of some of these ideas into the redesign of the Coq proof engine [Coq]. It also includes the treatment of η-conversion, a feature that is currently lacking in the PTS-based system Coq. We expect that, by adding *η-expansion* to our system, our approach to proof-search can be related to that of *uniform proofs* in logic programming.

Further work includes studying direct proofs of strong normalisation (such as Kikuchi's for propositional logic [Kik04]), and dealing with inductive types such as those used in Coq. Their specific proof-search tactics should also clearly appear in sequent calculus. Finally, given the importance of sequent calculi for classical logic, it would be interesting to build classical Pure Type Sequent Calculi.

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A Appendix

Definition 7 We write $\Gamma \vdash^\star M : A$ (resp. $\Gamma; B \vdash^\star l : A$) whenever we can derive $\Gamma \vdash M : A$ (resp. $\Gamma; B \vdash l : A$) and the last rule is not a conversion rule.

The following Lemma is easily derived by induction on the typing tree:

Lemma 24 (Generation Lemma)

1. (a) *If* $\Gamma \vdash_{PTSC} s : C$ *then there is* s' *such that* $\Gamma \vdash^\star s : s'$ *with* $C \longleftrightarrow^* s'$.
   (b) *If* $\Gamma \vdash_{PTSC} \Pi x^A.B : C$ *then there is* s *such that* $\Gamma \vdash^\star \Pi x^A.B : s$ *with* $C \longleftrightarrow^* s$.
   (c) *If* $\Gamma \vdash_{PTSC} \lambda x^A.M : C$ *then*
   there is B *such that* $C \longleftrightarrow^* \Pi x^A.B$ *and* $\Gamma \vdash^\star \lambda x^A.M : \Pi x^A.B$.
   (d) *If* $\Gamma \vdash_{PTSC} \langle M/x \rangle N : C$ *then there is* C' *such that* $\Gamma \vdash^\star \langle M/x \rangle N : C'$ *with* $C \longleftrightarrow^* C'$.
   (e) *If* M *is not of the above forms and* $\Gamma \vdash_{PTSC} M : C$, *then* $\Gamma \vdash^\star M : C$.
2. (a) *If* $\Gamma; B \vdash_{PTSC} [] : C$ *then* $B \longleftrightarrow^* C$.

(b) *If* $\Gamma; D \vdash_{PTSC} M \cdot l : C$ *then*
there are A, B *such that* $D \longleftrightarrow^* \Pi x^A.B$ *and* $\Gamma; \Pi x^A.B \vdash^\star M \cdot l : C$.

(c) *If* $\Gamma; B \vdash_{PTSC} \langle M/x \rangle l : C$ *then are* B', C' *such that*
$\Gamma; B' \vdash^\star \langle M/x \rangle l : C'$ *with* $C \longleftrightarrow^* C'$ *and* $B \longleftrightarrow^* B'$.

(d) *If* l *is not of the above forms and* $\Gamma; D \vdash_{PTSC} l : C$ *then* $\Gamma; D \vdash^\star l : C$.

Proof: Straightforward induction on the typing tree. □

Remark 25 The following rule is derivable, using a conversion rule:

$$\frac{\Gamma \vdash_{\mathsf{PTSC}} Q : A \qquad \Gamma, (x : A), \Delta \vdash_{\mathsf{PTSC}} M : C \qquad \Delta' \vdash_{\mathsf{PTSC}} \langle Q/x \rangle C : s \qquad \Gamma, \langle Q/x \rangle \Delta \sqsubseteq \Delta'}{\Delta' \vdash_{\mathsf{PTSC}} \langle Q/x \rangle M : \langle Q/x \rangle C}$$

Proving subject reduction relies on the following properties of $\longrightarrow_{\mathsf{Bx}}$:

Lemma 26

- *Two distinct sorts are not convertible.*
- *A* Π*-construct is not convertible to a sort.*
- $\Pi x^A.B \longleftrightarrow^* \Pi x^D.E$ *if and only if* $A \longleftrightarrow^* D$ *and* $B \longleftrightarrow^* E$.
- *If* $y \notin FV(P)$, *then* $M \longleftrightarrow^* \langle N/y \rangle P$.
- $\langle M/y \rangle \langle N/x \rangle P \longleftrightarrow^* \langle \langle M/y \rangle N/x \rangle \langle M/y \rangle P$ *(provided* $x \notin FV(M)$*).*

