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A guaranteed equilibrated error estimator for the $\mathbf{A} - \varphi$ and $\mathbf{T} - \Omega$ magnetodynamic harmonic formulations of the Maxwell system

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Abstract

In this paper, a guaranteed equilibrated error estimator is proposed for the harmonic magnetodynamic formulation of the Maxwell's system. This system is recast in two classical potential formulations, which are solved by a Finite Element method. The equilibrated estimator is built starting from these dual problems and is consequently available to estimate the error of both numerical resolutions. The estimator reliability and efficiency without generic constants are established. Then, numerical tests are performed, allowing to illustrate the obtained theoretical results.

Key Words: Maxwell equations, potential formulation, *a posteriori* estimators, finite element method.

AMS (MOS) subject classification 35Q61; 65N30; 65N15; 65N50.

1 Introduction

Nowadays, *a posteriori* error estimation techniques have become an indispensable tool for obtaining reliable results of any partial differential equation numerical resolution. There exists a vast amount of literature on locally defined *a posteriori* error estimators, and we refer to the monographs [6, 8, 25, 28] for a good overview on this topic in the simplest context of continuous approximations

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in finite element methods.

During the last decade, an important issue has been to derive some estimators which give rise to an upper bound where the constant is one up to higher order terms, leading to some so-called "guaranteed" *a posteriori* estimators. The first ideas on such estimators appeared nearly thirty years ago [5, 9, 24], and were based on the Prager-Synge identity [26]. Nevertheless, this research field remains always very active because of lots of challenges to overcome, depending of the approximation method as well as on the models under consideration. Among the most recent contributions on various stationary models, we can quote e.g. [3, 7, 13, 16, 29] for conforming methods or [2, 4, 12, 15, 20, 21, 22] for non-conforming ones.

In this paper, we are interested in a guaranteed and explicitly computable *a posteriori* error estimator based on an equilibration technique for the harmonic magnetodynamic formulations of the Maxwell system. Basically the equilibration is realized through the non-verification property of the constitutive laws equations. Some residual *a posteriori* error estimators have already been established for the $\mathbf{A} - \varphi$ magnetoharmonic formulation [18] as well as for the $\mathbf{T} - \Omega$ one [19]. Now, we aim to derive an equilibrated *a posteriori* estimator for both of these dual problems, in the same philosophy of [27], where an equilibrated estimator is proposed in the magnetostatic framework. In other words, the estimator will be built starting from both formulations and will be available to estimate the sum of the errors of both resolutions. The originality of this work resides in the fact that, neglecting some higher order terms, the equivalence between the error and the estimator is proved, without the interference of unknown constants. So, we not only derive a reliable upper bound for the error, but also a lower one, where the constant is one up to higher order terms. Moreover, a local in space efficiency without unknown constants is proved, this result being useful for developing some adaptive mesh refinement algorithms.

Let us recall the model of interest. For a given time $T > 0$ and an open connected bounded polyhedral domain $D \subset \mathbb{R}^3$ with a lipschitz connected boundary Γ , we consider the usual Maxwell's system in $D \times (0, T)$ given by:

$$\operatorname{curl} \mathbf{E} = -\partial_t \mathbf{B}, \quad (1)$$

$$\operatorname{curl} \mathbf{H} = \partial_t \mathbf{D} + \mathbf{J}, \quad (2)$$

with initial and boundary conditions to be specified. Here \mathbf{E} stands for the electrical field, \mathbf{B} for the magnetic flux density, \mathbf{H} for the magnetic field, \mathbf{D} for the displacement flux density and \mathbf{J} for the current flux density. We are interested in the "low-frequency approximation" [11], which consists in neglecting the temporal variation of the displacement flux density with respect to the current density, so that the propagation of electromagnetic waves is not taken into account. Formally the Maxwell-Ampère equation (2) becomes:

$$\operatorname{curl} \mathbf{H} = \mathbf{J}. \quad (3)$$

The current flux density can be decomposed in $\mathbf{J} = \mathbf{J}_s + \mathbf{J}_e$, where \mathbf{J}_s is a known divergence free distribution current density and \mathbf{J}_e represents the unknown eddy current, which from (3) also clearly satisfies $\operatorname{div} \mathbf{J}_e = 0$. Moreover, the fields are linked by the material constitutive laws:

$$\mathbf{B} = \mu \mathbf{H}, \quad (4)$$

$$\mathbf{J}_e = \sigma \mathbf{E}, \quad (5)$$

where μ stands for the magnetic permeability and σ for the electrical conductivity of the material. Let us note that from (1), we have $\operatorname{div} \mathbf{B} = 0$ provided that the initial condition on \mathbf{B} is divergence free.

In the following, $D_c \subset D$ denotes an open and simply connected domain with a lipschitz boundary $\Gamma_c = \partial D_c$ such that $\Gamma_c \cap \Gamma = \emptyset$ (an example of such a configuration is displayed in Figure 1). The electrical conductivity σ is different to zero in D_c where, consequently, eddy currents can be created, while it is identically equal to zero in the domain $D_e = D \setminus \overline{D_c}$.

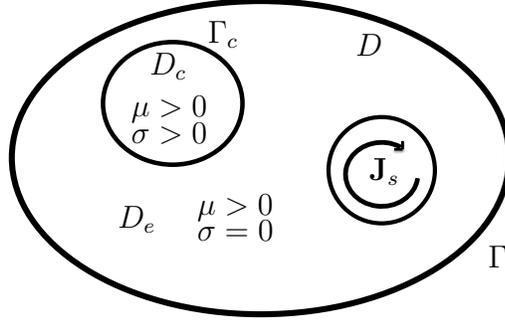


Figure 1: An example of domain configuration, where $\operatorname{supp} \mathbf{J}_s \cap D_c = \emptyset$.

Boundary conditions associated with the system (1) and (3) are given by $\mathbf{B} \cdot \mathbf{n} = 0$ on Γ and $\mathbf{J}_e \cdot \mathbf{n} = 0$ on Γ_c , where \mathbf{n} denotes the unit outward normal to D and D_c respectively.

Let us now explain how to derive the dual potential magnetodynamic formulations. They consist in recasting the Maxwell's system in the quasistatic approximation, that is the system (1) and (3), with the use of some potentials: \mathbf{A} and φ for the $\mathbf{A} - \varphi$ formulation and \mathbf{T} and Ω for the $\mathbf{T} - \Omega$ formulation.

From the properties of the magnetic flux density \mathbf{B} , a magnetic vector potential \mathbf{A} can be introduced such that

$$\mathbf{B} = \operatorname{curl} \mathbf{A} \text{ in } D. \quad (6)$$

The boundary condition $\mathbf{A} \times \mathbf{n} = 0$ on Γ allows to guarantee $\mathbf{B} \cdot \mathbf{n} = 0$ on Γ . To ensure the uniqueness of \mathbf{A} , it is necessary to impose a gauge condition. The most popular one is $\operatorname{div} \mathbf{A} = 0$ (so-called the Coulomb gauge). Moreover, from equations (1) and (6), an electrical scalar potential φ can be introduced in D_c so that the electrical field takes the form:

$$\mathbf{E} = -\partial_t \mathbf{A} - \nabla \varphi \text{ in } D_c. \quad (7)$$

Like the vector potential, it must be gauged and the averaged value of the potential φ in D_c is taken equal to zero to obtain uniqueness of the solution. From (4), (5), (6) and (7), equation (3) leads to the $\mathbf{A} - \varphi$ formulation:

$$\operatorname{curl} (\mu^{-1} \operatorname{curl} \mathbf{A}) + \sigma (\partial_t \mathbf{A} + \nabla \tilde{\varphi}) = \mathbf{J}_s \text{ in } D. \quad (8)$$

Let us point out that φ does not make sense in D_e , so that $\tilde{\varphi}$ denotes one fixed extension of φ in the whole domain D . This extension does not impact the problem since $\sigma \equiv 0$ in D_e . The classical

\mathbf{A} formulation used in the magnetostatic case is easily recovered remarking that in D_e , where σ is equal to zero, the second term vanishes. Recalling that the source \mathbf{J}_s is divergence free, and applying the divergence operator to (8), we get the divergence free condition on the eddy current:

$$\operatorname{div}(\sigma(\partial_t \mathbf{A} + \nabla \varphi)) = 0 \text{ in } D_c. \quad (9)$$

Once again, since $\operatorname{div} \mathbf{J}_s = 0$ in D , there exists a source term \mathbf{H}_s such that

$$\operatorname{curl} \mathbf{H}_s = \mathbf{J}_s \text{ in } D.$$

Moreover, since $\operatorname{div} \mathbf{J}_e = 0$ in the simply connected domain D_c , we can define a vector potential \mathbf{T} such that

$$\operatorname{curl} \mathbf{T} = \mathbf{J}_e \text{ in } D_c. \quad (10)$$

Similarly to the $\mathbf{A} - \varphi$ formulation, to ensure the uniqueness of \mathbf{T} we impose the $\operatorname{div} \mathbf{T} = 0$ in D_c and the boundary condition $\mathbf{T} \times \mathbf{n} = 0$ on Γ_c allows to guarantee $\mathbf{J}_e \cdot \mathbf{n} = 0$ on Γ_c . Hereafter, thanks to equation (3), we can introduce a scalar potential Ω and obtain

$$\mathbf{H} = \begin{cases} \mathbf{H}_s - \nabla \Omega & \text{in } D_e, \\ \mathbf{H}_s + \mathbf{T} - \nabla \Omega & \text{in } D_c. \end{cases} \quad (11)$$

