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REGULARITY AND B-FUNCTIONS FOR D-MODULES

YVES LAURENT

ABSTRACT

A holonomic \mathcal{D} -module on a complex analytic manifold admits always a b -function along any submanifold. If the module is regular, it admits also a regular b -function, that is a b -function with a condition on the order of the lower terms of the equation. We prove that these two b -functions are equal. In the same time we prove that a module which is regular along a submanifold admits a regular b -function along this submanifold.

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INTRODUCTION

Let f be a holomorphic function on an open set X of \mathbb{C}^n . A b -function for f is a polynomial $b \in \mathbb{C}[s]$ such that there exists a differential operator P on U with parameter s satisfying an equation:

$$(1) \quad b(s)f(x)^s = P(s, x, D_x)f(x)^{s+1}$$

The generator of the ideal of the b -functions associated to f is usually called the Bernstein-Sato polynomial of f (see [1] for details).

This definition has been extended by Kashiwara [3] to holonomic \mathcal{D} -modules. Let X be a complex manifold and Y a smooth hypersurface. Let \mathcal{D}_X be the sheaf of differential operators on X . Let \mathcal{M} be a holonomic \mathcal{D}_X -module and u a section of \mathcal{M} . A b -function for u along Y is an equation

$$(2) \quad b(tD_t)u = tP(t, x, tD_t, D_x)u$$

satisfied by u . Here (t, x) are local coordinates of X such that t is an equation for Y . A similar definition exists for submanifolds of X of any codimension.

The b -function is called **regular** if P may be chosen so that its order is not greater than the degree of b . "Order" means order as differential operator that is in (D_x, D_t) in equation (2) and order in (s, D_x) in equation (1).

It has been proved by Kashiwara [3] that a holonomic \mathcal{D} -module admits a b -function along any smooth hypersurface and by Kashiwara-Kawai [5] that a *regular* holonomic \mathcal{D} -module admits a *regular* b -function along any smooth hypersurface. It is an old question to know if these two b -functions are equal (see [13] for example but the question is older).

There is another notion of regularity, the regularity of a \mathcal{D} -module along a submanifold. The definition will be given in definitions 1.3 and 2.5. In the case of a hypersurface, this is equivalent to the fact that solutions in the formal completion transversally to the hypersurface are convergent. It has been proved that a regular holonomic module is regular along any submanifold in [6]. It is also a direct consequence of the definition that if a \mathcal{D} -module admits a regular b -function along a submanifold, it is regular along it. The question is to prove that the two properties are equivalent.

The aim of this paper is to solve these two questions by proving that if a \mathcal{D} -module is regular along a submanifold, its b -function is always regular.

The problem is better understood after microlocalization, that is for modules over the sheaf \mathcal{E}_X of microdifferential operators. Regularity and b -function may be defined for a coherent \mathcal{E}_X -module \mathcal{M} along a conic lagrangian submanifold Λ of the cotangent bundle [8]. All definitions are invariant under quantized canonical transformation. Concerning \mathcal{D} -modules, this will prove that the result is true not only for hypersurfaces but also for any smooth subvariety of X .

The b -function is the annihilator of a bi-graded module named $gr_V^{(\infty)}(\mathcal{M})$ and a regular b -function, if it exists, is the annihilator of a bi-graded module named $gr_V^{(1)}(\mathcal{M})$. Both are modules over the sheaf of holomorphic functions on the cotangent bundle to Λ . Moreover, there exists a surjective morphism $gr_V^{(1)}(\mathcal{M}) \rightarrow gr_V^{(\infty)}(\mathcal{M})$. We will get the result by proving that this morphism is an isomorphism if \mathcal{M} is regular along Λ .

If \mathcal{M} is regular along Λ , the analytic cycles defined by the $gr_V^{(\infty)}(\mathcal{M})$ and $gr_V^{(1)}(\mathcal{M})$ are equal. This is a consequence of the study of regularity in [10] and [12]. Hence the morphism $gr_V^{(1)}(\mathcal{M}) \rightarrow gr_V^{(\infty)}(\mathcal{M})$ is an isomorphism on a Zarisky open set.

The problem is now to prove that it is an isomorphism everywhere from the fact that it is generically an isomorphism. Such a result is well-known for holonomic \mathcal{E}_X -modules: the support of a holonomic \mathcal{E}_X -module is lagrangian, hence a morphism of holonomic \mathcal{E}_X -modules which is an isomorphism generically is an isomorphism. However, there is no such structure on the two graded rings. Here, the idea of the proof is to use the second microlocalization to define a structure of \mathcal{E}_X -module on the two graded rings. In fact, we get a structure of module on the sheaf $\mathcal{E}_X(0)$ of microdifferential operators of degree 0 but this is sufficient to get the result.

In the first section of this paper, we briefly recall the definitions of b -functions and state the result for \mathcal{D}_X -modules.

In the second section, we study the microlocal case. We define the filtrations and bifiltrations on holonomic \mathcal{E}_X -modules which define the b -function and the regularity.

In the third section we define the 2-microlocal operators and their filtrations. We prove an isomorphism between a 2-microlocal graded ring and a ring of microdifferential operators of order 0. Then we prove the main theorem.

In the last section we give an application to meromorphic b -function and an extension to non regular holonomic \mathcal{D} - or \mathcal{E} -modules.

1. THE CASE OF DIFFERENTIAL EQUATIONS

In this first section, we briefly recall the definitions of b -functions and regularity in the framework of \mathcal{D}_X -modules and state our main result in this case.

1.1. Filtrations and b -functions. Let X be a complex analytic manifold, \mathcal{O}_X the sheaf of holomorphic functions on X and \mathcal{D}_X the sheaf of differential operators on X with coefficients in \mathcal{O}_X . Let T^*X be the cotangent bundle to X with canonical projection $\pi : T^*X \rightarrow X$. Let Y be a submanifold of X .

The sheaf \mathcal{D}_X is provided with two filtrations. The first one is the filtration $(\mathcal{D}_{X,m})_{m \geq 0}$ by the usual order. The corresponding graded ring $gr\mathcal{D}_X$ is identified to $\pi_*\mathcal{O}_{[T^*X]}$ the sheaf of holomorphic functions on T^*X with polynomial coefficients in the fibers of π .

The second one is the V-filtration which has been defined by Kashiwara in [4] as:

$$(1.1) \quad V_k\mathcal{D}_X = \{ P \in \mathcal{D}_X \mid \forall \ell \in \mathbb{Z}, P\mathcal{I}_Y^\ell \subset \mathcal{I}_Y^{\ell-k} \}$$

where \mathcal{I}_Y is the ideal of definition of Y and $\mathcal{I}_Y^\ell = \mathcal{O}_X$ if $\ell \leq 0$.

Let $\tau : T_Y X \rightarrow Y$ be the normal bundle to Y in X and $\mathcal{O}_{[T_Y X]}$ the sheaf of holomorphic functions on $T_Y X$ which are polynomial in the fibers of τ . Let $\mathcal{O}_{[T_Y X]}[k]$ be the subsheaf of $\mathcal{O}_{[T_Y X]}$ of homogeneous functions of degree k in the fibers of τ . There are canonical isomorphisms between $\mathcal{I}_Y^k/\mathcal{I}_Y^{k-1}$ and $\tau_*\mathcal{O}_{[T_Y X]}[k]$, between $\bigoplus \mathcal{I}_Y^k/\mathcal{I}_Y^{k-1}$ and $\tau_*\mathcal{O}_{[T_Y X]}$. Hence the graded ring $gr^V\mathcal{D}_X$ associated to the V-filtration on \mathcal{D}_X acts naturally on $\mathcal{O}_{[T_Y X]}$. An easy calculation [16] shows that as a subring of $\mathcal{E}nd(\tau_*\mathcal{O}_{[T_Y X]})$ it is identified to $\tau_*\mathcal{D}_{[T_Y X]}$ the sheaf of differential operators on $T_Y X$ with coefficients in $\mathcal{O}_{[T_Y X]}$.

