# Approximation of Markov semigroups in total variation distance

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## Abstract

The first goal of this paper is to prove that, regularization properties of a Markov semigroup enable to prove convergence in total variation distance for approximation schemes for the semigroup. Moreover, using an interpolation argument we obtain estimates for the error in distribution sense (at the level of the densities of the semigroup with respect to the Lebesgue measure). In a second step, we build an abstract Malliavin calculus based on a splitting procedure, which turns out to be the suited instrument in order to prove the above mentioned regularization properties. Finally, we use these results in order to estimate the error in total variation distance for the Ninomiya Victoir scheme (which is an approximation scheme, of order 2, for diffusion processes).

## 1   Introduction

In this paper we study the total variation distance between two discrete time Markov semigroups and we give applications for the speed of convergence of approximation schemes. In order to do it we use an abstract Malliavin type calculus based on a splitting procedure which enables us to prove regularization properties of the semigroup - and it turns out that such regularization properties are crucial in order to be able to deal with measurable test functions. Moreover, we take a step further and we give estimates for the distance between the density function of the Markov semigroup and the density function of the approximation scheme. At this level we have to use an interpolation argument which has been recently obtained in [8]. Let us be more specific and describe the different steps of our approach. We consider the $d$ dimensional Markov chain

$$X_{k+1}^n = \psi_k(X_k^n, \frac{Z_{k+1}}{\sqrt{n}}), \qquad k \in \mathbb{N}, \tag{1}$$

where $\psi_k : \mathbb{R}^d \times \mathbb{R}^N \to \mathbb{R}^d$ is a smooth function such that $\psi_k(x,0) = x$ and $Z_k \in \mathbb{R}^N$, $k \in \mathbb{N}$ is a sequence of independent random variables. The semigroup of the Markov chain $X_k^n$ is denoted

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by $P_k^n$ and the transition probabilities are $\mu_k^n(x, dy) = \mathbb{P}(X_{k+1}^n(x) \in dy | X_k^n = x)$. Moreover we consider a Markov process in continuous time $(X_t)_{t \geqslant 0}$ with semigroup $P_t$ and we denote $\nu_k^n(x, dy) = \mathbb{P}(X_{t_k} \in dy | X_{t_k} = x)$ where $t_k = k\delta = \frac{k}{n}$ with $\delta = \frac{1}{n}$. A first standard result is the following: let us assume that there exists $h > 0$, $p \in \mathbb{N}$ such that for every $f \in \mathcal{C}^p(\mathbb{R}^d)$, $k \in \mathbb{N}$ and $x \in \mathbb{R}^d$,

$$\left| \mu_k^n f(x) - \nu_k^n f(x) \right| = \left| \int f(y) \mu_k^n(x, dy) - \int f(y) \nu_k^n(x, dy) \right| \leqslant C \|f\|_{p,\infty} \delta^{1+h} \tag{2}$$

where $\|f\|_{p,\infty}$ denotes the supremum norm of $f$ and of its derivatives up to order $p$. Then, for every $T > 0$,

$$\sup_{t_k \leqslant T} \|P_{t_k} f - P_k^n f\|_\infty \leqslant C \|f\|_{p,\infty} \delta^h. \tag{3}$$

It means that $(X_k^n)_{k \in \mathbb{N}}$ is an approximation scheme of weak order $h$ for the Markov process $(X_t)_{t \geqslant 0}$. In the case of the Euler scheme for diffusion processes, this result, with $h = 1$, has initially been proved in the seminal papers of Milstein [26] and of Talay and Tubaro [32] (see also [17]). Similar results were obtained in various situations: diffusion processes with jumps (see [31], [15]) or diffusion processes with boundary conditions (see [12], [11], [13]). See [16] for an overview of this subject. More recently, approximation schemes of higher orders (*e.g.*, $h = 2$), based on a cubature method, have been introduced and studied by Kusuoka [21], Lyons [25], Ninomiya, Victoir [27], Alfonsi [1], Kohatsu-Higa and Tankov [18].

Another result concerns convergence in total variation distance: we want to obtain (3) with $\|f\|_{p,\infty}$ replaced by $\|f\|_\infty$ when $f$ is a measurable function. In the case of the Euler scheme for diffusion processes, a first result of this type has been obtained by Bally and Talay [6], [7] using the Malliavin calculus (see also Guyon [14]). Afterwards Konakov, Menozzi and Molchanov [19], [20] obtained similar results using a parametrix method. Recently Kusuoka [22] obtained estimates of the error in total variation distance for the Victoir Ninomiya scheme (which corresponds to the case $h = 2$). We will obtain a similar result using our approach. Moreover, we give estimates of the rate of convergence of the density function and its derivatives.

**Regularization properties.**   We first remark that the crucial property which is used in order to replace $\|f\|_{p,\infty}$ by $\|f\|_\infty$ in (3), is the regularization property of the semigroup. Let us be more precise: let $\eta > 0$, $p \in \mathbb{N}$ be fixed. Given the time grid $t_k = k\delta$, we say that a semigroup $(P_k)_{k \in \mathbb{N}}$ satisfies $R_{p,\eta}$, if

$$R_{p,\eta} \qquad \|P_k f\|_{p,\infty} \leqslant \frac{C}{t_k^{\eta p}} \|f\|_\infty. \tag{4}$$

We also introduce a dual regularization property: we consider the dual semigroup $P_k^*$ (*i.e.* $\langle P_k^* g, f \rangle = \langle g, P_k f \rangle$ with the scalar product in $\mathrm{L}^2(\mathbb{R}^d)$) and we assume that

$$R_{p,\eta}^* \qquad \|P_k^* f\|_{p,1} \leqslant \frac{C}{t_k^{\eta p}} \|f\|_1, \tag{5}$$

where $\|f\|_{p,1}$ denotes the $\mathrm{L}^1$ norm of $f$ and of its derivatives up to order $p$. Finally, we consider the following stronger regularization property: for every multi-index $\alpha, \beta$ with $|\alpha| + |\beta| = p$,

$$\overline{R}_{p,\eta} \qquad \|\partial^\alpha P_k \partial^\beta f\|_\infty \leqslant \frac{C}{t_k^{\eta p}} \|f\|_\infty. \tag{6}$$

One notices that $\overline{R}_{p,\eta}$ implies both $R_{p,\eta}$ and $R_{p,\eta}^*$ and that a semigroup satisfying $\overline{R}_{p,\eta}$ is absolutely continuous with respect to the Lebesgue measure.

In addition to (2), we will also suppose that the following dual estimate of the error in short time holds:

$$| \langle g, (\mu_k^n - \nu_k^n)f \rangle | \leqslant C\|g\|_{p,1}\|f\|_\infty \delta^{1+h}. \tag{7}$$

Keeping these properties in mind, we can state the following result.

**Theorem 1.1.** *We fix $T, h > 0$, $p \in \mathbb{N}$ and we assume that the short time estimates (2) and (7) hold (with this $p$ and $h$). Moreover, we assume that (4) holds for $(P_{t_k})_{k\in\mathbb{N}}$ and that (5) holds for $(P_k^n)_{k\in\mathbb{N}}$ . Then,*

$$\forall 0 < S \leqslant T, \quad \sup_{S \leqslant t_k \leqslant T}\|P_{t_k}f - P_k^n f\|_\infty \leqslant \frac{C}{S^{\eta p}}\|f\|_\infty \delta^h. \tag{8}$$

**Integration by parts formulae.** Once we have this abstract result, the following step is to give sufficient conditions in order to obtain $R_{p,\eta}$, $R_{p,\eta}^*$ and $\overline{R}_{p,\eta}$. We will use Malliavin type integration by parts formulae based on the noise $Z_k \in \mathbb{R}^N$. In order to do it, we assume that the law of each $Z_k$ is locally lower bounded by the Lebesgue measure: there exists some $z_{*,k} \in \mathbb{R}^N$ and $r_*, \varepsilon_* > 0$ such that for every measurable set $A \subset B_{r_*}(z_{*,k})$ one has

$$\mathbb{P}(Z_k \in A) \geqslant \varepsilon_* \lambda(A) \tag{9}$$

where $\lambda$ is the Lebesgue measure. If this property holds then a "splitting method" can be used in order to represent $Z_k$ as

$$\frac{Z_k}{\sqrt{n}} = \chi_k U_k + (1 - \chi_k)V_k,$$

where $\chi_k, U_k, V_k$ are independent random variables, $\chi_k$ is a Bernoulli random variable and $\sqrt{n}U_k \sim \varphi_{r_*}(u)du$ with $\varphi_{r_*} \in \mathcal{C}^\infty(\mathbb{R}^N)$. Then we use the abstract Malliavin calculus based on $U_k$, developed in [5] and [3], in order to obtain integration by parts formulae. The crucial point is that the density $\varphi_{r_*}$ of $\sqrt{n}U_k$ is smooth and we control its logarithmic derivatives. Using this, we construct integration by parts formulae and obtain relevant estimates for the weights which appear in these formulae. It is worth mentioning that, a variant of the Malliavin calculus based on a similar splitting method has already been used by Nourdin and Poly [29] (see also [28] and [23]). They use the so called $\Gamma$ calculus introduced by Backry, Gentil and Ledoux [2]. Roughly speaking the difference between the approach in our paper and the one in [2] is the following: our construction is similar to the "simple functionals" approach in Malliavin calculus and has the derivative operator as basic object. In contrast, in the $\Gamma$ calculus, the basic object is the Ornstein Uhlenbeck operator.

In order to state the main result of our paper, we introduce some additional assumptions:

$$\forall p \in \mathbb{N}, \quad \sup_{k \in \mathbb{N}} \mathbb{E}[|Z_k|^p] < \infty, \tag{10}$$

$$\forall r \in \mathbb{N}^*, \quad \sup_{k \in \mathbb{N}} \sum_{1 \leqslant |\beta| \leqslant r} \sum_{0 \leqslant |\alpha| \leqslant r - |\beta|} \|\partial_x^\alpha \partial_z^\beta \psi_k\|_\infty < \infty, \tag{11}$$

$$\exists \lambda_* > 0, \ \forall k \in \mathbb{N}, \quad \inf_{x \in \mathbb{R}^d} \inf_{|\zeta|=1} \sum_{i=1}^N \langle \partial_{z_i} \psi_k(x,0), \zeta \rangle^2 \geqslant \lambda_*. \tag{12}$$

Moreover, we introduce the following regularized version of the approximation scheme $X_k^n$:

$$X_k^{n,\theta}(x) = \frac{1}{n^\theta} G + X_k^n(x), \tag{13}$$

with $G$ a standard normal random variable independent from $X_k^n$ and $\theta > h + 1$. Here $X_k^n(x)$ is the Markov chain which starts from $x$: $X_0^n(x) = x$. We denote

$$P^{n,\theta}(x, dy) = \mathbb{P}(X_k^{n,\theta}(x) \in dy) = p_k^{n,\theta}(x,y) dy. \tag{14}$$

**Theorem 1.2.** *Consider a Markov semigroup $(P_t)_{t \geqslant 0}$ and the approximation Markov chain $(P_k^n)_{k \in \mathbb{N}}$ defined in (1). We fix $T, h > 0$, $p \in \mathbb{N}$ and we assume that the short time estimates (2) and (7) hold (with this $p$ and $h$). Moreover, we assume (9), (10), (11) and (12).*
**A**. *For every $0 < S \leqslant T$, we have*

$$\sup_{S \leqslant t_k \leqslant T} \|P_{t_k} f - P_k^n f\|_\infty \leqslant \frac{C}{S^p} \|f\|_\infty \delta^h. \tag{15}$$

**B**. *For every $t > 0$, $P_t(x, dy) = p_t(x,y) dy$ with $(x,y) \to p_t(x,y)$ belonging to $\mathcal{C}^\infty(\mathbb{R}^d \times \mathbb{R}^d)$.*

**C**. *For every $R, \varepsilon > 0$ and every multi-index $\alpha, \beta$, we have*

$$\sup_{S \leqslant t_k \leqslant T} \sup_{|x|+|y| \leqslant R} |\partial_x^\alpha \partial_y^\beta p_{t_k}(x,y) - \partial_x^\alpha \partial_y^\beta p_k^{n,\theta}(x,y)| \leqslant C_\varepsilon \delta^{h(1-\varepsilon)}. \tag{16}$$

We notice that (15) gives the total variation distance between the semigroups $(P_t)_{t \geqslant 0}$ and $(P_k^n)_{k \in \mathbb{N}}$. Once the appropriate regularization properties are obtained (using the abstract Malliavin calculus), the proof of (15) is rather elementary. In contrast, the estimate (16) is based on a non trivial interpolation result recently obtained in [8]. Notice, however, that the estimate (16) is sub-optimal (because of $\varepsilon > 0$). We will illustrate (15) by taking $X_k^n$ to be the Ninomiya Victoir scheme of a diffusion process. This is a variant of the result already obtained by Kusuoka [22] in the case where $Z_k$ has a Gaussian distribution (and so the standard Malliavin calculus is available). Since in our paper $Z_k$ has an arbitrary distribution (except the property (9)), our result may be seen as an invariance principle as well.

The paper is organized as follows. In Section 2, we prove Theorem 1.1. In Section 3, we settle the abstract Malliavin calculus based on the splitting method and we use it in order to prove the regularization properties for the approximation scheme $X_k^n$ (in fact for the regularization $X_k^{n,\theta}$) and we prove Theorem 1.2. Finally, in Section 4, we use the previous results in order to give estimates of the total variation distance for the Ninomiya Victoir approximation scheme. In the Appendix, we prove some technical estimates concerning the Sobolev norms of $X_k^n$.

# 2 The distance between two Markov semigroups

Throughout this section the following notations will prevail. We fix $T > 0$ the horizon of the underlying processes and we denote $n \in \mathbb{N}^*$, the number of time step between $0$ and $T$. Then, we set $\delta := \delta_n = \frac{1}{n}$ and introduce the time homogeneous time grid $t_k = k T \delta = k T / n$. Notice that, all the results from this paper remains true with non homogeneous time step but, for sake of simplicity, we will not consider this case. First, we state some results for smooth test functions.

## 2.1 Regular test functions

We consider a sequence of finite transition measures $\mu_k(x, dy)$, $k \in \mathbb{N}$ from $\mathbb{R}^d$ to $\mathbb{R}^d$. This means that for each fixed $x$ and $k$, $\mu_k(x, dy)$ is a finite measure on $\mathbb{R}^d$ with the borelian $\sigma$ field and for each bounded measurable function $f : \mathbb{R}^d \to \mathbb{R}$, the application

$$x \mapsto \mu_k f(x) := \int_{\mathbb{R}^d} f(y) \mu_k(x, dy) \tag{17}$$

is measurable. We also denote

$$|\mu_k| := \sup_{x \in \mathbb{R}^d} \sup_{\|f\|_\infty \leqslant 1} \Big| \int_{\mathbb{R}^d} f(y) \mu_k(x, dy) \Big|, \tag{18}$$

and, we assume that all the sequences of measures we consider in this paper satisfies:

$$\sup_{k \in \mathbb{N}} |\mu_k| < \infty. \tag{19}$$

Although the main application concerns the case where $\mu_k(x, dy)$ is a probability measure, we do not assume this here: we allow $\mu_k(x, dy)$ to be a signed measure of finite (but arbitrary) total mass. This is because one may use the results from this section not only in order to estimate the distance between two semigroups but also in order to obtain a development of the error. To the sequence $\mu_k$, $k \in \mathbb{N}$ we associate the discrete semigroup

$$P_0 f(x) = f(x), \qquad P_{k+1} f(x) = \mu_{k+1} P_k f(x) = \int_{\mathbb{R}^d} P_k f(y) \mu_{k+1}(x, dy).$$

More generally, for $r \geqslant k$ we define $P_{k,r} f$ by

$$P_{k,k} f(x) = f(x), \qquad P_{k,r+1} f(x) = \mu_{r+1} P_{k,r} f(x).$$

For $f \in \mathcal{C}^\infty(\mathbb{R}^d)$ and for a multi-index $\alpha = (\alpha_1, \cdots, \alpha_d) \in \mathbb{N}^d$ we denote $|\alpha| = \alpha_1 + ... + \alpha_d$ and $\partial_\alpha f = \partial_{x_1}^{\alpha_1} ... \partial_{x_d}^{\alpha_d} f(x)$. We include the multi-index $\alpha = (0, ..., 0)$ and in this case $\partial_\alpha f = f$. We use the norms

$$\|f\|_{p,\infty} = \sup_{x \in \mathbb{R}^d} \sum_{0 \leqslant |\alpha| \leqslant p} |\partial_\alpha f(x)|, \qquad \|f\|_{p,1} = \sum_{0 \leqslant |\alpha| \leqslant p} \int_{\mathbb{R}^d} |\partial_\alpha f(x)| dx.$$

In particular $\|f\|_{0,\infty} = \|f\|_\infty$ is the usual supremum norm.

We will consider the following hypothesis: let $p \in \mathbb{N}$ and $0 \leqslant k \leqslant r$. If $f \in \mathcal{C}^p(\mathbb{R}^d)$ then $P_{k,r}f \in \mathcal{C}^p(\mathbb{R}^d)$ and

$$\sup_{0 \leqslant t_k \leqslant t_r} \|P_{k,r}f\|_{p,\infty} \leqslant C\|f\|_{p,\infty}. \tag{20}$$

We consider now a second sequence of finite transition measures $\nu_k(x, dy), k \in \mathbb{N}$ and the corresponding semigroup $Q_k$ defined as above. Our aim is to estimate the distance between $P_kf$ and $Q_kf$ in terms of the distance between the transition measures $\mu_k(x, dy)$ and $\nu_k(x, dy)$, so we denote

$$\Delta_k = \mu_k - \nu_k.$$

$P_k$ can be seen as a semigroup in continuous time considered on the time grid $t_k$, $k \in \mathbb{N}$, while $Q_k$ would be its approximation discrete semigroup. Let $p \in \mathbb{N}$, $h \geqslant 0$ be fixed. We introduce a short time error approximation assumption: there exists a constant $C > 0$ (depending on $p$ only) such that for every $k \in \mathbb{N}$, we have

$$E_n(h, p) \qquad \|\Delta_k f\|_\infty \leqslant C\|f\|_{p,\infty}\delta^{h+1}. \tag{21}$$

**Proposition 2.1.** *Let $p \in \mathbb{N}$ be fixed. Suppose that $\mu_k$ and $\nu_k$ satisfy (20) and (21) with $p = 0$. Then for every $f \in \mathcal{C}^p(\mathbb{R}^d)$,*

$$\sup_{t_m \leqslant T} \|P_mf - Q_mf\|_\infty \leqslant C\|f\|_{p,\infty}\delta^h. \tag{22}$$

*Proof.* We have

$$\|P_mf - Q_mf\|_\infty \leqslant \sum_{k=0}^{m-1} \|P_{k+1,m}P_{k,k+1}Q_kf - P_{k+1,m}Q_{k,k+1}Q_kf\|_\infty \tag{23}$$

$$= \sum_{k=0}^{m-1} \|P_{k+1,m}\Delta_{k+1}Q_kf\|_\infty.$$

Using (19) and (21), we obtain

$$\|P_{k+1,m}\Delta_{k+1}Q_kf\|_\infty \leqslant C\|\Delta_{k+1}Q_kf\|_\infty \leqslant C\delta^{1+h}\|Q_kf\|_{p,\infty} \leqslant C\delta^{1+h}\|f\|_{p,\infty}.$$

Summing over $k = 0, ..., m-1$, we conclude. $\qquad \square$

## 2.2   Measurable test functions (convergence in total variation distance)

The estimate (22) requires a lot of regularity for the test function $f$. Our aim is to show that, if the semigroups at work have a regularization property, then we may obtain estimates of the error for measurable test functions. In order to state this result we have to give some hypothesis on the adjoint semigroup. Let $p \in \mathbb{N}$. We assume that there exists a constant $C \geqslant 1$ such that for every measurable function $f$ and any $g \in \mathcal{C}^p(\mathbb{R}^d)$

$$E_n^*(h, p) \qquad |\langle g, \Delta_k f \rangle| \leqslant C\|g\|_{p,1}\|f\|_\infty\delta^{1+h}. \tag{24}$$

where $\langle g, f \rangle = \int g(x) f(x) dx$ is the scalar product in $\mathrm{L}^2(\mathbb{R}^d)$.