Proof: The first three properties are a consequence of the confluence of the rewrite system (Corollary 8). The last two rely on the fact that the system xsubst is terminating, so that only the case when P is an xsubst-normal form remains to be checked, which is done by structural induction. □

Using all of the results above, subject reduction can be proved:

Theorem 27 (Subject reduction in a PTSC)

1. *If* $\Gamma \vdash_{PTSC} M : F$ *and* $M \longrightarrow_{Bx} M'$, *then* $\Gamma \vdash_{PTSC} M' : F$
2. *If* $\Gamma; H \vdash_{PTSC} l : F$ *and* $l \longrightarrow_{Bx} l'$, *then* $\Gamma; H \vdash_{PTSC} l' : F$

Proof: By simultaneous induction on the typing tree. For every rule, if the reduction takes place within a sub-term that is typed by one of the premisses of the rule (e.g. the conversion rules), then we can apply the induction hypothesis on that premiss. In particular, this takes care of the cases where the last typing rule is a conversion rule.

So it now suffices to look at the root reductions. For lack of space we often do not display some minor premisses in following derivations, but we mention them before or after. We also drop the subscript PTSC from derivable statements.

B $(\lambda x^A.N)\ (P \cdot l_1) \longrightarrow (\langle P/x \rangle N)\ l_1$

By the Generation Lemma, 1.(c) and 2.(b), there exist B, D, E such that:

$$\dfrac{\dfrac{\Gamma \vdash \Pi x^A.B : s \quad \Gamma, x : A \vdash N : B}{\Gamma \vdash \lambda x^A.N : C} \qquad \dfrac{\Gamma \vdash P : D \quad \Gamma; \langle P/x \rangle E \vdash l_1 : F}{\Gamma; C \vdash P \cdot l_1 : F}}{\Gamma \vdash^\star (\lambda x^A.N)\ (P \cdot l_1) : F}$$

with $\Pi x^A.B \longleftrightarrow^* C \longleftrightarrow^* \Pi x^D.E$. Hence, $A \longleftrightarrow^* D$ and $B \longleftrightarrow^* E$.

Moreover, $\Gamma \vdash A : s_A$, $\Gamma, x : A \vdash B : s_B$ and Γ wf.
Hence, we get $\Gamma \vdash \langle P/x \rangle B : s_B$, so:

$$\dfrac{\dfrac{\dfrac{\Gamma \vdash P : D}{\Gamma \vdash P : A} \quad \Gamma, x : A \vdash N : B}{\Gamma \vdash \langle P/x \rangle N : \langle P/x \rangle B} \quad \dfrac{\Gamma ; \langle P/x \rangle E \vdash l_1 : F}{\Gamma ; \langle P/x \rangle B \vdash l_1 : F}}{\Gamma \vdash (\langle P/x \rangle N \; l_1) : F}$$

with $\langle P/x \rangle B \longleftarrow\!\!\!\!\!\rightarrow^* \langle P/x \rangle E$.

As A1 $(N \cdot l_1)@l_2 \longrightarrow N \cdot (l_1 @ l_2)$
By the Generation Lemma 2.(b), there are A and B such that $H \longleftarrow\!\!\!\!\!\rightarrow^* \Pi x^A . B$ and:

$$\dfrac{\dfrac{\Gamma \vdash \Pi x^A . B : s \quad \Gamma \vdash N : A \quad \Gamma ; \langle N/x \rangle B \vdash l_1 : C}{\Gamma ; H \vdash N \cdot l_1 : C} \quad \Gamma ; C \vdash l_2 : F}{\Gamma ; H \vdash^\star (N \cdot l_1)@l_2 : F}$$

Hence,

$$\dfrac{\Gamma \vdash H : s_H \quad \dfrac{\Gamma \vdash \Pi x^A . B : s \quad \Gamma \vdash N : A \quad \dfrac{\Gamma ; \langle N/x \rangle B \vdash l_1 : C \quad \Gamma ; C \vdash l_2 : F}{\Gamma ; \langle N/x \rangle B \vdash l_1 @ l_2 : C}}{\Gamma ; \Pi x^A . B \vdash N \cdot (l_1 @ l_2) : F}}{\Gamma ; H \vdash N \cdot (l_1 @ l_2) : F}$$

A2 $[]@l_1 \longrightarrow l_1$
By the Generation Lemma 2.(a), we have $A \longleftarrow\!\!\!\!\!\rightarrow^* H$ and

$$\dfrac{\Gamma ; H \vdash [] : A \quad \Gamma ; A \vdash l_1 : F}{\Gamma ; H \vdash^\star []@l_1 : F}$$