The Euler vector field θ of $T_Y X$ is the vector field which acts on $\mathcal{O}_{[T_Y X]}[k]$ by multiplication by k . Let ϑ be any differential operator in $V_0\mathcal{D}_X$ whose image in $gr_0^V\mathcal{D}_X$ is θ . Let \mathcal{M} be a coherent \mathcal{D}_X -module and u a section of \mathcal{M} .

Definition 1.1. A polynomial b is a b -function for u along Y if there exists a differential operator Q in $V_{-1}\mathcal{D}_X$ such that $(b(\vartheta) + Q)u = 0$.

Definition 1.2. A polynomial b is a *regular b-function* for u along Y if there exists a differential operator Q in $V_{-1}\mathcal{D}_X$ which is of usual order at most the degree of b such that $(b(\vartheta) + Q)u = 0$.

The set of b -functions is an ideal of $\mathbb{C}[T]$, if it is not zero we call a generator of this ideal "the" *b-function* of u along Y .

Remark that the set of regular b-function for u is not always an ideal of $\mathbb{C}[T]$.

1.2. Regularity. The definition of a regular holonomic module may be found in [6]. It is equivalent to the fact that formal solutions converges at each point. More precisely, if x is a point of X , $\mathcal{O}_{X,x}$ the ring of germs of \mathcal{O}_X at x and \mathfrak{m} the maximal ideal of $\mathcal{O}_{X,x}$, let us denote by $\widehat{\mathcal{O}}_{X,x}$ the formal completion of $\mathcal{O}_{X,x}$ for the \mathfrak{m} -topology. The holonomic \mathcal{D}_X -module \mathcal{M} is *regular* if and only if:

$$\forall j \geq 0, \quad \forall x \in X, \quad \mathcal{E}xt_{\mathcal{D}_X}^j(\mathcal{M}, \widehat{\mathcal{O}}_{X,x}) = \mathcal{E}xt_{\mathcal{D}_X}^j(\mathcal{M}, \mathcal{O}_{X,x})$$

This may be a definition as in Ramis [14] or a theorem as in Kashiwara-Kawaiï [6].

A weaker notion is the regularity along a submanifold. This has been studied in several papers [7],[8],[12]. The definition will be given in section 2.4 using microcharacteristic varieties. Here we give a more elementary definition.

Let Y be a submanifold of X , ϑ is the vector field of section 1.1 and \mathcal{M} be a holonomic \mathcal{D}_X -module \mathcal{M} defined in a neighborhood of Y .

Definition 1.3. The holonomic \mathcal{D}_X -module \mathcal{M} is *regular along* Y if any section u of \mathcal{M} is annihilated by a differential operator of the form $\vartheta^N + P + Q$ where P is in $\mathcal{D}_{X,N-1} \cap V_0\mathcal{D}_X$ and Q is in $\mathcal{D}_{X,N} \cap V_{-1}\mathcal{D}_X$

Let us denote by $\mathcal{O}_{\widehat{X|Y}}$ the formal completion of \mathcal{O}_X along Y , that is

$$\mathcal{O}_{\widehat{X|Y}} = \text{proj} \lim k\mathcal{O}_X / \mathcal{I}_Y^k$$

We proved in [9] that if \mathcal{M} is regular along Y then

$$(1.2) \quad \forall j \geq 0, \quad \mathcal{E}xt_{\mathcal{D}_X}^j(\mathcal{M}, \mathcal{O}_{\widehat{X|Y}}) = \mathcal{E}xt_{\mathcal{D}_X}^j(\mathcal{M}, \mathcal{O}_X)$$

We proved in [12] that the converse is true if Y is a hypersurface.

It has been proved first by Kashiwara in [3], that if \mathcal{M} is a holonomic \mathcal{D}_X -module, there exists a b -function for any section u of \mathcal{M} along any submanifold Y of X . Kashiwara and Kawaiï proved in [5] that if \mathcal{M} is a regular holonomic \mathcal{D}_X -module, there exists a **regular** b -function for any section u of \mathcal{M} along any submanifold Y of X .

The main result of this paper is:

Theorem 1.4. *If the holonomic \mathcal{D}_X -module \mathcal{M} is regular along Y , then for any section u of \mathcal{M} , the b -function of u along Y is a regular b -function for u along Y .*

This shows that if \mathcal{M} is regular along Y then it admits a regular b -function and also that if there is a regular b -function then the regular b -function of lowest degree is precisely the b -function.

Remark 1.5. Explicit calculations in coordinates of various filtrations and b -functions may be found in [11].

2. MICRODIFFERENTIAL EQUATIONS

In this section, we review basic definitions of filtration, V-filtration and bifiltrations on \mathcal{E}_X -modules. Details may be founded in [1] or [16], and [8].

2.1. V-filtration on microdifferential operators. We denote by \mathcal{E}_X the sheaf of microdifferential operators of [15]. It is filtered by the order, we will denote that filtration by $\mathcal{E}_X = \bigcup \mathcal{E}_{X,k}$ and call it the usual filtration.

Let $\mathcal{O}_{(T^*X)}$ be the sheaf of holomorphic functions on T^*X which are finite sums of homogeneous functions in the fibers of $\pi : T^*X \rightarrow X$. The graded ring $gr\mathcal{E}_X$ is isomorphic to $\pi_*\mathcal{O}_{(T^*X)}$ [15].

In [8], we extended the definitions of V-filtrations and b -functions to microdifferential equations and lagrangian subvarieties of the cotangent bundle. These definitions are invariant under quantized canonical transformations.

Let Λ be a lagrangian conic submanifold of the cotangent bundle T^*X and \mathcal{M}_Λ be a simple holonomic \mathcal{E}_X -module supported by Λ . By definition, such a module is generated by a non degenerate section u_Λ , that is such that the ideal of the principal symbols of the microdifferential operators annihilating u_Λ is the ideal of definition of Λ . It always exists locally [15].

Let $\mathcal{M}_{\Lambda,k} = \mathcal{E}_{X,k}u_\Lambda$. Then the V-filtration on \mathcal{E}_X along Λ is defined by:

$$(2.1) \quad V_k\mathcal{E}_X = \{ P \in \mathcal{E}_X \mid \forall \ell \in \mathbb{Z}, P\mathcal{M}_{\Lambda,\ell} \subset \mathcal{M}_{\Lambda,\ell+k} \}$$

This filtration is independent of the choices of \mathcal{M}_Λ and u_Λ , so it is globally defined.

Let $\mathcal{O}_\Lambda[k]$ be the sheaf of holomorphic functions on Λ homogeneous of degree k in the fibers of $\Lambda \rightarrow X$ and $\mathcal{O}_{(\Lambda)} = \bigoplus_{k \in \mathbb{Z}} \mathcal{O}_\Lambda[k]$. Then there is an isomorphism between $\mathcal{M}_{\Lambda,k}/\mathcal{M}_{\Lambda,k-1}$ and $\mathcal{O}_{\Lambda,k}$. By this isomorphism the graded ring $gr^V\mathcal{E}_X$ acts on $\mathcal{O}_{(\Lambda)}$ and may be identified to the sheaf $\mathcal{D}_{(\Lambda)}$ of differential operators on Λ with coefficients in $\mathcal{O}_{(\Lambda)}$.