Our regularization hypothesis is the following. Let $p \in \mathbb{N}$, $S > 0$ and $\eta \geqslant 0$ be given. We assume that there exists a constant $C \geqslant 1$ such that

$$R_{p,\eta}(S) \qquad \|P_{k,r}f\|_{p,\infty} \leqslant \frac{C}{S^{\eta p}} \|f\|_{\infty} \qquad \text{for} \qquad S \leqslant t_r - t_k. \tag{25}$$

We also consider the "adjoint regularization hypothesis". We assume that there exists an adjoint semigroup $P_{k,r}^*$, that is

$$\left\langle P_{k,r}^* g, f \right\rangle = \langle g, P_{k,r} f \rangle$$

for every bounded measurable function $f$ and every function $g \in \mathcal{C}_c^\infty(\mathbb{R}^d)$. We assume that $P_{k,r}^*$ satisfies (20) and moreover

$$R_{p,\eta}^*(S) \qquad \|P_{k,r}^* f\|_{p,1} \leqslant \frac{C}{S^{\eta p}} \|f\|_1 \qquad \text{for} \qquad S \leqslant t_r - t_k. \tag{26}$$

Notice that a sufficient condition in order that $R_{p,\eta}^*(S)$ holds is the following: for every multi index $\alpha$ with $|\alpha| \leqslant p$

$$\|P_{k,r} \partial_\alpha f\|_\infty \leqslant \frac{C}{S^{\eta p}} \|f\|_\infty \qquad \text{for} \qquad S \leqslant t_r - t_k. \tag{27}$$

Indeed:

$$\begin{aligned}
\|\partial_\alpha P_{k,r}^* f\|_1 &\leqslant \sup_{\|g\|_\infty \leqslant 1} |\langle \partial_\alpha P_{k,r}^* f, g \rangle| = \sup_{\|g\|_\infty \leqslant 1} |\langle f, P_{k,r}(\partial_\alpha g) \rangle| \\
&\leqslant \|f\|_1 \sup_{\|g\|_\infty \leqslant 1} \|P_{k,r}(\partial_\alpha g)\|_\infty \leqslant \frac{C}{S^{\eta p}} \|f\|_1.
\end{aligned}$$

**Proposition 2.2.** *Let $p \in \mathbb{N}$, $\eta > 0$, $h \geqslant 0$ and $0 < S \leqslant T/2$ be fixed. We suppose that (20), (21) and (24) hold for $P_m$ and $Q_m$. We also suppose that $P$ satisfies $R_{p,\eta}(S)$ (see (25)) and $Q$ satisfies $R_{p,\eta}^*(S)$ (see (26)). Then,*

$$\sup_{2S \leqslant t_m \leqslant T} \|P_m f - Q_m f\|_\infty \leqslant \frac{C}{S^{\eta p}} \|f\|_\infty \delta^h. \tag{28}$$

*Proof.* Using a density argument we may assume that $f \in \mathcal{C}^p(\mathbb{R}^d)$. Moreover, by (23), it is sufficient to prove that

$$\|Q_{k+1,m} \Delta_{k+1} P_k f\|_\infty \leqslant \frac{C}{S^{\eta p}} \|f\|_\infty \delta^{1+h}. \tag{29}$$

Since $t_m \geqslant 2S$ we have $t_k \geqslant S$ or $t_m - t_{k+1} \geqslant S$. Suppose first that $t_k \geqslant S$. Using (19) for $Q$, (21) and (25) for $P$,

$$\|Q_{k+1,m} \Delta_{k+1} P_k f\|_\infty \leqslant C \|\Delta_{k+1} P_k f\|_\infty \leqslant C \|P_k f\|_{p,\infty} \delta^{1+h} \leqslant C S^{-\eta p} \|f\|_\infty \delta^{1+h}.$$

Suppose now that $t_m - t_{k+1} \geqslant S$. We take $\phi_\varepsilon(x) = \varepsilon^{-d}\phi(\varepsilon^{-1}x)$ with $\phi \in \mathcal{C}_c(\mathbb{R}^d)$, $\phi \geqslant 0$. Then, for a fixed $x_0$, we define $\phi_{\varepsilon,x_0}(x) = \phi_\varepsilon(x - x_0)$. By (20) $Q_{k+1,m}\Delta_k P_k f \in \mathcal{C}^p(\mathbb{R}^d)$ so it is continuous. Then

$$|Q_{k+1,m}\Delta_k P_k f(x_0)| = \lim_{\varepsilon \to 0} |\langle \phi_{\varepsilon,x_0}, Q_{k+1,m}\Delta_k P_k f \rangle|.$$

Using (24), (26) and then (20), we obtain

$$|\langle \phi_{\varepsilon,x_0}, Q_{k+1,m}\Delta_{k+1} P_k f \rangle| = |\langle Q^*_{k+1,m}\phi_{\varepsilon,x_0}, \Delta_{k+1} P_k f \rangle| \leqslant C\|Q^*_{k+1,m}\phi_{\varepsilon,x_0}\|_{p,1}\|P_k f\|_\infty \delta^{1+h}$$
$$\leqslant CS^{-\eta p}\|\phi_{\varepsilon,x_0}\|_1\|f\|_\infty \delta^{1+h}$$

and since $\|\phi_{\varepsilon,x_0}\|_1 = \|\phi\|_1 \leqslant C$, the proof is completed. $\qquad\square$

In concrete applications the following slightly more general variant of the above proposition will be useful.

**Proposition 2.3.** *Let $p \in \mathbb{N}$, $\eta > 0$, $h \geqslant 0$ and $0 < S \leqslant T/2$ be fixed. We assume that (20), (21) and (24) hold for $P$ and $Q$ with these $p, \eta, h$ and $S$. Moreover, we assume that there exists some kernels $\overline{P}_{k,r}$ which satisfies $R_{p,\eta}(S)$ (see(25)) and $\overline{Q}_{k,r}$ which satisfies $R^*_{p,\eta}(S)$ (see (26)). We also assume that for every $0 \leqslant k \leqslant r$, with $t_r - t_k \geqslant S$*

$$\|Q_{k,r}f - \overline{Q}_{k,r}f\|_\infty + \|P_{k,r}f - \overline{P}_{k,r}f\|_\infty \leqslant CS^{-\eta p}\delta^{h+1}\|f\|_\infty. \tag{30}$$

*Then,*

$$\sup_{2S \leqslant t_m \leqslant T} \|P_m f - Q_m f\|_\infty \leqslant C \sup_{k \leqslant n}(|\mu_k| + |\nu_k|)S^{-\eta p}\delta^h\|f\|_\infty. \tag{31}$$

**Remark 2.1.** *Notice that $\overline{P}_{k,r}$ and $\overline{Q}_{k,r}$ are not supposed to satisfy the semigroup property and are not directly related to $\mu_k$ and $\nu_k$.*

*Proof.* The proof follows the same line as the one of the previous proposition. Suppose first that $t_k \geqslant S$. Then, (19) implies

$$\|Q_{k,m}\Delta_{k+1}P_k f\|_\infty \leqslant \|Q_{k,m}\Delta_{k+1}\overline{P}_k f\|_\infty + \|Q_{k,m}\Delta_{k+1}(P_k - \overline{P}_k)f\|_\infty$$
$$\leqslant \|\Delta_{k+1}\overline{P}_k f\|_\infty + \|\Delta_{k+1}(P_k - \overline{P}_k)f\|_\infty.$$

Since $\overline{P}_k$ verifies $R_{p,\eta}(S)$, we deduce from (21) that

$$\|\Delta_{k+1}\overline{P}_k f\|_\infty \leqslant C\delta^{1+h}\|\overline{P}_k f\|_{p,\infty} \leqslant CS^{-p\eta}\|f\|_\infty \delta^{1+h}.$$

Using (30), it follows

$$|\Delta_{k+1}(P_k - \overline{P}_k)f(x)| \leqslant |\int (P_k - \overline{P}_k)f(y)\nu_{k+1}(x,dy)| + |\int (P_k - \overline{P}_k)f(y)\mu_{k+1}(x,dy)|$$
$$\leqslant (|\nu_{k+1}| + |\mu_{k+1}|)\|(P_k - \overline{P}_k)f\|_\infty$$
$$\leqslant C(|\nu_{k+1}| + |\mu_{k+1}|)S^{-\eta p}\|f\|_\infty \delta^{h+1}.$$

Suppose now that $t_m - t_{k+1} \geqslant S$. We write

$$\|Q_{k+1,m}\Delta_{k+1}P_k f\|_\infty \leqslant \|\overline{Q}_{k+1,m}\Delta_{k+1}P_k f\|_\infty + \|(Q_{k+1,m} - \overline{Q}_{k+1,m})\Delta_{k+1}P_k f\|_\infty.$$

In order to bound $\|\overline{Q}_{k+1,m}\Delta_{k+1}P_k f\|_\infty$ we use the same reasoning as in the proof of the previous proposition. And the second term is bounded using (30). $\qquad\square$

## 2.3 Convergence of the density functions

In this section we will consider a Markov semigroup $(P_t)_{t \geq 0}$ and we will give an approximation result and a regularity criterion for it. The regularization property that we assume for the approximation processes is stronger than the one considered in the previous section and, instead of Proposition 2.2 we will use a general approximation result based on an interpolation inequality, proved in [8]. We recall that we have fixed $T > 0$ and that, $\delta = \delta^n = 1/n$ and we denote $t_k^n = t_k = \frac{kT}{n}$. For $k \in \mathbb{N}$, we consider $\mu_k^n(x, dy) = \mu^n(x, dy) = P_\delta(x, dy)$, for all $k \in \mathbb{N}$, the homogeneous sequence of finite transition measures which satisfy (20). Moreover we introduce a sequence of transition probability measures $\nu_k^n(x, dy)$, $k \in \mathbb{N}$, and the corresponding discrete semigroups $P^n(x, dy)$ defined by $P_{k,k}^n = Id$ and $P_{k,r+1}^n = \nu_{r+1}^n P_{k,r}^n$. We recall that $P_k^n f = P_{0,k}^n f$. We assume that for $f \in \mathcal{C}^p(\mathbb{R}^d)$, we have $P_k^n f \in \mathcal{C}^p(\mathbb{R}^d)$ and it verifies (20) :

$$\sup_{0 \leq t_k^n \leq t_r^n} \|P_{k,r}^n f\|_{p,\infty} \leq C\|f\|_{p,\infty}. \tag{32}$$

For $h > 0$ and $p \in \mathbb{N}$, we assume that for all $n \in \mathbb{N}$,

$$E_n(h, p) \qquad \|(\mu^n - \nu_k^n)f\|_\infty \leq C\delta^{1+h}\|f\|_{p,\infty}. \tag{33}$$

and,

$$E_n^*(h, p) \qquad |\langle g, (\mu^n - \nu_k^n)f\rangle| \leq C\|g\|_{p,1}\|f\|_\infty \delta^{1+h}. \tag{34}$$

We introduce now $(\overline{P}_k^n)_{k \in \mathbb{N}}$, a modification of $(P_k^n)_{k \in \mathbb{N}}$ in the sense that for every measurable and bounded function $f : \mathbb{R}^d \to \mathbb{R}$, we have

$$\sup_{S \leq t_r^n - t_k^n} \|P_{k,r}^n f - \overline{P}_{k,r}^n f\|_\infty \leq \frac{C}{S^{\eta p}}\delta^{h+1}\|f\|_\infty. \tag{35}$$

We assume that $(\overline{P}_k^n)_{k \in \mathbb{N}}$ satisfies the following strong regularization property. We fix $q \in \mathbb{N}$ $S, \eta > 0$, and we assume that for every multi-index $\alpha, \beta$ with $|\alpha| + |\beta| \leq q$ and $f \in \mathcal{C}^q(\mathbb{R}^d)$ one has

$$\overline{R}_{p,\eta}(S) \qquad \|\partial^\alpha \overline{P}_{k,r}^n \partial^\beta f\|_\infty \leq CS^{-\eta p}\|f\|_\infty, \qquad t_r^n - t_k^n \geq S. \tag{36}$$

Notice that if $\overline{R}_{q+d,\eta}(S)$ holds, then there exists $\overline{p}_k^n \in \mathcal{C}^q(\mathbb{R}^d \times \mathbb{R}^d)$ such that $\overline{P}_k^n(x, dy) = \overline{p}_k^n(x, y)dy$ and, for every $R \geq 1$, $t_k^n \geq S$ and $|\alpha| + |\beta| \leq q$, then

$$\sup_{|x|+|y| \leq R} |\partial_x^\alpha \partial_y^\beta \overline{p}_k^n(x, y)| \leq CS^{-\eta q}. \tag{37}$$

Moreover, the regularization properties $R_{p,\eta}(S)$ and $R_{p,\eta}^*(S)$ hold when $\overline{R}_{p,\eta}(S)$ is satisfied.

**Theorem 2.1. A**. *We fix $p \in \mathbb{N}$, $p$ and $h, \eta > 0$, $0 \leq S \leq T/2$, and we assume that (20) holds for $P$ and that (32), (33), (34), (35) and (36) hold for this $p, h, \eta$ and $S$, and for every $n \in \mathbb{N}$. Then*

$$\sup_{2S \leq t_k^n \leq T} \|P_{t_k^n}f - P_k^n f\|_\infty \leq \frac{CS^{-\eta p}}{n^h}\|f\|_\infty. \tag{38}$$

***B.*** *Suppose that (36) holds for every* $p \in \mathbb{N}$*. Then, for every* $t > 0$*,* $P_t(x, dy) = p_t(x, y)dy$ *with* $(x, y) \rightarrow p_t(x, y)$ *belonging to* $\mathcal{C}^\infty(\mathbb{R}^d \times \mathbb{R}^d)$*.*
***C.*** *For every* $R, \varepsilon > 0$ *and every multi-index* $\alpha, \beta$ *we have*

$$\sup_{2S \leqslant t_k^n \leqslant T} \sup_{|x|+|y| \leqslant R} |\partial_x^\alpha \partial_y^\beta p_{t_k^n}(x, y) - \partial_x^\alpha \partial_y^\beta \overline{p}_k^n(x, y)| \leqslant \frac{C}{n^{h(1-\varepsilon)}} \tag{39}$$

*with a constant* $C$ *which depends on* $R, S, T, \varepsilon$ *and on* $|\alpha| + |\beta|$ *(and may go to infinity as* $\varepsilon \downarrow 0$*).*

**Remark 2.2.** *The inequality (38) is essentially a consequence of Proposition 2.3. However, we may not use directly this result, because we do not assume that the semigroup* $(P_t)_{t \geqslant 0}$ *has the regularization property (25). This is pleasant, because we have to check the regularization property on the approximation scheme* $(P_k^n)_{k \in \mathbb{N}}$ *only.*

**Remark 2.3.** *The estimate (39) is sub-optimal because of* $\varepsilon > 0$*. One may wonder if optimal estimates (with* $n^h$ *instead of* $n^{h(1-\varepsilon)}$*) may be obtained - as it was the case in the paper of Bally and Talay [6] concerning the Euler scheme. Notice that, in the above paper, specific properties related to the dynamics of the diffusion process which gives the semigoup are used, and in particular properties of the tangent flow. For example, if* $X_t(x)$ *denotes the diffusion process starting from* $x$ *then we have* $\mathbb{E}[f'(X_t(x))] = \partial_x \mathbb{E}[f(X_t(x))(\partial_x X_t(x))^{-1}] - \mathbb{E}[f(X_t(x))\partial_x(\partial_x X_t(x))^{-1})]$*. Such properties are crucial in the above paper - but are difficult to express in terms of general semigroups.*

*Proof.* Let $m = \zeta n$, $\zeta \in \mathbb{N}^*$. Using (33), (34), we obtain $\|(P_{k,k+1}^n - P_{\zeta k, \zeta(k+1)}^m)f\|_\infty \leqslant C\|f\|_{p,\infty}n^{-h-1}$ and $|\langle g, (P_{k,k+1}^n - P_{\zeta k, \zeta(k+1)}^m)f\rangle| \leqslant C\|g\|_{p,1}\|f\|_\infty n^{-h-1}$. Then Proposition 2.3 implies that: $\forall 2S \leqslant t_k^n \leqslant T$, $\|P_k^m f - P_k^n f\|_\infty \leqslant CS^{-\eta p}n^{-h}\|f\|_\infty$. So the sequence $(P^n)_{n \in \mathbb{N}}$ is Cauchy and then converges with rate $CS^{-\eta p}n^{-h}\|f\|_\infty$. It remains to identify the limit. By Proposition 2.1, this limit is $(P_t f)_{t \geqslant 0}$ for $f \in \mathcal{C}_b^p$, so we conclude.

Let us prove **B**. We are going to use a result from [8]. First, we introduce some notations. Let $d_p$ be the distance defined by

$$d_p(\mu, \nu) = \sup \left\{|\int f d\mu - \int f d\nu\| : \|f\|_{p,\infty} \leqslant 1\right\}.$$

For $q, l \in \mathbb{N}$, $r > 1$ and $f \in \mathcal{C}^q(\mathbb{R}^d \times \mathbb{R}^d)$, we denote

$$\|f\|_{q,l,r} = \sum_{0 \leqslant |\alpha| \leqslant q} \left(\int\int(1 + |x|^l + |y|^l)|\partial_\alpha f(x, y)|^r dx dy\right)^{1/r}.$$

We have the following result which is Theorem 2.11 from [8].