Since $\Gamma \vdash H : s_H$, we get

$$\dfrac{\Gamma ; A \vdash l_1 : F}{\Gamma ; H \vdash l_1 : F}$$

A3 $(l_1 @ l_2)@l_3 \longrightarrow l_1 @ (l_2 @ l_3)$
By the Generation Lemma 2.(d),

$$\dfrac{\dfrac{\Gamma ; H \vdash l_1 : B \quad \Gamma ; B \vdash l_2 : A}{\Gamma ; H \vdash^\star l_1 @ l_2 : A} \quad \Gamma ; A \vdash l_3 : F}{\Gamma ; H \vdash^\star (l_1 @ l_2)@l_3 : F}$$

Hence,

$$\dfrac{\Gamma ; H \vdash l_1 : B \quad \dfrac{\Gamma ; B \vdash l_2 : A \quad \Gamma ; A \vdash l_3 : F}{\Gamma ; B \vdash l_2 @ l_3 : F}}{\Gamma ; H \vdash l_1 @ (l_2 @ l_3) : F}$$

Bs B1 $N \; [] \longrightarrow N$

$$\dfrac{\Gamma \vdash N : A \quad \Gamma ; A \vdash [] : F}{\Gamma \vdash^\star N \; [] : F}$$

By the Generation Lemma 2.(a), we have $A \longleftarrow\!\!\!\!\!\rightarrow^* F$.
Since $\Gamma \vdash F : s_F$, we get

$$\dfrac{\Gamma \vdash N : A}{\Gamma \vdash N : F}$$

B2 $(x\ l_1)\ l_2 \longrightarrow\ x\ (l_1@l')$

By the Generation Lemma 1.(e),

$$\dfrac{\dfrac{\Gamma; A \vdash l_1 : B \quad (x:A) \in \Gamma}{\Gamma \vdash^\star x\ l : B} \qquad \Gamma; B \vdash l_2 : F}{\Gamma \vdash^\star (x\ l_1)\ l_2 : F}$$

Hence,

$$\dfrac{(x:A) \in \Gamma \qquad \dfrac{\Gamma; A \vdash l_1 : B \quad \Gamma; B \vdash l_2 : F}{\Gamma; A \vdash l_1@l_2 : F}}{\Gamma \vdash x\ (l_1@l_2) : F}$$

B3 $(N\ l_1)\ l_2 \longrightarrow\ N\ (l_1@l_2)$

By the Generation Lemma 1.(e),

$$\dfrac{\dfrac{\Gamma \vdash N : A \quad \Gamma; A \vdash l_1 : B}{\Gamma \vdash^\star N\ l_1 : B} \qquad \Gamma; B \vdash l_2 : F}{\Gamma \vdash^\star (N\ l_1)\ l_2 : F}$$

Hence,

$$\dfrac{\Gamma \vdash N : A \qquad \dfrac{\Gamma; A \vdash l_1 : B \quad \Gamma; B \vdash l_2 : F}{\Gamma; A \vdash l_1@l_2 : F}}{\Gamma \vdash N\ (l_1@l_2) : F}$$

Cs We have a redex of the form $\langle Q/y\rangle R$ typed by:

$$\dfrac{\Delta' \vdash Q : E \quad \Delta', y : E, \Delta \vdash R : F' \quad \Delta', \langle Q/y\rangle\Delta \sqsubseteq \Gamma\ \ \mathsf{wf}}{\Gamma \vdash^\star \langle Q/y\rangle R : F}$$

with either $F = F' \in \mathcal{S}$ or $F = \langle Q/y\rangle F'$.
In the latter case, $\Gamma \vdash F : s_F$ for some $s_F \in \mathcal{S}$. We also have $\Gamma\ \mathsf{wf}$.
Let us consider each rule:

C1 $\langle Q/y\rangle\lambda x^A.N \longrightarrow\ \lambda x^{\langle Q/y\rangle A}.\langle Q/y\rangle N$

