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HARDY-LITTLEWOOD FIRST APPROXIMATION THEOREM FOR QUASI L-FUNCTIONS

BY

R.BALASUBRAMANIAN and K.RAMACHANDRA

(DEDICATED WITH DEEPEST REGARDS TO

THE MEMORY OF LEONARD EULER)

(accepted on 12-06-2005)

§1. **INTRODUCTION.** We begin by stating the following theorem due to G.H.HARDY and J.E.LITTLEWOOD (see Theorem 4.11 of [ECT]).

THEOREM 1. *We have*

$$\zeta(s) = \sum_{n \leq x} n^{-s} - \frac{x^{1-s}}{1-s} + O(x^{-\sigma}) \quad (1)$$

uniformly for $\sigma \geq \sigma_0 > 0$, $|t| \leq \pi x C^{-1}$ which $C > 1$ is any given constant.

REMARK. We call (1) as the first approximation theorem of HARDY and LITTLEWOOD. In [RB, KR]₁ we proved (1) with the conditions $t > 2$ and $x > (\frac{1}{2} + \delta)t$, where $\delta > 0$ is any constant.

The object of the present note (which is an addendum to [RB, KR]₁) is to make a few remarks on the results of [RB, KR]₁ and prove the following theorem.

THEOREM 2. *We define (quasi L-functions) by*

$$L(s, \chi) = \sum_{n=1}^{\infty} \chi(n)(n + \alpha)^{-s} F(n + \alpha) \quad (2)$$

where $\chi(n)$ is any periodic sequence of complex numbers (not all zero) whose sum over any period is zero. $\alpha > 0$ is any constant and $F(X)$ is any complex valued function of X which is infinitely often continuously differentiable in $X \geq 1$ with $F^{(k)}(X) = O(X^{-k+\epsilon})$ for every $k \geq 0$ and every $\epsilon > 0$. As usual $s = \sigma + it$, $\sigma \geq \sigma_0 > 0$. Then $L(s, \chi)$ defined above is uniformly convergent in any compact sub-set. Moreover $L(s, \chi)$ can be continued as an entire

function and

$$L(s, \chi) = \sum_{n \leq x} \chi(n)(n + \alpha)^{-s} F(n + \alpha) + E \quad (3)$$

where

$$|E| \leq C_\epsilon x^{-\sigma + \epsilon}, \sigma \geq -K, \quad (4)$$

provided only that $x \geq x_0 = D_\epsilon(|t| + 2)^{1+\epsilon}$, $D_\epsilon > 0$ being a certain constant depending on $K (> 0)$ and ϵ (and of course on constants depending upon the quasi L function). In (4) $C_\epsilon > 0$ depends only on $\epsilon (> 0)$ and K which are arbitrary.

REMARK 1. In (2) and (3) we have written $F(n + \alpha)$ for some convenience. Certainly we can write $F(n)$ instead.

REMARK 2. The proof that certain functions are entire (see [RB, KR]₂) is a special case of theorem 2 obtained by treating $\sum_{n=1}^{\infty} (-1)^n \int_0^1 f'(n+x) dx$.

REMARK 3. Examples of $F(n)$ are $\exp(\sqrt{\log n})$ and so on.

§2. **PROOF OF THEOREM 2.** We begin by stating Theorem 2 of [RB, KR]₁ as a lemma.

LEMMA. Let a and b integers with $a < b$, k a non-negative integer and $f(x)$, a function of x which is k -times continuously differentiable in $a \leq x \leq b - 1 + k$. Then

$$\begin{aligned} \sum_{a \leq n < b} f(n) &= \int_a^b f(x) dx + \left\{ -\frac{1}{2} \int_0^1 \int_0^1 (f(b + u_1 \nu_1^{\frac{1}{2}}) - f(a + u_1 \nu_1^{\frac{1}{2}})) du_1 d\nu_1 \right. \\ &+ \frac{1}{2^2} \int_0^1 \int_0^1 \int_0^1 (f'(b + u_1 \nu_1^{\frac{1}{2}} + u_2 \nu_2^{\frac{1}{2}}) - f'(a + u_1 \nu_1^{\frac{1}{2}} + u_2 \nu_2^{\frac{1}{2}})) du_1 d\nu_1 du_2 d\nu_2 \\ &\dots + \frac{(-1)^{k-1}}{2^{k-1}} \int_0^1 \dots \int_0^1 (f^{(k-2)}(b + u_1 \nu_1^{\frac{1}{2}} + \dots + u_{k-1} \nu_{k-1}^{\frac{1}{2}}) \\ &\left. - f^{(k-2)}(a + u_1 \nu_1^{\frac{1}{2}} + \dots + u_{k-1} \nu_{k-1}^{\frac{1}{2}})) du_1 d\nu_1 \dots du_{k-1} d\nu_{k-1} \right\} \\ &+ \frac{(-1)^k}{2^k} \int_0^1 \dots \int_0^1 \sum_{a \leq n < b} f^{(k)}(n + u_1 \nu_1^{\frac{1}{2}} + \dots + u_k \nu_k^{\frac{1}{2}}) du_1 d\nu_1 \dots du_k d\nu_k. \end{aligned}$$