**Theorem 2.2.** *Let* $q, p, l, m \in \mathbb{N}$ *and* $r > 1$ *be given and let* $r^*$ *be the conjugate of* $r$*. Consider some measures* $\mu(dx, dy)$ *and* $\mu_n(dx, dy) = g_n(x, y)dxdy$ *with* $g_n \in \mathcal{C}^{q+2m}(\mathbb{R}^d \times \mathbb{R}^d)$*. Suppose that for some* $\alpha > (q + p + d/r^*)/m$*, we have*

$$d_p(\mu, \mu_n)\|g_n\|_{q+2m,2m,r}^\alpha \leqslant C \qquad \forall n \in \mathbb{N}. \tag{40}$$

*Then $\mu(dx, dy) = g(x, y)dxdy$ with $g \in W^{q,r}(\mathbb{R}^d)$ and*

$$\|g - g_n\|_{W^{q,r}(\mathbb{R}^d)} \leqslant C(d_p^{1/\alpha}(\mu, \mu_n) + d_p^{1-(q+p+d/r_*)/\alpha m}(\mu, \mu_n)). \tag{41}$$

*In particular, if (40) is satisfied for all $m \in \mathbb{N}^*$ and any $\alpha > (q + p + d/r^*)/m$, then for every $\varepsilon > 0$ we have*

$$\|g - g_n\|_{W^{q,r}(\mathbb{R}^d)} \leqslant C d_0^{1-\varepsilon}(\mu, \mu_n). \tag{42}$$

*with a constant $C$ which depends on $\varepsilon$ and may go to infinity as $\varepsilon \downarrow 0$.*

We come back to our framework. We fix $R > 0$, $2S \leqslant t \leqslant T$ and choose $k(n, t) \in \mathbb{N}$ such that $t_{k(n,t)}^n = t$. Let us consider a function $\Phi_R \in \mathcal{C}_b^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ such that $\mathbb{1}_{B_R \times B_R}(x, y) \leqslant \Phi_R(x, y) \leqslant \mathbb{1}_{B_{R+1} \times B_{R+1}}(x, y)$ and denote

$$g_{k(n,t)}^{n,R}(x, y) = \Phi_R(x, y)\overline{p}_{k(n,t)}^n(x, y).$$

We use the result above for the sequence $g_n := g_{k(n,t)}^{n,R}$, $n \in \mathbb{N}$ and $\mu(dx, dy) := P_t(x, dy)dx$. In our specific case (35) and (38) give $d_0(g, g_n) \leqslant Cn^{-h}$ and the hypothesis (36) ensures that $\sup_n \|g_n\|_{q+2m,2m,r} < \infty$ so (40) holds for every $\alpha \in \mathbb{R}_+$ and $r > 1$. Using Sobolev's embedding theorem, for $u \leqslant q - d/r$ we have

$$\|g - g_n\|_{u,\infty} \leqslant C\|g - g_n\|_{W^{q,r}(\mathbb{R}^d)} \leqslant Cn^{-h(1-\varepsilon)}$$

and we conclude. $\qquad\square$

# 3 A class of Markov chains

## 3.1 Integration by parts using a splitting method

In this section we consider a sequence of independent random variables $Z_k = (Z_k^1, \cdots, Z_k^N) \in \mathbb{R}^N$, $k \in \{1, \cdots, n\}$ and we denote $Z = (Z_1, ..., Z_n)$. The number $n$ is fixed throughout this section (so there is no asymptotic procedure going on; but morally $n$ is large because we are interested in estimating the error as $n \to \infty$). Our aim is to settle an integration by parts formula based on the law of $Z$. The basic assumption is the following: there exists $z_{*,k} \in \mathbb{R}^N$ and $\varepsilon_*, r_* > 0$ such that for every Borel set $A \subset \mathbb{R}^N$ and every $k \in \{1, \cdots, n\}$

$$L_{z_*}(\varepsilon_*, r_*) \qquad \mathbb{P}(Z_k \in A) \geqslant \varepsilon_* \lambda(A \cap B_{r_*}(z_{*,k})) \tag{43}$$

where $\lambda$ is the Lebesgue measure on $\mathbb{R}^N$. We also define

$$M_p(Z) := 1 \vee \sup_{k \leqslant n} \mathbb{E}[|Z_k|^p] \tag{44}$$

and assume that $M_p(Z) < \infty$ for every $p \geqslant 1$.

It is easy to check that (43) holds if and only if there exists some non negative measures $\mu_k$ with total mass $\mu_k(\mathbb{R}^N) < 1$ and a lower semi-continuous function $\varphi \geqslant 0$ such that $\mathbb{P}(Z_k \in$

$dz) = \mu_k(dz) + \varphi(z - z_{*,k})dz$. Notice that the random variables $Z_1, \cdots, Z_n$ are not assumed to be identically distributed. However, the fact that $r_* > 0$ and $\varepsilon_* > 0$ are the same for all $k$ represents a mild substitute of this property. In order to construct $\varphi$ we have to introduce the following function: For $a > 0$, set $\varphi_a : \mathbb{R}^N \to \mathbb{R}$ defined by

$$\varphi_a(z) = \mathbb{1}_{|z| \leqslant a} + \exp\left(1 - \frac{a^2}{a^2 - (|z| - a)^2}\right)\mathbb{1}_{a < |z| < 2a}. \tag{45}$$

Then $\varphi_a \in \mathcal{C}_b^\infty(\mathbb{R}^N)$, $0 \leqslant \varphi_a \leqslant 1$ and we have the following crucial property: for every $p, k \in \mathbb{N}$ there exists a universal constant $C_{q,p}$ such that for every $x \in \mathbb{R}^N$, $q \in \mathbb{N}$ and $i_1, \cdots, i_q \in \{1, \cdots, N\}$, we have

$$\varphi_a(z)|\frac{\partial^q}{\partial z^{i_1} \cdot \partial z^{i_q}}(\ln \varphi_a)(z)|^p \leqslant \frac{C_{q,p}}{a^{pq}}, \tag{46}$$

with the convention $\ln \varphi_a(z) = 0$ for $|z| \geqslant 2a$. As an immediate consequence of (43), for every non negative function $f : \mathbb{R}^N \to \mathbb{R}_+$

$$\mathbb{E}[f(Z_k)] \geqslant \varepsilon_* \int_{\mathbb{R}^N} \varphi_{r_*/2}(\ z - z_{*,k}\ )f(z)dz. \tag{47}$$

By a change of variable

$$\mathbb{E}[f(\frac{1}{\sqrt{n}}Z_k)] \geqslant \varepsilon_* \int_{\mathbb{R}^N} n^{N/2}\varphi_{r_*/2}\left(\sqrt{n}(z - \frac{z_{*,k}}{\sqrt{n}})\right)f(z)dz. \tag{48}$$

We denote

$$m_* = \varepsilon_* \int_{\mathbb{R}^N} \varphi_{r_*/2}(z)dz = \varepsilon_* \int_{\mathbb{R}^N} \varphi_{r_*/2}(z - z_{*,k})dz \tag{49}$$

and

$$\phi_n(z) = n^{N/2}\varphi_{r_*/2}(\sqrt{n}z) \tag{50}$$

and we notice that $\int \phi_n(z)dz = m_*\varepsilon_*^{-1}$.

We consider a sequence of independent random variables $\chi_k \in \{0,1\}$, $U_k, V_k \in \mathbb{R}^N$, $k \in \{1, \cdots, n\}$ with laws given by

$$\begin{aligned} \mathbb{P}(\chi_k &= 1) = m_*, \qquad \mathbb{P}(\chi_k = 0) = 1 - m_*, \\ \mathbb{P}(U_k &\in dz) = \frac{\varepsilon_*}{m_*}\phi_n(z - \frac{z_{*,k}}{\sqrt{n}})dz, \\ \mathbb{P}(V_k &\in dz) = \frac{1}{1 - m_*}(\mathbb{P}(\frac{1}{\sqrt{n}}Z_k \in dz) - \varepsilon_*\phi_n(z - \frac{z_{*,k}}{\sqrt{n}})dz). \end{aligned} \tag{51}$$

Notice that (48) guarantees that $\mathbb{P}(V_k \in dz) \geqslant 0$. Then a direct computation shows that

$$\mathbb{P}(\chi_k U_k + (1 - \chi_k)V_k \in dz) = \mathbb{P}(\frac{1}{\sqrt{n}}Z_k \in dz). \tag{52}$$

This is the splitting procedure for $\frac{1}{\sqrt{n}}Z_k$. Now on we will work with this representation of the law of $\frac{1}{\sqrt{n}}Z_k$. So, we always take

$$\frac{1}{\sqrt{n}}Z_k = \chi_k U_k + (1 - \chi_k)V_k.$$

**Remark 3.1.** *The above splitting procedure has already been widely used in the litterature: in [30] and [24], it is used in order to prove convergence to equilibrium of Markov processes. In [9], [10] and [33], it is used to study the Central Limit Theorem. Last but not least, in [29], the above splitting method (with $\mathbb{1}_{B_{r_*}(z_{*,k})}$ instead of $\phi_n(z - \frac{z_{*,k}}{\sqrt{n}})$) is used in a framework which is similar to the one in this paper.*

In the following we denote $\chi = (\chi_1, \cdots, \chi_n)$, $U = (U_1, \cdots, U_n)$ and $V = (V_1, \cdots, V_n)$ and we consider the following class of random variables:

$$\mathcal{S} = \{F = f(\chi, U, V) : f \text{ is measurable and } u \to f(\chi, u, v) \in \mathcal{C}_b^\infty(\mathbb{R}^n \times \mathbb{R}^N), \forall \chi, v\}. \quad (53)$$

For a multi index $\alpha = (\alpha_1, \cdots, \alpha_q)$ with $\alpha_j = (k_j, i_j)$, $k_j \in \{1, \cdots, n\}$, $i_j \in \{1, \cdots, N\}$, we denote $|\alpha| = q$ the length of $\alpha$ and

$$\partial_u^\alpha f(\chi, u, v) = \frac{\partial^q}{\partial u_{k_1}^{i_1} \cdots \partial u_{k_q}^{i_q}} f(\chi, u, v).$$

We construct now a differential calculus based on the laws of the random variables $U_k$, $k = 1, \cdots, n$ which mimics the Malliavin calculus, following the ideas from [5], [3] and [4]. In order to be self contained we shortly present the results that we need. For $F = f(\chi, U, V) \in \mathcal{S}$ we define the Malliavin derivatives

$$D_{(k,i)}F = \chi_k \frac{1}{\sqrt{n}} \frac{\partial F}{\partial U_k^i} = \chi_k \frac{1}{\sqrt{n}} \frac{\partial f}{\partial u_k^i}(\chi, U, V), \qquad k = 1, \cdots, n, \ \ i = 1, \cdots, N. \quad (54)$$

We denote by $\langle \circ, \circ \rangle$ the usual scalar product on $\mathbb{R}^N \times \mathbb{R}^n$. The Malliavin covariance matrix for a multi dimensional functional $F = (F^1, \cdots, F^d)$ is defined as

$$\sigma_F^{i,j} = \left\langle DF^i, DF^j \right\rangle = \sum_{k=1}^n \sum_{r=1}^N D_{(k,r)}F^i \times D_{(k,r)}F^j, \qquad i, j = 1, \cdots, d. \quad (55)$$

The higher order derivatives are defined by iterating $D$:

$$D_\alpha F = D_{\alpha_1} \cdots D_{\alpha_m} F. \quad (56)$$

Now we define the Ornstein Uhlenbeck operator $L : \mathcal{S} \to \mathcal{S}$. We denote

$$\Gamma_k = \ln \phi_n(U_k - \frac{z_{*,k}}{\sqrt{n}}) \in \mathcal{S} \quad (57)$$

and we notice that

$$
\begin{aligned}
D_{(k,i)}\Gamma_k &= \frac{1}{\sqrt{n}} \chi_k \partial_{u_k^i} \ln \phi_n(U_k - \frac{z_{*,k}}{\sqrt{n}}) \\
&= \chi_k \partial_{z^i}(\ln \varphi_{r_*/2})\big(\sqrt{n}(U_k - \frac{z_{*,k}}{\sqrt{n}})\big).
\end{aligned}
$$

Finally, we define

$$-LF = \sum_{k=1}^n \sum_{i=1}^N D_{(k,i)} D_{(k,i)} F + \sum_{k=1}^n \sum_{i=1}^N D_{(k,i)} F \times D_{(k,i)}\Gamma_k. \quad (58)$$

**Remark 3.2.** *The basic random variables in our calculus are $Z_k$, $k = 1, \cdots, n$ so we precise the way in which the differential operators act on them. Since $\chi_k Z_k = \sqrt{n}\chi_k U_k$ it follows that*

$$
\begin{aligned}
D_{(m,j)} Z_k^i &= \chi_k \delta_{m,k} \delta_{i,j}, & (59)\\
L Z_k^i &= -\chi_k \partial_{z^i}(\ln \varphi_{r_*/2})\big(\sqrt{n}(U_k - \tfrac{z_{*,k}}{\sqrt{n}})\big). & (60)
\end{aligned}
$$

In our framework, the duality formula in Malliavin calculus reads as follows: for each $F, G \in \mathcal{S}$

$$
\mathbb{E}[FLG] = \mathbb{E}[\langle DF, DG \rangle] = \mathbb{E}[GLF]. \tag{61}
$$

This follows immediately using the independence structure and standard integration by parts on $\mathbb{R}^N$: indeed, if $f, g \in \mathcal{C}_b^1(\mathbb{R}^N)$ and $k \in \{1, \cdots, n\}$, then

$$
\begin{aligned}
&\sum_{i=1}^{N} \mathbb{E}[\partial_{u_k^i} f(U_k)\partial_{u_k^i} g(U_k)]\\
&= \frac{\varepsilon_*}{m_*} \sum_{i=1}^{N} \int_{\mathbb{R}^N} \partial_{u_k^i} f(u)\partial_{u_k^i} g(u)\phi_n(u - \tfrac{z_{*,k}}{\sqrt{n}})du\\
&= -\frac{\varepsilon_*}{m_*} \sum_{i=1}^{N} \int_{\mathbb{R}^N} f(u)\big(\partial_{u_k^i}^2 g(u) + \partial_{u_k^i} g(u)\frac{\partial_{u_k^i}\phi_n(u - \tfrac{z_{*,k}}{\sqrt{n}})}{\phi_n(u - \tfrac{z_{*,k}}{\sqrt{n}})}\big)\phi_n(u - \tfrac{z_{*,k}}{\sqrt{n}})du\\
&= -\mathbb{E}\Big[f(U_k)\sum_{i=1}^{N} \partial_i^2 g(U_k) + \partial_{u_k^i} g(U_k)\partial_{u_k^i}\ln\phi_n(U_k - \tfrac{z_{*,k}}{\sqrt{n}})\Big].
\end{aligned}
$$

It follows that

$$
\begin{aligned}
&\sum_{k=1}^{n}\sum_{i=1}^{N} \mathbb{E}[D_{(k,i)}F \times D_{(k,i)}G]\\
&= \frac{1}{n}\sum_{k=1}^{n}\sum_{i=1}^{N} \mathbb{E}[\chi_k \partial_{u_k^i} f(\chi, U, V) \times \partial_{u_k^i} g(\chi, U, V)]\\
&= -\mathbb{E}\Big[f(\chi, U, V)\sum_{k=1}^{n}\chi_k \sum_{i=1}^{N}\frac{1}{n}\partial_{u_k^i}^2 g(\chi, U, V) + \frac{1}{\sqrt{n}}\partial_{u_k^i} g(\chi, U, V)\frac{1}{\sqrt{n}}\partial_{u_k^i}\ln\phi_n(U_k - \tfrac{z_{*,k}}{\sqrt{n}})\Big]\\
&= -\mathbb{E}\Big[f(\chi, U, V)\sum_{k=1}^{n}\chi_k \sum_{i=1}^{N} D_{(k,i)}D_{(k,i)}G + D_{(k,i)}GD_{(k,i)}\Gamma_k\Big]\\
&= \mathbb{E}[FLG],
\end{aligned}
$$

which is exactly (61). We have the following standard chain rule: for $\phi \in \mathcal{C}^1(\mathbb{R}^d)$ and $F \in \mathcal{S}^d$

$$
D\phi(F) = \sum_{j=1}^{d} \partial_j \phi(F)DF^j. \tag{62}
$$

Moreover, one may prove, using (62) and the duality relation (or direct computation), that

$$L\phi(F) = \sum_{j=1}^{d} \partial_j \phi(F) LF^j + \sum_{i,j=1}^{d} \partial_i \partial_j \phi(F) \left\langle DF^i, DF^j \right\rangle. \tag{63}$$

In particular for $F, G \in \mathcal{S}$,

$$L(FG) = FLG + GLF + 2 \left\langle DF, DG \right\rangle. \tag{64}$$

We are now able to give the Malliavin integration by parts formula:

**Theorem 3.1.** *Let $F \in \mathcal{S}^d$ and $G \in \mathcal{S}$ be such that $\mathbb{E}[(\det \sigma_F)^{-p}] < \infty$ for every $p \geqslant 1$. We denote $\gamma_F = \sigma_F^{-1}$. Then for every $\phi \in \mathcal{C}_c^{\infty}(\mathbb{R}^d)$ and every $i = 1, \cdots, d$*

$$\mathbb{E}[\partial_i \phi(F) G] = \mathbb{E}[\phi(F) H_i(F, G)] \tag{65}$$

*with*

$$-H(F, G) = G \gamma_F LF + \left\langle D(G\gamma_F), DF \right\rangle \tag{66}$$

*and*

$$H_i(F, G) = -\sum_{j=1}^{d} G \gamma_F^{i,j} LF^j + D(G\gamma_F^{i,j}) DF^j. \tag{67}$$

*Moreover, for every multi index $\alpha = (\alpha_1, \cdots, \alpha_m) \in \{1, \cdots, d\}^m$*

$$\mathbb{E}[\partial_\alpha \phi(F) G] = \mathbb{E}[\phi(F) H_\alpha(F, G)] \tag{68}$$

*with $H_\alpha(F, G)$ defined by the recurrence relation $H_{(\alpha_1, \cdots, \alpha_m)}(F, G) = H_{\alpha_m}(F, H_{(\alpha_1, \cdots, \alpha_{m-1})}(F, G))$.*

*Proof.* Using the chain rule $D\phi(F) = \nabla \phi(F) DF$ we have

$$\left\langle D\phi(F), DF \right\rangle = \nabla \phi(F) \left\langle DF, DF \right\rangle = \nabla \phi(F) \sigma_F.$$

It follows that $\nabla \phi(F) = \gamma_F \left\langle D\phi(F), DF \right\rangle$. Then, using (64) and the duality formula (61),

$$\begin{aligned}
\mathbb{E}[G \nabla \phi(F)] &= \mathbb{E}[G \gamma_F \left\langle D\phi(F), DF \right\rangle] = \frac{1}{2} \mathbb{E}[G \gamma_F (L(\phi(F)F) - \phi(F)LF - FL\phi(F))] \\
&= \frac{1}{2} \mathbb{E}[\phi(F)(FL(G\gamma_F) - G\gamma_F LF - L(G\gamma_F F))].
\end{aligned}$$

We use once again (64) in order to obtain $H(F, G)$ in (66). $\qquad \square$

We give now estimates of the weights $H_\alpha(F, G)$ which appear in the above integration by parts formulas. We will work with the norms:

$$|F|_{1,m}^2 = \sum_{1 \leqslant |\alpha| \leqslant m} |D_\alpha F|^2, \qquad |F|_m^2 = |F|^2 + |F|_{1,m}^2, \tag{69}$$

and

$$\begin{aligned}
\|F\|_{1,m,p} &= \left\| |F|_{1,m} \right\|_p = \mathbb{E}[|F|_{1,m}^p]^{1/p} \tag{70} \\
\|F\|_{m,p} &= \|F\|_p + \left\| |F|_{1,m} \right\|_p.
\end{aligned}$$

**Proposition 3.1.** *For each $m, q \in \mathbb{N}$, there exists a universal constant $C \geqslant 1$ (depending on $d, m, q$ only) such that for every multi index $\alpha$ with $|\alpha| \leqslant q$ and every $F \in \mathcal{S}^d$ and $G \in \mathcal{S}$ on has*

$$|H_\alpha(F, G)|_m \leqslant C(1 \vee (\det \sigma_F)^{-1})^{q(q+m+1)}(1 + |F|_{1,m+q+1}^{2dq(q+m+2)} + |LF|_{m+q-1}^{2q})|G|_{m+q}. \tag{71}$$

The proof is long but straightforward so we skip it. The reader may find the detailed proof in [5] and in [3], Proposition 3.3.