All these definitions are invariant under quantized canonical transformations [8].

The restriction to the zero section T_X^*X of T^*X of the sheaf \mathcal{E}_X is the sheaf \mathcal{D}_X of differential operators. If Λ is the conormal bundle T_Y^*X to a submanifold Y of X then the V-filtration induced on \mathcal{D}_X by the V-filtration of \mathcal{E}_X is the same than the V-filtration defined in section 1.1. The associated graded ring is the restriction to Y of $\mathcal{D}_{(\Lambda)}$ which is the sheaf $\mathcal{D}_{[\Lambda]}$ of differential operators with coefficients polynomial in the fibers of $\Lambda \rightarrow X$.

The correspondence between the isomorphism $gr^V\mathcal{D}_X \simeq \mathcal{D}_{[T_Y X]}$ of section 1.1 and the isomorphism $gr^V\mathcal{D}_X \simeq \mathcal{D}_{[T_Y^* X]}$ is given by the partial Fourier transform associated to the duality between the normal bundle $T_Y X$ and the conormal bundle $T_Y^* X$.

2.2. Filtrations on \mathcal{E} -modules. Let \mathcal{M} be a (left) coherent \mathcal{E}_X -module. A filtration (resp. a V-filtration) of \mathcal{M} is a filtration compatible with the usual filtration (resp. V-filtration) of \mathcal{E}_X .

A good filtration of \mathcal{M} is a filtration which is locally of the form $\mathcal{M}_k = \sum_{i=1}^N \mathcal{E}_{X,k-k_i} u_i$ for (u_1, \dots, u_N) local sections of \mathcal{M} and (k_1, \dots, k_N) integers. In the same way, a good V-filtration of \mathcal{M} is locally of the form $V_k\mathcal{M} = \sum_{i=1}^N V_{k-k_i} \mathcal{E}_X u_i$.

If \mathcal{M} is provided with a good filtration, the associated graded module $gr\mathcal{M}$ is a coherent module over $gr\mathcal{E}_X = \pi_*\mathcal{O}_{(T^*X)}$. Then $\mathcal{O}_{T^*X} \otimes_{\pi^{-1}gr\mathcal{E}_X} \pi^{-1}gr\mathcal{M}$ is a coherent \mathcal{O}_{T^*X} -module which defines a positive analytic cycle on T^*X independent of the good filtration. This cycle is called the *characteristic cycle* of \mathcal{M} and denoted by $\widetilde{Ch}(\mathcal{M})$. Its support is the

characteristic variety of \mathcal{M} denoted $Ch(\mathcal{M})$, it is equal to the support of the module \mathcal{M} itself.

The characteristic variety of a coherent \mathcal{E}_X -module is a *homogeneous involutive* subvariety of T^*X . When its dimension is minimal, that is when it is a lagrangian, the module is *holonomic*.

If \mathcal{M} is provided with a good V-filtration, the graded module $gr^V \mathcal{M}$ is a coherent module over $gr^V \mathcal{E}_X = \mathcal{D}_{(\Lambda)}$ hence $gr^V \mathcal{M}$ is a coherent $\mathcal{D}_{(\Lambda)}$ -module. The characteristic cycle of this module is a positive analytic cycle on $T^*\Lambda$ and its support is involutive. According to [8] and [12], this characteristic cycle is called the microcharacteristic cycle of \mathcal{M} of type ∞ and is denoted by $\widetilde{Ch}_{\Lambda(\infty)}(\mathcal{M})$. The corresponding microcharacteristic variety is $Ch_{\Lambda(\infty)}(\mathcal{M})$ (sometimes also denoted by $Ch_{\Lambda(\infty, \infty)}(\mathcal{M})$). We have the following fundamental result:

Theorem 2.1 (Theorem 4.1.1 of [8]). *The dimension of the characteristic variety of $gr^V \mathcal{M}$ is less or equal to the dimension of the characteristic variety of \mathcal{M} . In particular, $gr^V \mathcal{M}$ is holonomic if \mathcal{M} is holonomic.*

As $gr_k^V \mathcal{E}_X$ is identified to the subsheaf of $\mathcal{D}_{(\Lambda)}$ of differential operators P satisfying $[\theta, P] = kP$, the Euler vector field θ acts on $gr^V \mathcal{M}$, so we may define a morphism Θ by $\Theta = \theta - k$ on $gr_k^V \mathcal{M}$. As $[\theta, P] = +kP$ on $gr_k^V \mathcal{E}_X$, Θ commutes with the action of $gr^V \mathcal{E}_X$ that is defines a section of $\mathcal{E}nd_{gr^V \mathcal{E}_X}(gr^V \mathcal{M})$.

Definition 2.2. The set of polynomials b satisfying

$$b(\Theta)gr^V(\mathcal{M}) = 0$$

is an ideal of $\mathbb{C}[T]$. When this ideal is not zero, a generator is called *the b-function of \mathcal{M} along Λ* .

Theorem 2.3. *If \mathcal{M} is a holonomic \mathcal{E}_X -module, there exists a b-function for \mathcal{M} along any lagrangian submanifold Λ of T^*X .*

This theorem has been proved first by Kashiwara for \mathcal{D} -modules in [3]. It has been proved for \mathcal{E}_X -module in [8] as a corollary of theorem 2.1.

The b -function depends on the good V-filtration, in fact its roots are shifted by integers by change of V-filtration.

2.3. Bifiltration. From the two filtrations on \mathcal{E}_X , we get a bifiltration:

$$(2.2) \quad \forall (k, j) \in \mathbb{Z} \times \mathbb{Z} \quad F_{k,j} \mathcal{E}_X = \mathcal{E}_{X,j} \cap V_k \mathcal{E}_X$$

To this bifiltration is associated the bigraded ring:

$$(2.3) \quad gr^F \mathcal{E}_X = \bigoplus_{(k,j)} gr_{k,j}^F \mathcal{E}_X \quad \text{with} \quad gr_{k,j}^F \mathcal{E}_X = F_{k,j} \mathcal{E}_X / (F_{k-1,j} \mathcal{E}_X + F_{k,j-1} \mathcal{E}_X)$$

This bigraded ring is equal to the graded ring of $gr^V \mathcal{E}_X = \mathcal{D}_{(\Lambda)}$ (this latter with the standard filtration). So it is isomorphic to $\pi_{\Lambda*} \mathcal{O}_{[T^*\Lambda]}$, the sheaf of holomorphic functions on $T^*\Lambda$ which are polynomial in the fibers of $\pi_{\Lambda} : T^*\Lambda \rightarrow \Lambda$ and sum of homogeneous functions for the second action of \mathbb{C}^* on $T^*\Lambda$. "Second action" means action induced on $T^*\Lambda$ by the action of \mathbb{C}^* on Λ .

Let \mathcal{M} be a (left) coherent \mathcal{E}_X -module. A good bifiltration of \mathcal{M} is a bifiltration compatible with the bifiltration of \mathcal{E}_X which is locally equal to $F_{k,j}\mathcal{M} = \sum_{\nu=1}^N F_{k-k\nu, j-j\nu} \mathcal{E}_X u_\nu$ for (u_1, \dots, u_N) local sections of \mathcal{M} .

