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# Singular solutions of conformal Hessian equation

Nikolai Nadirashvili\* Serge Vlăduț†

*Abstract.* We show that for any  $\varepsilon \in ]0, 1[$  there exists an analytic outside zero solution to a uniformly elliptic conformal Hessian equation in a ball  $B \subset \mathbb{R}^5$  which belongs to  $C^{1,\varepsilon}(B) \setminus C^{1,\varepsilon+}(B)$ .

AMS 2000 Classification: 35J60, 53C38

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## 1 Introduction

In this paper we study a class of fully nonlinear second-order elliptic equations of the form

$$F(D^2u, Du, u) = 0 \quad (1)$$

defined in a domain of  $\mathbb{R}^n$ . Here  $D^2u$  denotes the Hessian of the function  $u$ ,  $Du$  being its gradient. We assume that  $F$  is a Lipschitz function defined on a domain in the space  $\text{Sym}_2(\mathbb{R}^n) \times \mathbb{R}^n \times \mathbb{R}$ ,  $\text{Sym}_2(\mathbb{R}^n)$  being the space of  $n \times n$  symmetric matrices and that  $F$  satisfies the uniform ellipticity condition, i.e. there exists a constant  $C = C(F) \geq 1$  (called an *ellipticity constant*) such that

$$C^{-1}\|N\| \leq F(M + N) - F(M) \leq C\|N\|$$

for any non-negative definite symmetric matrix  $N$ ; if  $F \in C^1(\text{Sym}_2(\mathbb{R}^n))$  then this condition is equivalent to

$$\frac{1}{C'}|\xi|^2 \leq F_{u_{ij}}\xi_i\xi_j \leq C'|\xi|^2, \forall \xi \in \mathbb{R}^n.$$

Here,  $u_{ij}$  denotes the partial derivative  $\partial^2u/\partial x_i\partial x_j$ . A function  $u$  is called a *classical* solution of (1) if  $u \in C^2(\Omega)$  and  $u$  satisfies (1). Actually, any classical solution of (1) is a smooth ( $C^{\alpha+3}$ ) solution, provided that  $F$  is a smooth ( $C^\alpha$ ) function of its arguments.

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More precisely, we are interested in conformal Hessian equations (see, e.g. [9], pp. 5-6) i.e. those of the form

$$F[u] := f(\lambda(A^u)) = \psi(u, x) \quad (2)$$

$f$  being a Lipschitz function on  $\mathbb{R}^n$  invariant under permutations of the coordinates and

$$\lambda(A^u) = (\lambda_1, \dots, \lambda_n)$$

being the eigenvalues of the conformal Hessian in  $\mathbb{R}^n$ :

$$A^u := uD^2u - \frac{1}{2}|Du|^2I_n \quad (3)$$

where  $n \geq 3, u > 0$ .

In this case  $F$  is invariant under conformal mappings  $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ , i.e. transformations which preserve angles between curves. In contrast to the case  $n = 2$ , for  $n \geq 3$  any conformal transformation of  $\mathbb{R}^n$  is decomposed into a finitely many Möbius transformations, that is mappings of the form

$$Tx = y + \frac{kA(x-z)}{|x-z|^a},$$

with  $x, z \in \mathbb{R}^n, k \in \mathbb{R}, a \in \{0, 2\}$  and an orthogonal matrix  $A$ . In other words, each  $T$  is a composition of a translation, a homothety, a rotation and (may be) an inversion. If  $T$  is a conformal mapping and  $v(x) = J_T^{-1/n}u(Tx)$ , where  $J_T$  denotes the Jacobian determinant of  $T$  then  $F[v] = F[u]$ . Note that this class of equations is very important in geometry, see [4] and references therein.

We are interested in the Dirichlet problem

$$\begin{cases} F(D^2u, Du, u) = 0, u > 0 & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega, \end{cases} \quad (4)$$

where  $\Omega \subset \mathbb{R}^n$  is a bounded domain with a smooth boundary  $\partial\Omega$  and  $\varphi$  is a continuous function on  $\partial\Omega$ .