We finish this section with an estimate of $\|LZ_k^i\|_{q,p}$ :

**Lemma 3.1. *A*.** *For every $k = 1, \cdots, n$ and $i = 1, \cdots, N$, we have*

$$\mathbb{E}[LZ_k^i] = 0. \tag{72}$$

***B*.** *For every $q \in \mathbb{N}$ and $p \geqslant 2$ there exists a constant $C$ depending on $q$, $p$ only*

$$\|LZ_k^i\|_{q,p} \leqslant \frac{Cm_*^{1/p}}{r_*}(1 + r_*^{-q}) \tag{73}$$

*Proof. **A**.* Using the duality relation we have $\mathbb{E}[1 \times LZ_k^i] = \mathbb{E}[\langle D1, DZ_k^i \rangle] = 0$. In order to prove **B** we recall (see (60)) that

$$LZ_k^i = -\chi_k \partial_{z^i}(\ln \varphi_{r_*/2})\left(\sqrt{n}(U_k - \frac{z_{*,k}}{\sqrt{n}})\right).$$

Let $\Lambda_{k,q}$ be the set of the multi-index $\alpha = (\alpha_1, \cdots, \alpha_q)$ such that $\alpha_j = (k, i_j)$. Notice that for a multi-index $\alpha$ of length $q$, such that $\alpha \notin \Lambda_{k,q}$, we have $D_\alpha LZ_k^i = 0$. Suppose now that $\alpha \in \Lambda_{k,q}$ and let $\overline{\alpha} = (i_1, \cdots, i_q)$. It follows

$$D_\alpha LZ_k^i = -\chi_k \partial_z^{\overline{\alpha}} \partial_{z^i}(\ln \varphi_{r_*/2})\left(\sqrt{n}(U_k - \frac{z_{*,k}}{\sqrt{n}})\right).$$

Using (46), we obtain

$$\begin{aligned}
\|D_\alpha LZ_k^i\|_p^p &= \frac{\varepsilon_* \|\chi_k\|_p^p}{m_*} \int_{|v| \leqslant r_*} n^{N/2} \left|\partial_z^{\overline{\alpha}} \partial_{z^i}(\ln \varphi_{r_*/2})\left(\sqrt{n}(u - \frac{z_{*,k}}{\sqrt{n}})\right)\right|^p \varphi_{r_*/2}\left(\sqrt{n}(u - \frac{z_{*,k}}{\sqrt{n}})\right) du \\
&= \frac{\varepsilon_* \|\chi_k\|_p^p}{m_*} \int_{r_*/2 \leqslant |v| \leqslant r_*} \left|\partial_z^{\overline{\alpha}} \partial_{z^i}(\ln \varphi_{r_*/2})(v)\right|^p \varphi_{r_*/2}(v) dv \\
&\leqslant \frac{C_{q+1,p} m_*}{r_*^{p(q+1)}}.
\end{aligned}$$

and then

$$\|LZ_k^i\|_{q,p} \leqslant C \sup_{l \leqslant q} \sup_{\alpha \in \Lambda_{k,l}} \|D_\alpha LZ_k^i\|_p \leqslant \frac{Cm_*^{1/p}}{r_*}(1 + r_*^{-q}).$$

$\square$

### 3.1.1   Localizaton

In the following, we will not work under $\mathbb{P}$, but under a localized probability measure defined as follows. We fix $M \leqslant n$ and we consider the set

$$\Lambda_M = \{\frac{1}{M} \sum_{k=1}^{M} \chi_k \geqslant \frac{m_*}{2}\}. \tag{74}$$

Using Hoeffding's inequality and the fact that $\mathbb{E}[\chi_k] = m_*$, it can be checked that

$$\mathbb{P}(\Lambda_M^c) \leqslant C \exp(-CMm_*^2) \tag{75}$$

We consider also the localization function $\varphi_{n^{1/4}/2}$, defined in (45), and we construct the random variable

$$\Theta = \Theta_{M,n} = \mathbb{1}_{\Lambda_M} \times \prod_{k=1}^{n} \varphi_{n^{1/4}/2}(Z_k). \tag{76}$$

Since $Z_k$ has finite moments of any order, the following inequality can be shown: for every $q \in \mathbb{N}$ there exists $C$ such that

$$\mathbb{P}(\Theta_{M,n} = 0) \leqslant \mathbb{P}(\Lambda_M^c) + \sum_{k=1}^{n} \mathbb{P}(|Z_k| \geqslant n^{1/4}) \leqslant C \exp(-CMm_*^2) + \frac{M_{4(q+1)}(Z)}{n^q}. \tag{77}$$

We define the probability measure

$$d\mathbb{P}_\Theta = \frac{1}{\mathbb{E}[\Theta]} \Theta d\mathbb{P}. \tag{78}$$

**Corollary 3.1.** *Let $F \in \mathcal{S}^d$ and $G \in \mathcal{S}$ be such that $\mathbb{E}_\Theta[(\det \sigma_F)^{-p}] < \infty$ for every $p \geqslant 1$. We denote $\gamma_F = \sigma_F^{-1}$. Then, for every $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^d)$ and every $i = 1, \cdots, d$*

$$\mathbb{E}_\Theta[\partial_i \phi(F)G] = \mathbb{E}_\Theta[\phi(F)H_i^\Theta(F,G)] \tag{79}$$

*with*

$$-H^\Theta(F,G) = G\gamma_F LF + \langle D(G\gamma_F, DF) + G\gamma_F \langle D \ln \Theta, DF \rangle \tag{80}$$

*and*

$$H_i^\Theta(F,G) = -\sum_{j=1}^{d} G\gamma_F^{i,j} LF^j + D(G\gamma_F^{i,j})DF^j + G\gamma_F^{i,j} \langle D \ln \Theta, DF^j \rangle. \tag{81}$$

*And for every multi index $\alpha = (\alpha_1, \cdots, \alpha_m) \in \{1, \cdots, d\}^m$,*

$$\mathbb{E}_\Theta[\partial_\alpha \phi(F)G] = \mathbb{E}_\Theta[\phi(F)H_\alpha^\Theta(F,G)], \tag{82}$$

*with $H_\alpha^\Theta(F,G)$ defined by the recurrence relation $H_{(\alpha_1,\cdots,\alpha_m)}^\Theta(F,G) = H_{\alpha_m}^\Theta(F, H_{(\alpha_1,\cdots,\alpha_{m-1})}^\Theta(F,G))$. Moreover there exists an universal constant $C$ such that for every multi index $\alpha$ with $|\alpha| = q$*

$$\mathbb{E}_\Theta[|H_\alpha^\Theta(F,G)|_m^p] \leqslant C \times C_{q,\Theta}(F,G) \tag{83}$$

*with*

$$C_{q,\Theta}(F,G) = \mathbb{E}_\Theta[(1 \vee (\det \sigma_F)^{-1})^{2pq(q+m+1)}]^{1/2}$$
$$\times (1 + \mathbb{E}[|F|_{1,m+q+1}^{8pqd(q+m+2)}]^{1/4} + \mathbb{E}[|LF|_{m+q-1}^{8pq}]^{1/4})\mathbb{E}[|G|_{m+q}^{4p}]^{1/4}. \tag{84}$$

*Proof.* Using (66) with $G$ replaced by $G\Theta$ we obtain $\mathbb{E}[\partial_i \phi(F) G\Theta] = \mathbb{E}[\phi(F)\overline{H}]$ with

$$\overline{H} = -\Theta G \gamma_F LF - \langle D(\Theta G \gamma_F), DF \rangle = \Theta H_i(F, G) - G\gamma_F \langle D\Theta, DF \rangle.$$

It follows that

$$\begin{aligned} \mathbb{E}_{\Theta}[\partial_i \phi(F) G] &= \frac{1}{\mathbb{E}[\Theta]} \mathbb{E}[\partial_i \phi(F) G\Theta] = \frac{1}{\mathbb{E}[\Theta]} \mathbb{E}[\phi(F)(\Theta H_i(F, G) - G\gamma_F \langle D\Theta, DF \rangle)] \\ &= \mathbb{E}_{\Theta}[\phi(F)(H_i(F, G) - G\gamma_F \langle D\ln\Theta, DF \rangle)]. \end{aligned}$$

So (79) is proved and (82) follows by recurrence. Notice that by (46) we have

$$\mathbb{E}_{\Theta}[|G\gamma_F \langle D\ln\Theta, DF \rangle|^p] \leqslant C\mathbb{E}_{\Theta}[|G\gamma_F DF|^p].$$

Then (83) follows from (71). □

## 3.2   Markov chains

Throughout this section, $n \in \mathbb{N}$ will still be fixed and will be the number of time step between $0$ and $T$ and also the number of increments that we consider in our abstract Malliavin calculus. We consider two sequences of independent random variables $Z_k \in \mathbb{R}^N, \kappa_k \in \mathbb{R}, \ k \in \mathbb{N}$ and we assume that $Z_k$ verifies (43). We also assume that $Z_k$ has finite moments of any order and we recall that

$$M_p(Z) = 1 \vee \sup_{k \leqslant n} \mathbb{E}[|Z_k|^p].$$

We construct the $\mathbb{R}^d$ valued Markov chain

$$X_{k+1}^n = \psi(\kappa_k, X_k^n, \frac{Z_{k+1}}{\sqrt{n}}), \qquad k \in \mathbb{N} \tag{85}$$

where

$$\psi \in \mathcal{C}^\infty(\mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^N; \mathbb{R}^d) \quad \text{and} \quad \psi(\kappa, x, 0) = x. \tag{86}$$

We denote

$$\|\psi\|_{1,r,\infty} = \sum_{0 \leqslant |\alpha| \leqslant r} \sum_{1 \leqslant |\beta| \leqslant r - |\alpha|} \|\partial_x^\alpha \partial_z^\beta \psi\|_\infty \tag{87}$$

and, for $C, r \geqslant 1$ we denote

$$N_\psi(C, r) = (1 + \|\psi\|_{1,r,\infty}) \exp(C\|\psi\|_{1,3,\infty}^2). \tag{88}$$

Since $|\beta| \geqslant 1$ in the above definition, we have at least one derivative with respect to $z$. All our estimates will be done in terms of $\|\psi\|_{1,r,\infty}$ so we may assume without loss of generality that

$$\mathbb{E}[Z_k^i] = 0, \qquad k \in \mathbb{N}, \ i = 1, \cdots, N. \tag{89}$$

Indeed, if this is not true, we denote by $m_k = (\mathbb{E}[Z_k^1], \cdots, \mathbb{E}[Z_k^N])$ and we work with $\widetilde{Z}_k = Z_k - m_k$ instead of $Z_k$ and with $\widetilde{\psi}(x, z) = \psi(\kappa, x, z + m_k)$ instead of $\psi$. Since $\nabla_z \psi = \nabla_z \widetilde{\psi}$ all the results remain true.

**Remark 3.3.** *The reason to consider the random variables $\kappa_k$ is the following. In the Victoir Ninomiya scheme, at each time step $k$, one throws a coin $\kappa_k \in \{1, -1\}$ and employs different form of the function $\psi$ according to the fact that $\kappa_k$ is equal to $1$ or to $-1$.*

In order to simplify the notation we denote

$$\psi_k(x, z) = \psi(\kappa_k, x, z).$$

Our aim is to give sufficient conditions under which the above Markov chain has the regularization property (36). In order to do it, we consider the following new representation of $X_k$. Let us introduce some notations. We denote

$$H_k = \frac{Z_k}{\sqrt{n}} = \chi_k U_k + (1 - \chi_k)V_k. \tag{90}$$

Using a Taylor development of order one, we write

$$
\begin{aligned}
X_{k+1}^n &= X_k + \sum_{i=1}^N \partial_{z_i}\psi_k(X_k^n, 0)H_{k+1}^i \\
&\quad + \frac{1}{2}\sum_{i,j=1}^N H_{k+1}^i H_{k+1}^j \int_0^1 (1-\lambda)\partial_{z_i}\partial_{z_j}\psi_k(X_k^n, \lambda H_{k+1})d\lambda.
\end{aligned}
$$

We denote

$$a_k^i = \partial_{z_i}\psi_k(X_k^n, 0), \qquad b_k^{i,j} = \int_0^1 (1-\lambda)\partial_{z_i}\partial_{z_j}\psi_k(X_k^n, \lambda\frac{Z_{k+1}}{\sqrt{n}})d\lambda,$$

and then, we write

$$X_m^n = x + \sum_{i=1}^N \sum_{k=0}^{m-1} a_k^i H_{k+1}^i + \frac{1}{2}\sum_{i,j=1}^N \sum_{k=0}^{m-1} b_k^{i,j} H_{k+1}^i H_{k+1}^j. \tag{91}$$

Moreover we denote by $X_m^n(x)$ the Markov chain which starts from $x$ (i.e. $X_0^n(x) = x$) and we denote by $\partial^\alpha X_m^n$ the derivative with respect to the starting point $x$. We will use the results from the previous section for $X_m^n$. In order to do it we have to estimate the Sobolev norms of $X_m^n$ :

**Theorem 3.2.** *For every $q, q' \in \mathbb{N}$ with $q \geqslant q'$, and $p \geqslant 2$ there exists some constants $C \geqslant 1$, $l \in \mathbb{N}$ (depending on $r_*, \varepsilon_*, m_*, q, p$ and the moments of $Z$, but not on $n$) such that*

$$\sup_{t_m \leqslant T} \sup_{0 \leqslant |\alpha| \leqslant q-q'} \|\partial_x^\alpha X_m^n(x)\|_{q',p} \leqslant C(1 + \|\psi\|_{1,q+2,\infty}^l)\exp(C\|\psi\|_{1,3,\infty}^2), \tag{92}$$

$$\sup_{t_m \leqslant T} \|LX_m^n\|_{q,p} \leqslant C(1 + \|\psi\|_{1,q+4,\infty}^l)\exp(C\|\psi\|_{1,3,\infty}^2). \tag{93}$$

The proof is long and technical so we postpone it to the Appendix.

### 3.2.1   The Malliavin covariance matrix

We turn now to the covariance matrix. We will work under the probability $\mathbb{P}_\Theta$ defined in (78). We recall that $M \leqslant n$ are given and we have denoted $\Lambda_M = \{\frac{1}{M}\sum_{k=1}^M \chi_k \geqslant \frac{m_*}{2}\}$. The localization random variable $\Theta = \Theta_{M,n}$ is defined in (76) and we have proved in (77) that, for every $q \in \mathbb{N}$,

$$\mathbb{P}(\Theta_{M,n} = 0) \leqslant C\exp(-CMm_*^2) + \frac{M_{4(q+1)}(Z)}{n^q}.$$

We also have

$$\{\Theta \neq 0\} \subset \{\frac{1}{n}\sum_{k=1}^M \chi_k \geqslant \frac{m_*}{2}\frac{M}{n}\} \cap \{|Z_k| \leqslant n^{1/4},\ k = 1, \cdots, n\}. \tag{94}$$

Using the computational rules for $k \in \{0, \cdots, m-1\}$ and $m \leqslant n$, we obtain

$$D_{(k+1,i)}X_m^n = I_{k,i} + \sum_{l=k+1}^{m-1} J_l D_{(k+1,i)}X_l^n \tag{95}$$

with

$$I_{k,i} = \frac{1}{\sqrt{n}}\chi_{k+1}(a_k^i + \sum_{j=1}^N H_{k+1}^j b_k^{i,j} + \sum_{j,q=1}^N H_{k+1}^j H_{k+1}^q c_k^{i,j,q}) \quad \text{and} \tag{96}$$

$$c_k^{i,j,q} = \frac{1}{\sqrt{n}}\chi_{k+1}\int_0^1 \lambda(1-\lambda)\partial_{z_i}\partial_{z_j}\partial_{z_q}\psi_k(X_k^n, \lambda\frac{Z_{k+1}}{\sqrt{n}})d\lambda$$

and the $d \times N$ dimensional matrices $J_l$, defined by

$$J_l^{p,r} = \sum_{j=1}^N J_l^{p,r}(j) + \sum_{j,q=1}^N J_l^{p,r}(j,q), \tag{97}$$

with

$$J_l^{p,r}(j) = \chi_{k+1}H_{l+1}^j \partial_{x_p}\partial_{z_j}\psi_l^r(X_l^n, 0),$$

$$J_l^{p,r}(j,q) = \chi_{k+1}H_{l+1}^j H_{l+1}^q \int_0^1 (1-\lambda)\partial_{x_p}\partial_{z_j}\partial_{z_q}\psi_l^r(X_l^n, \lambda H_{l+1})d\lambda.$$

We first aim to express $D_{(k+1,i)}X_m^n$ using the variance of constants method. We consider the tangent flow $Y_m = \nabla_x X_m^n(x)$ which is the $d \times d$ dimensional matrix solution of

$$Y_m = I + \sum_{l=0}^{m-1} J_l Y_l, \tag{98}$$

where $I$ is the identity matrix. The explicit solution of the above equation is given by $Y_m = \prod_{k=1}^m (I + J_k)$. If each of the matrices $I + J_k$, $k = 1, \cdots, m$, is invertible then, $Y_m$ is also invertible. On the set $\{\Theta \neq 0\}$, we have $|H_k| = |n^{-1/2}Z_k| \leqslant n^{-1/4}$ so that $\|J_k\|_\infty :=$

$\sup_{i,j\leqslant d} \|J_k^{i,j}\|_\infty \leqslant 2\|\psi\|_{1,3,\infty} n^{-1/4}$. It follows that, if $\|\psi\|_{1,3,\infty} n^{-1/4} \leqslant \frac{1}{4}$, then the lower eigenvalue of $I + J_k$ is larger then $\frac{1}{2}$, so we have the invertibility property. We denote by $\widehat{Y}_m$ the inverse of $Y_m$ and it is easy to check that $\widehat{Y}_m$ solves the equation:

$$\widehat{Y}_m = I - \sum_{l=0}^{m-1} \widehat{Y}_l (I + J_l)^{-1} J_l. \tag{99}$$

The following representation of the Malliavin derivative, known as the "variance of constants method", is given by

$$D_{(k+1,i)} X_m^n = Y_m \widehat{Y}_{k+1} I_{k,i}. \tag{100}$$

We will use the following estimate.