The bigraded module associated to the bifiltration is defined by the same formula than (2.3). It is proved in §3.3. of [8], that the bigraded module associated to a good bifiltration is a coherent $gr^F \mathcal{E}_X = \pi_{\Lambda*} \mathcal{O}_{[T^*\Lambda]}$ module.

Let $\mathcal{O}_{T^*\Lambda}$ be the sheaf of holomorphic functions on $T^*\Lambda$. Then $\mathcal{O}_{T^*\Lambda} \otimes_{\mathcal{O}_{[T^*\Lambda]}} \pi^{-1} gr^F \mathcal{M}$ is a coherent $\mathcal{O}_{T^*\Lambda}$ -module. The associated analytic cycle and analytic variety are called the microcharacteristic cycle and microcharacteristic variety of \mathcal{M} of type $(\infty, 1)$ and are denoted respectively by $\widetilde{Ch}_{\Lambda(\infty,1)}(\mathcal{M})$ and $Ch_{\Lambda(\infty,1)}(\mathcal{M})$. This microcharacteristic variety is an involutive bihomogeneous subvariety of $T^*\Lambda$ [7],[8]. (The same microcharacteristic variety has been defined with different notations by Teresa Monteiro Fernandes [2].)

Definition 2.4. Let \mathcal{M} be a coherent \mathcal{E}_X -module with a good bifiltration $F\mathcal{M}$. A *regular b-function* for \mathcal{M} is a polynomial b such that:

$$\forall (i, j) \in \mathbb{Z} \times \mathbb{Z}, \quad b(\Theta)F_{i,j}\mathcal{M} \subset F_{i-1, j+n}\mathcal{M}$$

where n is the degree of b .

To end this section, let us remark that we may recover the microcharacteristic cycle $\widetilde{Ch}_{\Lambda(\infty)}(\mathcal{M})$ from a good bifiltration. Indeed, if $F\mathcal{M}$ is a good bifiltration, we may define a V-filtration by $V_k\mathcal{M} = \bigcup_j F_{k,j}\mathcal{M}$ and a good filtration on $gr_k^V \mathcal{M}$ is induced by the images of the sheaves $F_{k,j}\mathcal{M}$. In this way, the cycle $\widetilde{Ch}_{\Lambda(\infty)}(\mathcal{M})$ as defined in section 2.2 is the analytic cycle associated to the bigraded module:

$$(2.4) \quad gr^W \mathcal{M} = \bigoplus_{(k,j)} gr_{k,j}^W \mathcal{M} \text{ with} \\ gr_{k,j}^W \mathcal{M} = F_{k,j}\mathcal{M} \Big/ \left(F_{k,j-1}\mathcal{M} + F_{k,j}\mathcal{M} \bigcap \bigcup_{\ell} F_{k-1,\ell}\mathcal{M} \right)$$

2.4. Regularity along a lagrangian conic submanifold. The Euler vector field θ of Λ is a differential operator on Λ , its characteristic variety is a canonical subvariety of $T^*\Lambda$ which will be denoted by S_Λ . As in section 2.1, \mathcal{M}_Λ is a simple holonomic \mathcal{E}_X -module supported by Λ .

Definition 2.5. Let Λ be a lagrangian conic submanifold of T^*X and \mathcal{M} be a holonomic \mathcal{E}_X -module. The module \mathcal{M} is **regular along** Λ if and only if it satisfies the following equivalent properties:

- i) The microcharacteristic variety $Ch_{\Lambda(\infty,1)}(\mathcal{M})$ is contained in S_Λ .
- ii) The microcharacteristic variety $Ch_{\Lambda(\infty,1)}(\mathcal{M})$ is lagrangian.
- iii) The microcharacteristic variety $Ch_{\Lambda(\infty,1)}(\mathcal{M})$ is equal to $Ch_{\Lambda(\infty)}(\mathcal{M})$
- iv) The microcharacteristic cycle $\widetilde{Ch}_{\Lambda(\infty,1)}(\mathcal{M})$ is equal to $\widetilde{Ch}_{\Lambda(\infty)}(\mathcal{M})$
- v) $\forall j \geq 0, \quad \mathcal{E}xt_{\mathcal{E}_X}^j(\mathcal{M}, \mathcal{M}_\Lambda) = \mathcal{E}xt_{\mathcal{E}_X}^j(\mathcal{M}, \mathcal{E}_X^\infty \otimes_{\mathcal{E}_X} \mathcal{M}_\Lambda)$

We recall that \mathcal{E}_X^∞ is the sheaf of microdifferential operators of infinite order.

Proof. This equivalence between the items of this definition has been proved in [8], [10] and [12], more precisely:

- (iv) \Rightarrow (iii) is straightforward while (iii) \Rightarrow (ii) comes from theorem 2.1.
- (ii) \Rightarrow (i) because any lagrangian bihomogeneous subvariety of $T^*\Lambda$ is contained in S_Λ [8].
- (i) \Rightarrow (v) by corollary 4.4.2. of [10].
- (v) \Rightarrow (iv) by theorem 2.4.2 of [12].

□

If Λ is the conormal bundle to a submanifold Y of X , we may take $\mathcal{M}_\Lambda = \mathcal{C}_{Y|X}$ the sheaf of holomorphic microfunctions of [15]. Then (v) is reformulated as:

$$(v)' \quad \forall j \geq 0, \quad \mathcal{E}xt_{\mathcal{E}_X}^j(\mathcal{M}, \mathcal{C}_{Y|X}) = \mathcal{E}xt_{\mathcal{E}_X}^j(\mathcal{M}, \mathcal{C}_{Y|X}^\infty)$$

If Y has codimension 1, let $\mathcal{B}_{Y|X}^\infty = \mathcal{H}_Y^1(\mathcal{O}_X)$ be the cohomology of \mathcal{O}_X and $\mathcal{B}_{Y|X} = \mathcal{H}_{[Y]}^1(\mathcal{O}_X)$ the corresponding algebraic cohomology. As we are in codimension 1, we have:

$$\mathcal{C}_{Y|X}^\infty / \mathcal{C}_{Y|X} = \pi^{-1}(\mathcal{B}_{Y|X}^\infty / \mathcal{B}_{Y|X}) \quad \text{with} \quad \pi : T_Y^*X \rightarrow Y$$

So if \mathcal{M} is a holonomic \mathcal{D}_X -module condition (v) is equivalent to

$$(v)'' \quad \forall j \geq 0, \quad \mathcal{E}xt_{\mathcal{D}_X}^j(\mathcal{M}, \mathcal{B}_{Y|X}) = \mathcal{E}xt_{\mathcal{D}_X}^j(\mathcal{M}, \mathcal{B}_{Y|X}^\infty)$$

and by [12] this is equivalent to formula 1.2.

2.5. The main result for \mathcal{E}_X -module. Let Λ be conic lagrangian submanifold of T^*X . Remark that if \mathcal{M} has a regular b -function along Λ , it satisfies definition 2.5 hence is regular along Λ .

Theorem 2.6. *If the holonomic \mathcal{E}_X -module \mathcal{M} is regular along Λ , then the b -function of \mathcal{M} along Λ is a regular b -function.*

As stated in the introduction, this theorem contains two results:

- \mathcal{M} is regular along Λ if and only if it admits a regular b -function along Λ .
- If \mathcal{M} admits a regular b -function along Λ , then any usual b -function (including "the" b -function) is regular.