The averaged value of the potential Ω in D is taken equal to zero in order to get its uniqueness. From equations (10) and (11) and from the material laws (4) and (5), (1) becomes:

$$\operatorname{curl}(\sigma^{-1} \operatorname{curl} \mathbf{T}) + \partial_t(\mu(\mathbf{T} - \nabla \Omega)) = -\partial_t(\mu \mathbf{H}_s) \text{ in } D_c. \quad (12)$$

Taking the divergence of (12) in D_c and recalling that $\operatorname{div} \mathbf{B} = 0$ in D , we get the second equation of the $\mathbf{T} - \Omega$ formulation:

$$\begin{cases} \operatorname{div}(\mu \nabla \Omega) = \operatorname{div}(\mu \mathbf{H}_s) & \text{in } D_e, \\ \operatorname{div}(\mu(\mathbf{T} - \nabla \Omega)) = -\operatorname{div}(\mu \mathbf{H}_s) & \text{in } D_c. \end{cases} \quad (13)$$

The two potential formulations (8)-(9) and (12)-(13) constitute a key ingredient to establish the proposed equilibrated error estimator for the numerical resolution of the harmonic formulation of the quasistatic Maxwell's system (1) and (3). In section 2 the harmonic $\mathbf{A} - \varphi$ and $\mathbf{T} - \Omega$ formulations and their weak formulations (continuous and discrete ones) are recalled. In section 3 a guaranteed equilibrated error estimator for both classical formulations in term of potentials is developed. Finally, Section 4 is devoted to some numerical results to illustrate the equivalence between the error and the estimator.

2 Formulations of the continuous and discretized problems

2.1 Continuous weak formulations

In the following, we suppose that $\mu \in L^\infty(D)$ and that there exists $\mu_0 \in \mathbb{R}_+^*$ such that $\mu > \mu_0$ in D . We also assume that $\sigma \in L^\infty(D)$, $\sigma|_{D_e} \equiv 0$, and that there exists $\sigma_0 \in \mathbb{R}_+^*$ such that $\sigma > \sigma_0$ in D_c . At last, we recall the Gauge conditions. Like mentioned in Section 1, we choose the Coulomb one for the vector potentials \mathbf{A} and \mathbf{T} , and we ask for the averaged value to be equal to zero for

the scalar potentials φ and Ω .

On a given domain \mathcal{D} , the $L^2(\mathcal{D})$ norm is denoted by $\|\cdot\|_{\mathcal{D}}$, and the corresponding $L^2(\mathcal{D})$ inner product by $(\cdot, \cdot)_{\mathcal{D}}$. The usual norm and semi-norm on $H^1(\mathcal{D})$ are respectively denoted by $\|\cdot\|_{1,\mathcal{D}}$ and $|\cdot|_{1,\mathcal{D}}$. In the case $\mathcal{D} = D$, the index D is dropped. Recall that $H_0^1(\mathcal{D})$ is the subspace of $H^1(\mathcal{D})$ with vanishing trace on $\partial\mathcal{D}$. Moreover, let us introduce the following spaces:

$$\begin{aligned} X(\mathcal{D}) &= H_0(\text{curl}, \mathcal{D}) = \left\{ \mathbf{F} \in L^2(\mathcal{D})^3; \text{curl } \mathbf{F} \in L^2(\mathcal{D})^3 \text{ and } \mathbf{F} \times \mathbf{n} = \mathbf{0} \text{ on } \partial\mathcal{D} \right\}, \\ \tilde{X}(\mathcal{D}) &= \left\{ \mathbf{F} \in X(\mathcal{D}); (\mathbf{F}, \nabla \xi)_{\mathcal{D}} = 0 \forall \xi \in H_0^1(\mathcal{D}) \right\}, \\ \widetilde{H}^1(\mathcal{D}) &= \left\{ f \in H^1(\mathcal{D}); \int_{\mathcal{D}} f \, d\mathbf{x} = 0 \right\}, \end{aligned}$$

where $H_0(\text{curl}, \mathcal{D})$ is equipped with its usual norm:

$$\|\mathbf{F}\|_{H(\text{curl}, \mathcal{D})}^2 = \|\mathbf{F}\|_{\mathcal{D}}^2 + \|\text{curl } \mathbf{F}\|_{\mathcal{D}}^2.$$

The harmonic formulation of (8)-(9) is obtained taking $\mathbf{A}(t, \mathbf{x}) = \overline{\mathbf{A}}(\mathbf{x}) e^{j\omega t}$ and $\varphi(t, \mathbf{x}) = \overline{\varphi}(\mathbf{x}) e^{j\omega t}$, where $j^2 = -1$ denotes the unit imaginary number and ω is a given real number corresponding to the frequency. Writing from now \mathbf{A} and φ instead of $\overline{\mathbf{A}}$ and $\overline{\varphi}$, the harmonic $\mathbf{A} - \varphi$ formulation reads:

$$\text{curl } (\mu^{-1} \text{curl } \mathbf{A}) + \sigma (j\omega \mathbf{A} + \nabla \tilde{\varphi}) = \mathbf{J}_s \text{ in } D,$$

$$\text{div } (\sigma (j\omega \mathbf{A} + \nabla \varphi)) = 0 \text{ in } D_c,$$

$$\mathbf{A} \times \mathbf{n} = 0 \text{ on } \Gamma,$$

$$\sigma (j\omega \mathbf{A} + \nabla \varphi) \cdot \mathbf{n} = 0 \text{ on } \Gamma_c.$$

The corresponding weak formulation is given by (see [18]):

Find $(\mathbf{A}, \varphi) \in \tilde{X}(D) \times \widetilde{H}^1(D_c)$ such that

$$a_{A,\varphi}((\mathbf{A}, \varphi), (\mathbf{A}', \varphi')) = l_{A,\varphi}((\mathbf{A}', \varphi')) \quad \forall (\mathbf{A}', \varphi') \in \tilde{X}(D) \times \widetilde{H}^1(D_c), \quad (14)$$

with $a_{A,\varphi}$ and $l_{A,\varphi}$ the bilinear and linear forms respectively defined by:

$$a_{A,\varphi}((\mathbf{A}, \varphi), (\mathbf{A}', \varphi')) = (\mu^{-1} \text{curl } \mathbf{A}, \text{curl } \mathbf{A}') + \frac{j}{\omega} (\sigma^{1/2} (j\omega \mathbf{A} + \nabla \varphi), \sigma^{1/2} (j\omega \mathbf{A}' + \nabla \varphi'))_{D_c},$$

$$l_{A,\varphi}((\mathbf{A}', \varphi')) = (\mathbf{J}_s, \mathbf{A}').$$

Theorem 2.1 of [18] ensures the existence and uniqueness of the solution $(\mathbf{A}, \varphi) \in \tilde{X}(D) \times \widetilde{H}^1(D_c)$ of problem (14). Let remark that (14) occurs even for any $(\mathbf{A}', \varphi') \in X(D) \times H^1(D_c)$, that is to say that the Gauge conditions are not needed for the test functions (see Lemma 2.2 of [18]).

Likewise, the harmonic formulation of (12)-(13) is obtained taking $\mathbf{T}(t, \mathbf{x}) = \overline{\mathbf{T}}(\mathbf{x}) e^{j\omega t}$ and $\Omega(t, \mathbf{x}) = \overline{\Omega}(\mathbf{x}) e^{j\omega t}$. Like previously, writing \mathbf{T} and Ω instead of $\overline{\mathbf{T}}$ and $\overline{\Omega}$ respectively, the

harmonic $\mathbf{T} - \Omega$ formulation reads:

$$\operatorname{curl}(\sigma^{-1} \operatorname{curl} \mathbf{T}) + j\omega \mu (\mathbf{T} - \nabla \Omega) = -j\omega \mu \mathbf{H}_s \text{ in } D_c.$$

$$\operatorname{div}(\mu(\tilde{\mathbf{T}} - \nabla \Omega)) = -\operatorname{div}(\mu \mathbf{H}_s) \text{ in } D.$$

$$\mathbf{T} \times \mathbf{n} = 0 \text{ on } \Gamma_c,$$

$$\mu(\mathbf{H}_s - \nabla \Omega) \cdot \mathbf{n} = 0 \text{ on } \Gamma,$$

where we have used a slight abuse of notation $\tilde{\mathbf{T}}$ to denote a fixed extension of \mathbf{T} in the non conductor domain D_e , like what we did for $\tilde{\varphi}$. The corresponding weak formulation is given by (see [19]):

Find $(\mathbf{T}, \Omega) \in \tilde{X}(D_c) \times \tilde{H}^1(D)$ such that

$$a_{T,\Omega}((\mathbf{T}, \Omega), (\mathbf{T}', \Omega')) = l_{T,\Omega}((\mathbf{T}', \Omega')) \quad \forall (\mathbf{T}', \Omega') \in \tilde{X}(D_c) \times \tilde{H}^1(D), \quad (15)$$

with $a_{T,\Omega}$ and $l_{T,\Omega}$ the bilinear and linear forms respectively defined by:

$$\begin{aligned} a_{T,\Omega}((\mathbf{T}, \Omega), (\mathbf{T}', \Omega')) &= (\sigma^{-1} \operatorname{curl} \mathbf{T}, \operatorname{curl} \mathbf{T}')_{D_c} + (j\omega \mu (\mathbf{T} - \nabla \Omega), \mathbf{T}' - \nabla \Omega')_{D_c} \\ &\quad + (j\omega \mu \nabla \Omega, \nabla \Omega')_{D_e}, \end{aligned}$$

$$l_{T,\Omega}((\mathbf{T}', \Omega')) = -(j\omega \mu \mathbf{H}_s, \mathbf{T}' - \nabla \Omega')_{D_c} + (j\omega \mu \mathbf{H}_s, \nabla \Omega')_{D_e}.$$

By Theorem 2.2 of [19], problem (15) admits a unique solution $(\mathbf{T}, \Omega) \in \tilde{X}(D_c) \times \tilde{H}^1(D)$. Once again (see Lemma 2.3 of [19]), (15) occurs even for any $(\mathbf{T}', \Omega') \in X(D_c) \times H^1(D)$.