**Lemma 3.2.** *Let $p \geqslant 1$. There exists some constants $C_1 \geqslant 1$, $C_2 \geqslant 2$, and $C_3 \geqslant 1$ which depends on $M_8(Z)$ and $\|\psi\|_{1,3,\infty}$, such that the following holds. Suppose that $M$ and $n$ are sufficiently large in order to have*

$$\frac{2\|\psi\|_{1,3,\infty}}{n^{1/4}} + \frac{M_8(Z)}{n} + C_1 \exp(-C_1 M m_*^2) \leqslant \frac{1}{C_2}. \tag{101}$$

*Then*

$$\mathbb{E}_\Theta \big[ \sup_{t_m \leqslant T} \|Y_m\|^p \big] + \mathbb{E}_\Theta \big[ \sup_{t_m \leqslant T} \|\widehat{Y}_m\|^p \big] \leqslant C_3, \tag{102}$$

*with*

$$\|Y_m\| := \sup_{i,j\leqslant d} |Y_m^{i,j}|.$$

*Proof.* **Step 1**. We notice that on the set $\{\Theta \neq 0\}$ we have $H_l = \overline{H}_l := H_l \mathbb{1}_{\{|Z_l|\leqslant n^{1/4}\}}$. Consequently $J_l = \overline{J}_l := J_l \mathbb{1}_{\{|Z_{l+1}|\leqslant n^{1/4}\}}$ and $\widehat{Y}_l = \overline{Y}_l$ where $\overline{Y}_l$ is the solution of the equation

$$\overline{Y}_m = I - \sum_{l=0}^{m-1} \overline{Y}_l (I + \overline{J}_l)^{-1} \overline{J}_l.$$

It follows that

$$\mathbb{E}_\Theta \big[ \sup_{t_m \leqslant T} \|\widehat{Y}_m\|^p \big] \leqslant \frac{1}{\mathbb{E}[\Theta]} \mathbb{E}\big[ \sup_{t_m \leqslant T} \|\overline{Y}_m\|^p \big] \leqslant C\mathbb{E}\big[ \sup_{t_m \leqslant T} \|\overline{Y}_m\|^p \big]$$

, the last inequality is a consequence of (77). Indeed

$$\mathbb{E}[\Theta] \geqslant 1 - \mathbb{P}(\Theta = 0) \geqslant 1 - C\exp(-CM m_*^2) - \frac{M_8(Z)}{n} \geqslant 1 - \frac{1}{C_2}.$$

The last inequality is true under the hypothesis (101). So, our task is now to estimate $\mathbb{E}[\sup_{t_m\leqslant T} \|\overline{Y}_m\|^p]$.
**Step 2**. Let

$$\mathcal{F}_l = \sigma(\chi_i, U_i, V_i, i = 1, \cdots, l).$$

We have

$$
\begin{aligned}
|\mathbb{E}[\overline{J}_l^{p,r}(j) \mid \mathcal{F}_l]| & \leqslant \frac{C}{\sqrt{n}}\|\psi\|_{1,2,\infty}\mathbb{E}[|Z_{l+1}^j|\mathbb{1}_{\{|Z_{l+1}|\geqslant n^{1/4}\}} \mid \mathcal{F}_l] \\
& \leqslant \frac{C}{n}\|\psi\|_{1,2,\infty}\mathbb{E}[|Z_{l+1}^j|^3] \leqslant \frac{C\|\psi\|_{1,2,\infty}M_3(Z)}{n}.
\end{aligned}
$$

Moreover, using the Hölder inequality, we obtain

$$
|\mathbb{E}[\overline{J}_l^{p,r}(j,q) \mid \mathcal{F}_l]| \leqslant \frac{CM_4^{1/2}(Z)\|\psi\|_{1,3,\infty}}{n}.
$$

Since $\overline{Y}_l$ is $\mathcal{F}_l$ measurable, we obtain

$$
\|\mathbb{E}[\overline{Y}_l\overline{J}_l \mid \mathcal{F}_l]\| \leqslant \|\overline{Y}_l\|\frac{CM_4(Z)\|\psi\|_{1,3,\infty}}{n}. \tag{103}
$$

**Step 3**. Using the above estimate we write

$$
\|\mathbb{E}[\overline{Y}_l(I+\overline{J}_l)^{-1}\overline{J}_l \mid \mathcal{F}_l]\| \leqslant \|\mathbb{E}[\overline{Y}_l((I+\overline{J}_l)^{-1}-I)\overline{J}_l \mid \mathcal{F}_l]\| + \|\overline{Y}_l\|\frac{C}{n}.
$$

We write $(I+\overline{J}_l)^{-1}-I = -\overline{J}_l(I+\overline{J}_l)^{-1}$ and we notice that, $\|(I+\overline{J}_l)^{-1}\| \leqslant 2$ (because the lower eigenvalue of $(I+\overline{J}_l)$ is larger than $1-C_2^{-1} \geqslant \frac{1}{2}$). It follows that $\|\overline{Y}_l((I+\overline{J}_l)^{-1}-I)\overline{J}_l\| \leqslant 2\|Y_l\|\|\overline{J}_l\|^2$ and consequently

$$
\|\mathbb{E}[\overline{Y}_l((I+\overline{J}_l)^{-1}-I)\overline{J}_l \mid \mathcal{F}_l]\| \leqslant 2\|Y_l\|\mathbb{E}[\|\overline{J}_l\|^2 \mid \mathcal{F}_l].
$$

The same reasoning as above shows that $\mathbb{E}[\|\overline{J}_l\|^2 \mid \mathcal{F}_l] \leqslant \frac{C}{n}M_8(Z)\|\psi\|_{1,3,\infty}^2$ so, finally, we obtain

$$
\|\mathbb{E}[\overline{Y}_l(I+\overline{J}_l)^{-1}\overline{J}_l \mid \mathcal{F}_l]\| \leqslant \frac{CM_8(Z)\|\psi\|_{1,3,\infty}^2}{n}\|\overline{Y}_l\|. \tag{104}
$$

**Step 4**. We are now ready to start our proof. We write

$$
\overline{Y}_m^{i,j} = \delta_{i,j} - \sum_{l=0}^{m-1}\theta_l^{i,j} \tag{105}
$$

with

$$
\theta_l^{i,j} = (\overline{Y}_l((I+\overline{J}_l)^{-1}\overline{J}_l)^{i,j}.
$$

We denote

$$
\widehat{\theta}_l = \mathbb{E}[\theta_l \mid \mathcal{F}_l], \qquad \widetilde{\theta}_l = \theta_l - \widehat{\theta}_l
$$

and we write

$$
\begin{aligned}
\overline{Y}_m & = M_m + A_m \qquad \text{with} \\
M_m & = -\sum_{l=0}^{m-1}\widetilde{\theta}_l, \qquad A_m^{i,j} = \delta_{i,j} - \sum_{l=0}^{m-1}\widehat{\theta}_l^{i,j}.
\end{aligned}
$$

By (104) we have $\|n\widehat{\theta}_k\| \leqslant C\|\overline{Y}_k\|$ and using the triangle inequality, we deduce that

$$\sup_{t_k \leqslant t_m} \|A_k\| \leqslant 1 + \frac{C}{n} \sum_{l=0}^{m-1} \|\overline{Y}_l\|.$$

So that,

$$\mathbb{E}[\sup_{t_k \leqslant t_m} \|A_k\|^p]^{1/p} \leqslant 1 + \frac{C}{n} \sum_{l=0}^{m-1} \|\overline{Y}_l\|_p.$$

We notice that that

$$\|\widetilde{\theta}_l\| \leqslant \|\theta_k\| + \|\widehat{\theta}_l\| \leqslant \frac{C}{n^{1/2}} \|\overline{Y}_l\|(|Z_{l+1}| + 1),$$

and then,

$$\|\widetilde{\theta}_l\|_p^2 \leqslant \frac{C M_p^{2/p}(Z)}{n} \|\overline{Y}_l\|_p^2.$$

Moreover, $M_m$ is a martingale so, using Burkholder's inequality (see (144)), we have

$$\mathbb{E}[\sup_{t_k \leqslant t_m} \|M_k\|^p]^{1/p} \leqslant C\Big(\sum_{l=0}^{m-1} \|\widetilde{\theta}_l\|_p^2\Big)^{1/2}.$$

We conclude that

$$\mathbb{E}[\sup_{t_k \leqslant t_m} \|\overline{Y}_k\|^p]^{1/p} \leqslant C M_p^{1/p}(Z)\Big(\frac{1}{n} \sum_{l=0}^{m-1} \|\overline{Y}_l\|_p^2\Big)^{1/2}.$$

Now, we are going to use the Gronwall's lemma. We put $Q_l = \|\overline{Y}_l\|^2$, so that, $\|\overline{Y}_l\|_p^2 = \|Q_l\|_{p/2}$. It follows that

$$\mathbb{E}[\sup_{t_k \leqslant t_m} Q_k^{p/2}]^{1/p} \leqslant C + \Big(\frac{C}{n} \sum_{l=0}^{m-1} \|Q_l\|_{p/2}\Big)^{1/2},$$

which gives,

$$\|\sup_{t_k \leqslant t_m} Q_k\|_{p/2} \leqslant C + \frac{C}{n} \sum_{l=0}^{m-1} \|Q_l\|_{p/2} \leqslant C + \frac{C}{n} \sum_{l=0}^{m-1} \|\sup_{t_k \leqslant t_l} Q_k\|_{p/2}.$$

Then, by Gronwall's lemma,

$$\|\sup_{t_k \leqslant t_m} Q_k\|_{p/2} \leqslant C e^C,$$

where $C$ depends on $\|\psi\|_{1,3,\infty}$ and the moments of $Z$. The estimate of $\mathbb{E}_\Theta[\|Y_m\|^p]$ is similar but simpler, so we leave it out.

$\square$

We have the following estimate for the covariance matrix of $X_m^n$ :

**Proposition 3.2.** *Suppose that there exists $\lambda_* > 0$ such that*

$$\inf_{\kappa \in \mathbb{R}} \inf_{x \in \mathbb{R}^d} \inf_{|\xi|=1} \sum_{i=1}^{N} \langle \partial_{z_i} \psi(\kappa, x, 0), \xi \rangle^2 \geqslant \lambda_* \tag{106}$$

*Assume also that $M$ and $n$ are sufficiently large in order that (101) holds and that*

$$n^{1/2} \geqslant \frac{2^{N+4}}{\lambda_*} \|\psi\|_{1,3,\infty}^2. \tag{107}$$

*Let $\sigma_{X_M^n}$ be the Malliavin covariance matrix of $X_M^n$ defined in (55). There exists a universal constant $C$ such that*

$$\mathbb{E}_\Theta[(\det \sigma_{X_M^n})^{-p}]^{1/p} \leqslant C \frac{C_3^2}{\lambda_* m_*} \frac{n}{M}. \tag{108}$$

*with $C_3$ defined in (102).*

*Proof.* By (100), $\sigma_{X_m^n} = Y_M \widehat{\sigma} Y_M^*$, with $\widehat{\sigma} = \sum_{k=1}^{M} (\widehat{Y}_k I_{k-1}) \times (\widehat{Y}_k I_{k-1})^*$ so that $\det \sigma_{X_M^n} = (\det Y_M)^2 \det \widehat{\sigma}$. It follows that

$$\mathbb{E}_\Theta[(\det \sigma_{X_M^n})^{-p}] \leqslant \mathbb{E}_\Theta[(\det Y_M)^{-4p}]^{1/2} \mathbb{E}_\Theta[(\det \widehat{\sigma})^{-2p}]^{1/2}.$$

Since $(\det Y_M)^{-1} = \det \widehat{Y}_M$, we use (102) and we obtain $\mathbb{E}_\Theta[(\det Y_M)^{-4p}]^{1/2} \leqslant C_3$. We estimate now the lower eigenvalue of $\widehat{\sigma}$ given by

$$\widehat{\lambda} = \inf_{|\xi|=1} \sum_{k=1}^{M} \sum_{i=1}^{N} \left\langle (\widehat{Y}_k I_{k-1,i}) \times (\widehat{Y}_k I_{k-1,i})^* \xi, \xi \right\rangle = \inf_{|\xi|=1} \sum_{k=1}^{M} \sum_{i=1}^{N} \left\langle (I_{k-1,i} I_{k-1,i})^* \widehat{Y}_k^* \xi, \widehat{Y}_k^* \xi \right\rangle. \tag{109}$$

Recall that, $I_{k,i}$ is given in (96):

$$I_{k,i} = \frac{1}{\sqrt{n}} \chi_{k+1} \left( a_k^i + \sum_{j=1}^{N} H_{k+1}^j b_k^{i,j} + \frac{1}{\sqrt{n}} \sum_{j=1}^{N} H_{k+1}^j H_{k+1}^q c_k^{i,j,q} \right).$$

Then, for $\eta \in \mathbb{R}^d$ and $k \in \{0, \cdots, M-1\}$ we have

$$\begin{aligned}
\sum_{i=1}^{N} \langle (I_{k,i} I_{k,i})^* \eta, \eta \rangle &= \sum_{i=1}^{N} \langle I_{k,i}, \eta \rangle^2 \\
&\geqslant \frac{1}{2n} \sum_{i=1}^{N} \chi_{k+1} \langle a_k, \eta \rangle^2 \\
&\quad - \frac{2^N}{n} \sum_{i,j=1}^{N} \chi_{k+1} \left\langle H_{k+1}^j b_k^{i,j}, \eta \right\rangle^2 - \frac{2^N}{n} \sum_{i,j,q=1}^{N} \chi_{k+1} \left\langle H_{k+1}^j H_{k+1}^q c_k^{i,j,q}, \eta \right\rangle^2.
\end{aligned}$$

Since we are on the set $\{\Theta \neq 0\}$, we have $\sup_{k \in \{1,\cdot,n-1\}} |H_k| \leqslant n^{-1/4}$. We also have $|b_k^{i,j}| + |c_k^{i,j,q}| \leqslant 2\|\psi\|_{1,3,\infty}$, for all $k \in \{0, \cdots, n-1\}$, so that

$$|\left\langle H_{k+1}^j b_k^{i,j}, \eta \right\rangle| + |\left\langle H_{k+1}^j H_{k+1}^q c_k^{i,j,q}, \eta \right\rangle| \leqslant \frac{2}{n^{1/4}} \|\psi\|_{1,3,\infty} |\eta|.$$

Moreover by our hypothesis (106)

$$\sum_{i=1}^{N} \left\langle a_k^i, \eta \right\rangle^2 = \sum_{i=1}^{N} \left\langle \partial_{z_i} \psi_k(X_k^n, 0), \eta \right\rangle^2 \geqslant \lambda_* |\eta|^2.$$

Using (107), we have $\frac{\lambda_*}{2} - \frac{2^{N+2}\|\psi\|_{1,3,\infty}^2}{n^{1/2}} \geqslant \frac{\lambda_*}{4}$, and we obtain

$$\sum_{i=1}^{N} \left\langle (I_{k,i}I_{k,i})^*\eta, \eta \right\rangle \geqslant \frac{\chi_{k+1}}{n}\left(\frac{\lambda_*}{2} - \frac{2^{N+2}\|\psi\|_{1,3,\infty}^2}{n^{1/2}}\right)|\eta|^2 \geqslant \chi_{k+1}\frac{1}{4n}\lambda_*|\eta|^2$$

We come back to (109) and we take $\eta = \widehat{Y}_k^* \xi$. Since on the set $\{\Theta \neq 0\}$ we have $\frac{1}{M}\sum_{k=1}^{M}\chi_k \geqslant \frac{1}{2}m_*$, it follows that

$$
\begin{aligned}
\widehat{\lambda} &\geqslant \frac{\lambda_*}{4}\frac{1}{n}\inf_{|\xi|=1}\sum_{k=1}^{M}\chi_k\|\widehat{Y}_k^*\xi\|^2 \geqslant \frac{\lambda_*}{4n}\sum_{k=1}^{M}\chi_k \inf_{|\xi|=1}\|\widehat{Y}_k^*\xi\|^2 \\
&\geqslant \frac{\lambda_* m_* M}{8n}\min_{k=1,\cdots,n}\inf_{|\xi|=1}\|\widehat{Y}_k^*\xi\|^2 \geqslant \frac{\lambda_* m_* M}{8n}(\sup_{k=1,\cdots,n}\|Y_k\|)^{-2}
\end{aligned}
$$

Since we have (101), (102) follows and we conclude that

$$\mathbb{E}_\Theta[\widehat{\lambda}^{-p}]^{1/p} \leqslant \frac{8n}{\lambda_* m_* M}\mathbb{E}_\Theta[\sup_{t_k\leqslant T}\|Y_k\|^{2p}]^{1/p} \leqslant C\frac{C_3^2 n}{\lambda_* m_* M}.$$

$\square$

### 3.2.2   The regularization property

We still fix $n$ and we consider the Markov chain $X_M^n, M \in \mathbb{N}$, defined in (85). We also recall that $\Theta_{M,n}$ is defined in (76) and we introduce

$$P_M^{\Theta,n}f(x) := \mathbb{E}_{\Theta_{M,n}}[f(X_M^n(x))] = \frac{1}{\mathbb{E}[\Theta_{M,n}]}\mathbb{E}[\Theta_{M,n}f(X_M^n(x))], \quad M \in \mathbb{N}. \tag{110}$$

Notice that $P_M^{\Theta,n}, M \in \mathbb{N}$, is not a semigroup, but this is not necessary. We will not be able to prove the regularization property for $P_M^n$ but for $P_M^{\Theta,n}$.

**Proposition 3.3.  *A*.** *Assume that (106) holds true. There exists some constants $C_1 \geqslant 1$, $C_2 \geqslant 2$ such that the following holds: suppose that $n$ and $M$ are sufficiently large in order to have (101) :*

$$\frac{2\|\psi\|_{1,3,\infty}}{n^{1/4}} + \frac{M_8(Z)}{n} + C_1\exp(-C_1 M m_*^2) \leqslant \frac{1}{C_2},$$

*and (107). Then for every $q \in \mathbb{N}$ and multi index $\alpha, \beta$ with $|\alpha| + |\beta| \leqslant q$, there exists $l \in \mathbb{N}^*$ and $C$ which depends on $m_*, r_*$ and $M_l(Z)$ such that*

$$
\begin{aligned}
\|\partial_\alpha P_M^{\Theta,n}\partial_\beta f\|_\infty &\leqslant C_{\psi,l}\|f\|_\infty\left(\frac{Cn}{\lambda_* m_* M}\right)^q \qquad with \\
C_{\psi,l} &= C(1 + \|\psi\|_{1,q+4,\infty}^l)\exp(C\|\psi\|_{1,3,\infty}^2).
\end{aligned}
\tag{111}
$$

In particular, $P_M^{\Theta,n}(x, dy) = p_M^{\Theta,n}(x, y) dy$ with $(x, y) \mapsto p_M^{\Theta,n}(x, y)$, a function that belongs to $\mathcal{C}^\infty(\mathbb{R}^d \times \mathbb{R}^d)$.

**B**. For every $l \in \mathbb{N}$ there exists $C \geqslant 1$ such that

$$\|P_M^n f - P_M^{\Theta,n} f\|_\infty \leqslant \|f\|_\infty (C \exp(-CMm_*^2) + \frac{M_{4(l+1)}(Z)}{n^l}). \tag{112}$$

**Remark 3.4.** *Recall that* $t_M = \frac{MT}{n}$. *Then (111) means that the strong regularization property* $\overline{R}_{q,\eta}$, *with* $\eta = 1$, *holds for* $P_M^{\Theta,n}$.

*Proof.* We fix $M$ and $n$, and we denote $\Theta = \Theta_{M,n}$.