REMARK 1. The terms in the curly brackets are absent if $k = 1$.

REMARK 2. We have corrected the following. In Theorem 2 of [RB, KR]₁ $a < n \leq b$ is not correct. It should read $a \leq n < b$.

REMARK 3. Actually the proof of the lemma is simple and runs as follows.

$$\sum_{a \leq n < b} f(n) = \sum_{a \leq n < b} (f(n) - \int_n^{n+1} f(u) du) + \int_a^b f(u) du$$

and here the first term on the RHS is

$$\begin{aligned} & \sum_{a \leq n < b} \int_0^1 (f(n) - f(n+u)) du = - \sum_{a \leq n < b} \int_0^1 \int_0^u f'(n+v) dv du \\ & = - \sum_{a \leq n < b} \int_0^1 \int_0^1 u f'(n+uv) dudv \\ & = -\frac{1}{2} \sum_{a \leq n < b} \int_0^1 \int_0^1 f'(n+uv^{\frac{1}{2}}) dudv \end{aligned}$$

(by a change of variable). This gives the case $k = 1$ of the lemma. The lemma follows by a k -fold application of this case.

We now resume the proof of theorem 2. We define $\chi(0)$ to be $\chi(p)$ where p is the length of the period. Let X be any positive integer. We have

$$\begin{aligned} & \sum_{n \geq Xp} \chi(n)(n+\alpha)^{-s} F(n+\alpha) \\ & = \sum_{\nu=0}^{p-1} \chi(\nu) \sum_{n \equiv \nu \pmod{p}, n \geq Xp} (n+\alpha)^{-s} F(n+\alpha) \\ & = \sum_{\nu=0}^{p-1} \chi(\nu) \left\{ \sum_{n \equiv \nu \pmod{p}, n \geq Xp} (n+\alpha)^{-s} F(n+\alpha) - \sum_{n \equiv 0 \pmod{p}, n \geq Xp} (n+\alpha)^{-s} F(n+\alpha) \right\} \\ & = \sum_{\nu=0}^{p-1} \chi(\nu) \left\{ \sum_{lp+\nu \geq Xp} (lp+\nu+\alpha)^{-s} F(lp+\nu+\alpha) - \sum_{l \geq X} (lp+\alpha)^{-s} F(lp+\alpha) \right\} \\ & = \sum_{\nu=0}^{p-1} \chi(\nu) \left\{ \sum_{l \geq X} (lp+\nu+\alpha)^{-s} F(lp+\nu+\alpha) - \sum_{l \geq X} (lp+\alpha)^{-s} F(lp+\alpha) \right\} \\ & = \sum_{\nu=0}^{p-1} \chi(\nu) \left[\int_X^\infty (xp+\nu+\alpha)^{-s} F(xp+\nu+\alpha) dx - \int_X^\infty (xp+\alpha)^{-s} F(xp+\alpha) dx \right] \\ & + \frac{1}{2} \frac{d}{dx} \left((xp+\nu+\alpha)^{-s} F(xp+\nu+\alpha) - (xp+\alpha)^{-s} F(xp+\alpha) \right)_{x=X} \\ & - \frac{1}{2^2} \frac{d^2}{dx^2} \left((xp+\nu+\alpha)^{-s} F(xp+\nu+\alpha) - (xp+\alpha)^{-s} F(xp+\alpha) \right)_{x=X} \\ & + \dots \end{aligned}$$

$$\begin{aligned}
& -\frac{(-1)^{k-2}}{2^{k-2}} \frac{d^{(k-2)}}{dx^{k-2}} \left((xp + \nu + \alpha)^{-s} F(xp + \nu + \alpha) - (xp + \alpha)^{-s} F(xp + \alpha) \right)_{x=X} \\
& + \frac{(-)^k}{2^k} \int_0^1 \cdots \int_0^1 \left\{ \frac{d^k}{dx^k} \left(\sum_{n \geq 1} ((X + x + n + \alpha + y + \nu)^{-s} F(X + x + n + \alpha + y + \nu)) \right. \right. \\
& \left. \left. - \sum_{n \geq 1} ((X + x + n + \alpha + y)^{-s} F(X + x + n + \alpha + y)) \right) \right\}_{x=0} du_1 d\nu_1 \dots du_k d\nu_k \quad (5)
\end{aligned}$$

where $y = u_1 \nu_1^{\frac{1}{2}} + \dots + u_k \nu_k^{\frac{1}{2}}$.