$R = \lambda x^A.N$

By the Generation Lemma 1.(b), there is s_3 such that $C \longleftrightarrow^* s_3$ and:

$$\dfrac{\dfrac{\Delta', y : E, \Delta \vdash A : s_1 \qquad \Delta', y : E, \Delta, x : A \vdash B : s_2}{\Delta', y : E, \Delta \vdash \Pi x^A.B : C} \qquad \Delta', y : E, \Delta, x : A \vdash N : B}{\Delta', y : E, \Delta \vdash \lambda x^A.N : F'}$$

with $(s_1, s_2, s_3) \in \mathcal{R}$ and $F' \equiv \Pi x^A.B$. Hence, $F' \notin \mathcal{S}$, so $F = \langle Q/y\rangle F' \longleftrightarrow^* \langle Q/y\rangle\Pi x^A.B \longleftrightarrow^* \Pi x^{\langle Q/y\rangle A}.\langle Q/y\rangle B$. We have:

$$\dfrac{\Delta' \vdash Q : E \quad \Delta', y : E, \Delta \vdash A : s_1}{\Gamma \vdash \langle Q/y\rangle A : s_1}$$

Hence, $\Gamma, x : \langle Q/y\rangle A\ \mathsf{wf}$ and $\Delta', \langle Q/y\rangle\Delta, x : \langle Q/y\rangle A \sqsubseteq \Gamma, x : \langle Q/y\rangle A$, so:

$$\dfrac{\Delta' \vdash Q : E \quad \Delta', y : E, \Delta, x : A \vdash B : s_2}{\Gamma, x : \langle Q/y\rangle A \vdash \langle Q/y\rangle B : s_2}$$

so that $\Gamma \vdash \Pi x^{\langle Q/y\rangle A}.\langle Q/y\rangle B : s_3$ and

$$\dfrac{\dfrac{\dfrac{\Delta' \vdash Q : E \quad \Delta', y : E, \Delta, x : A \vdash N : B}{\Gamma, x : \langle Q/y\rangle A \vdash \langle Q/y\rangle N : \langle Q/y\rangle B}}{\Gamma \vdash \lambda x^{\langle Q/y\rangle A}.\langle Q/y\rangle N : \Pi x^{\langle Q/y\rangle A}.\langle Q/y\rangle B} \qquad F \longleftrightarrow^* \Pi x^{\langle Q/y\rangle A}.\langle Q/y\rangle B}{\Gamma \vdash \lambda x^{\langle Q/y\rangle A}.\langle Q/y\rangle N : F}$$

C2 $\langle Q/y\rangle(y\ l_1) \longrightarrow\ Q\ \langle Q/y\rangle l_1$
$R = y\ l_1$
By the Generation Lemma 1.(e), $\Delta', y : E, \Delta; E \vdash l_1 : F'$. Now notice that $y \notin FV(E)$, so $\langle Q/y\rangle E \longleftrightarrow^* E$ and $\Delta' \vdash E : s_E$. Also, $\Delta' \sqsubseteq \Gamma$, so

$$\dfrac{\dfrac{\Delta' \vdash Q : E}{\vdots}}{\Gamma \vdash Q : E} \qquad \dfrac{\dfrac{\Delta' \vdash Q : E \quad \Delta', y : E, \Delta; E \vdash l_1 : F'}{\Gamma; \langle Q/y\rangle E \vdash \langle Q/y\rangle l_1 : F} \qquad \dfrac{\Delta' \vdash E : s_E}{\vdots}}{\Gamma \vdash E : s_E}}{\Gamma; E \vdash \langle Q/y\rangle l_1 : F}$$

$$\Gamma \vdash Q\ \langle Q/y\rangle l_1 : F$$

C3 $\langle Q/y\rangle(x\ l_1) \longrightarrow\ x\ \langle Q/y\rangle l_1$
$R = x\ l_1$
By the Generation Lemma 1.(e), $\Delta', y : E, \Delta; A \vdash l_1 : F'$ with $(x : A) \in \Delta', \Delta$. Let B be the type of x in Γ. We have

$$\dfrac{\dfrac{\dfrac{\Delta' \vdash Q : E \quad \Delta', y : E, \Delta; A \vdash l_1 : F'}{\Gamma; \langle Q/y\rangle A \vdash \langle Q/y\rangle l_1 : F} \qquad \Gamma \vdash B : s_B}{\Gamma; B \vdash \langle Q/y\rangle l_1 : F}}{\Gamma \vdash x\ \langle Q/y\rangle l_1 : F}$$

Indeed, if $x \in \mathsf{Dom}(\Delta)$ then $B \longleftrightarrow^* \langle Q/y\rangle A$, otherwise $B \longleftrightarrow^* A$ with $y \notin FV(A)$, so in both cases $B \longleftrightarrow^* \langle Q/y\rangle A$. Besides, Γ wf so $\Gamma \vdash B : s_B$.