2.2 Discrete weak formulations

We consider now a conforming mesh \mathcal{T}_h made of tetrahedra, each element T of \mathcal{T}_h belonging either to D_c or to D_e . We denote h_T the diameter of the element T and ρ_T the diameter of its largest inscribed ball. We suppose that for any element T , the ratio h_T/ρ_T is bounded by a constant $\alpha > 0$ independent of T and of the mesh size $h = \max_{T \in \mathcal{T}_h} h_T$. Finally, the conductivity σ and the permeability μ are supposed to be constant on each tetrahedron: we write $\mu_{\max} = \max_{T \in \mathcal{T}_h} \mu|_T$, $\sigma_{\max} = \max_{T \in \mathcal{T}_h, T \in D_c} \sigma|_T$ and $\sigma_{\min} = \min_{T \in \mathcal{T}_h, T \in D_c} \sigma|_T$.

The vector fields \mathbf{A} and \mathbf{T} are approximated by first order edge elements and the scalar fields φ and Ω by first order nodal elements. Thus the considered approximation spaces are the following:

$$\begin{aligned} \mathcal{ND}_1(T) &= \{ \mathbf{F}_h : T \rightarrow \mathbb{C}^3, \mathbf{x} \rightarrow \mathbf{a} + \mathbf{b} \times \mathbf{x}, \mathbf{a}, \mathbf{b} \in \mathbb{C}^3 \}, \\ X_h(\mathcal{D}) &= \{ \mathbf{F}_h \in X(\mathcal{D}); \mathbf{F}_h|_T \in \mathcal{ND}_1(T), \forall T \in \mathcal{T}_h \}, \\ \Theta_h^0(\mathcal{D}) &= \{ \xi_h \in H_0^1(\mathcal{D}); \xi_h|_T \in \mathbb{P}_1(T) \forall T \in \mathcal{T}_h \}, \\ \tilde{X}_h(\mathcal{D}) &= \{ \mathbf{F}_h \in X_h(\mathcal{D}); (\mathbf{F}_h, \nabla \xi_h) = 0 \forall \xi_h \in \Theta_h^0(\mathcal{D}) \}, \\ \tilde{\Theta}_h(\mathcal{D}) &= \{ f_h \in \tilde{H}^1(\mathcal{D}); f_h|_T \in \mathbb{P}_1(T) \forall T \in \mathcal{T}_h \}. \end{aligned}$$

The discrete $\mathbf{A} - \varphi$ formulation consists in looking for $(\mathbf{A}_h, \varphi_h) \in \widetilde{X}_h(D) \times \widetilde{\Theta}_h(D_c)$ such that

$$a_{A,\varphi}((\mathbf{A}_h, \varphi_h), (\mathbf{A}'_h, \varphi'_h)) = l_{A,\varphi}((\mathbf{A}'_h, \varphi'_h)) \quad \forall (\mathbf{A}'_h, \varphi'_h) \in \widetilde{X}_h(D) \times \widetilde{\Theta}_h(D_c). \quad (16)$$

Theorem 2.2 of [18] ensures that problem (16) admits a unique solution $(\mathbf{A}_h, \varphi_h) \in \widetilde{X}_h(D) \times \widetilde{\Theta}_h(D_c)$. On the other hand, the discrete $\mathbf{T} - \Omega$ formulation consists in looking for $(\mathbf{T}_h, \Omega_h) \in \widetilde{X}_h(D_c) \times \widetilde{\Theta}_h(D)$ such that

$$a_{T,\Omega}((\mathbf{T}_h, \Omega_h), (\mathbf{T}'_h, \Omega'_h)) = l_{T,\Omega}((\mathbf{T}'_h, \Omega'_h)) \quad \forall (\mathbf{T}'_h, \Omega'_h) \in \widetilde{X}_h(D_c) \times \widetilde{\Theta}_h(D). \quad (17)$$

Theorem 2.4 of [19] ensures that problem (17) admits a unique solution $(\mathbf{T}_h, \Omega_h) \in \widetilde{X}_h(D_c) \times \widetilde{\Theta}_h(D)$. Like for the continuous cases, the test functions arising in (16) and (17) do not need to be gauged (see Lemma 2.3 of [18] and Lemma 2.6 of [19] respectively).

2.3 Definition of the error

On one hand, we want to estimate the $\mathbf{A} - \varphi$ error $e_{A,\varphi}$ given by:

$$e_{A,\varphi}^2 = \|\mu^{-1/2} \operatorname{curl} \mathbf{e}_A\|^2 + \|\omega^{-1/2} \sigma^{1/2} (j \omega \mathbf{e}_A + \nabla e_\varphi)\|_{D_c}^2, \quad (18)$$

where $\mathbf{e}_A = \mathbf{A} - \mathbf{A}_h$ and $e_\varphi = \varphi - \varphi_h$. From the FE resolution of the $\mathbf{A} - \varphi$ system, we define:

$$\begin{aligned} \mathbf{B}_h &= \operatorname{curl} \mathbf{A}_h, \\ \mathbf{E}_h &= -(j \omega \mathbf{A}_h + \nabla \varphi_h), \end{aligned}$$

so that, since $\mathbf{B} = \operatorname{curl} \mathbf{A}$ and $\mathbf{E} = -j \omega \mathbf{A} - \nabla \varphi$, the $\mathbf{A} - \varphi$ error (18) can be reformulated as:

$$e_{A,\varphi}^2 = \|\mu^{-1/2} (\mathbf{B} - \mathbf{B}_h)\|^2 + \|\omega^{-1/2} \sigma^{1/2} (\mathbf{E} - \mathbf{E}_h)\|_{D_c}^2. \quad (19)$$

On the other hand, we want to estimate the $\mathbf{T} - \Omega$ error $e_{T,\Omega}$ given by:

$$e_{T,\Omega}^2 = \|(\omega \sigma)^{-1/2} \operatorname{curl} \mathbf{e}_T\|_{D_c}^2 + \|\mu^{1/2} (\tilde{e}_T - \nabla e_\Omega)\|^2, \quad (20)$$

where $\mathbf{e}_T = \mathbf{T} - \mathbf{T}_h$ and $e_\Omega = \Omega - \Omega_h$. We recall the abuse of notation $\tilde{\mathbf{T}}$ to denote a fixed extension of \mathbf{T} in the non conductor domain D_e , so that relation (11) can be written as:

$$\mathbf{H} = \mathbf{H}_s + \tilde{\mathbf{T}} - \nabla \Omega \text{ in } D, \quad (21)$$

and $\tilde{e}_T = \tilde{\mathbf{T}} - \tilde{\mathbf{T}}_h$ in D . In this case, from the FE resolution of the $\mathbf{T} - \Omega$ system, we define:

$$\begin{aligned} \mathbf{H}_h &= \mathbf{H}_s + \tilde{\mathbf{T}}_h - \nabla \Omega_h, \\ \mathbf{J}_{e,h} &= \operatorname{curl} \mathbf{T}_h, \end{aligned}$$

Therefore, recalling the continuous relations (10) and (21), the $\mathbf{T} - \Omega$ error (20) can be reformulated as:

$$e_{T,\Omega}^2 = \|(\omega \sigma)^{-1/2} (\mathbf{J}_e - \mathbf{J}_{e,h})\|_{D_c}^2 + \|\mu^{1/2} (\mathbf{H} - \mathbf{H}_h)\|^2. \quad (22)$$

The following section is devoted to derive an *a posteriori* equilibrated error estimator in order to control the total error e defined as

$$e^2 = e_{A,\varphi}^2 + e_{T,\Omega}^2. \quad (23)$$

3 A posteriori error analysis

3.1 Technical results

This subsection is dedicated to establish some technical results needed to derive the equivalence between the estimator and the error, up to some oscillation terms. Let us define

$$\mathbf{e} := -(\mathbf{e}_T - \nabla\alpha), \quad (24)$$

where $\mathbf{e}_T = \mathbf{T} - \mathbf{T}_h$ (as already defined in subsection 2.3) and $\alpha \in H_0^1(D_c)$ is defined as the unique solution of

$$\int_{D_c} \nabla\alpha \cdot \nabla\bar{\chi} = \int_{D_c} \mathbf{e}_T \cdot \nabla\bar{\chi} \quad \forall \chi \in H_0^1(D_c). \quad (25)$$