Consider the problem of existence and regularity of solutions to the Dirichlet problem (4) which has always a unique viscosity (weak) solution for fully nonlinear elliptic equations. The viscosity solutions satisfy the equation (1) in a weak sense, and the best known interior regularity ([1],[2],[8]) for them is  $C^{1+\varepsilon}$  for some  $\varepsilon > 0$ . For more details see [2], [3]. Recall that in [5] the authors constructed a homogeneous singular viscosity solution in 5 dimensions for Hessian equations of order  $1 + \delta$  for any  $\delta \in ]0, 1]$ , that is, of any order compatible with the mentioned interior regularity results. In fact we proved in [5] the following result.

**Theorem 1.1.**

The function

$$w_{5,\delta}(x) = P_5(x)/|x|^{1+\delta}, \quad \delta \in [0, 1[$$

is a viscosity solution to a uniformly elliptic Hessian equation  $F(D^2w) = 0$  with a smooth functional  $F$  in a unit ball  $B \subset \mathbb{R}^5$  for the isoparametric Cartan cubic form

$$P_5(x) = x_1^3 + \frac{3x_1}{2} (z_1^2 + z_2^2 - 2z_3^2 - 2x_2^2) + \frac{3\sqrt{3}}{2} (x_2z_1^2 - x_2z_2^2 + 2z_1z_2z_3)$$

with  $x = (x_1, x_2, z_1, z_2, z_3)$ .

which proves the optimality of the interior  $C^{1+\varepsilon}$ -regularity of viscosity solutions to fully nonlinear equations in 5 and more dimensions.

In the present paper we show that the same singularity result remains true for conformal Hessian equations.

**Theorem 1.2.**

Let  $\delta \in ]0, 1[$ . The function

$$u(x) := c + w_{5,\delta}(x) = c + \frac{P_5(x)}{|x|^{1+\delta}},$$

is a viscosity solution to a uniformly elliptic conformal Hessian equation (1) in a unit ball  $B \subset \mathbb{R}^5$  for a sufficiently large positive constant  $c$  ( $c = 240000$  is sufficient for  $\delta = \frac{1}{2}$ ).

Notice also that the result does *not* hold for  $\delta = 0$  and we do not know how to construct a non-classical  $C^{1,1}$ -solution to a uniformly elliptic conformal Hessian equation.

The rest of the paper is organized as follows: in Section 2 we recall some necessary preliminary results and we prove our main results in Section 3; to simplify the notation we suppose that  $\delta = \frac{1}{2}$  in Section 3; for any  $\delta$  the proof is along the same line, but more cumbersome. The proof in Section 3 uses MAPLE to verify some algebraic identities but is completely rigorous (and is human-controlled for  $\delta = \frac{1}{2}$ ).

## 2 Preliminary results

*Notation:* for a real symmetric matrix  $A$  we denote by  $|A|$  the maximum of the absolute value of its eigenvalues.

Let  $u$  be a strictly positive function on  $B_1$ . Define the map

$$\Lambda : B_1 \longrightarrow \lambda(S) \in \mathbb{R}^n .$$

$\lambda(S) = \{\lambda_1 \geq \dots \geq \lambda_n\} \in \mathbb{R}^n$  being the (ordered) set of eigenvalues of the conformal Hessian

$$A^u := uD^2u - \frac{1}{2}|Du|^2I_n.$$

The following ellipticity criterion can be proved similarly to Lemma 2.1 of [6].