This follows by choosing $f(x)$ (in the lemma) suitably and letting $b \rightarrow \infty$ (provided $\sigma \geq 2$). Thus we get analytic continuation in $\sigma > 0$. By choosing $k = 1, 2, 3, \dots, [K] + 10$ (successively) we see that $L(s, \chi)$ is entire (since K is arbitrary). We have

$$\int_X^\infty (xp + \nu + \alpha)^{-s} F(xp + \nu + \alpha) dx = \int_{Xp + \nu + \alpha}^\infty W^{-s} F(W) \frac{dW}{p}$$

(by putting $xp + \nu + \alpha = W$) and

$$\int_X^\infty (xp + \alpha)^{-s} F(xp + \alpha) dx = \int_{Xp + \alpha}^\infty W^{-s} F(W) \frac{dW}{p}.$$

Hence their difference is entire and is $O(X^{-\sigma+\epsilon})$.

In (5) all terms except the last are entire and are $O(X^{-\sigma+\epsilon})$. We give some details. For $k \geq l \geq 0$ and $x \geq A \geq 10$ and $\beta (> 0)$ bounded above by B we have

$$\frac{d^l}{dx^l} \left((x + \beta)^{-s} f(x) \right) \quad (= O(x^{-\sigma+\epsilon}) \text{ if } l = 0)$$

and if $l \geq 1$ it is

$$\begin{aligned}
& = O \left(\frac{|s|(|s| + 1) \dots (|s| + l - 1)}{x^{\sigma+l}} |f(x)| + l \frac{|s|(|s| + 1) \dots (|s| + l - 2)}{x^{\sigma+l-1}} |f'(x)| \right) \\
& + \frac{l(l-1)}{2!} \frac{|s|(|s| + 1) \dots (|s| + l - 3)}{x^{\sigma+l-2}} |f''(x)| + \dots \text{to } l+1 \text{ terms} \\
& = O \left(2^l (l+1) \max_{1 \leq j \leq l+1} \frac{|s|(|s| + 1) \dots (|s| + l - j)}{x^{\sigma+l-j+1}} |f^{(j-1)}(x)| \right) \\
& = O \left(2^l (l+1) \max_{1 \leq j \leq l+1} \left\{ \frac{(|s| + l - j + 1)^{l-j+1}}{x^{l-j+1}} x^{-\sigma} x^{-(j-1)+\epsilon} \right\} \right) \\
& = O \left(2^l (l+1) \max_{1 \leq j \leq l+1} \left\{ \frac{(|\sigma| + |t| + k)^l}{x^l} x^{-\sigma+\epsilon} \right\} \right)
\end{aligned}$$

$$= O(x^{-\sigma+\epsilon}) \text{ if } x \geq 4(|\sigma| + |t| + k).$$

On the other hand the last but one inequality shows us that the last multiple integral in (5) is

$$\begin{aligned} & O \left(\sum_{n \geq 1} 2^k (k+1) \max_{1 \leq j \leq k+1} \left\{ \frac{(|\sigma| + |t| + k)^k}{(X+n)^k} (X+n)^{-\sigma+\epsilon} \right\} \right) \\ &= O \left(2^k (k+1) \left(\frac{|\sigma| + |t| + k}{X} \right)^k X^{-\sigma+\epsilon+1} \right) \\ &= O \left(X^{-\sigma+\epsilon} \right) \end{aligned}$$

provided $k + \sigma \geq 3$ and $X \geq (|\sigma| + |t| + k + 200)^{1+\epsilon}$ and also $k = \left\lceil \frac{100}{\epsilon} \right\rceil$. This completes the proof of Theorem 2.

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P.S. Sharper results are known for $\zeta(s)$ and $L(s, \chi)$ ($\chi - a$ character *mod* k). See the booklet 'RIEMANN ZETA-FUNCTION' published by Ramanujan Institute, Chennai (1979).