Let us note that, taking $\chi = \alpha$ in (25) and thanks to the Cauchy-Schwarz inequality, the estimation

$$\|\nabla\alpha\|_{D_c} \leq \|\mathbf{e}_T\|_{D_c}$$

is straightforward. Let us moreover point out that \mathbf{e} belongs to $X(D_c)$ and satisfies

$$\operatorname{div} \mathbf{e} = 0. \quad (26)$$

Moreover, since $\nabla\alpha \in X(D_c)$ and $\mathbf{e}_A \in H(\operatorname{curl}, D_c)$, from the divergence theorem, we obtain

$$\int_{D_c} \nabla\alpha \cdot \operatorname{curl} \mathbf{e}_A = \int_{D_c} \operatorname{curl} \nabla\alpha \cdot \mathbf{e}_A + \int_{\Gamma_c} \nabla\alpha \times \mathbf{n} \cdot \mathbf{e}_A = 0,$$

which implies

$$(-\mathbf{e}_T, \operatorname{curl} \mathbf{e}_A)_{D_c} = (\mathbf{e}, \operatorname{curl} \mathbf{e}_A)_{D_c}. \quad (27)$$

Let a^* be the bilinear form defined as:

$$\begin{aligned} a^*((\mathbf{T}, \Omega), (\mathbf{T}', \Omega')) &= \overline{a_{T,\Omega}((\mathbf{T}', \Omega'), (\mathbf{T}, \Omega))} \\ &= (\sigma^{-1} \operatorname{curl} \mathbf{T}, \operatorname{curl} \mathbf{T}')_{D_c} - (j\omega\mu(\tilde{\mathbf{T}} - \nabla\Omega), \tilde{\mathbf{T}}' - \nabla\Omega'). \end{aligned} \quad (28)$$

We now introduce the weak problem that consists in finding $(\mathbf{T}_e, \Omega_e) \in \tilde{X}(D_c) \times \tilde{H}^1(D)$ such that

$$a^*((\mathbf{T}_e, \Omega_e), (\mathbf{T}', \Omega')) = (\mathbf{e}, \mathbf{T}') \quad \forall (\mathbf{T}', \Omega') \in \tilde{X}(D_c) \times \tilde{H}^1(D). \quad (29)$$

This problem has clearly a unique solution since the form a^* is coercive (by multiplication by an appropriate factor, see Theorem 2.1 of [18]).

Lemma 3.1. Relation (29) is satisfied for any test function $(\mathbf{T}', \Omega') \in X(D_c) \times H^1(D)$.

Proof: By Theorem 3.40 of [25, p. 66], any $\mathbf{T}' \in X(D_c)$ admits the following Helmholtz decomposition

$$\mathbf{T}' = \psi + \nabla\tau,$$

where $\psi \in \tilde{X}(D_c)$ and $\tau \in H_0^1(D_c)$.

In the following $\tilde{\tau}$ denotes the extension of τ by zero outside D_c . Similarly to Lemma 2.3 of [19],

defining $\hat{\Omega}' = \Omega' + \tilde{\tau} - 1/|D| \int_D \Omega' - 1/|D_c| \int_{D_c} \tau$, we have:

$$\begin{aligned} a^*((\mathbf{T}_e, \Omega_e), (\mathbf{T}', \Omega')) &= a^*((\mathbf{T}_e, \Omega_e), (\psi + \nabla\tau, \Omega')) \\ &= (\sigma^{-1} \operatorname{curl} \mathbf{T}_e, \operatorname{curl} \psi)_{D_c} - (j\omega\mu(\tilde{\mathbf{T}}_e - \nabla\Omega_e), \psi + \nabla(\Omega' - \tilde{\tau})) \\ &= a^*((\mathbf{T}_e, \Omega_e), (\psi, \hat{\Omega}')). \end{aligned}$$

Thanks to (29) we obtain:

$$a^*((\mathbf{T}_e, \Omega_e), (\mathbf{T}', \Omega')) = \int_{D_c} \mathbf{e} \cdot \bar{\psi}. \quad (30)$$

The divergence theorem, joined to the relation (26), give that $\int_{D_c} \mathbf{e} \cdot \nabla \bar{\tau} = 0$ for any $\tau \in H_0^1(D_c)$. This property and (30) leads to the conclusion. \blacksquare

Corollary 3.2. Let $(\mathbf{T}_e, \Omega_e) \in \tilde{X}(D_c) \times \tilde{H}^1(D)$ be the unique solution of (29), then we have:

$$\begin{cases} \operatorname{div}(\mu(\tilde{\mathbf{T}}_e - \nabla \Omega_e)) = 0 & \text{in } D, \quad (i) \\ \mu(\tilde{\mathbf{T}}_e - \nabla \Omega_e) \cdot \mathbf{n} = 0 & \text{on } \Gamma, \quad (ii) \end{cases} \quad (31)$$

and

$$\operatorname{curl} \sigma^{-1} \operatorname{curl} \mathbf{T}_e - j \omega \mu (\mathbf{T}_e - \nabla \Omega_e) = \mathbf{e} \text{ in } D_c. \quad (32)$$

Proof: Let $\mathcal{D}(D)$ be the space of infinitely differentiable functions with compact support in D .

I) By Lemma 3.1, let us take $\mathbf{T}' = 0$ and $\Omega' \in \mathcal{D}(D)$ in (29). We have :

$$\int_D \mu (\tilde{\mathbf{T}}_e - \nabla \Omega_e) \cdot \nabla \Omega' = 0 \quad \forall \Omega' \in \mathcal{D}(D),$$

so that (31) (i) holds.

II) Let us take now $\mathbf{T}' = 0$ and $\Omega' \in H^1(D)$ in (29), so that by Lemma 3.1 we have:

$$\int_D \mu (\tilde{\mathbf{T}}_e - \nabla \Omega_e) \cdot \nabla \Omega' = 0 \quad \forall \Omega' \in H^1(D).$$

Now, applying Green's theorem we obtain:

$$\begin{aligned} 0 &= - \int_D \operatorname{div}(\mu (\tilde{\mathbf{T}}_e - \nabla \Omega_e)) \Omega' + \langle \mu (\tilde{\mathbf{T}}_e - \nabla \Omega_e) \cdot \mathbf{n}, \Omega' \rangle_{\Gamma} \\ &= \langle \mu (\tilde{\mathbf{T}}_e - \nabla \Omega_e) \cdot \mathbf{n}, \Omega' \rangle_{\Gamma} \\ &= 0 \quad \forall \Omega' \in H^1(D), \end{aligned}$$

so that (31) (ii) is proved.

III) Finally, let us take $\mathbf{T}' \in \mathcal{D}(D_c)^3 \subset X(D_c)$ and $\Omega' = 0$ in (29), so that

$$\int_{D_c} (\sigma^{-1} \operatorname{curl} \mathbf{T}_e \cdot \operatorname{curl} \mathbf{T}' - j \omega \mu (\mathbf{T}_e - \nabla \Omega_e) \cdot \mathbf{T}') = 0 \quad \forall \mathbf{T}' \in \mathcal{D}(D)^3,$$

which leads to (32) in the distributional sense. \blacksquare

Corollary 3.3. Let us recall that D and D_c are both simply connected and that the boundary Γ_c is connected. Let \mathbf{e} defined by (24) and (\mathbf{T}_e, Ω_e) solution of (29). Then there exists $\epsilon \in (0, 1/2)$ such that

$$\mathbf{T}_e \in H^{1/2+\epsilon}(D_c), \quad (33)$$

$$\operatorname{curl} \mathbf{T}_e \in H^{1/2+\epsilon}(D_c), \quad (34)$$

$$\Omega_e \in H^{1+\epsilon}(D), \quad (35)$$

with

$$\|\mathbf{T}_e\|_{H^{1/2+\epsilon}(D_c)} + \|\operatorname{curl} \mathbf{T}_e\|_{H^{1/2+\epsilon}(D_c)} + \|\Omega_e\|_{H^{1+\epsilon}(D)} \leq c \|\mathbf{e}\|_{D_c}, \quad (36)$$

where c represents a general constant independent of h .

Proof: I) First, let us prove (33).