C4 $\langle Q/y\rangle(N\ l_1) \longrightarrow\ \langle Q/y\rangle N\ \langle Q/y\rangle l_1$
$R = N\ l_1$
By the Generation Lemma 1.(e),

$$\dfrac{\Delta', y : E, \Delta \vdash N : A \quad \Delta', y : E, \Delta; A \vdash l_1 : F'}{\Delta', y : E, \Delta \vdash^\star N\ l_1 : F'}$$

Also, we have

$$\dfrac{\Delta' \vdash Q : E \quad \Delta', y : E, \Delta \vdash A : s_A}{\Gamma \vdash \langle Q/y\rangle A : s_A}$$

Hence,

$$\dfrac{\dfrac{\Delta' \vdash Q : E \quad \Delta', y : E, \Delta \vdash N : A}{\Gamma \vdash \langle Q/y\rangle N : \langle Q/y\rangle A} \qquad \dfrac{\Delta' \vdash Q : E \quad \Delta', y : E, \Delta; A \vdash l_1 : F'}{\Gamma; \langle Q/y\rangle A \vdash \langle Q/y\rangle l_1 : F}}{\Gamma \vdash \langle Q/y\rangle N\ \langle Q/y\rangle l_1 : F}$$

C5 $\langle Q/y\rangle \Pi x^A.B \longrightarrow\ \Pi x^{\langle Q/y\rangle A}.\langle Q/y\rangle B$
$R = \Pi x^A.B$
By the Generation Lemma 1.(b), there exist s_3 such that $F' \longleftrightarrow^* s_3$ and:

$$\dfrac{\Delta', y : E, \Delta \vdash A : s_1 \qquad \Delta', y : E, \Delta, x : A \vdash B : s_2}{\Delta', y : E, \Delta \vdash \Pi x^A.B : F'}$$

with $(s_1, s_2, s_3) \in \mathcal{R}$.

$$\dfrac{\Delta' \vdash Q : E \quad \Delta', y : E, \Delta \vdash A : s_1}{\Gamma \vdash \langle Q/y\rangle A : s_1}$$

Hence, $\Gamma, x : \langle Q/y\rangle A$ wf and $\Delta', \langle Q/y\rangle \Delta, x : \langle Q/y\rangle A \sqsubseteq \Gamma, x : \langle Q/y\rangle A$, so:

$$\dfrac{\Delta' \vdash Q : E \quad \Delta', y : E, \Delta, x : A \vdash B : s_2}{\Gamma, x : \langle Q/y\rangle A \vdash \langle Q/y\rangle B : s_2}$$

so that $\Gamma \vdash \Pi x^{\langle Q/y\rangle A}.\langle Q/y\rangle B : s_3$. Now if $F' \in \mathcal{S}$, then $F = F' = s_3$ and we are done. Otherwise $F = \langle Q/y\rangle F' \longleftrightarrow^* \langle Q/y\rangle s_3 \longleftrightarrow^* s_3$, and we conclude using a conversion rule (because $\Gamma \vdash F : s_F$).

C6 $\langle Q/y\rangle s \longrightarrow s$
$R = s$
By the Generation Lemma 1.(a), we get $F' \longleftarrow\!\!\!\longrightarrow^* s'$ for some s' with $(s, s') \in \mathcal{A}$. Since Γ wf, we get $\Gamma \vdash s : s'$. If $F' \in \mathcal{S}$, then $F = F' = s'$ and we are done. Otherwise $F = \langle Q/y\rangle F' \longleftarrow\!\!\!\longrightarrow^* \langle Q/y\rangle s' \longleftarrow\!\!\!\longrightarrow^* s'$ and we conclude using a conversion rule (because $\Gamma \vdash F : s_F$).

Ds We have a redex of the form $\langle Q/y\rangle l_1$ typed by:

$$\frac{\Delta' \vdash Q : E \quad \Delta', y : E, \Delta; H' \vdash l_1 : F' \quad \Delta', \langle Q/y\rangle\Delta \sqsubseteq \Gamma \ \text{wf}}{\Gamma; H \vdash^\star \langle Q/y\rangle l_1 : F}$$

with $F = \langle Q/y\rangle F'$ and $H = \langle Q/y\rangle H'$. We also have Γ wf and $\Gamma \vdash H : s_H$ and $\Gamma \vdash F : s_F$.