**A**. We have

$$\partial_\alpha P_M^{\Theta,n} \partial_\beta f(x) = \frac{1}{\mathbb{E}[\Theta]} \sum_{|\beta| \leqslant |\gamma| \leqslant q} \mathbb{E}_\Theta[\partial_\gamma f(X_M^n(x)) Q_\gamma(X_M^n)], \tag{113}$$

where $Q_\gamma(X_m^n)$ is a universal polynomial of $\partial_x^\rho X_m^n(x), 0 \leqslant |\rho| \leqslant |\alpha|$. Using the integration by parts formula (79) and the estimate (83) we obtain

$$
\begin{aligned}
|\mathbb{E}_\Theta[\partial_\gamma f(X_M^n(x)) Q_\gamma(X_M^n(x))]| &= |\mathbb{E}_\Theta[f(X_M^n(x)) H_\gamma^\Theta(X_M^n(x), Q_\gamma(X_M^n(x))]| \tag{114} \\
&\leqslant \|f\|_\infty \mathbb{E}_\Theta[|H_\gamma^\Theta(X_m^n(x), Q_\gamma(X_M^n(x))|] \\
&\leqslant \|f\|_\infty \times A_1 \times A_2 \times A_3
\end{aligned}
$$

with

$$
\begin{aligned}
A_1 &= 1 \vee \mathbb{E}_\Theta[((\det \sigma_{X_M^n(x)})^{-1})^{2q}]^{1/2} \\
A_2 &= 1 + \mathbb{E}[|X_M^n(x)|_{1,q+2}^{16qd}]^{1/4} + \mathbb{E}[|LX_M^n(x)|_q^{4q}]^{1/4} \\
A_3 &= \mathbb{E}[|Q_\gamma(X_M^n(x))|_q^4]^{1/4}.
\end{aligned}
$$

Using the results from Theorem 3.2, we obtain

$$A_2 \times A_3 \leqslant C(1 + \|\psi\|_{1,q+4,\infty}^l) \exp(C\|\psi\|_{1,3,\infty}^2).$$

We use now (108) and we obtain

$$A_1 = \mathbb{E}_\Theta[(\det \sigma_{X_M^n(x)})^{-2q}]^{1/2} \leqslant C \left( \frac{C_3^2 n}{\lambda_* m_* M} \right)^q.$$

So we have proved that

$$|\partial_\alpha P_M^{\Theta,n} \partial_\beta f(x))| \leqslant C \left( \frac{n}{\lambda_* m_* M} \right)^q (1 + \|\psi\|_{1,q+4,\infty}^l) \exp(C\|\psi\|_{1,3,\infty}^2).$$

**B**. We have

$$
\begin{aligned}
|P_M^n f(x) - P_M^{\Theta,n} f(x)| &\leqslant |P_M^n f(x)||1 - \frac{1}{\mathbb{E}[\Theta]}| + \frac{1}{\mathbb{E}[\Theta]} |\mathbb{E}[f(X_m^n)(1 - \Theta)]| \\
&\leqslant 2\|f\|_\infty \frac{\mathbb{E}[|1 - \Theta|]}{\mathbb{E}[\Theta]} \leqslant 2\|f\|_\infty \frac{\mathbb{P}(\Theta = 0)}{1 - \mathbb{P}(\Theta = 0)}.
\end{aligned}
$$

By (77) we have, for every $l \in \mathbb{N}$, $\mathbb{P}(\Theta = 0) \leqslant C \exp(-CMm_*^2) + M_{4(l+1)}(Z) n^{-l}$. $\qquad\square$

We give now an alternative way to regularize the semigroup $P_k^n$ (by convolution). We consider a $d$ dimensional standard normal random variable $G$ which is independent from $Z_k, k \in \mathbb{N}$, and for $\theta > 0$, we introduce

$$X_k^{\theta,n}(x) = \frac{1}{n^\theta}G + X_k^n(x). \tag{115}$$

We denote by $p_k^{\theta,n}(x,y)$ the density of the law of $X_k^{\theta,n}(x)$ and we define

$$P_k^{\theta,n}f(x) := \mathbb{E}[f(\frac{1}{n^\theta}G + X_k^n(x))]. \tag{116}$$

**Corollary 3.2.** *Under the hypothesis of the previous proposition we have:*
**A**. *For every multi index $\alpha, \beta$ with $|\alpha| + |\beta| \leqslant q$, and every $q' \in \mathbb{N}^*$, there exists $C, l \geqslant 1$ such that*

$$\|\partial_\alpha P_M^{\theta,n}\partial_\beta f\|_\infty \leqslant \|f\|_\infty C\big(C_{\psi,l}(\frac{n}{\lambda_* m_* M})^q + n^{q\theta/2}(\exp(-CMm_*^2) + \frac{M_{4(q'+1)}(Z)}{n^{q'}})\big) \tag{117}$$

*with $C_{\psi,l}$ given in (111).*
**B**. *For every $q' \in \mathbb{N}^*$, there exists $C, l \geqslant 1$, such that*

$$\|P_M^n f(x) - P_M^{\theta,n}f(x)\|_\infty \leqslant \frac{1}{n^\theta}C_{\psi,l}\|f\|_\infty \frac{Cn}{\lambda_* m_* M} + 2\|f\|_\infty(C\exp(-CMm_*^2) + \frac{M_{4(q'+1)}(Z)}{n^{q'}}). \tag{118}$$

*Proof.* We fix $M$ and $n$, and we denote $\Theta = \Theta_{M,n}$.
**A**. As in (113), we write

$$\partial_\alpha P_M^{\theta,n}\partial_\beta f(x) = \sum_{|\beta| \leqslant |\gamma| \leqslant q} \mathbb{E}[(\partial_\gamma f)(n^{-\theta}G + X_M^n(x))Q_\gamma(X_M^n(x))],$$

where $Q_\gamma(X_m^n)$ is a universal polynomial of $\partial_x^\rho X_m^n(x), 0 \leqslant |\rho| \leqslant |\alpha|$. We decompose

$$\mathbb{E}[(\partial_\gamma f)(n^{-\theta}G + X_M^n(x))Q_\gamma(X_M^n(x))] = I + J$$

with

$$\begin{aligned} I &= \mathbb{E}[\Theta]\mathbb{E}_\Theta[\partial_\gamma f(n^{-\theta}G + X_M^n(x))Q_\gamma(X_M^n(x))], \\ J &= \mathbb{E}[\partial_\gamma f(n^{-\theta}G + X_M^n(x))Q_\gamma(X_M^n(x))(1 - \Theta)]. \end{aligned}$$

The reasoning from the previous proof shows that

$$I \leqslant C_{\psi,l}\|f\|_\infty\big(\frac{Cn}{\lambda_* m_* M}\big)^q.$$

And since $G$ follows the standard normal law, standard integration by parts give

$$|J| \leqslant Cn^{|\gamma|\theta/2}\|f\|_\infty\mathbb{E}[1 - \Theta] \leqslant Cn^{|\gamma|\theta/2}\|f\|_\infty(\exp(-CMm_*^2) + \frac{M_{4(q'+1)}(Z)}{n^{q'}})$$

the last inequality being a consequence of (77).

**B.** Let $q' \in \mathbb{N}^*$. Using (77) and (111), there exists $C, l \geqslant 1$ such that

$$
\begin{aligned}
|P_M^n f(x) - P_M^{\theta,n} f(x)| \;\leqslant\;& \mathbb{E}[\Theta]|\mathbb{E}_\Theta[f(X_M^n(x)) - f(X_M^n(x) + n^{-\theta}G)]| + 2\|f\|_\infty \mathbb{E}[1 - \Theta] \\
\leqslant\;& n^{-\theta} \sum_{j=1}^d \int_0^1 |\mathbb{E}[\partial_j f(X_M^n(x) + \lambda n^{-\theta}G)G^j]|d\lambda + 2\|f\|_\infty \mathbb{E}[1 - \Theta] \\
\leqslant\;& n^{-\theta} C_{\psi,l} \|f\|_\infty \frac{Cn}{\lambda_* m_* M} + 2\|f\|_\infty (C \exp(-CMm_*^2) + \frac{M_{4(q'+1)}(Z)}{n^{q'}}).
\end{aligned}
$$

$\square$

## 3.3   Approximation result

In this section we give the approximation result for a Markov semigroup $(P_t)_{t \geqslant 0}$. For $T > 0$ and $n \in \mathbb{N}$, we denote $\delta_n = \frac{1}{n}$, $t_k = kT\delta_n$ and $\mu_k^n(x, dy) = P_{T\delta_n}(x, dy)$ for all $k \in \mathbb{N}$. We consider now an approximation scheme based on the Markov chain introduced in the previous section. So we consider two sequences of independent random variables $Z_k \in \mathbb{R}^N$, $\kappa_k \in \mathbb{R}$, $k \in \mathbb{N}$. We assume that $Z_1, \cdots, Z_n$ verifies (43) and have finite moments of any order: for every $p \geqslant 1$,

$$
M_p(Z) = 1 \vee \sup_{k \leqslant n} \mathbb{E}[|Z_k|^p] < \infty. \tag{119}
$$

Moreover, we take $\psi \in \mathcal{C}^\infty(\mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^N; \mathbb{R}^d)$ such that $\psi(\kappa, x, 0) = x$ and we construct $X_{k+1}^n(x) = \psi(\kappa_k, X_k^n(x), \frac{Z_{k+1}}{\sqrt{n}})$ with $X_0^n(x) = x$. We denote $\nu_{k+1}^n(x, dy) = \mathbb{P}(X_{k+1}^n \in dy \mid X_k^n = x)$ and we construct the discrete semigroup $P_{k+1}^n = \nu_{k+1}^n P_k^n$. We recall that the notation $\|\psi\|_{1,r,\infty}$ is introduced in (87) and we assume that, for every $r \in \mathbb{N}$,

$$
\|\psi\|_{1,r,\infty} < \infty. \tag{120}
$$

We also assume that there exists $\lambda_* > 0$ such that

$$
\inf_{\kappa \in R} \inf_{x \in \mathbb{R}^d} \inf_{|\xi|=1} \sum_{i=1}^N \langle \partial_{z_i}\psi(\kappa, x, 0), \xi \rangle^2 \geqslant \lambda_*. \tag{121}
$$

Now we are able to prove our main result.

**Theorem 3.3. *A.*** *Consider a Markov semigrop $P_t$, $t \geqslant 0$, and the approximation Markov chains $P_k^n, k \in \mathbb{N}$, defined above. We fix $0 < S \leqslant T/2$, $p \in \mathbb{N}$ and $h > 0$, and we assume that (20) holds for $P$ and that (32), (33), (34), (119), (120) and (121) hold for this $p$ and $h$, and for every $n \in \mathbb{N}$. There exists $l, n_* \in \mathbb{N}$ and $C$ which depends on $\|\psi\|_{1,p+3,\infty}$, $m_*$, $r_*$ and $M_l(Z)$ such that, for $n \geqslant n_*$, we hace*

$$
\sup_{2S \leqslant t_k \leqslant T} \|P_{t_k}f - P_k^n f\|_\infty \leqslant \frac{C}{S^p} \|f\|_\infty \frac{1}{n^h}. \tag{122}
$$

***B.*** *Moreover, for every $t > 0$, $P_t(x, dy) = p_t(x, y)dy$ with $(x, y) \to p_t(x, y)$ belonging to $\mathcal{C}^\infty(\mathbb{R}^d \times \mathbb{R}^d)$.*

***C***. *We recall the $P_k^{\Theta,n}$ is defined in (110) and verifies $P_k^{\Theta,n}(x,dy) = p_k^{\Theta,n}(x,y)dy$. For every $R, \varepsilon > 0$ and every multi-index $\alpha, \beta$ we have*

$$\sup_{2S \leqslant t_k \leqslant T} \sup_{|x|+|y| \leqslant R} |\partial_x^\alpha \partial_y^\beta p_{t_k}(x,y) - \partial_x^\alpha \partial_y^\beta p_k^{\Theta,n}(x,y)| \leqslant \frac{C}{n^{h(1-\varepsilon)}}, \tag{123}$$

*with a constant $C$ which depends on $R, S, \varepsilon$ and on $|\alpha| + |\beta|$ (and may go to infinity as $\varepsilon \downarrow 0$).*
***D***. *Let $\theta > h+1$. We recall the $P_k^{\theta,n}$ is defined in (116) and verifies $P_k^{\theta,n}(x,dy) = p_k^{\theta,n}(x,y)dy$. For every $R, \varepsilon > 0$ and every multi-index $\alpha, \beta$ we have*

$$\sup_{2S \leqslant t_k \leqslant T} \sup_{|x|+|y| \leqslant R} |\partial_x^\alpha \partial_y^\beta p_{t_k}(x,y) - \partial_x^\alpha \partial_y^\beta p_k^{\theta,n}(x,y)| \leqslant \frac{C}{n^{h(1-\varepsilon)}} \tag{124}$$

*Proof.* **A-B**. We use Proposition 2.3: we have proved in Proposition 3.3 that $P_k^{\Theta,n}$ verifies the regularization properties. The proof of (122) and (123) is an immediate consequence of Theorem 2.1. **C**. In order prove (124) one employs Corollary 3.2 instead of Proposition 3.3. $\square$

**Remark 3.5.** *The simulation of an approximation scheme given by $P^{\Theta,n}$ may be cumbersome, so the estimate obtained in (123) is not very useful. This is why we propose the regularized scheme $X_k^{\theta,n}$ which is easier to simulate.*

# 4 The Ninomiya Victoir scheme

We illustrate this theorem when $X^n$ is the Ninomiya Victoir scheme for a diffusion process. This is a variant of the result already obtained by Kusuoka [22] in the case where $Z_k$ has a Gaussian distribution (and so the standard Malliavin calculus is available). Since in our paper $Z_k$ has an arbitrary distribution (except the property (43)) our result may be seen as an invariance principle as well. We consider the $d$ dimensional diffusion process

$$dX_t = \sum_{i=1}^N V_i(X_t) \circ dW_t^i + V_0(X_t)dt \tag{125}$$

with $V_0, V_i \in \mathcal{C}_b^\infty(\mathbb{R}^d; \mathbb{R}^d)$, $i = 1, \cdots, N$ and $W = (W^1, \cdots, W^N)$ a Brownian motion and $\circ dW_t^i$ denotes the Stratonovich integral with respect to $W^i$. The infinitesimal operator of this Markov process is

$$A = V_0 + \frac{1}{2} \sum_{k=1}^N V_k^2 \tag{126}$$

with the notation $Vf(x) = \langle V(x), \nabla f(x) \rangle$. Let us define $\exp(V)(x) := \Phi_V(x, 1)$ where $\Phi_V$ solves the deterministic equation

$$\Phi_V(x, t) = x + \int_0^t V(\Phi_V(x, s))ds. \tag{127}$$

By a change of variables one obtains $\Phi_{\varepsilon V}(x, t) = \Phi_V(x, \varepsilon t)$ so we have

$$\exp(\varepsilon V)(x) := \Phi_{\varepsilon V}(x, 1) = \Phi_V(x, \varepsilon).$$

We also notice that the semigroup of the above Markov process is given by $P_t^V f(x) = f(\Phi_V(x,t))$ and has the infinitesimal operator $A_V f(x) = V f(x)$. In particular the relation $P_t^V A_V = A_V P_t^V$ reads

$$V f(\Phi_V(x,t)) = A_V P_t^V f = P_t^V A_V f = \langle V(x), \nabla_x \left( f(\Phi_V(x,t)) \right) \rangle.$$

Using $m$ times Dynkin's formula $P_t^V f(x) = f(x) + \int_0^t P_s^V A_V f(x) ds$ we obtain

$$f(\Phi_V(x,t))) = f(x) + \sum_{r=1}^m \frac{t^r}{r!} V^r f(x) + \frac{1}{m!} \int_0^t (t-s)^m V^{m+1} P_s^V f(x) ds. \tag{128}$$

We present now the Ninomiya Victoir scheme. We consider a sequence $\rho_k$, $k \in \mathbb{N}$ of independent Bernoulli random variables and we define $\psi_k : \mathbb{R}^d \times \mathbb{R}^{N+1} \to \mathbb{R}^d$ in the following way

$$\psi_k(x,w) = \exp(w^0 V_0) \circ \exp(w^1 V_1) \circ \cdot \circ \exp(w^N V_N) \circ \exp(w^0 V_0)(x), \quad \text{if } \rho_k = 1, \tag{129}$$

$$\psi_k(x,w) = \exp(w^0 V_0) \circ \exp(w^N V_N) \circ \cdot \circ \exp(w^1 V_1) \circ \exp(w^0 V_0)(x), \quad \text{if } \rho_k = -1. \tag{130}$$

Here $w = (w^0, w^1, \cdots, w^N)$. with $w_k^0 = T/2n$, $w_k^i = \sqrt{T} Z_k^i / \sqrt{n}$, for $i = 1, \cdots, N$. Moreover $Z_k^i$, $i = 1, \cdots, d$, $k \in \mathbb{N}$ are independent random variables which verify (43) and moreover satisfy the following moment conditions:

$$\mathbb{E}[Z_k^i] = \mathbb{E}[(Z_k^i)^3] = 0, \qquad \mathbb{E}[(Z_k^i)^2] = 1, \qquad \mathbb{E}[(Z_k^i)^4] = 6. \tag{131}$$

In the original paper of Ninomiya Victoir, the random variables $Z_k^i$ are standard normal distributed, and then verify (43). The new point here is that we do not require that $Z_k$ follows this particular law anymore but only the weaker assumptions (43) and (131). We also denote $t_k = Tk/n$. One step of our scheme is given by

$$X_{k+1}^n = \psi_k(X_k^n, w_{k+1}). \tag{132}$$

We have the first following result.

**Theorem 4.1.** *Suppose that $V_i \in \mathcal{C}_b^\infty(\mathbb{R}^d; \mathbb{R}^d)$, $i = 0, \cdots, N$. There exists some universal constants $C, q \geqslant 1$ such that for every $f \in \mathcal{C}_b^6(\mathbb{R}^d)$ one has*

$$\sup_{t_k \leqslant T} |\mathbb{E}[f(X_{t_k})) - \mathbb{E}[f(X_k^n)]| \leqslant \frac{C}{n^2} C_{6(N+1)}(V) \|f\|_{6N,\infty} \tag{133}$$

*with $C_k(V) := \sup_{i=0,\cdot,N} \|V_i\|_{k,\infty}$.*

**Remark 4.1.** *A slightly more precise estimate has already been proved by Alfonsi [1] : he obtained (133) with $\|f\|_{6,\infty}$ instead of $\|f\|_{6N,\infty}$. Since in the following Theorem we will replace it by $\|f\|_\infty$, the estimate in (133) is sufficient for us (and the proof is simpler).*

Under an ellipticity condition we are able to give an estimate of the total variation distance between a diffusion process of the form (125) and its Ninomiya Victoir scheme.