It is easy to see that definitions 1.1, 1.2 and 1.3 and theorem 1.4 are obtained as restriction to the zero section of T^*X of definitions 2.2, 2.4 and 2.5 and theorem 2.6.

3. SECOND MICROLOCALIZATION

3.1. 2-microdifferential operators. Let Λ be a lagrangian conic submanifold of T^*X . The sheaf \mathcal{E}_Λ^2 of 2-microdifferential operators is a sheaf of rings on $T^*\Lambda$ which has been defined in [7]. In this paper, we will use it in a very specific situation, so we just give the definition in local coordinates.

We assume that $Y = \{(x, t) \in X \mid t = 0\}$ is a smooth hypersurface of X , that $\Lambda = T_Y^*X = \{(x, t, \xi, \tau) \in T^*X \mid t = 0, \xi = 0\}$ and that $T^*\Lambda$ has coordinates (x, τ, x^*, τ^*) . We denote by π_Λ the projection $T^*\Lambda \rightarrow \Lambda$.

\mathcal{E}_Λ^2 is a bifiltered sheaf of rings $\mathcal{E}_\Lambda^2 = \bigcup \mathcal{E}_\Lambda^2[k, \ell]$ where a section of $\mathcal{E}_\Lambda^2[k, \ell]$ on an open set Ω of $T^*\Lambda$ is represented by a formal series:

$$(3.1) \quad P = \sum_{\substack{(i,j) \in \mathbb{Z}^2 \\ j-i \leq k, j \leq \ell}} P_{ij}(x, \tau, x^*, \tau^*)$$

where P_{ij} is a holomorphic function on Ω , homogeneous of degree i in (x^*, τ^*) and of degree j in (τ, x^*) , with the growth condition:

$$(3.2) \quad \forall K \subset\subset U, \exists C > 0, \forall (x, \tau, x^*, \tau^*) \in K, |P_{ij}(x, \tau, x^*, \tau^*)| < C^{k+\ell+i-2j}(\ell-j)!$$

The product of two operators $P = \sum P_{ij}$ and $Q = \sum Q_{ij}$ is given by $R = \sum R_{ij}$ with

$$(3.3) \quad R_{\lambda\mu}(x, \tau, x^*, \tau^*) = \sum_{\substack{\lambda=i+k-|\alpha|-|\beta| \\ \mu=j+\ell-|\alpha|-|\beta|}} \frac{1}{\alpha!} \frac{1}{\beta!} \left(\frac{\partial}{\partial x^*} \right)^\alpha \left(\frac{\partial}{\partial \tau} \right)^\beta P_{ij}(x, \tau, x^*, \tau^*) \left(\frac{\partial}{\partial x} \right)^\alpha \left(\frac{\partial}{\partial \tau^*} \right)^\beta Q_{k\ell}(x, \tau, x^*, \tau^*)$$

The restriction of \mathcal{E}_Λ^2 to the zero section of $T^*\Lambda$ is denoted by \mathcal{D}_Λ^2 . Sections of \mathcal{D}_Λ^2 have a symbol P which is polynomial in (x^*, τ^*) . In fact, \mathcal{D}_Λ^2 is canonically isomorphic as a sheaf of rings to $\mathcal{E}_X|_\Lambda$. Explicitly the morphism is given as follows:

A section of $\mathcal{E}_X|_\Lambda$ is represented by a symbol $P = \sum_{j \leq j_0} P_j(x, t, \xi, \tau)$ where P_j is a holomorphic function homogeneous of degree j in (ξ, τ) and defined near $\{t = 0, \xi = 0\}$. Each P_j develops in Taylor series as:

$$P_j(x, t, \xi, \tau) = \sum_{\alpha, \beta} P_{j\alpha\beta}(x, \tau) t^\alpha \xi^\beta$$

and its image in \mathcal{D}_Λ^2 is given by $\sum P_{ij}$ with

$$P_{ij}(x, \tau, x^*, \tau^*) = \sum_{|\alpha|+|\beta|=i} P_{j\alpha\beta}(x, \tau) (-\tau^*)^\alpha (x^*)^\beta$$

The isomorphism $\mathcal{E}_X|_\Lambda \rightarrow \mathcal{D}_\Lambda^2$ is compatible with the bifiltrations. We denote $F_{ij}\mathcal{E}_\Lambda^2 = \mathcal{E}_\Lambda^2[i, j]$ and define the associated bigraded ring by the same formula than (2.3). This bigraded ring is isomorphic to the sheaf $\mathcal{O}_{(T^*\Lambda)} = \bigoplus \mathcal{O}_{(T^*\Lambda)}[i, j]$ where $\mathcal{O}_{(T^*\Lambda)}[i, j]$ is the sheaf of holomorphic functions on $T^*\Lambda$ which are homogeneous of degree i in (x^*, τ^*) and of degree j in (τ, x^*) .

The sheaf of rings \mathcal{E}_Λ^2 is coherent, noetherian and flat on $\pi_\Lambda^{-1}\mathcal{E}_X|_\Lambda$ [7]. (Remark that in [7], the sheaf denoted here by \mathcal{E}_Λ^2 is denoted by $\mathcal{E}_\Lambda^2(\infty, 1)$.)

3.2. Bifiltrations and V-filtration on \mathcal{E}_Λ^2 -modules. Let \mathcal{M} be a coherent \mathcal{E}_Λ^2 -module with a good bifiltration $\mathcal{M} = \bigcup F_{ij}\mathcal{M}$. The associated bigraded module is

$$(3.4) \quad gr^F \mathcal{M} = \bigoplus_{(i,j)} gr_{i,j}^F \mathcal{M} \quad \text{with} \quad gr_{i,j}^F \mathcal{M} = F_{i,j}\mathcal{M} / (F_{i-1,j}\mathcal{M} + F_{i,j-1}\mathcal{M})$$

As in the case of \mathcal{E}_X -modules, we may also define $gr^W \mathcal{M}$ by formula (2.4).

From [7, §2.6.], we know that $gr^F \mathcal{M}$ and $gr^W \mathcal{M}$ are coherent modules over $gr^F \mathcal{E}_\Lambda^2 = \mathcal{O}_{(T^*\Lambda)}$. The module $\mathcal{O}_{T^*\Lambda} \otimes_{\mathcal{O}_{(T^*\Lambda)}} gr^F \mathcal{M}$ defines an analytic cycle $\widetilde{Ch}_\Lambda(\infty, 1)(\mathcal{M})$ whose support is equal to the support of \mathcal{M} and is denoted by $Ch_\Lambda(\infty, 1)(\mathcal{M})$. In the same way, $\mathcal{O}_{T^*\Lambda} \otimes_{\mathcal{O}_{(T^*\Lambda)}} gr^W \mathcal{M}$ defines the cycle $\widetilde{Ch}_\Lambda(\infty)(\mathcal{M})$ and the microcharacteristic variety $Ch_\Lambda(\infty)(\mathcal{M})$. These microcharacteristic varieties are involutive [7].

Let \mathcal{M} be a coherent \mathcal{E}_X -module with a good bifiltration. As \mathcal{E}_Λ^2 is flat on $\pi_\Lambda^{-1}\mathcal{E}_X|_\Lambda$, the module $\widetilde{\mathcal{M}} = \mathcal{E}_\Lambda^2 \otimes_{\pi_\Lambda^{-1}\mathcal{E}_X|_\Lambda} \pi_\Lambda^{-1}\mathcal{M}|_\Lambda$ is coherent and the bifiltration given by

$$(3.5) \quad F_{ij}\widetilde{\mathcal{M}} = \sum_{k+p=i, \ell+q=j} \mathcal{E}_\Lambda^2[k, \ell] \otimes \pi_\Lambda^{-1}F_{pq}\mathcal{M}$$

is a good bifiltration.