Let us consider each rule:

D1 $\langle Q/y\rangle [] \longrightarrow []$
$l_1 = []$
By the Generation Lemma 2.(a), $H' \longleftarrow\!\!\!\longrightarrow^* F'$, so $H \longleftarrow\!\!\!\longrightarrow^* F$.

$$\frac{\dfrac{\Gamma \vdash H : s_H}{\Gamma; H \vdash [] : H} \qquad \Gamma \vdash F : s_F}{H \vdash [] : F}$$

D2 $\langle Q/y\rangle(N \cdot l_2) \longrightarrow (\langle Q/y\rangle N) \cdot (\langle Q/y\rangle l_2)$
$l_1 = N \cdot l_2$
By the Generation Lemma 2.(b), there are A, B such that $H' \longleftarrow\!\!\!\longrightarrow^* \Pi x^A.B$ and:

$$\frac{\Delta', y : E, \Delta \vdash \Pi x^A.B : s \quad \Delta', y : E, \Delta \vdash N : A \quad \Delta', y : E, \Delta; \langle N/x\rangle B \vdash l_2 : F'}{\Delta', y : E, \Delta; \Pi x^A.B \vdash^\star l_1 : F'}$$

From $\Delta', y : E, \Delta; \langle N/x\rangle B \vdash l_2 : F'$ we get

$$\Gamma; \langle Q/y\rangle\langle N/x\rangle B \vdash \langle Q/y\rangle l_2 : F$$

From $\Delta', y : E, \Delta \vdash N : A$ we get $\Gamma \vdash \langle Q/y\rangle N : \langle Q/y\rangle A$.
From $\Delta', y : E, \Delta \vdash \Pi x^A.B : s$ the Generation Lemma 1.(b) provides $\Delta', y : E, \Delta \vdash A : s_A$ and $\Delta', y : E, \Delta, x : A \vdash B : s_B$. Hence we get

$$\frac{\Delta', y : E, \Delta \vdash A : s_A}{\Gamma \vdash \langle Q/y\rangle A : s_A}$$

and thus $\Gamma, x : \langle Q/y\rangle A$ wf and then

$$\frac{\Delta', y : E, \Delta, x : A \vdash B : s_B}{\Gamma, x : \langle Q/y\rangle A \vdash \langle Q/y\rangle B : s_B}$$

From that we get both $\Gamma \vdash \Pi x^{\langle Q/y\rangle A}.\langle Q/y\rangle B : s$ and
$\Gamma \vdash \langle\langle Q/y\rangle N/x\rangle\langle Q/y\rangle B : s_B$.
Note that $\Pi x^{\langle Q/y\rangle A}.\langle Q/y\rangle B \longleftarrow\!\!\!\longrightarrow^* \langle Q/y\rangle \Pi x^A.B \longleftarrow\!\!\!\longrightarrow^* \langle Q/y\rangle H' = H$. We get

$$\frac{\dfrac{\Gamma \vdash \langle Q/y\rangle N : \langle Q/y\rangle A \qquad \dfrac{\Gamma; \langle Q/y\rangle\langle N/x\rangle B \vdash \langle Q/y\rangle l_2 : F}{\Gamma; \langle\langle Q/y\rangle N/x\rangle\langle Q/y\rangle B \vdash \langle Q/y\rangle l_2 : F}}{\Gamma; \Pi x^{\langle Q/y\rangle A}.\langle Q/y\rangle B \vdash (\langle Q/y\rangle N) \cdot (\langle Q/y\rangle l_2) : F}}{\Gamma; H \vdash (\langle Q/y\rangle N) \cdot (\langle Q/y\rangle l_2) : F}$$

D3 $\langle Q/y\rangle(l_2@l_3) \longrightarrow (\langle Q/y\rangle l_2)@(\langle Q/y\rangle l_3)$
$l_1 = l_2@l_3$
By the Generation Lemma 2.(d),

$$\frac{\Delta', y : E, \Delta; H' \vdash l_2 : A \quad \Delta', y : E, \Delta; A \vdash l_3 : F'}{\Delta', y : E, \Delta; H' \vdash^{\star} l_2@l_3 : F'}$$

Hence,

$$\frac{\Gamma; H \vdash \langle Q/y\rangle l_2 : \langle Q/y\rangle A \qquad \Gamma; \langle Q/y\rangle A \vdash \langle Q/y\rangle l_3 : F}{\Gamma; H \vdash (\langle Q/y\rangle l_2)@(\langle Q/y\rangle l_3) : F}$$

□