**Lemma 2.1.** *Suppose that the family*

$$\{A^u(a) - O^{-1} \cdot A^u(b) \cdot O : a, b \in B_1, O \in SO(n)\} \setminus \{0\}$$

*is uniformly hyperbolic, i.e. if  $\{\mu_1(a, b, O) \geq \dots \geq \mu_n(a, b, O)\}$  is the ordered spectrum of  $A^u(a) - O^{-1} \cdot A^u(b) \cdot O \neq 0$  then*

$$\forall a, b \in B_1, \forall O \in SO(n), \quad C^{-1} \leq -\frac{\mu_1(a, b, O)}{\mu_n(a, b, O)} \leq C$$

*for some constant  $C > 1$ . Then  $u$  is a viscosity solution in  $B_1$  of a uniformly elliptic conformal Hessian equation (1).*

We recall then some properties of the function  $w := w_{5,\delta}(x) = \frac{P_5(x)}{|x|^{1+\delta}}$ , and its Hessian  $D^2w$  proved in [5].

**Lemma 2.2.**

*There exists a 3-dimensional Lie subgroup  $G_P$  of  $SO(5)$  such that  $P$  is invariant under its natural action and the orbit  $G_P \mathbb{S}_1^4$  of the circle*

$$\mathbb{S}_1^4 = \{(\cos(\chi), 0, \sin(\chi), 0, 0) : \chi \in \mathbb{R}\} \subset \mathbb{S}_1^4$$

*under this action is the whole  $\mathbb{S}_1^4$ .*

**Lemma 2.3.**

(i) *Let  $x \in \mathbb{S}_1^4$ , and let  $x \in G_P(p, 0, r, 0, 0)$  with  $p^2 + r^2 = 1$ . Then*

$$\text{Spec}(D^2w_{5,\delta}(x)) = \{\mu_{1,\delta}, \mu_{2,\delta}, \mu_{3,\delta}, \mu_{4,\delta}, \mu_{5,\delta}\}$$

*for*

$$\begin{aligned} \mu_{1,\delta} &= \frac{p(p^2\delta + 6 - 3\delta)}{2}, \\ \mu_{2,\delta} &= \frac{p(p^2\delta - 3 - 3\delta) + 3\sqrt{12 - 3p^2}}{2}, \\ \mu_{3,\delta} &= \frac{p(p^2\delta - 3 - 3\delta) - 3\sqrt{12 - 3p^2}}{2}, \\ \mu_{4,\delta} &= -\frac{p\delta(6 - \delta)(3 - p^2) + \sqrt{D(p, \delta)}}{4}, \\ \mu_{5,\delta} &= -\frac{p\delta(6 - \delta)(3 - p^2) - \sqrt{D(p, \delta)}}{4}, \end{aligned}$$

*and*

$$D(p, \delta) := (6 - \delta)(4 - \delta)(2 - \delta)\delta(p^2 - 3)^2p^2 + 144(\delta - 2)^2 > 0.$$

(ii) Let  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_5$  be the ordered eigenvalues of  $D^2w_{5,\delta}(x)$ . Then

$$\begin{aligned}\lambda_1 &= \mu_{2,\delta}, & \lambda_5 &= \mu_{3,\delta}, \\ \lambda_2 &= \begin{cases} \mu_{4,\delta} & \text{for } p \in [-1, p_0(\delta)], \\ \mu_{1,\delta} & \text{for } p \in [p_0(\delta), 1], \end{cases} \\ \lambda_3 &= \begin{cases} \mu_{5,\delta} & \text{for } p \in [-1, -p_0(\delta)], \\ \mu_{1,\delta} & \text{for } p \in [-p_0(\delta), p_0(\delta)], \\ \mu_{4,\delta} & \text{for } p \in [p_0(\delta), 1], \end{cases} \\ \lambda_4 &= \begin{cases} \mu_{1,\delta} & \text{for } p \in [-1, -p_0(\delta)], \\ \mu_{5,\delta} & \text{for } p \in [-p_0(\delta), 1], \end{cases}\end{aligned}$$

where

$$p_0(\delta) := \frac{3^{1/4}\sqrt{1-\delta}}{(3+2\delta-\delta^2)^{1/4}} = \frac{3^{1/4}\sqrt{\varepsilon}}{(4-\varepsilon^2)^{1/4}} \in ]0, 1].$$