Lemma 3.1. *Let \mathcal{M} be a coherent \mathcal{E}_X -module and $\widetilde{\mathcal{M}} = \mathcal{E}_\Lambda^2 \otimes_{\pi_\Lambda^{-1}\mathcal{E}_X|_\Lambda} \pi_\Lambda^{-1}\mathcal{M}|_\Lambda$. The microcharacteristic cycles of \mathcal{M} and $\widetilde{\mathcal{M}}$ are equal:*

$$\widetilde{Ch}_{\Lambda(\infty,1)}(\mathcal{M}) = \widetilde{Ch}_{\Lambda(\infty,1)}(\widetilde{\mathcal{M}}) \quad \text{and} \quad \widetilde{Ch}_{\Lambda(\infty)}(\mathcal{M}) = \widetilde{Ch}_{\Lambda(\infty)}(\widetilde{\mathcal{M}})$$

Proof. Let \mathcal{I} be a coherent ideal of \mathcal{E}_X . Theorem 2.6.3. of [7] shows that $gr^F(\mathcal{E}_\Lambda^2\pi_\Lambda^{-1}\mathcal{I}) = gr^F(\mathcal{E}_\Lambda^2)gr^F(\pi_\Lambda^{-1}\mathcal{I})$ and $gr^W(\mathcal{E}_\Lambda^2\pi_\Lambda^{-1}\mathcal{I}) = gr^W(\mathcal{E}_\Lambda^2)gr^W(\pi_\Lambda^{-1}\mathcal{I})$.

Using the same argument than in the proof of proposition 2.6.1. in [16] we get the result. \square

Lemma 3.2. *Let \mathcal{M} be a coherent \mathcal{E}_Λ^2 -module such that $\widetilde{Ch}_{\Lambda(\infty,1)}(\mathcal{M}) = \widetilde{Ch}_{\Lambda(\infty)}(\mathcal{M})$. Then there exists a hypersurface L of $Ch_{\Lambda(\infty,1)}(\mathcal{M})$ such that the morphism $gr^F\mathcal{M} \rightarrow gr^W\mathcal{M}$ is an isomorphism on the complementary of L .*

Proof. Remark first that from the definition of graded modules, $gr^F\mathcal{M} \rightarrow gr^W\mathcal{M}$ is always well defined and surjective.

The microcharacteristic cycles are by definition the analytic cycles associated to the coherent $\mathcal{O}_{T^*\Lambda}$ -modules $\mathcal{O}_{T^*\Lambda} \otimes_{\mathcal{O}_{(T^*\Lambda)}} gr^F\mathcal{M}$ and $\mathcal{O}_{T^*\Lambda} \otimes_{\mathcal{O}_{(T^*\Lambda)}} gr^W\mathcal{M}$. If these cycles are equal the surjective morphism between them is an isomorphism almost everywhere that is outside some hypersurface L .

Then the fact that $\mathcal{O}_{T^*\Lambda} \otimes_{\mathcal{O}_{(T^*\Lambda)}} gr^F\mathcal{M} \rightarrow \mathcal{O}_{T^*\Lambda} \otimes_{\mathcal{O}_{(T^*\Lambda)}} gr^W\mathcal{M}$ is an isomorphism implies that the morphism $gr^F\mathcal{M} \rightarrow gr^W\mathcal{M}$ is an isomorphism. This has been proved for graded modules in lemma 2.2.2. of [16] and the proof for bigraded modules is similar. \square

3.3. V-filtration. We may define a filtration $V\mathcal{E}_\Lambda^2$ which is an extension of the V-filtration on $\mathcal{E}_X|_\Lambda$ by:

$$V_k\mathcal{E}_\Lambda^2 = \bigcup_{\ell} \mathcal{E}_\Lambda^2[k, \ell]$$

Let $\widetilde{Y} = Y \times \mathbb{C}$ with coordinates (x, s) . We denote by (x, s, x^*, s^*) the corresponding coordinates of $T^*Y \times T^*\mathbb{C}$ and define $\mathcal{E}_{\widetilde{Y}}^*$ as the subsheaf of $\mathcal{E}_{\widetilde{Y}}$ of microdifferential operators commuting with the derivation D_s (that is with a symbol independent of s).

Let $\gamma : T^*Y \times T^*\mathbb{C} \rightarrow T^*Y$ be the canonical map $(x, x^*, s, 0) \mapsto (x, x^*)$.

From now on in this section, we assume that $\tau \neq 0$, that is we work outside of the zero section of T^*X . Then $S_\Lambda = \{(x, \tau, x^*, \tau^*) \in T^*\Lambda \mid \tau^* = 0\}$ and we set $\varrho : S_\Lambda \rightarrow T^*Y$.

Lemma 3.3. *There is an isomorphism of sheaves of rings*

$$\varrho_* (gr_0^V \mathcal{E}_\Lambda^2|_{S_\Lambda}) \simeq \gamma_* \left(\mathcal{E}_{\widetilde{Y}}^*|_{T^*Y \times T^*\mathbb{C}} \right)$$

which induces isomorphisms for any $j \in \mathbb{Z}$

$$\varrho_* (gr_0^V \mathcal{E}_\Lambda^2[0, j]|_{S_\Lambda}) \simeq \gamma_* \left(\mathcal{E}_{\widetilde{Y}}^*(j)|_{T^*Y \times T^*\mathbb{C}} \right)$$

Proof. From definition (3.1), we see that a section of $gr_0^V \mathcal{E}_\Lambda^2$ is represented by a series $P = \sum_{j \leq \ell} P_j(x, \tau, x^*, \tau^*)$ where P_j is holomorphic and homogeneous of degree j in (x^*, τ^*) and also j in (τ, x^*) . If it is defined in a neighborhood of $\{\tau^* = 0\}$, each P_j develops in Taylor series in τ^* and by homogeneity may be rewritten as

$$P_j(x, \tau, x^*, \tau^*) = \sum_{k \geq 0} P_{jk}(x, x^*) (\tau \tau^*)^k$$

where P_{jk} is defined on an homogeneous subset V of T^*Y and homogeneous of degree $j - k$ in x^* . Inequalities (3.2) give:

$$(3.6) \quad \forall K \subset\subset V, \exists C > 0, \exists C_1 > 0, \forall (x, x^*) \in K, |P_{jk}(x, x^*)| < C_1^k C^{\ell-j} (\ell - j)!$$

Through the isomorphism $\mathcal{E}_X|_\Lambda \rightarrow \mathcal{D}_\Lambda^2$ of section 3.1, $(\tau \tau^*)^k$ is the symbol of the operator $(-1)^k t^k D_t^k$ which may be written as a polynomial in tD_t . Explicitly,

$$(3.7) \quad t^k D_t^k = \sum_{j=1}^k \alpha_{j,k} (tD_t)^j \quad \text{with} \quad \sum_{j=1}^k \alpha_{j,k} T^j = T(T-1) \dots (T-k+1)$$