**Theorem 4.2.** *Suppose that $V_i \in \mathcal{C}_b^\infty(\mathbb{R}^d; \mathbb{R}^d), i = 0, \cdots, N$, and moreover*

$$\sum_{i=1}^N \langle V_i(x), \xi \rangle^2 \geqslant \lambda_* > 0 \qquad \forall x \in \mathbb{R}^d. \tag{134}$$

*Then for every $0 < S \leqslant T/2$ and every bounded and measurable function $f : \mathbb{R}^d \to \mathbb{R}$*

$$\sup_{2S \leqslant t_k \leqslant T} |\mathbb{E}[f(X_{t_k})] - \mathbb{E}[f(X_k^n)]| \leqslant \frac{C}{S^{1/2}} \frac{1}{n^2} \|f\|_\infty. \tag{135}$$

**Remark 4.2.** *This estimate has already been proved by Kusoucka [22] (with a different approach). He considers a much more general non degeneracy assumptions (of Hörmander type) and uses Malliavin calculus in order to prove his result. Here the noise $Z_k^i$ is no more Gaussian so the standard Malliavin calculus does not work anymore, but, since we have the property (43), we may use the abstract integration by parts formula introduced in the first section.*

*Proof of Theorem 4.1.* In order to simplify the notation, we fix $T = 1$ without loss of generality. We denote

$$\begin{aligned} T_i f(x) &= f(\exp(\frac{Z^i}{\sqrt{n}} V_i)(x)) & i = 1, \cdots, N \\ &= f(\exp(\frac{1}{2n} V_0)(x)) & i = 0 \text{ and } i = N+1. \end{aligned}$$

Notice that, with the notation introduced in the beginning of this section, $T_i f(x) = P_1^{U_i} f(x)$ with $U_i = n^{-1/2} Z^i V_i, i = 1, \cdots, N$. so that, using (128) with $t = 1$ and $V = U_i$ we obtain

$$T_i f(x) = f(x) + \sum_{r=1}^m \frac{(Z^i)^r}{n^{r/2}} \frac{1}{r!} V_i^r f(x) + \frac{(Z^i)^{m+1}}{n^{(m+1)/2}} R_{m,i} f(x) \tag{136}$$

with

$$R_{m,i} f(x) = \frac{1}{m!} \int_0^1 (1-s)^m V_i^{(m+1)} P_s^{U_i} f(x))) ds \tag{137}$$

and we recall that $P_s^{U_i} f(x) = f(\exp(\frac{sZ^i}{\sqrt{n}} V_i))$. For $i = 0$ or $N+1$, we have a similar development with $U_0 = U_{N+1} = \frac{1}{2n} V_0$. Our aim is to give a development of order 3 (with respect to $T$) for $\mathbb{E}[T_0 T_1 \cdots T_{N+1} f(x)]$ (see (138) bellow). We replace each $T_i$ with the development of order $m = 5$ given above and calculate the products. All the terms containing $\frac{1}{n^r}$, $r \geqslant 3$ go in the remainder. Moreover one notices that $\mathbb{E}[(Z^i)^r] = 0$ for odd $r$ so a lot of terms cancel. Finally $\mathbb{E}[(Z^i)^2] = 1$ and $\mathbb{E}[(Z^i)^4] = 6$ and this permits to achieve the computation and to obtain:

$$\mathbb{E}[T_0 T_1 \cdots T_{N+1} f(x)] \tag{138}$$

$$\begin{aligned} = \ &f(x) + \frac{1}{n}(V_0 f(x) + \frac{1}{2} \sum_{i=1}^N V_i^2 f(x)) + \frac{1}{2n^2} V_0^2 f(x) + \frac{1}{8n^2} \sum_{i=1}^N V_i^4 f(x) \\ &+ \frac{1}{4n^2} \sum_{i<j} V_i^2 V_j^2 f(x) + \frac{1}{4n^2} \sum_{i=1}^N (V_0 V_i^2 f(x) + V_i^2 V_0 f(x)) + \frac{1}{n^3} R f(x). \end{aligned}$$

The remainder $R$ is a sum of terms of the following form:

$$\frac{C}{n^{|\alpha|-3}}T_{0,\alpha_0}T_{1,\alpha_1}\cdots T_{N+1,\alpha_{N+1}}f(x) \tag{139}$$

with $\alpha = (\alpha_0,\cdots,\alpha_{N+1}) \in \{1,\cdots,6\}^{N+2}$, $|\alpha| = \alpha_0 + \cdots + \alpha_{N+1}$ and for all $i \in \{0,\cdots,N+1\}$,

$$\begin{aligned} T_{i,k} &= V_i^k, \qquad k = 1,\cdots,5, \tag{140}\\ &= \int_0^1 (1-s)^5 V_i^6 P_s^{U_i} f(x)))ds \qquad k = 6. \end{aligned}$$

It is easy to check that for every $g \in \mathcal{C}^{k+p}(\mathbb{R}^d)$, one has

$$\|T_{i,k}g\|_{p,\infty} \leqslant CC_{k+p}^q(V)\|g\|_{k+p,\infty}$$

for some constants $C, q \geqslant 1$. So

$$\|Rf\|_\infty \leqslant CC_{6(N+1)}^q(V)\|f\|_{6(N+1),\infty}. \tag{141}$$

We turn now to the diffusion process $X_t$. We have the development

$$\mathbb{E}[f(X_t(x))] = f(x) + tAf(x) + \frac{1}{2}t^2 A^2 f(x) + \frac{t^3}{6}R_t'f(x).$$

with

$$R_t'f(x) = t^{-1}\int_0^t P_s A^3 f(x)(1 - \frac{s}{t})^2 ds. \tag{142}$$

We take $t = \delta$ and we compute $Af$ and $A^2 f$. Then we make the difference between (142) and (138). All the terms cancel except for the remainders so we obtain

$$\forall k \in \{0,\cdots,n-1\}, \quad \mathbb{E}[f(X_{t_k+\delta}(x))] - \mathbb{E}[f(X_{k+1}^n) \mid X_{t_k} = X_k^n = x] = \delta^3(\frac{1}{6}R_\delta'f(x) - Rf(x)).$$

We clearly have $\|R'f\|_\infty \leqslant C \times C_6^q(V)\|f\|_{6,\infty}$. This together with (141) proves that the hypothesis (21) is verified. So (133) is a consequence of Proposition 2.1 with $p = 6(N+1)$ and constant $C \times C_6^q(V)$. $\qquad\square$

*Proof of Theorem 4.2.* This will be a consequence of Theorem 3.3 as soon as we check that the ellipticity assumption (106) holds true. We fix $k$ and we look at $\psi_k(x,w)$ defined in (130). We supose that $\rho_k = 1$ (the proof for $\rho_k = -1$ is similar). We denote $w = (w_1,\cdots,w_N\}$ and $T_k = k$ and we consider the process $x_t(w), 0 \leqslant t \leqslant T_{N+2}$ solution of the following equation:

$$\begin{aligned} x_t(w) &= x + \frac{1}{2n}\int_{T_0}^t V_0(x_s(w))ds, \qquad T_0 \leqslant t \leqslant T_1,\\ x_t(w) &= x_{T_k}(w) + w_k \int_{T_k}^t V_k(x_s(w))ds, \qquad T_k \leqslant t \leqslant T_{k+1}, \qquad k = 1,\cdots,N,\\ x_t(w) &= x_{T_{N+1}}(w) + \frac{1}{2n}\int_{T_{N+1}}^t V_0(x_s(w))ds, \qquad T_{N+1} \leqslant t \leqslant T_{N+2}. \end{aligned}$$

Then, $\psi_k(x, w) = x_{T_{N+2}}(w)$ and consequently for $r \in \{1, \cdots, N\}$, we have $\partial_{w_r}\psi_k(x, w) = \partial_{w_r}x_{T_{N+2}}(w)$. Moreover $\partial_{w_r}x_t(w) = 0$ for $t \leqslant T_r$ and

$$\partial_{w_r}x_t(w) = \partial_{w_r}x_{T_{r+1}}(w) + \sum_{i=r+1}^{N+1} w_i \int_{T_i \vee t}^{T_{i+1} \vee t} \nabla V_i(x_s(w))\partial_{w_r}x_s(w)ds,$$

for $t \geqslant T_{r+1}$, in particular for $t = T_{N+1}$. For $T_r < t \leqslant T_{r+1}$, $\partial_{w_r}x_t(w)$ solves the equation

$$\partial_{w_r}x_t(w) = \int_{T_r}^t V_r(x_s(w))ds + w_r \int_{T_r}^t \nabla V_r(x_s(w))\partial_{w_r}x_s(w)ds.$$

It follows that

$$\partial_{w_r}x_t(w)\mid_{w=0} = \int_{T_r}^t V_r(x_s(0))ds = V_r(x_{T_1}(0))(t - T_r).$$

Notice that $T_{r+1} - T_r = 1$. Then, we have

$$\partial_{w_r}x_{T_{N+1}}(w)\mid_{w=0} = \partial_{w_r}x_{T_{r+1}}(w)\mid_{w=0} = V_r(x_{T_1}(0)).$$

and then, by (134),

$$\sum_{r=1}^N \langle \partial_{w_r}x_{T_{N+1}}(0), \xi \rangle^2 \geqslant \lambda_*|\xi|^2.$$

Notice that

$$|\nabla_z\psi_k(x, 0) - \partial_{w_r}x_{T_{r+1}}(0)| = |\partial_{w_r}x_{T_{r+2}}(0) - \partial_{w_r}x_{T_{r+1}}(0)|$$
$$\leqslant \frac{C}{n}C_2(V).$$

So, for $n$ sufficiently large, we obtain

$$\sum_{r=1}^N \langle \partial_{w_r}x_{T_{N+2}}(0), \xi \rangle^2 \geqslant \frac{\lambda_*}{2}|\xi|^2.$$

$\square$

# 5   Sobolev Norms

We consider a separable Hilbert space $U$, we denote $|a|_U$ the norm of $U$ and, for a random variable $F \in U$, we denote $\|F\|_{U,p} = (\mathbb{E}[|F|_U^p])^{1/p}$. Moreover we consider a martingale $M_n \in U$, $n \in \mathbb{N}$ and we recall Burkholder's inequality in this framework: for each $p \geqslant 2$ there exists a constant $b_p \geqslant 1$ such that

$$\forall n \in \mathbb{N}, \quad \|M_n\|_{U,p} \leqslant b_p\mathbb{E}[(\sum_{k=1}^n |M_k - M_{k-1}|_U^2)^{p/2}]^{1/p}. \tag{143}$$

As an immediate consequence

$$\|M_n\|_{U,p} \leqslant b_p (\sum_{k=1}^{n} \|M_k - M_{k-1}\|_{U,p}^2)^{1/2}. \tag{144}$$

Indeed

$$
\begin{aligned}
\|M_n\|_{U,p}^2 &\leqslant b_p^2 \mathbb{E}[(\sum_{k=1}^{n} |M_k - M_{k-1}|_U^2)^{p/2}]^{2/p} = b_p^2 \|\sum_{k=1}^{n} |M_k - M_{k-1}|_U^2\|_{p/2} \\
&\leqslant b_p^2 \sum_{k=1}^{n} \||M_k - M_{k-1}|_U^2\|_{p/2} = b_p^2 \sum_{k=1}^{n} \|M_k - M_{k-1}\|_{U,p}^2.
\end{aligned}
$$

We consider the scheme defined in the previous sections (see (91)) :

$$X_{k+1}^n = X_k^n + \sum_{i=1}^{N} H_{k+1}^i a_k^i(X_k^n) + \frac{1}{2} \sum_{i,j=1}^{N} H_{k+1}^i H_{k+1}^j b_k^{i,j}(X_k^n, H_{k+1}) \tag{145}$$

with $H_k = n^{-1/2} Z_k$ and

$$a_k^i(x) = \partial_{z_i}\psi(\kappa_k, x, 0), \qquad b_k^{i,j}(x,z) = \int_0^1 (1-\lambda)\partial_{z_i}\partial_{z_j}\psi(\kappa_k, x, \lambda z)d\lambda.$$

We also denote

$$A_k = \sum_{i=1}^{N} H_{k+1}^i \nabla_x a_k^i(X_k) + \frac{1}{2} \sum_{i,j=1}^{N} H_{k+1}^i H_{k+1}^j \nabla_x b_k^{i,j}(X_k, H_{k+1}).$$

Notice that $X_k, a_k^i, b_k^{i,j} \in \mathbb{R}^d$ and $A_k$ is a $d \times d$ dimensional matrix.
Our aim is to obtain estimates of the Sobolev norms of $X_k$. Before doing it, we give some abstract estimates. As before, $U$ is a separable Hilbert space. We say that, a $U$ valued random variable $F$ belongs to $\mathcal{S}(U)$ if for every $h \in U$ we have $\langle h, F \rangle \in \mathcal{S}$ (see (53)) and we define $DF$ by $\langle h, DF \rangle = D\langle h, F \rangle$. Then, we define the norms (see (69) and (70))

$$|F|_{U,m}^2 = \sum_{0 \leqslant |\alpha| \leqslant m} |D_\alpha F|_U^2, \qquad \|F\|_{U,m,p} = \||F|_{U,m}\|_p = (\mathbb{E}[|F|_{U,m}^m])^{1/p}.$$

The Hilbert space $U$ being given, we denote $V = U^d$ (recall that $X_k^n \in \mathbb{R}^d$ so, in this case, $U = \mathbb{R}$ and $V = \mathbb{R}^d$). We consider now some processes $(\alpha_k)_{k\in\mathbb{N}}, (\beta_k)_{k\in\mathbb{N}}, (\Gamma_k)_{k\in\mathbb{N}}$ with $\alpha_k = (\alpha_k^1, \cdots, \alpha_k^N) \in V^N$, $\beta_k = (\beta_k^1, \cdots, \beta_k^N) \in V^N$, $\Gamma_k \in V$. We assume that $\alpha_k^i = \alpha_k^i(Z_1, \cdots, Z_k)$ and $\langle h, \alpha_k^i \rangle \in \mathcal{C}_b^\infty(\mathbb{R}^{kN})$ for every $h \in V, i = 1, \cdots, N$ (we recall that $Z_k \in \mathbb{R}^N$). So $\alpha_k \in \mathcal{S}(V)$. The same is assumed on $\beta_k$ and $\Gamma_k$. We look at a process $Y_k \in V = U^d$, $k \in \mathbb{N}$ which satisfies the equation

$$Y_m = Y_0 + \sum_{k=0}^{m-1} A_k Y_k + \sum_{i=1}^{N}\sum_{k=0}^{m-1} H_{k+1}^i \alpha_k^i + \sum_{i=1}^{N}\sum_{k=0}^{m-1} L H_{k+1}^i \beta_k^i + \Gamma_m. \tag{146}$$

Notice that we do not discuss about existence and uniqueness of the solution of such an equation. We just suppose that, the process $Y$ at hand satisfies this equation (which naturally appears in our calculus). Our aim is to estimate the Sobolev norms of $Y_m$. Let $q \in \mathbb{N}$ and $p \geqslant 2$. We denote

$$C_{q,p}(\alpha, \beta, \Gamma) = \sup_{0 \leqslant m \leqslant n-1} \sup_{i=1,\cdot,N} (1 + \|\alpha_m^i\|_{V,q,p} + \|\beta_m^i\|_{V,q,p} + \|\Gamma_{m+1}\|_{V,q,p}) \qquad (147)$$

**Proposition 5.1.** *For every $q \in \mathbb{N}$ and $p \geqslant 2$ there exists some constants $C \geqslant 1$, $l \in \mathbb{N}$ (depending on $q$ and $p$) such that*

$$\sup_{m \leqslant n} \|Y_m\|_{V,q,p} \leqslant C(M_l^{1/l}(Z) + \frac{m_*^{1/l}}{r_*}(1 + r_*^{-q}))^l M_l^{1/l}(Z) N_\psi^l(CM_l^{4/l}(Z), q+2) C_{q,l}(\alpha, \beta, \Gamma). \quad (148)$$

*with $N_\psi(C, l)$ and $M_l(Z)$ defined in (88) and (44).*

*Proof.* **Step 1**. Let $q = 0$, so that $\|Y_m\|_{V,q,p} = \|Y_m\|_{V,p}$. We will check that

$$\sup_{m \leqslant n} \|Y_m\|_{V,p} \leqslant C(M_p^{1/p}(Z) + \frac{m_*^{1/p}}{r_*}) C_{0,p}(\alpha, \beta, \Gamma) \exp(CM_{2p}^{2/p}(Z)\|\psi\|_{1,3,\infty}^2). \qquad (149)$$

We study the terms which appear in the right hand side of (146). Notice that $\beta_k^i$ is $\sigma(Z_1, \cdots, Z_k)$ measurable and $\mathbb{E}[LH_{k+1}^i] = 0$ (see (72)). It follows that, $M_m = \sum_{k=0}^{m-1} LH_{k+1}^i \beta_k^i$ is a martingale and consequently, by (144)

$$\|M_m\|_{V,p} \leqslant b_p(\sum_{k=0}^{m-1} \|LH_k^i \beta_k^i\|_{V,p}^2)^{1/2}.$$

Since $LH_k^i$ and $\beta_k^i$ are independent, using (73) we obtain

$$\|LH_k^i \beta_k^i\|_{V,p}^2 = \|LH_k^i\|_p^2 \|\beta_k^i\|_{V,p}^2 \leqslant \frac{Cm_*^{2/p}}{r_*^2} \frac{1}{n} \|\beta_k^i\|_{V,p}^2.$$

We conclude that

$$\sup_{m \leqslant n} \|M_m\|_{V,p} \leqslant \frac{C}{r_*}(\frac{1}{n}\sum_{k=0}^{n-1} \|\beta_k^i\|_{V,p}^2)^{1/2} \leqslant \frac{Cm_*^{1/p}}{r_*} \sup_{k \leqslant n-1} \|\beta_k^i\|_{V,p}.$$

Since $H_{k+1}^i$ is independent of $\alpha_k^i$ and $\mathbb{E}[H_k^i] = 0$, it follows that $M_m = \sum_{k=0}^{m-1} H_{k+1}^i \alpha_k^i$ is a martingale. We have $\|H_k^i\|_p \leqslant n^{-1/2} M_p^{1/p}(Z)$ so the same reasoning as above proves that the previous inequality holds for $M_m$ (with $m_*^{1/p} r_*^{-1}$ replaced by $M_p^{1/p}(Z)$ and $\|\beta_k^i\|_{V,p}$ replaced by $\|\alpha_k^i\|_{V,p}$).
We use the same reasoning for $M_m = \sum_{k=0}^{m-1} H_k^i \nabla_x a_k^i(X_k))Y_k \in V$ and we obtain

$$\|M_m\|_{V,p} \leqslant b_p(\sum_{k=1}^m \|H_{k+1}^i \nabla_x a_k^i(X_k^n))Y_k\|_{V,p}^2)^{1/2} \leqslant CM_p^{1/p}(Z)\|\psi\|_{1,2,\infty}(\frac{1}{n}\sum_{k=1}^m \|Y_k\|_{V,p}^2)^{1/2}.$$

Finally, using the triangle inequality

$$\|\sum_{k=0}^{m-1} H_{k+1}^i H_{k+1}^j \nabla_x b_k^{i,j}(X_k^n, H_{k+1}) Y_k\|_{V,p} \quad \leqslant \quad \sum_{k=0}^{m-1} \|H_{k+1}^i H_{k+1}^j \nabla_x b_k^{i,j}(X_k^n, H_{k+1}) Y_k\|_{V,p}$$

$$\leqslant \quad C M_{2p}^{1/p}(Z) \|\psi\|_{1,3,\infty} \frac{1}{n} \sum_{k=0}^{m-1} \|Y_k\|_{V,p}.$$

We gather all the terms and we obtain

$$\|Y_m\|_{V,p} \quad \leqslant \quad \|Y_0\|_{V,p} + C M_{2p}^{1/p}(Z) \|\psi\|_{1,3,\infty} (\frac{1}{n} \sum_{k=0}^{m-1} \|Y_k\|_{V,p}^2)^{1/2}$$

$$+ C(M_p^{1/p}(Z) + \frac{m_*^{1/p}}{r_*}) \sup_{k \leqslant n-1} (\|\alpha_k^i\|_{V,p} + \|\beta_k^i\|_{V,p}) + \|\Gamma_m\|_{V,p}$$

Using Gronwall's lemma we obtain (149).
**Step 2**. Let

$$H = \{h : \{1, \cdots, n\} \times \{1, \cdots, N\} \to \mathbb{R} : |h|_H^2 = \sum_{k=1}^n \sum_{i=1}^N h^2(k,i) < \infty\}.$$

so that $DX_m^n \in H^d$. We prove that

$$\sup_{m \leqslant n} \|DX_m^n\|_{H^d,p} \leqslant C(M_p^{1/p}(Z) + \frac{m_*^{1/p}}{r_*}) M_p^{1/p}(Z) \|\psi\|_{1,3,\infty} \exp(C M_{2p}^{2/p} \|\psi\|_{1,3,\infty}^2). \tag{150}$$

For $h \in H$ we denote

$$D_h F = \langle DF, h \rangle = \sum_{k=1}^n \sum_{i=1}^N h(k,i) D_{k,i} F.$$

Since

$$D_{r,j} H_k^i = \frac{1}{\sqrt{n}} \delta_{r,k} \delta_{j,i} \chi_k,$$

we use (145) to obtain

$$D_h X_{k+1}^n \quad = \quad D_h X_k^n + A_k D_h X_k^n + \frac{1}{\sqrt{n}} \sum_{i=1}^N \chi_k h(k+1,i) a_k^i(X_k^n)$$

$$+ \frac{1}{\sqrt{n}} \sum_{i \leqslant j} \chi_k h(k+1,i) H_{k+1}^j b_k^{i,j}(X_k^n, H_{k+1})$$

$$+ \frac{1}{\sqrt{n}} \sum_{i \leqslant j} \chi_{k+1} H_{k+1}^i H_{k+1}^j \langle \nabla_z b_k^{i,j}(X_k^n, H_{k+1}), h(k+1,\circ) \rangle.$$