Note the oddness property of the spectrum:

$$\lambda_{1,\delta}(-p) = -\lambda_{5,\delta}(p), \quad \lambda_{2,\delta}(-p) = -\lambda_{4,\delta}(p), \quad \lambda_{3,\delta}(-p) = -\lambda_{3,\delta}(p).$$

**Proposition 2.1.**

Let  $N_\delta(x) = D^2w_\delta(x)$ ,  $0 \leq \delta < 1$ . Suppose that  $a \neq b \in B_1 \setminus \{0\}$  and let  $O \in O(5)$  be an orthogonal matrix s.t.

$$N_\delta(a, b, O) := N_\delta(a) - {}^tO \cdot N_\delta(b) \cdot O \neq 0.$$

Denote  $\Lambda_1 \geq \Lambda_2 \geq \dots \geq \Lambda_5$  the eigenvalues of the matrix  $N_\delta(a, b, O)$ . Then

$$\frac{1}{C} \leq -\frac{\Lambda_1}{\Lambda_5} \leq C$$

for  $C := C(\delta) := \frac{1000(\delta+1)(3-\delta)}{3(1-\delta)^2}$ ; for  $k \in [\frac{1}{2}, 1]$  one can choose  $C = 1000$ .

**Corollary 2.1.**

$$\Lambda_1 \geq \frac{|N_\delta(a, b, O)|}{C(\delta)}, \quad |\Lambda_5| \geq \frac{|N_\delta(a, b, O)|}{C(\delta)}.$$

We need also the following classical Weyl's result:

**Lemma 2.4.**

Let  $A, B$  be two real symmetric matrices with the eigenvalues  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$  and  $\lambda'_1 \geq \lambda'_2 \geq \dots \geq \lambda'_n$  respectively. Then for the eigenvalues  $\Lambda_1 \geq \Lambda_2 \geq \dots \geq \Lambda_n$  of the matrix  $A - B$  we have

$$\Lambda_1 \geq \max_{i=1, \dots, n} (\lambda_i - \lambda'_i), \quad \Lambda_n \leq \min_{i=1, \dots, n} (\lambda_i - \lambda'_i).$$

### 3 Proofs

Let  $n = 5, u(x) = c + w_{5,\delta}(x)$ . We begin with  $\delta = 0$  and show that the result is false in this case. Indeed let  $a = (1, 0, 0, 0, 0), b = (\frac{1}{2}, 0, 0, 0, 0), O = I_5$ . Then

$$w(a) = 1, w(b) = \frac{1}{2}, |Du(a)| = |Dw(a)| = 9, |Du(b)|^2 = |Dw(b)|^2 = \frac{9}{4},$$

$$D^2u(a) = D^2w(a) = D^2u(b) = D^2w(b),$$

and

$$A^u(a) - A^u(b) = \frac{1}{2}D^2w(a) - \frac{27}{4}I_5$$

which is negative since the spectrum of  $D^2w(a)$  is  $(2, 2, 2, -7, -7)$ . The reason is clearly that  $D^2w(a)$  for  $\delta = 0$  is homogeneous order 0 and depends only on the direction vector  $a/|a|$ .

Suppose now that  $\delta \in ]0, 1[$ . As we mentioned before, we set  $\delta = \frac{1}{2}$ ; in this case  $c = 240000$ . First we spell out Lemma 2.3 for  $\delta = \frac{1}{2}$ .

**Lemma 3.1.**

(i) Let  $x \in \mathbb{S}_1^4$ , and let  $x \in G_P(p, 0, r, 0, 0)$  with  $p^2 + r^2 = 1$ . Then

$$\text{Spec}(D^2u(x)) = \text{Spec}(D^2w(x)) = \{\mu_1, \mu_2, \mu_3, \mu_4, \mu_5\}$$