The numbers $\alpha_{j,k}$ are determined by the relation $\alpha_{j,k+1} = \alpha_{j-1,k} - k\alpha_{j,k}$ hence the numbers $\beta_{j,k} = (-1)^{k-j} (k-1)! / (i-1)! \alpha_{j,k}$ satisfy $\beta_{j,k+1} = (j-1)/(k-1) \beta_{j-1,k} + \beta_{j,k}$. So, $\beta_{j,k}$ is smaller than the binomial coefficient hence than 2^k , this gives

$$(3.8) \quad |\alpha_{j,k}| < 2^k k! / j! \leq 4^k (k-j)!$$

The operator P of symbol $\sum P_j(x, \tau, x^*, \tau^*)$ may be written as $\sum Q_{j,k}(x, x^*) \sigma^k$ where σ is the symbol of the operator tD_t and $Q_{j,k}(x, x^*)$ is homogeneous of degree $j - k$ in x^* . From (3.7) we get

$$Q_{j,k}(x, x^*) = \sum_{\ell \geq 0} P_{j+\ell-k, \ell}(x, x^*) \alpha_{\ell,k}$$

Combining inequalities (3.6) and (3.8) we have:

$$(3.9) \quad \forall K \subset\subset V, \exists C > 0, \exists C_1 > 0, \forall (x, x^*) \in K, |Q_{jk}(x, x^*)| < C_1^k C^{\ell-j} (\ell - j)!$$

The function $Q_j(x, x^*, s^*) = \sum_{k \geq 0} Q_{j-k, k}(x, x^*) (s^*)^k$ is homogeneous of degree j in (x^*, s^*) and defined near $s^* = 0$, that is on a conic open subset of $T^*\tilde{Y}$ of the form $V = \{x \in V_0, |s^*| < \varepsilon |x^*|\}$ and the condition (3.8) shows that $Q = \sum Q_j$ is the symbol of a microdifferential operator on \tilde{Y} independent of s and defined near $s^* = 0$.

As tD_t commutes with itself, formula (3.3) implies that the product of P by Q when they are written as $P(x, x^*, \sigma)$ and $Q(x, x^*, \sigma)$ is

$$(3.10) \quad R(x, x^*, \sigma) = \sum \left(\frac{\partial}{\partial x^*} \right)^\alpha P(x, x^*, \sigma) \left(\frac{\partial}{\partial x} \right)^\alpha Q(x, x^*, \sigma)$$

which is precisely the product of two operators of $\mathcal{E}_{\tilde{Y}}^*$ (when σ is replaced by s^*) hence the morphism $\varrho_* (gr_0^V \mathcal{E}_\Lambda^2|_{S_\Lambda}) \simeq \gamma_* \left(\mathcal{E}_{\tilde{Y}}^*|_{T^*Y \times T^*\mathbb{C}} \right)$ is a morphism of sheaves of rings. \square

Remark 3.4. On the zero section of T^*X this result is equivalent to the well-known fact that $gr_V^0 \mathcal{D}_X \simeq \mathcal{D}_Y[\theta]$.

Let D_s be the derivation in s , that is the section of $\mathcal{E}_{\tilde{Y}}^*$ of symbol s^* . We have $\mathcal{E}_{\tilde{Y}}^* / \mathcal{E}_{\tilde{Y}}^* D_s \simeq \gamma^{-1} \mathcal{E}_Y$.

Lemma 3.5. *Let \mathcal{M} be a coherent \mathcal{E}_Y^* -module whose support is contained in $\{s^* = 0\}$. Then \mathcal{M} is a coherent $\gamma^{-1}\mathcal{E}_Y$ -module and a good \mathcal{E}_Y^* -filtration for \mathcal{M} is also a good $\gamma^{-1}\mathcal{E}_Y$ -filtration.*

Proof. Let $P \in \mathcal{E}_Y^*$ whose principal symbol is $\sigma(P) = (s^*)^N$ for some integer N . By the division theorem [15, Th 2.2.1 Ch II], $\mathcal{E}_Y^*/\mathcal{E}_Y^*P$ is isomorphic to $(\gamma^{-1}\mathcal{E}_Y)^N$.

If the support of \mathcal{M} is contained in $\{s^* = 0\}$, each section of \mathcal{M} is annihilated by an operator P with $\sigma(P) = (s^*)^N$ hence there is a surjective morphism

$$\bigoplus_{k=1 \dots q} \mathcal{E}_Y^*/\mathcal{E}_Y^*P_k \rightarrow \mathcal{M} \rightarrow 0$$

and the kernel of this morphism has his support contained in $\{s^* = 0\}$ so we may iterate the process and get (locally) a resolution of \mathcal{M} by free $(\gamma^{-1}\mathcal{E}_Y)^N$ -modules. This proves that \mathcal{M} is a coherent $\gamma^{-1}\mathcal{E}_Y$ -module.

If $(\mathcal{M}_j)_{j \in \mathbb{Z}}$ is a good filtration, there are locally sections (u_1, \dots, u_N) of \mathcal{M} and integers (j_1, \dots, j_N) such that $\mathcal{M}_j = \sum \mathcal{E}_Y^*(j - j_k)u_k$. We choose P_k as before for each u_k and the map $\bigoplus \mathcal{E}_Y^*/\mathcal{E}_Y^*P_k \rightarrow \mathcal{M}$ define the same filtration which is then a good $\gamma^{-1}\mathcal{E}_Y$ filtration. \square

The previous lemma is still true for microdifferential operators of order 0:

Lemma 3.6. *Let \mathcal{M} be a coherent $\mathcal{E}_Y^*(0)$ -module whose support is contained in $\{s^* = 0\}$. Then \mathcal{M} is a coherent $\gamma^{-1}\mathcal{E}_Y(0)$ -module and a good $\mathcal{E}_Y^*(0)$ -filtration for \mathcal{M} is also a good $\gamma^{-1}\mathcal{E}_Y(0)$ -filtration.*

Proof. On the zero section of T^*Y , that is near point where $x^* = 0$, $\mathcal{E}_Y^*(0)$ and $\mathcal{E}_Y(0)$ are both isomorphic to \mathcal{O}_Y so the result is trivial.

If $x^* \neq 0$, there is locally an invertible operator Δ of order 1 in \mathcal{E}_Y and the proof is the same than the proof of lemma 3.5 replacing D_s by $\Delta^{-1}D_s$ which is of order 0. \square

Proposition 3.7. *Let \mathcal{M} be a coherent \mathcal{E}_Λ^2 -module with a good bifiltration $F\mathcal{M}$. Assume that the support of \mathcal{M} is contained in S_Λ . Then for all $(i, j) \in \mathbb{Z} \times \mathbb{Z}$, the two quotients:*

$$F_{ij}\mathcal{M}/F_{i-1,j}\mathcal{M} \quad \text{and} \quad F_{ij}\mathcal{M} \Big/ \left(F_{ij}\mathcal{M} \cap \bigcup_k F_{i-1,k}\mathcal{M} \right)$$

are coherent $\varrho^{-1}\mathcal{E}_Y(0)$ -modules.

Proof. We consider the sheaf of rings $\mathcal{E}_\Lambda^2[0, 0]$ filtered by the subsheaves $\mathcal{E}_\Lambda^2[i, 0]$ for $i \leq 0$.

It has been proved in [7, theorem 2.5.3] that this filtration is a "good noetherian filtration". This denomination is due to Björk [1] while the same property is called "zariskian" by Schapira [16]. Moreover the associated graded ring is coherent [7, lemma 2.6.2].