Iterating this formula over $k$ we obtain

$$
\begin{aligned}
D_h X_m^n &= \sum_{k=0}^{m-1} A_k D_h X_k^n + \frac{2}{\sqrt{n}} \sum_{i=1}^{N} \sum_{k=0}^{m-1} \chi_{k+1} h(k+1,i) a_k^i(X_k^n) \\
&\quad + \frac{1}{\sqrt{n}} \sum_{i,j=1}^{N} \sum_{k=0}^{m-1} \chi_{k+1} H_{k+1}^j \Big( h(k+1,i) b_k^{i,j}(X_k^n, H_{k+1}) + H_{k+1}^i \sum_{l=1}^{N} \partial_{z^l} b_k^{i,j}(X_k^n, H_{k+1}) h(k+1,l) \Big) \\
&= \sum_{k=0}^{m-1} A_k D_h X_k^n + \langle h, \Gamma_m \rangle
\end{aligned}
$$

with $\Gamma_m(k,i) = 0$ for $k > m$ and, for $k \leqslant m$

$$
\Gamma_m(k,i) = \frac{\chi_k}{\sqrt{n}} \Big( a_k^i(X_{k-1}^n) + \sum_{j=1}^{N} H_k^j b_k^{i,j}(X_{k-1}^n, H_k) + \sum_{1 \leqslant j,l \leqslant N} H_k^j H_k^l \partial_{z^i} b_k^{l,j}(X_{k-1}^n, H_k) \Big).
$$

One has

$$
|\Gamma_m|_H^2 = \sum_{k=m}^{n} \sum_{i=1}^{N} |\Gamma_m(k,i)|^2 \leqslant N \|\psi\|_{1,3,\infty}^2 \frac{1}{n} \sum_{k=1}^{n} (1 + |Z_k|^2)
$$

so, using (149) (with $V = H^d, \alpha_k = \beta_k = 0$), we obtain

$$
\begin{aligned}
\sup_{m \leqslant n} \|DX_m^n\|_{H^d,p} &\leqslant C \Big( M_p^{1/p}(Z) + \frac{m_*^{1/p}}{r_*} \Big) \sup_{m \leqslant n} \|\Gamma_m\|_{H \times U, p} \exp(C M_{2p}^{2/p} \|\psi\|_{1,3,\infty}^2) \\
&\leqslant C \Big( M_p^{1/p}(Z) + \frac{m_*^{1/p}}{r_*} \Big) M_p^{1/p}(Z) \|\psi\|_{1,3,\infty} \exp(C M_{2p}^{2/p} \|\psi\|_{1,3,\infty}^2).
\end{aligned}
$$

**Step 3**. We estimate the derivatives of $Y_m$, solution of (146). We have

$$
DY_m = A_k DY_m + \sum_{i=1}^{N} \sum_{k=0}^{m-1} H_{k+1}^i \overline{\alpha}_k^i + \sum_{i=1}^{N} \sum_{k=0}^{m-1} L H_{k+1}^i \overline{\beta}_k^i + \overline{\Gamma}_m
$$

with

$$
\begin{aligned}
\overline{\alpha}_k^i &= \nabla_x \nabla_x a_k^i(X_k) DX_k^n Y_k + D\alpha_k^i, \\
\overline{\beta}_k^i &= D\beta_k^i
\end{aligned}
$$

and

$$
\begin{aligned}
\overline{\Gamma}_m &= \sum_{k=0}^{m-1} \nabla_x a_k^i(X_k^n) Y_k DH_{k+1}^i + \frac{1}{2} \sum_{i,j=1}^{N} \sum_{k=0}^{m-1} D(H_{k+1}^i H_{k+1}^j \nabla_x b_k^{i,j}(X_k^n, H_{k+1})) Y_k \\
&\quad + \sum_{i=1}^{N} \sum_{k=0}^{m-1} \alpha_k^i DH_{k+1}^i + \sum_{i=1}^{N} \sum_{k=0}^{m-1} \beta_k^i DLH_{k+1}^i + D\Gamma_m.
\end{aligned}
$$

Notice that $DY_m$ is a process with values in $H \times V$. We will prove that

$$
\begin{aligned}
C_{0,p}(\overline{\alpha}, \overline{\beta}, \overline{\Gamma}) \;&=\; \sup_{m \leqslant n} \sup_{i=1,N} (1 + \|\overline{\alpha}_m^i\|_{H \times V, p} + \|\overline{\beta}_m^i\|_{H \times V, p} + \|\overline{\Gamma}_m\|_{H \times V, p}) \qquad (151) \\
&\leqslant\; C(M_{2p}^{1/2p}(Z) + \frac{m_*^{1/2p}}{r_*}(1 + r_*^{-1}))^2 M_{2p}^{1/2p}(Z) C_{1,2p}(\alpha, \beta, \Gamma) \\
&\quad \times (1 + \|\psi\|_{1,4,\infty}^4) \exp(C M_{4p}^{1/p} \|\psi\|_{1,3,\infty}^2)
\end{aligned}
$$

Once (151) is proved, the whole proof is concluded. Indeed, using (151) and the result from the first step (that is (148) with $q = 0$ and $Y_m$ replaced by $DY_m$), we obtain (148) with $q = 1$. And using recursively the same reasoning we obtain (148) for every $q \in \mathbb{N}$.

We estimate each of the terms which appear in the right hand side of (151). To begin we write

$$
\begin{aligned}
\|\nabla_x \nabla_x a_k^i(X_k^n) DX_k^n Y_k\|_{H \times V, p} \;&\leqslant\; \|\psi\|_{1,3,\infty} \|DX_k^n Y_k\|_{H \times V, p} = \|\psi\|_{1,3,\infty} \big\||DX_k^n|_{H \times \mathbb{R}^d} |Y_k|_V\big\|_p \\
&\leqslant\; \|\psi\|_{1,3,\infty} \|DX_k^n\|_{H \times \mathbb{R}^d, 2p} \|Y_k\|_{V, 2p} \\
&\leqslant\; C(M_{2p}^{1/2p}(Z) + \frac{m_*^{1/2p}}{r_*})^2 M_{2p}^{1/2p}(Z) C_{0,2p}(\alpha, \beta, \Gamma) \\
&\quad \times \|\psi\|_{1,3,\infty}^3 \exp(C M_{4p}^{1/p} \|\psi\|_{1,3,\infty}^2),
\end{aligned}
$$

the last inequality being a consequence of (149) and (150). It follows that

$$
\|\overline{\alpha}_k^i\|_{H \times V, p} \leqslant C(M_{2p}^{1/2p}(Z) + \frac{m_*^{1/2p}}{r_*})^2 M_{2p}^{1/2p}(Z) C_{0,2p}(\alpha, \beta, \Gamma) \|\psi\|_{1,3,\infty}^3 \exp(C M_{4p}^{1/p} \|\psi\|_{1,3,\infty}^2)
$$
$$
+ \|D\alpha_k^i\|_{H \times V, p}.
$$

And

$$
\|\overline{\beta}_k^i\|_{H \times V, p} = \|D\beta_k^i\|_{H \times V, p}.
$$

We conclude that

$$
\sup_{m \leqslant n}(\|\overline{\alpha}_k^i\|_{U,p} + \|\overline{\beta}_k^i\|_{U,p}) \leqslant C_{1,p}(\alpha, \beta, \Gamma)
$$
$$
+ C(M_{2p}^{1/2p}(Z) + \frac{m_*^{1/2p}}{r_*})^2 M_{2p}^{1/2p}(Z) C_{0,2p}(\alpha, \beta, \Gamma) \|\psi\|_{1,3,\infty}^3 \exp(C M_{4p}^{1/p} \|\psi\|_{1,3,\infty}^2).
$$

We analyse now $\overline{\Gamma}_m$. We treat first $I_m := \sum_{k=0}^{m-1} \beta_k^i DLH_{k+1}^i$. Since $\beta_k^i D_{p,j} LH_{k+1}^i = 0$ if $p \neq k+1$, we obtain

$$
|I_m|_{H \times V}^2 \leqslant \sum_{j=1}^{N} \sum_{k=0}^{m-1} |D_{k+1,j} LH_{k+1}^i|^2 |\beta_k^i|_V^2
$$

so that, using (73), and the independency of $LH_{k+1}$ and $\beta_k$, we have

$$
\begin{aligned}
\||I_m|_{H \times V}\|_p &= \||I_m|^2_{H \times V}\|^{1/2}_{p/2} \leqslant (\sum_{j=1}^{N} \sum_{k=0}^{m-1} \||D_{k+1,j}LH^i_{k+1}|^2 |\beta^i_k|^2_V\|_{p/2})^{1/2} \\
&= (\sum_{j=1}^{N} \sum_{k=0}^{m-1} \||D_{k+1,j}LH^i_{k+1}||\beta^i_k|_V\|^2_p)^{1/2} = (\sum_{j=1}^{N} \sum_{k=0}^{m-1} \|D_{k+1,j}LH^i_{k+1}\|^2_p \||\beta^i_k|_V\|^2_p)^{1/2} \\
&\leqslant \frac{Cm^{1/p}_*}{r_*}(1 + r^{-1}_*) \sup_{k \leqslant m-1} \||\beta^i_k|_V\|_p = \frac{Cm^{1/p}_*}{r_*}(1 + r^{-1}_*) \sup_{k \leqslant m} \|\beta^i_k\|_{V,p}.
\end{aligned}
$$

Since $DH^i_k$ has properties which are similar to the ones of $DLH^i_k$, the same reasoning as above gives

$$
\|\sum_{k=0}^{m-1} \alpha^i_k DH^i_{k+1}\|_{H \times V,p} \leqslant C \sup_{k \leqslant m-1} \|\alpha^i_k\|_{V,p}
$$

and we have

$$
\begin{aligned}
|\sum_{k=0}^{m-1} \nabla_x a^i_k(X^n_k) Y_k DH^i_{k+1}|^2_{H \times V} &\leqslant \|\psi\|^2_{1,2,\infty} \sum_{k=0}^{m-1} \sum_{j=1}^{N} |Y_k|^2_V |D_{k+1,j}H^i_{k+1}|^2 \\
&\leqslant \frac{N}{n} \|\psi\|^2_{1,2,\infty} \sum_{k=0}^{n-1} |Y_k|^2_V.
\end{aligned}
$$

Using (149), we obtain

$$
\begin{aligned}
\|\sum_{k=0}^{m-1} \nabla_x a^i_k(X^n_k) Y_k DH^i_{k+1}\|_{H \times V,p} &\leqslant \frac{C}{n} \|\psi\|_{1,2,\infty} \sum_{k=0}^{n-1} \|Y_k\|_{V,p} \\
&\leqslant C(M^{1/p}_p(Z) + \frac{m^{1/p}_*}{r_*})C_{0,p}(\alpha, \beta, \Gamma) \\
&\quad \times \|\psi\|_{1,2,\infty} \exp(CM^{2/p}_{2p}(Z)\|\psi\|^2_{1,3,\infty}).
\end{aligned}
$$

We write now

$$
\sum_{k=0}^{m-1} D(H^i_{k+1}H^j_{k+1}\nabla_x b^{i,j}_k(X_k, H_{k+1}))Y_k = I + J
$$

with

$$
\begin{aligned}
I &= \sum_{k=0}^{m-1}(H^i_{k+1}DH^j_{k+1} + H^j_{k+1}DH^i_{k+1})\nabla_x b^{i,j}_k(X^n_k, H_{k+1})Y_k, \\
J &= \sum_{k=0}^{m-1} H^i_{k+1}H^j_{k+1}D\nabla_x b^{i,j}_k(X^n_k, H_{k+1})Y_k.
\end{aligned}
$$

We have

$$|I|_{H \times V} \leqslant \|\psi\|_{1,3,\infty} \frac{1}{n} \sum_{k=0}^{m-1} (|Z_{k+1}^i| + |Z_{k+1}^j|)|Y_k|_V$$

so that

$$\|I\|_{H \times V,p} \leqslant C M_p^{1/p}(Z)(M_p^{1/p}(Z) + \frac{m_*^{1/p}}{r_*}(1 + r_*^{-1}))C_{0,p}(\alpha,\beta,\Gamma)\|\psi\|_{1,3,\infty} \exp(C M_{2p}^{2/p}(Z)\|\psi\|_{1,3,\infty}^2).$$

And using the estimates of $DX_k$ we obtain in a similar way the same inequality for $\|J\|_{H \times V,p}$. So a similar estimate holds for $\sum_{k=0}^{m-1} D(H_{k+1}^i H_{k+1}^j \nabla_x b_{i,j}(X_k))Y_k$. We put all these terms together and we obtain (151).

$\square$

**Theorem 5.1.** *For every* $q, q' \in \mathbb{N}$, $q' \leqslant q$, *and* $p \geqslant 2$ *there exists some constants* $C \geqslant 1$, $l \in \mathbb{N}$ *(depending on* $r_*, \varepsilon_*, m_*, q, p$ *and the moments of* $Z$ *but not on* $n$*) such that*

$$\sup_{m \leqslant n} \sup_{0 \leqslant |\alpha| \leqslant q-q'} \|\partial_x^\alpha X_m^n(x)\|_{q',p} \leqslant C(1 + \|\psi\|_{1,q+2,\infty}^l) \exp(C\|\psi\|_{1,3,\infty}^2), \tag{152}$$

$$\sup_{m \leqslant n} \|LX_m^n\|_{q,p} \leqslant C(1 + \|\psi\|_{1,q+4,\infty}^l) \exp(C\|\psi\|_{1,3,\infty}^2). \tag{153}$$

*Proof.* . We estimate first $\|X_m^n\|_{q,p}$. We have already checked that

$$DX_m^n = \sum_{k=0}^{m-1} A_k DX_k^n + \Gamma_m$$

with

$$\Gamma_m(k,i) = \mathbb{1}_{\{k \leqslant m\}} \frac{\chi_k}{\sqrt{n}} \Big( a_k^i(X_{k-1}^n) + \sum_{j=1}^N H_k^j b_k^{i,j}(X_{k-1}^n, H_k) + \sum_{j,l=1}^N H_k^j H_k^l \partial_{z^i} b_k^{l,j}(X_{k-1}^n, H_k) \Big).$$

So, in view of (148), the only thing to prove is that $\|\Gamma_m\|_{q-1,p} \leqslant C(1 + \|\psi\|_{1,q+2,\infty}^l) \exp(C\|\psi\|_{3,\infty}^2)$. We have already done this for the first order derivatives (that is $q = 1$). For higher order derivatives the proof follows the same line (one employs a reccurence argument).
We look now to $\nabla_x X_m^n(x)$ which solves the equation

$$\nabla_x X_m^n(x) = I + \sum_{k=1}^{m-1} A_k \nabla_x X_k^n(x).$$

This equation si of type (146) so the upper bound of $\|\nabla_x X_m^n(x)\|_{q,p}$ follows from (149). For higher order derivatives the reasoning is the same.
Let us now deal with $LX_m^n$. Notice that $\langle DH_k^j, DH_k^i \rangle = 0$ for $i \neq j$. Then, using the computational rules (see (63)) we obtain

$$LX_{k+1}^n = A_k LX_k^n + \sum_{i=1}^N H_{k+1}^i \alpha_k^i + \sum_{i=1}^N LH_{k+1}^i \beta_k^i + \sum_{i,j=1}^N \gamma_k^{i,j}$$

with

$$\alpha_k^i = \sum_{l,r=1}^{d} \partial_{x_l} \partial_{x_r} a_k^i(\kappa_k, X_k) \left\langle (DX_k^n)^r, (DX_k^n)^l \right\rangle, \qquad \beta_k^i = a_k^i(X_k^n)$$

and

$$
\begin{aligned}
\gamma_k^{i,j} &= \frac{1}{2} L H_{k+1}^i H_{k+1}^j b_k^{i,j}(\kappa_k, X_k^n, H_{k+1}) + \frac{1}{2} L H_{k+1}^i H_{k+1}^j b_k^{i,j}(\kappa_k, X_k^n, H_{k+1}) \\
&\quad + \frac{1}{2} H_{k+1}^i H_{k+1}^j \Big( \sum_{l,r=1}^{d} \partial_{x_l} \partial_{x_r} b_k^{i,j}(\kappa_k, X_k^n) \left\langle (DX_k^n)^l, (DX_k^n)^r \right\rangle + \sum_{r=1}^{N} \partial_{z_r} b_k^{i,j}(\kappa_k, X_k^n, H_{k+1}) L H_{k+1}^r \\
&\quad + \frac{\chi_{k+1}}{n} \sum_{r=1}^{N} \partial_{z_r}^2 b_k^{i,j}(\kappa_k, X_k^n, H_{k+1}) \Big) + \mathbb{1}_{i=j} \frac{\chi_{k+1}}{n} b_k^{i,i}(\kappa_k, X_k^n, H_{k+1}) \\
&\quad + \frac{\chi_{k+1}}{n} \big( H_{k+1}^i \partial_{z_j} b_k^{i,j}(\kappa_k, X_k^n, H_{k+1}) + H_{k+1}^j \partial_{z_i} b_k^{i,j}(\kappa_k, X_k^n, H_{k+1}) \big).
\end{aligned}
$$

We have

$$\|\alpha_k^i\|_{q,p} \leqslant C\|\psi\|_{1,q+3,\infty} \|X_k^n\|_{q+1,p}^l \leqslant C(1 + \|\psi\|_{1,q+3,\infty}^l) \exp(C\|\psi\|_{1,3,\infty}^2)$$

and a similar estimate holds for $\|\beta_k^i\|_{q,p}$.

Moreover, we have $\Gamma_m = \sum_{i,j=1}^{N} \sum_{k=0}^{m-1} \gamma_k^{i,j}$ so we have to analyse each of the terms in $\gamma_k^{i,j}$. We look first at

$$
\begin{aligned}
\|L H_{k+1}^i H_{k+1}^j b_k^{i,j}(\kappa_k, X_k^n, H_{k+1})\|_{q,p} &\leqslant \|L H_{k+1}^i H_{k+1}^j\|_{q,2p} \|b_k^{i,j}(\kappa_k, X_k^n, H_{k+1})\|_{q,2p} \\
&\leqslant \|L H_{k+1}^i\|_{q,4p} \|H_{k+1}^j\|_{q,4p} \|\psi\|_{1,q+2,\infty} (\|X_k^n\|_{q,2p}^l + \|H_k^j\|_{q,2p}^l) \\
&\leqslant \frac{C}{n}(1 + \|\psi\|_{1,q+2,\infty}^l) \exp(C\|\psi\|_{1,3,\infty}^2).
\end{aligned}
$$

The other terms in $\gamma_k^{i,j}$ verify similar estimates. So we obtain

$$\|\Gamma_m\|_{q,p} \leqslant \sum_{i,j=1}^{N} \sum_{k=0}^{m-1} \|\gamma_k^{i,j}\|_{q,p} \leqslant C(1 + \|\psi\|_{1,q+4,\infty}^l) \exp(C\|\psi\|_{1,3,\infty}^2).$$

We conclude that

$$C_{q,p}(\alpha, \beta, \Gamma) \leqslant C(1 + \|\psi\|_{1,q+4,\infty}^l) \exp(C\|\psi\|_{1,3,\infty}^2)$$

and we are done.

$\square$

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