Since $\mathbf{T}_e \in \tilde{X}(D_c) \hookrightarrow H^{1/2+\epsilon}(D_c)$ (see Theorem 3.50 page 71 of [25]), and using the Friedrichs inequality (see Corollary 3.51 of [25]) we have:

$$\begin{aligned} \|\mathbf{T}_e\|_{H^{1/2+\epsilon}(D_c)} &\leq c(\|\operatorname{curl} \mathbf{T}_e\|_{D_c} + \underbrace{\|\operatorname{div} \mathbf{T}_e\|_{D_c}}_{=0} + \|\mathbf{T}_e\|_{D_c}) \\ &\leq c \|\operatorname{curl} \mathbf{T}_e\|_{D_c}. \end{aligned} \quad (37)$$

Taking as test functions $\mathbf{T}' = \mathbf{T}_e$ and $\Omega' = \Omega_e$ in (29), we get that

$$\|\sigma^{1/2} \operatorname{curl} \mathbf{T}_e\|_{D_c}^2 \leq |a^*((\mathbf{T}_e, \Omega_e), (\mathbf{T}_e, \Omega_e))| = |(\mathbf{e}, \mathbf{T}_e)_{D_c}| \leq c \|\mathbf{e}\|_{D_c} \|\operatorname{curl} \mathbf{T}_e\|_{D_c},$$

where Cauchy-Schwarz and Friedrichs inequalities have been used in the last step of the estimation. This result implies that

$$\|\operatorname{curl} \mathbf{T}_e\|_{D_c} \leq c \sigma_{\min}^{-1} \|\mathbf{e}\|_{D_c}. \quad (38)$$

With the help of the estimate (37) we get

$$\|\mathbf{T}_e\|_{H^{1/2+\epsilon}(D_c)} \leq c \sigma_{\min}^{-1} \|\mathbf{e}\|_{D_c}. \quad (39)$$

II) Let us prove (34). From Corollary 3.2, one can deduce that $\operatorname{curl} \mathbf{T}_e \in H(\operatorname{div}, D)$ verifies, in the distributional sense,

$$\operatorname{curl} \operatorname{curl} \mathbf{T}_e = \sigma j \omega \mu (\mathbf{T}_e - \nabla \Omega_e) + \sigma \mathbf{e} \in L^2(D)^3. \quad (40)$$

Moreover $\operatorname{div} \operatorname{curl} \mathbf{T}_e = 0$ in D_c and, as long as $\mathbf{T}_e \times \mathbf{n} = 0$ on Γ_c , $\operatorname{curl} \mathbf{T}_e \cdot \mathbf{n} = 0$ on Γ_c . In other words,

$$\operatorname{curl} \mathbf{T}_e \in X_T(D_c) := \{\mathbf{F} \in H(\operatorname{curl}, D_c) \cap H(\operatorname{div}, D_c); \mathbf{F} \cdot \mathbf{n} = 0 \text{ on } \Gamma_c\}.$$

Theorem 3.50 of [25] ensures that $X_T(D_c) \hookrightarrow H^{1/2+\epsilon}(D_c)^3$, so that $\operatorname{curl} \mathbf{T}_e \in H^{1/2+\epsilon}(D_c)^3$ with :

$$\begin{aligned} \|\operatorname{curl} \mathbf{T}_e\|_{H^{1/2+\epsilon}(D_c)} &\leq c \|\operatorname{curl} \mathbf{T}_e\|_{X_T(D_c)} \\ &\leq c (\|\operatorname{curl} \operatorname{curl} \mathbf{T}_e\|_{D_c} + \|\operatorname{curl} \mathbf{T}_e\|_{D_c}) \end{aligned} \quad (41)$$

Let us remark that equation (40) brings to the estimation

$$\|\operatorname{curl} \operatorname{curl} \mathbf{T}_e\|_{D_c} \leq \sigma_{\max} \|\omega \mu (\mathbf{T}_e - \nabla \Omega_e)\|_{D_c} + \sigma_{\max} \|\mathbf{e}\|_{D_c},$$

and in particular, similarly to the point I), taking in (29) the test functions $\mathbf{T}' = \mathbf{T}_e$ and $\Omega' = \Omega_e$, and using the coerciveness of a^* and (37) we get

$$\|\omega \mu (\mathbf{T}_e - \nabla \Omega_e)\|_{D_c}^2 \leq |a^*((\mathbf{T}_e, \Omega_e), (\mathbf{T}_e, \Omega_e))| = |(\mathbf{e}, \mathbf{T}_e)_{D_c}| \leq c \|\mathbf{e}\|_{D_c} \|\operatorname{curl} \mathbf{T}_e\|_{D_c},$$

the norm $\|\operatorname{curl} \mathbf{T}_e\|_{D_c}$ is now estimate by (38), so that

$$\|\omega \mu (\mathbf{T}_e - \nabla \Omega_e)\|_{D_c} \leq c \sigma_{\min}^{-1/2} \|\mathbf{e}\|_{D_c}.$$

It is now easy to deduce that (41) becomes:

$$\|\operatorname{curl} \mathbf{T}_e\|_{H^{1/2+\epsilon}(D_c)} \leq c \max(\sigma_{\max} \sigma_{\min}^{-1/2}, \sigma_{\max}, \sigma_{\min}^{-1}) \|\mathbf{e}\|_{D_c}. \quad (42)$$

III) Let us prove (35).

- By Corollary 3.2 and the fact that $\text{curl}(\tilde{\mathbf{T}}_e - \nabla\Omega_e) \in L^2(D)^3$, clearly $\tilde{\mathbf{T}}_e - \nabla\Omega_e \in X_T(D, \mu)$, where

$$X_T(D, \mu) = \{ \mathbf{F} \in H(\text{curl}, D); \text{div}(\mu \mathbf{F}) \in L^2(D) \text{ and } \mathbf{F} \cdot \mathbf{n} = 0 \text{ on } \Gamma_c \}.$$

From Theorem 3.5 of [17] we have $X_T(D, \mu) \hookrightarrow H^\epsilon(D)$, as well as

$$\| \tilde{\mathbf{T}}_e - \nabla\Omega_e \|_{H^\epsilon(D)} \leq c \| \tilde{\mathbf{T}}_e - \nabla\Omega_e \|_{X_T(D, \mu)}.$$

Consequently, using the Friedrichs inequality as well as (42),

$$\begin{aligned} \| \tilde{\mathbf{T}}_e - \nabla\Omega_e \|_{H^\epsilon(D)} &\leq c \| \text{curl}(\tilde{\mathbf{T}}_e - \nabla\Omega_e) \|_{L^2(D)} \\ &\leq c \| \text{curl} \mathbf{T}_e \|_{L^2(D_c)} \\ &\leq c \| \text{curl} \mathbf{T}_e \|_{H^{1/2+\epsilon}(D_c)} \\ &\leq c \max(\sigma_{\max} \sigma_{\min}^{-1/2}, \sigma_{\max}, \sigma_{\min}^{-1}) \| \mathbf{e} \|_{D_c}. \end{aligned} \quad (43)$$

- Due to point I), \mathbf{T}_e belongs to $H^{1/2+\epsilon}(D_c)$. In particular this implies that $\mathbf{T}_e \in H^\epsilon(D_c)$, and, by Corollary 1.4.4.5 of [23], $\tilde{\mathbf{T}}_e \in H^\epsilon(D)$ with the property that

$$\| \tilde{\mathbf{T}}_e \|_{H^\epsilon(D)} \leq c \| \mathbf{T}_e \|_{H^\epsilon(D_c)} \leq c \| \mathbf{T}_e \|_{H^{1/2+\epsilon}(D_c)} \leq c \| \mathbf{e} \|_{D_c}, \quad (44)$$

where the last inequality is derived from (39).

As a result of (43) and (44), $\nabla\Omega_e \in H^\epsilon(D)$ and it is such that $\| \nabla\Omega_e \|_{H^\epsilon(D)} \leq c \| \mathbf{e} \|_{D_c}$, so that (35) holds.

IV) Since the averaged value of Ω_e is zero, we get the estimation

$$\| \Omega_e \|_{H^{1+\epsilon}(D)} \cong \| \Omega_e \|_{H^1(D)} + | \Omega_e |_{H^{1+\epsilon}(D)} \cong \| \Omega_e \|_{H^1(D)} + | \nabla\Omega_e |_{H^\epsilon(D)} \leq c \| \nabla\Omega_e \|_{H^{1+\epsilon}(D)}$$

which, joined with (39) and (42), leads directly to (36). \blacksquare

Theorem 3.4. Under the assumptions of Corollary 3.3, there exists $\epsilon \in (0, 1/2)$ such that

$$\| \mathbf{e} \|_{D_c} \leq c h^\epsilon e_{T, \Omega}, \quad (45)$$

where c represents a general constant independent of h .

Proof: Lemma 3.1 allows us to take $\mathbf{T}' = \mathbf{e}$ and $\Omega' = \Omega_h - \Omega = -e_\Omega$ in (29). Thus we get

$$\| \mathbf{e} \|_{D_c}^2 = -a^*((\mathbf{T}_e, \Omega_e), (\mathbf{e}_T, e_\Omega)) + a^*((\mathbf{T}_e, \Omega_e), (\nabla\alpha, 0)).$$

From (26) and (29) we remark that $a^*((\mathbf{T}_e, \Omega_e), (\nabla\alpha, 0)) = 0$. Hence recalling the definition (28) of a^* with respect to the bilinear form $a_{T, \Omega}$, we get

$$\| \mathbf{e} \|_{D_c}^2 = -a^*((\mathbf{T}_e, \Omega_e), (\mathbf{e}_T, e_\Omega)) \quad (46)$$

$$\begin{aligned} &= \frac{-a_{T, \Omega}((\mathbf{e}_T, e_\Omega), (\mathbf{T}_e, \Omega_e))}{-a_{T, \Omega}((\mathbf{T}_e - \mathbf{T}_{e, h}, \Omega_e - \Omega_{e, h}), (\mathbf{e}_T, e_\Omega))}, \end{aligned} \quad (47)$$

where the last equality is possible thanks to Lemma 2.7 of [19], having taken $\mathbf{T}_{e, h} \in X_h(D_c)$ and $\Omega_{e, h} \in \Theta_h(D)$.