for

$$\mu_1 = \frac{3p(p^2 + 1)}{4},$$

$$\mu_2 = \frac{3p(p^2 - 5) + 6\sqrt{12 - 3p^2}}{4},$$

$$\mu_3 = \frac{3p(p^2 - 5) - 6\sqrt{12 - 3p^2}}{4},$$

$$\mu_4 = \frac{27p(p^2 - 3) + 3\sqrt{105p^6 - 630p^4 + 945p^2 + 64}}{16},$$

$$\mu_5 = \frac{27p(p^2 - 3) - 3\sqrt{105p^6 - 630p^4 + 945p^2 + 64}}{16}.$$

(ii) Let  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_5$  be the ordered eigenvalues of  $\text{Spec}(D^2u(x)) = \text{Spec}(D^2w(x))$ . Then

$$\lambda_1 = \mu_2, \quad \lambda_5 = \mu_3,$$

$$\lambda_2 = \begin{cases} \mu_4 & \text{for } p \in [-1, p_0], \\ \mu_1 & \text{for } p \in [p_0, 1], \end{cases}$$

$$\lambda_3 = \begin{cases} \mu_5 & \text{for } p \in [-1, -p_0], \\ \mu_1 & \text{for } p \in [-p_0, p_0], \\ \mu_4 & \text{for } p \in [p_0, 1], \end{cases}$$

$$\lambda_4 = \begin{cases} \mu_1 & \text{for } p \in [-1, -p_0], \\ \mu_5 & \text{for } p \in [-p_0, 1], \end{cases}$$

where

$$p_0 = 5^{-1/4} \simeq 0.6687403050.$$

We will need also the derivatives of the eigenvalues.

**Lemma 3.2.** Let  $d_i(p) := \frac{d(\mu_i)}{dp}$ . Then

$$\begin{aligned} d_1(p) &= \frac{3(3p^2 + 1)}{4}, \\ d_2(p) &= -\frac{3(5 - 3p^2)}{4} + \frac{9p}{2\sqrt{12 - 3p^2}}, \\ d_3(p) &= -\frac{3(5 - 3p^2)}{4} - \frac{9p}{2\sqrt{12 - 3p^2}}, \\ d_4(p) &= \frac{81(1 - p^2)}{16} \left( \frac{35p(3 - p^2)}{3\sqrt{105p^6 - 630p^4 + 945p^2 + 64}} - 1 \right), \\ d_5(p) &= -\frac{81(1 - p^2)}{16} \left( \frac{35p(3 - p^2)}{3\sqrt{105p^6 - 630p^4 + 945p^2 + 64}} + 1 \right). \end{aligned}$$

Simple calculus gives

**Corollary 3.1.**

$$D := \max\{|d_i(p)| : p \in [-1, 1], i = 1, \dots, 5\} < 10.$$

Below we denote  $D_i(p) := \frac{d(\lambda_i)}{dp}$ ; the relation of  $D_i(p)$  and  $d_i(p)$  is clear from Lemma 3.1 (ii); for example,  $D_1(p) = d_2(p)$ ,  $D_5(p) = d_3(p)$ .

The proof of Theorem 1.2 is based on the following lemmas. Let

$$a, b \in B_1 \setminus \{0\}, |a| = s \leq 1, |b| = t \leq 1, O \in O(5),$$

$$a' := \frac{a}{s} \in G_P(p, 0, r, 0, 0), b' := \frac{b}{t} \in G_P(q, 0, r', 0, 0).$$

Below we denote

$$K := K(p, q, s, t) = |s - t| + |p - q|,$$

$$M_1 := M_1(a, b, O) := D^2u(a) - O^{-1}D^2u(b) \cdot O,$$

$$M_2 := M_2(a, b, O) := w(a)D^2u(a) - O^{-1}w(b)D^2u(b) \cdot O.$$

**Lemma 3.3.**

$$\left| |Du(a)|^2 - |Du(b)|^2 \right| \leq 16K.$$



*Proof.* First,  $|Du(a)|^2 = |Dw(a)|^2$ ,  $|Du(b)|^2 = |Dw(b)|^2$ . Since  $P = P_5(x)$  can be represented as the generic traceless norm in the Jordan algebra  $\text{Sym}_3(\mathbb{R})$  it verifies the eiconal equation  $|DP|^2 = |x|^4$ , see e.g. [7]. Therefore, an easy calculation gives

$$\begin{aligned} |Du(a)|^2 &= \frac{9s(16 - 3p^2(p^2 - 3)^2)}{32}, \quad |Du(b)|^2 = \frac{9t(16 - 3q^2(q^2 - 3)^2)}{32}, \\ \left| |Du(a)|^2 - |Du(b)|^2 \right| &\leq \left| \frac{9s(16 - 3p^2(p^2 - 3)^2)}{32} - \frac{9t(16 - 3p^2(p^2 - 3)^2)}{32} \right| + \\ &\quad + \left| \frac{9t(16 - 3p^2(p^2 - 3)^2)}{32} - \frac{9t(16 - 3q^2(q^2 - 3)^2)}{32} \right| = \\ \left| \frac{9(s-t)(16 - 3p^2(p^2 - 3)^2)}{32} \right| &+ \left| \frac{27t(p-q)(p+q)((q^2 - 3)^2 - (p^2 - 3)^2)}{32} \right| \leq \\ &\quad \left| \frac{9(s-t)}{2} \right| + \left| \frac{243(p-q)}{16} \right| \leq 16K. \end{aligned}$$

**Lemma 3.4.** *Let  $M := |M_1| = |D^2u(a) - O^{-1} \cdot D^2u(b) \cdot O|$ . Then*

$$M \geq \frac{K}{8}.$$

*Proof.* If one replaces  $a$  by  $a' = a/s$  and  $b$  by  $b'' = b/s$  the quantity  $M$  gets bigger and  $K$  gets smaller. Therefore we can suppose that  $|a| = s = 1$ . Then we have

$$D^2u(a) - O^{-1} \cdot D^2u(b) \cdot O = D^2u(a) - \frac{O^{-1} \cdot D^2u(b') \cdot O}{\sqrt{t}}.$$

By Lemma 2.4 we have

$$\begin{aligned} M &\geq \max \left\{ \lambda_i(p) - \frac{\lambda_i(q)}{\sqrt{t}} : i = 1, \dots, 5 \right\}, \\ M &\geq \left| \min \left\{ \lambda_i(p) - \frac{\lambda_i(q)}{\sqrt{t}} : i = 1, \dots, 5 \right\} \right|. \end{aligned}$$

Suppose first  $p \geq q$ . If  $q \geq -\frac{24}{25} = -0.96$  then

$$\forall p' \in [q, p], \quad D_1(p') < -1/4 = -0.25, \quad \lambda_1(p) > \frac{3}{2}$$

(by a simple calculation using the explicit formulas for  $D_1, \lambda_1$ ). Therefore

$$\lambda_1(p) - \frac{\lambda_1(q)}{\sqrt{t}} = \lambda_1(p) - \lambda_1(q) + \lambda_1(q) - \frac{\lambda_1(q)}{\sqrt{t}} \leq -\frac{p-q}{4} - \frac{3}{2} \left( \frac{1}{\sqrt{t}} - 1 \right) < -\frac{K}{4}.$$

If  $q < -0.96$  but  $p \geq -\frac{23}{25} = -0.92$  then

$$\begin{aligned} \lambda_1(p) - \frac{\lambda_1(q)}{\sqrt{t}} &= \lambda_1(p) - \lambda_1(q) + \lambda_1(q) - \frac{\lambda_1(q)}{\sqrt{t}} \leq \lambda_1(p) - \lambda_1\left(\frac{24}{25}\right) + \lambda_1(q) - \frac{\lambda_1(q)}{\sqrt{t}} \\ &= -\frac{p+0.96}{4} - \frac{3}{2}\left(\frac{1}{\sqrt{t}} - 1\right) < -\frac{p-q}{8} - \frac{3}{2}\left(\frac{1}{\sqrt{t}} - 1\right) < -\frac{K}{8}. \end{aligned}$$