- Thanks to Corollary 3.3 and to Section 2.5.4 of [10], we can define $\mathbf{T}_{e,h} := \mathcal{I}_{\mathcal{ND}}\mathbf{T}_e$ (the interpolation operator already specified in Section 3.2) and the estimate (2.5.52) of Proposition 2.5.7 in [10] give:

$$\|\mathbf{T}_e - \mathbf{T}_{e,h}\|_{D_c} \leq c h^{1/2+\epsilon} (\|\mathbf{T}_e\|_{H^{1/2+\epsilon}(D_c)} + \|\mathbf{T}_{e,h}\|_{L^p(D_c)})$$

for $p > 2$. On the other hand we have $H^{1/2+\epsilon}(D_c) \hookrightarrow L^p(D_c)$ for any $p \leq 3/(1-\epsilon)$. Therefore $H^{1/2+\epsilon}(D_c) \hookrightarrow L^p(D_c)$ for any $p \in (2, 3/(1-\epsilon)]$. It is now immediate from Corollary 3.3 that

$$\|\mathbf{T}_e - \mathbf{T}_{e,h}\|_{D_c} \leq c h^{1/2+\epsilon} \|\mathbf{e}\|_{D_c}. \quad (48)$$

Similarly, the estimate (2.5.53) of Proposition 2.5.7 in [10] and Corollary 3.3 give

$$\|\operatorname{curl}(\mathbf{T}_e - \mathbf{T}_{e,h})\|_{D_c} \leq c h^{1/2+\epsilon} \|\operatorname{curl} \mathbf{T}_e\|_{H^{1/2+\epsilon}(D_c)} \leq c h^{1/2+\epsilon} \|\mathbf{e}\|_{D_c}. \quad (49)$$

- Let us define $\Omega_{e,h} = \mathcal{I}_{Cl}\Omega_e$, where \mathcal{I}_{Cl} denotes the Clément interpolation operator (see [14] for more details). Interpolating the results of Theorem 1 of [14] we gain the following inequality:

$$\|\Omega_e - \Omega_{e,h}\|_{H^1(D)} \leq c h^\epsilon \|\Omega_e\|_{H^{1+\epsilon}(D)}. \quad (50)$$

Corollary 3.3 together with (50) leads easily to

$$\|\Omega_e - \Omega_{e,h}\|_{H^1(D)} \leq c h^\epsilon \|\mathbf{e}\|_{D_c}. \quad (51)$$

Finally (46) becomes:

$$\begin{aligned} \|\mathbf{e}\|_{D_c}^2 &= -(\sigma^{-1} \operatorname{curl}(\mathbf{T}_e - \mathbf{T}_{e,h}), \operatorname{curl} \mathbf{e}_T)_{D_c} - j \omega (\mu(\mathbf{T}_e - \mathbf{T}_{e,h} - \nabla(\Omega_e - \Omega_{e,h})), \tilde{\mathbf{e}}_T - \nabla e_\Omega) \\ &\leq |\omega|^{1/2} \sigma_{\min}^{-1/2} \|\omega^{1/2} \sigma^{-1/2} \operatorname{curl} \mathbf{e}_T\|_{D_c} \|\operatorname{curl}(\mathbf{T}_e - \mathbf{T}_{e,h})\|_{D_c} \\ &\quad + |\omega| \mu_{\max}^{1/2} \|\mu^{1/2} (\tilde{\mathbf{e}}_T - \nabla e_\Omega)\| \|\tilde{\mathbf{T}}_e - \tilde{\mathbf{T}}_{e,h} - \nabla(\Omega_e - \Omega_{e,h})\|, \end{aligned}$$

so that :

$$\|\mathbf{e}\|_{D_c} \leq c \max(\sigma_{\min}^{-1/2} |\omega|^{1/2}, |\omega| \mu_{\max}^{1/2}) h^\epsilon e_{T,\Omega}.$$

where we have used substantially the Cauchy-Schwarz inequality and the estimates (48), (49) and (51). \blacksquare

3.2 Definition of the estimator

Let us suppose that the magnetic source term \mathbf{H}_s belongs to $H^{1+\delta}(D)^3$ with $0 < \delta \leq 1$, and let us define $\mathbf{H}_{s,h} := \mathcal{I}_{\mathcal{ND}} \mathbf{H}_s$, where $\mathcal{I}_{\mathcal{ND}}$ represents the global interpolation operator onto the Nedelec space $\mathcal{ND}_1(D, \mathcal{T}_h) = \{\mathbf{F} \in H(\operatorname{curl}, D); \mathbf{F}|_T \in \mathcal{ND}_1(T) \forall T \in \mathcal{T}_h\}$, as defined in Section 2.5.4 of [10]. Then, from Proposition 2.5.7 of [10], there exists a constant c independent of h such that

$$\|\mathbf{H}_s - \mathbf{H}_{s,h}\| \leq c h^{1+\delta} \|\mathbf{H}_s\|_{H^{1+\delta}(D)^3}. \quad (52)$$

The *a posteriori* error estimator is defined by:

$$\eta^2 = \sum_{T \in \mathcal{T}_h} \eta_{magnetic,T}^2 + \sum_{T \in \mathcal{T}_h, T \subset D_c} \eta_{electric,T}^2, \quad (53)$$

with

$$\eta_{magnetic,T}^2 = \|\mu^{1/2}(\mathbf{H}_h - \mu^{-1}\mathbf{B}_h + \mathbf{H}_{s,h} - \mathbf{H}_s)\|_T^2 = \|\mu^{1/2}(\mathbf{H}_{s,h} + \tilde{\mathbf{T}}_h - \nabla\Omega_h - \mu^{-1}\text{curl}\mathbf{A}_h)\|_T^2,$$

and

$$\eta_{electric,T}^2 = \|(\omega\sigma)^{-1/2}(\mathbf{J}_{e,h} - \sigma\mathbf{E}_h)\|_T^2 = \|(\omega\sigma)^{-1/2}(\text{curl}\mathbf{T}_h - \sigma(j\omega\mathbf{A}_h + \nabla\varphi_h))\|_T^2$$

as local error indicators.

3.3 Equivalence between the error and the estimator

Theorem 3.5. Under the assumptions of Theorem 3.4, the following relation holds:

$$\eta^2 = e^2 + h.o.t., \quad (54)$$

where *h.o.t.* denotes higher-order terms (with respect to the error e^2) which are specified in (58).

Proof: Thanks to the material constitutive laws (4) and (5), we can split the estimator terms as follows:

$$\begin{aligned} \eta^2 &= \|\mu^{1/2}(\mathbf{H}_h - \mathbf{H} + \mu^{-1}(\mathbf{B} - \mathbf{B}_h) + \mathbf{H}_{s,h} - \mathbf{H}_s)\|_D^2 + \|(\omega\sigma)^{-1/2}(\mathbf{J}_{e,h} - \mathbf{J}_e + \sigma(\mathbf{E} - \mathbf{E}_h))\|_{D_c}^2 \\ &= \|\mu^{1/2}(\mathbf{H}_h - \mathbf{H})\|_D^2 + \|\mu^{-1/2}(\mathbf{B} - \mathbf{B}_h)\|_D^2 + \|\mu^{1/2}(\mathbf{H}_s - \mathbf{H}_{s,h})\|_D^2 \\ &\quad + 2\Re((\mathbf{H}_s - \mathbf{H}_{s,h}, (\mathbf{B} - \mathbf{B}_h)) + (\mathbf{H}_s - \mathbf{H}_{s,h}, \mu(\mathbf{H}_h - \mathbf{H})) + (\mathbf{H}_h - \mathbf{H}, \mathbf{B} - \mathbf{B}_h)) \\ &\quad + \|(\omega\sigma)^{-1/2}(\mathbf{J}_{e,h} - \mathbf{J}_e)\|_{D_c}^2 + \|\omega^{-1/2}\sigma^{1/2}(\mathbf{E} - \mathbf{E}_h)\|_{D_c}^2 + 2\Re(\omega^{-1}(\mathbf{J}_{e,h} - \mathbf{J}_e, \mathbf{E} - \mathbf{E}_h)_{D_c}). \end{aligned} \quad (55)$$

From the divergence theorem joined to the boundary conditions on \mathbf{e}_A we remark that

$$\begin{aligned} (\mathbf{H}_h - \mathbf{H}, \mathbf{B} - \mathbf{B}_h) &= \int_D (\mathbf{H}_s + \tilde{\mathbf{T}}_h - \nabla\Omega_h - \mathbf{H}_s - \tilde{\mathbf{T}} + \nabla\Omega) \cdot \text{curl}(\overline{\mathbf{A} - \mathbf{A}_h}) \\ &= - \int_D (\mathbf{e}_T - \nabla e_\Omega) \cdot \text{curl}\bar{\mathbf{e}}_A = - \int_{D_c} \mathbf{e}_T \cdot \text{curl}\bar{\mathbf{e}}_A \end{aligned} \quad (56)$$

and, from the divergence theorem joined to the boundary conditions on \mathbf{e}_T , we remark that

$$\begin{aligned} \omega^{-1}(\mathbf{J}_{e,h} - \mathbf{J}_e, \mathbf{E} - \mathbf{E}_h)_{D_c} &= \omega^{-1} \int_{D_c} \text{curl}(\tilde{\mathbf{T}}_h - \tilde{\mathbf{T}}) \cdot \overline{(j\omega(\mathbf{A} - \mathbf{A}_h) + \nabla(\varphi - \varphi_h))} \\ &= -\omega^{-1} \int_{D_c} \text{curl}\mathbf{e}_T \cdot \overline{(j\omega\mathbf{e}_A + \nabla e_\varphi)} = j \int_{D_c} \mathbf{e}_T \cdot \text{curl}\bar{\mathbf{e}}_A. \end{aligned} \quad (57)$$

Recalling the definition of the error (23), the definition of the potentials and relations (56) and (57), the identity (55) takes the form:

$$\eta^2 = e^2 + r,$$

with

$$r = 2 \Re \left((\mathbf{H}_s - \mathbf{H}_{s,h}, (\mathbf{B} - \mathbf{B}_h)) + (\mathbf{H}_s - \mathbf{H}_{s,h}, \mu(\mathbf{H}_h - \mathbf{H})) + (1-j)(-\mathbf{e}_T, \text{curl } \mathbf{e}_A)_{D_c} \right) + \|\mu^{1/2}(\mathbf{H}_s - \mathbf{H}_{s,h})\|_D^2. \quad (58)$$

Let us show that r is an higher-order term with respect to the error e^2 , so that the conclusion (54) directly follows. We recall that the constant c will represent a generic constant, independent of h .