Suppose then that  $q < -0.96$ ,  $p < -0.92$ . In this case we have

$$\forall p' \in [q, p], \quad d_2(p') > \frac{5}{2}, \quad \lambda_2(p') < -\frac{3}{2}$$

and thus

$$\lambda_2(p) - \frac{\lambda_2(q)}{\sqrt{t}} = \lambda_2(p) - \lambda_2(q) + \lambda_2(q) - \frac{\lambda_2(q)}{\sqrt{t}} \geq \frac{5(p-q)}{2} + \frac{3}{2}\left(\frac{1}{\sqrt{t}} - 1\right) \geq \frac{3K}{4}$$

which finishes the proof for  $p \geq q$ . The case  $q \geq p$  is treated similarly (replace  $\lambda_1$  by  $\lambda_5$  and  $\lambda_2$  by  $\lambda_4$ ).

**Lemma 3.5.**

$$|M_2| = |w(a)D^2u(a) - O^{-1}w(b)D^2u(b) \cdot O| \leq 10K$$

*Proof.* Indeed, let  $a' := a/s$ ,  $b' := b/s$  then by homogeneity

$$\begin{aligned} |w(a)D^2w(a) - O^{-1}w(b)D^2w(b) \cdot O| &= |sD^2w(a') - O^{-1} \cdot tD^2w(b') \cdot O| \leq \\ &\leq s|D^2w(a') - O^{-1} \cdot D^2w(b') \cdot O| + |s-t| \cdot |O^{-1} \cdot D^2w(b')| \leq \\ &\leq \max_{p,i}\{|D_i(p)|\}|p-q| + 7|s-t| = \max_{p,i}\{|d_i(p)|\}|p-q| + 7|s-t| \leq 10K. \end{aligned}$$

*Remark 3.1.* These results remain true for any  $\delta \in ]0, 1[$  if one replaces the respective constants 16,  $1/8$  and 10 in Lemmas 3.3, 3.4 and 3.5 by appropriate positive constants depending on  $\delta$ . On the contrary, Lemma 3.4 is false for  $\delta = 0$ .

We can now prove the uniform hyperbolicity of  $M(a, b, O)$  and thus the theorem. In fact we show that one can take  $C = 6007$  in Lemma 3.1.

Indeed,

$$|M(a, b, O)| = |A^u(a) - O^{-1} \cdot A^u(b) \cdot O| = |cM_1 + M_2 - (|Du(a)|^2 - |Du(b)|^2) I_5|.$$

Therefore,

$$\begin{aligned} |\Lambda_5| &\geq c|\Lambda_5(M_1)| - 10K - 16K \geq \frac{c|M_1|}{1000} - 26K \geq 240|M_1| - 26K \geq 4K, \\ |\Lambda_1| &\geq c\Lambda_1(M_1) - 10K - 16K \geq \frac{c|M_1|}{1000} - 26K \geq 240|M_1| - 26K \geq 4K, \end{aligned}$$

$$|M(a, b, O)| \leq c|M_1| + |M_2| + \left| |Du(a)|^2 - |Du(b)|^2 \right| \leq c|M_1| + 26K.$$

Thus

$$\frac{1}{C} < \frac{4}{240026} \leq \frac{240|M_1| - 26K}{c|M_1| + 26K} \leq \frac{|\Lambda_5|}{|\Lambda_1|} \leq \frac{c|M_1| + 26K}{240|M_1| - 26K} \leq \frac{240026}{4} < C$$

which finishes the proof. Notice that we can take  $C = 1000 + \varepsilon$  for  $\delta \leq \frac{1}{2}$  if  $c$  is sufficiently large; in the case  $\frac{1}{2} < \delta < 1$  for sufficiently large  $c$  one gets  $C = C(\delta) + \varepsilon = \frac{1000(\delta+1)(3-\delta)}{3(1-\delta)^2} + \varepsilon$ .

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