- Let us estimate $(\mathbf{H}_s - \mathbf{H}_{s,h}, \mathbf{B} - \mathbf{B}_h)$ and $(\mathbf{H}_s - \mathbf{H}_{s,h}, \mu(\mathbf{H}_h - \mathbf{H}))$. The Cauchy-Schwarz inequality and relation (52) give

$$\begin{aligned} |(\mathbf{H}_s - \mathbf{H}_{s,h}, (\mathbf{B} - \mathbf{B}_h))| &\leq \|\mathbf{H}_s - \mathbf{H}_{s,h}\| \mu_{\max}^{1/2} \|\mu^{-1/2}(\mathbf{B} - \mathbf{B}_h)\| \\ &\leq c \mu_{\max}^{1/2} \|\mathbf{H}_s\|_{H^{1+\delta}(D)^3} h^{1+\delta} e_{A,\varphi}, \end{aligned} \quad (59)$$

$$\begin{aligned} |(\mathbf{H}_s - \mathbf{H}_{s,h}, \mu(\mathbf{H}_h - \mathbf{H}))| &\leq \|\mathbf{H}_s - \mathbf{H}_{s,h}\| \mu_{\max}^{1/2} \|\mu^{1/2}(\mathbf{H}_h - \mathbf{H})\| \\ &\leq c \mu_{\max}^{1/2} \|\mathbf{H}_s\|_{H^{1+\delta}(D)^3} h^{1+\delta} e_{T,\Omega}, \end{aligned} \quad (60)$$

where we recall that $e_{A,\varphi}$ (resp. $e_{T,\Omega}$) is defined by (19) (resp. by (22)).

- Let us estimate now $(-\mathbf{e}_T, \text{curl } \mathbf{e}_A)_{D_c}$. From definition (24) and relation (27), we can directly apply the result (45) of Theorem 3.4, that gives:

$$|(\mathbf{e}_T, \text{curl } \mathbf{e}_A)_{D_c}| = |(\mathbf{e}, \text{curl } \mathbf{e}_A)_{D_c}| \leq \|\mathbf{e}\| \|\text{curl } \mathbf{e}_A\| \leq c h^\epsilon e_{T,\Omega} e_{A,\varphi}. \quad (61)$$

Joining the estimates (52), (59), (60) and (61), we gain the following estimation for the term (58):

$$|r| \leq c (h^{1+\delta} e_{A,\varphi} + h^{1+\delta} e_{T,\Omega} + h^\epsilon e_{T,\Omega} e_{A,\varphi} + h^{2+2\delta}). \quad (62)$$

This estimate shows that r is a higher order term. Indeed if the error decays as h^s with $0 < s \leq 1$, then the estimate (62) and $h^{1+\delta} \leq c h^s h^\delta$ yield

$$|r| \leq c (h^\delta + h^\epsilon + h^{2\delta}) h^{2s},$$

and leads to the fact that $|r| = o(h^{2s})$. ■

Let us state now a result concerning the local in space efficiency of the estimator.

Theorem 3.6. Let $\mathbf{H}_s \in H^{1+\delta}(D)^3$ with $0 < \delta \leq 1$ and $\mathbf{H}_{s,h} = \mathcal{I}_{\mathcal{N}\mathcal{D}} \mathbf{H}_s$, as defined in Section 3.2. According to the notation of Section 2.3, let

$$e_T^2 = \|\mu^{-1/2}(\mathbf{B} - \mathbf{B}_h)\|_T^2 + \|\omega^{-1/2} \sigma^{1/2} (\mathbf{E} - \mathbf{E}_h)\|_T^2 + \|(\omega \sigma)^{-1/2} (\mathbf{J}_e - \mathbf{J}_{e,h})\|_T^2 + \|\mu^{1/2}(\mathbf{H} - \mathbf{H}_h)\|_T^2.$$

Denoting $\eta_T := (\eta_{magnetic,T}^2 + \eta_{electric,T}^2)^{1/2}$, the following estimate holds:

$$\eta_T \leq 2 e_T + h. o. t. . \quad (63)$$

Proof: Thanks to the material constitutive laws (4) and (5), the triangular inequality and the property $(a + b)^2 \leq 2(a^2 + b^2)$, we get

$$\begin{aligned}
\eta_T^2 &= \eta_{magnetic,T}^2 + \eta_{electric,T}^2 \\
&= \left\| \mu^{1/2}(\mathbf{H}_h - \mathbf{H} + \mu^{-1}\mathbf{B} - \mu^{-1}\mathbf{B}_h) + \mu^{1/2}(\mathbf{H}_{s,h} - \mathbf{H}_s) \right\|_T^2 + \left\| (\omega\sigma)^{-1/2}(\mathbf{J}_{e,h} - \mathbf{J}_e + \sigma\mathbf{E} - \sigma\mathbf{E}_h) \right\|_T^2 \\
&\leq 2 \left\| \mu^{1/2}(\mathbf{H}_h - \mathbf{H}) + \mu^{1/2}(\mathbf{H}_{s,h} - \mathbf{H}_s) \right\|_T^2 + 2 \left\| \mu^{-1/2}(\mathbf{B} - \mathbf{B}_h) \right\|_T^2 \\
&\quad + 2 \left\| (\omega\sigma)^{-1/2}(\mathbf{J}_{e,h} - \mathbf{J}_e) \right\|_T^2 + 2 \left\| \omega^{-1/2}\sigma^{1/2}(\mathbf{E} - \mathbf{E}_h) \right\|_T^2 \\
&\leq 2 \left(\left\| (\omega\sigma)^{-1/2}(\mathbf{J}_{e,h} - \mathbf{J}_e) \right\|_T^2 + \left\| \omega^{-1/2}\sigma^{1/2}(\mathbf{E} - \mathbf{E}_h) \right\|_T^2 + \left\| \mu^{-1/2}(\mathbf{B} - \mathbf{B}_h) \right\|_T^2 \right) \\
&\quad + 4 \left\| \mu^{1/2}(\mathbf{H}_h - \mathbf{H}) \right\|_T^2 + 4 \left\| \mu^{1/2}(\mathbf{H}_{s,h} - \mathbf{H}_s) \right\|_T^2 \\
&\leq 4e_T^2 + 4ch^{2+2\delta} \mu_{\max} \left\| \mathbf{H}_s \right\|_{H^{1+\delta}(D)^3},
\end{aligned}$$

where the estimate (52) is used for the last step. The conclusion (63) directly follows remembering that $(a^2 + b^2)^{1/2} \leq a + b$. \blacksquare

4 Numerical results

This section is devoted to the numerical validation of the error estimator. The following numerical experiments are performed with the software CARMEL3D [1]. The aim consists in illustrating the equality (54) between the error and the equilibrated estimator up to higher order terms, as well as the local efficiency result (63). This test is inspired by the one proposed in [18] devoted to a residual error estimator for the $\mathbf{A} - \varphi$ harmonic formulation. Let us notice that here we have also to manage the $\mathbf{T} - \Omega$ formulation in order to compute our equilibrated estimator.

Let us recall the framework. As displayed in Figure 2, the geometrical domains under consideration are $D = [-2.5, 5] \times [-2, 2] \times [-2, 2]$ and $D_c = [2, 4] \times [-1, 1] \times [-1, 1]$. The physical parameters are defined by $\mu \equiv 1$ in D , $\sigma \equiv 1$ in D_c , $\sigma \equiv 0$ in D_e and $\omega = 2\pi$.

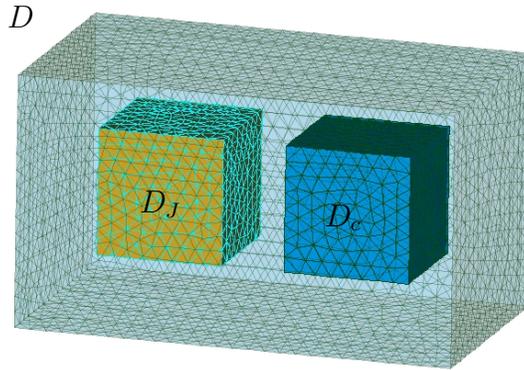


Figure 2: Configuration and regular mesh of the domains D , D_c and D_J .

Thanks to (8), we define \mathbf{J}_s so that the exact solution (\mathbf{A}, φ) of (14) is given by:

$$\mathbf{A} = \text{curl} \begin{pmatrix} f(x, y, z) \\ 0 \\ 0 \end{pmatrix} \quad \text{in } D,$$

where

$$f(x, y, z) = \begin{cases} (x^2 - 1)^4 (y^2 - 1)^4 (z^2 - 1)^4 & \text{in } D_J = [-1, 1]^3, \\ 0 & \text{otherwise,} \end{cases}$$

and $\varphi \equiv 0$ in D_c . Let us remark that D_J corresponds to the support of \mathbf{J}_s . The FE resolution of (16) and (17) gives the numerical solutions $(\mathbf{A}_h, \varphi_h)$ and (\mathbf{T}_h, Ω_h) respectively, allowing to evaluate the errors $e_{A,\varphi}$ and $e_{T,\Omega}$ given by (19) and (22), where we recall that:

$$\begin{aligned} \mathbf{B} - \mathbf{B}_h &= \text{curl}(\mathbf{A} - \mathbf{A}_h), \\ \mathbf{E} - \mathbf{E}_h &= -(j\omega(\mathbf{A} - \mathbf{A}_h) + \nabla(\varphi - \varphi_h)), \\ \mathbf{J}_e - \mathbf{J}_{e,h} &= -\sigma(j\omega\mathbf{A} + \nabla\varphi) - \text{curl}\mathbf{T}_h, \\ \mathbf{H} - \mathbf{H}_h &= \mu^{-1}\text{curl}\mathbf{A} - \mathbf{H}_s - \tilde{\mathbf{T}}_h + \nabla\Omega_h. \end{aligned}$$

We are then able to compute the exact error e given by (23), the equilibrated estimator η given by (53) as well as, for each tetrahedron T of the mesh, the local error e_T and the local estimator η_T defined in Theorem 3.6.

First of all, we check the convergence of the error e . Figure 3 shows the rate of decay of e as a function of the total number of degrees of freedom DoF corresponding to the \mathbf{A}/φ formulation. The theoretical *a priori* expected order of convergence is obtained, namely $-1/3$, corresponding to the order one in h for the three dimensional case, since regular meshes are used.

Then, we investigate the behavior of the equilibrated estimator η . The order of convergence of η is the same as the one of the error (see Figure 3). Moreover, we see in Figure 4 that the effectivity index $E_I = \frac{e}{\eta}$ is of the order one, this illustrates the equality (54) of Theorem 3.5 and corresponds to the so-called asymptotic exactness property of the equilibrated estimator.

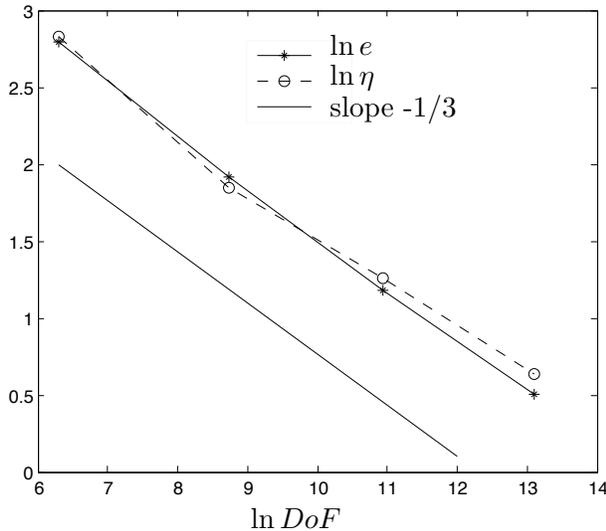


Figure 3: Convergence of the exact error e and the equilibrated estimator η with respect to the DoF.

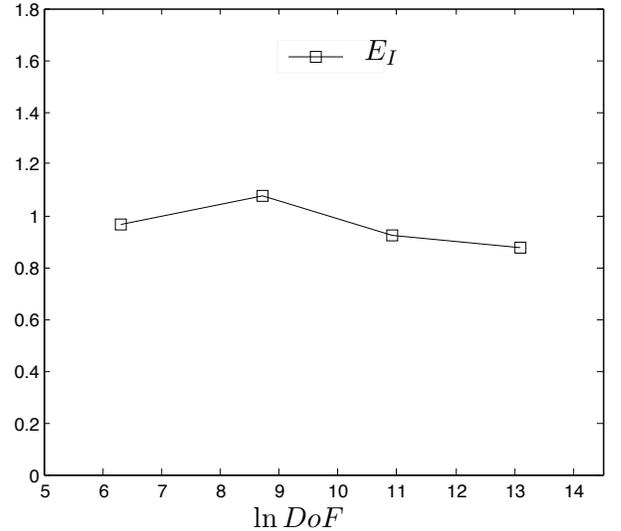


Figure 4: Evolution of the effectivity index $E_I = \frac{e}{\eta}$ with respect to the DoF.

Finally, for each tetrahedron T of the mesh we define its local effectivity index by:

$$(E_I)_T = \frac{e_T}{\eta_T},$$

what allows to define the quantity

$$(E_I)_{\min} = \min_{T \in \mathcal{T}_h} (E_I)_T.$$

From (63), we clearly expect that $(E_I)_{\min} \geq 0.5$ up to higher order terms, this is nearly the case as we can see in Figure 6, since its value oscillates between 0.5207 and 0.4827. Since the smallest value corresponds to a mesh with around 500000 degrees of freedom, we believe that it is smaller than 0.5 due to computing errors. That is why we also introduce the averaged local efficiency index $(E_I)_{\min\text{avg}}$, defined as the averaged value of $(E_I)_T$ only for the 0.1% of the tetrahedra in the mesh for which the values of $(E_I)_T$ are the smallest. Clearly, we see in Figure 6 that this quantity is this time always larger than 0.5, as theoretically expected. Since $(E_I)_{\min}$ is very close to 0.5 and since $(E_I)_{\min\text{avg}}$ is always larger than 0.5, the use of η_T for an adaptive mesh-refinement strategy seems very promising.

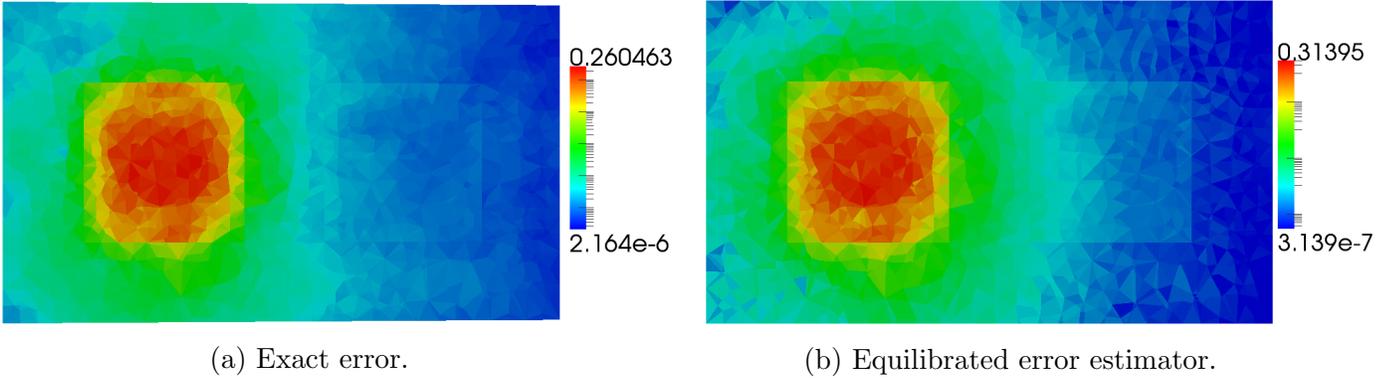


Figure 5: Local error maps in the plane $z = 0$.

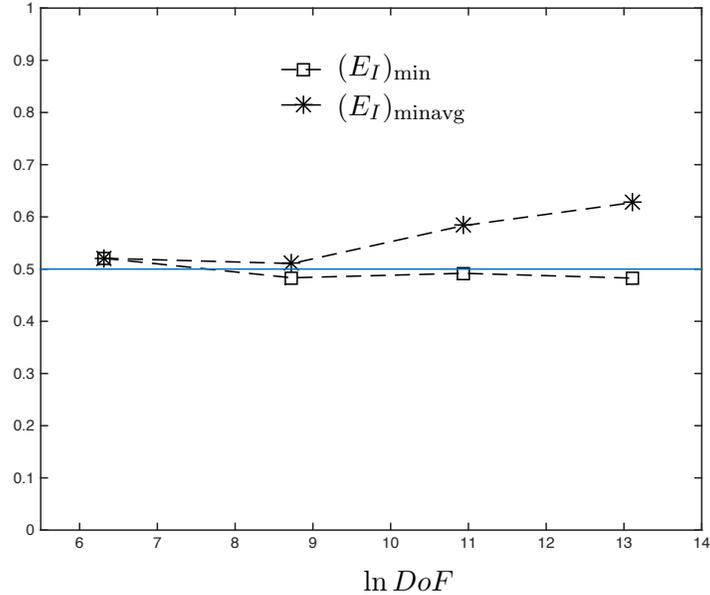


Figure 6: Evolution of $(E_I)_{\min}$ and $(E_I)_{\min\text{avg}}$ with respect to the DoF.

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