So we may apply [16, proposition 1.4.1] to the filtered coherent $\mathcal{E}_\Lambda^2[0, 0]$ -module $F_{ij}\mathcal{M}$ and we get that $F_{ij}\mathcal{M}/F_{i-1,j}\mathcal{M}$ is a coherent $\mathcal{E}_\Lambda^2[0, 0]/\mathcal{E}_\Lambda^2[-1, 0]$ -module.

The same results are true for \mathcal{E}_Λ^2 with the V-filtration [7, theorem 2.5.1] and we get that $F_{ij}\mathcal{M}/\bigcup_k F_{i-1,k}\mathcal{M}$ is also a coherent $\mathcal{E}_\Lambda^2[0, 0]/\mathcal{E}_\Lambda^2[-1, 0]$ -module.

So $F_{ij}\mathcal{M}/F_{i-1,j}\mathcal{M}$ and $F_{ij}\mathcal{M}/\bigcup_k F_{i-1,k}\mathcal{M}$ are coherent $(\mathcal{E}_\Lambda^2[0, 0]/\mathcal{E}_\Lambda^2[-1, 0])$ -modules supported by S_Λ . We conclude immediately by using lemmas 3.3 and 3.6. \square

Proposition 3.8. *If the support of \mathcal{M} is contained in S_Λ , the submodules $F_{i,k}\mathcal{M}$ for $k \leq j$ define a good filtration of $\varrho^{-1}\mathcal{E}_Y(0)$ -modules*

$$\text{on } F_{ij}\mathcal{M}/F_{i-1,j}\mathcal{M} \quad \text{and on } F_{ij}\mathcal{M} \Big/ \left(F_{ij}\mathcal{M} \cap \bigcup_k F_{i-1,k}\mathcal{M} \right)$$

Proof. The good bifiltration of \mathcal{M} is given locally by sections (u_1, \dots, u_N) :

$$\forall(\ell, k) \in \mathbb{Z} \times \mathbb{Z}, \quad F_{\ell k}\mathcal{M} = \sum_\lambda \mathcal{E}_\Lambda^2[\ell - \ell_\lambda, k - k_\lambda]u_\lambda$$

We fix (i, j) . As we have assumed that $\tau \neq 0$, the operator D_t is invertible in \mathcal{E}_Λ^2 . As D_t is of order $(1, 1)$, we may multiply each u_λ by a power of D_t and assume that the integers (ℓ_1, \dots, ℓ_N) are all equal to i so that

$$F_{ik}\mathcal{M} = \sum_\lambda \mathcal{E}_\Lambda^2[0, k - k_\lambda]u_\lambda$$

Let us assume first that $x^* \neq 0$. Then there is an operator in \mathcal{E}_Λ^2 which is of order $(0, 1)$, hence we may assume that all (k_1, \dots, k_N) are all equal to 0 and so $F_{ik}\mathcal{M} = \sum_\lambda \mathcal{E}_\Lambda^2[0, k]u_\lambda$. Then the image of $F_{ik}\mathcal{M}$ in $F_{ij}\mathcal{M}/F_{i-1,j}\mathcal{M}$ and $F_{ij}\mathcal{M}/\bigcup_k F_{i-1,k}\mathcal{M}$ are generated by $\sum gr_0^V \mathcal{E}_\Lambda^2[0, k]\bar{u}_\lambda$ where \bar{u}_λ is the class of u_λ . Then by lemmas 3.3 and 3.5, these images define good filtrations.

At a point p where $x^* = 0$, $\mathcal{E}_\Lambda^2[0, 0]/\mathcal{E}_\Lambda^2[-1, 0]$ is isomorphic to \mathcal{O}_Y by lemma 3.3, the filtration induced by $F_{i,k}\mathcal{M}$ is finite because $\mathcal{E}_\Lambda^2[0, k] = 0$ for $k < 0$. So the filtration is good if its graduate is \mathcal{O}_Y -coherent. This a consequence of the fact that the bigraded module $gr\mathcal{M}$ is coherent. \square

Remark 3.9. If Λ is not the conormal bundle to a hypersurface, the same results may be proved if Y is changed to $\mathbb{P}\Lambda = \Lambda/\mathbb{C}^*$.

3.4. Proof of the main result. Let Λ be a smooth conic lagrangian submanifold of T^*X . Let \mathcal{M} be a holonomic \mathcal{E}_X -module with a good bifiltration $F\mathcal{M}$ relative to Λ .

The aim of this section is to prove the following result:

Theorem 3.10. *If \mathcal{M} is regular along Λ , the canonical map*

$$F_{ij}\mathcal{M}/F_{i-1,j}\mathcal{M} \rightarrow F_{ij}\mathcal{M} \Big/ \left(F_{ij}\mathcal{M} \cap \bigcup_k F_{i-1,k}\mathcal{M} \right)$$

is an isomorphism for each $(i, j) \in \mathbb{Z} \times \mathbb{Z}$.

To prove the theorem, we will use the following lemma:

Lemma 3.11. *Let Y be a complex manifold of dimension m and \mathcal{K} be a coherent $\mathcal{E}_Y(0)$ -module which is globally defined on T^*Y . If the support of \mathcal{K} is of dimension $\leq m - 1$, then $\mathcal{K} = 0$.*

Proof. The module $\mathcal{E}_Y \otimes_{\mathcal{E}_Y(0)} \mathcal{K}$ is a coherent \mathcal{E}_Y -module which is supported by the support of \mathcal{K} . But the support of a coherent \mathcal{E}_Y -module is an involutive subvariety of T^*Y hence must be of dimension $\geq m$ or be empty. So $\mathcal{E}_Y \otimes_{\mathcal{E}_Y(0)} \mathcal{K} = 0$.

The restriction of \mathcal{E}_Y to the zero section of T^*Y is $\mathcal{E}_Y|_Y = \mathcal{D}_Y$ while $\mathcal{E}_Y(0)|_Y = \mathcal{O}_Y$, hence $\mathcal{K}|_Y$ is a coherent \mathcal{O}_Y -module such that $\mathcal{D}_Y \otimes_{\mathcal{O}_Y} (\mathcal{K}|_Y) = 0$.

The sheaf \mathcal{D}_Y is a locally free \mathcal{O}_Y -module hence is faithfully flat so $\mathcal{K}|_Y = 0$. As \mathcal{K} is globally defined on T^*Y , it is invariant under the action of \mathbb{C} on the fibers of T^*Y . So, if $\mathcal{K}|_Y = 0$ then $\mathcal{K} = 0$. \square

Proof of theorem 3.10: first step. We will first prove the result under the conditions of section 3. That is we assume that Λ is the conormal to a smooth hypersurface Y with local coordinates such that $Y = \{(x, t) \in X \mid t = 0\}$ and $\Lambda = \{(x, t, \xi, \tau) \in T^*X \mid t = 0, \xi = 0\}$. We also assume $\tau \neq 0$.

Let $\widetilde{\mathcal{M}} = \mathcal{E}_\Lambda^2 \otimes_{\pi^{-1}\mathcal{E}_X|_\Lambda} \pi^{-1}\mathcal{M}|_\Lambda$ with the bifiltration $F_{ij}\widetilde{\mathcal{M}}$ given by formula 3.5.

We fix some $(i, j) \in \mathbb{Z} \times \mathbb{Z}$ for which we will proof the theorem. We denote

$$(3.11) \quad \mathcal{N}_1 = F_{ij}\widetilde{\mathcal{M}} / F_{i-1,j}\widetilde{\mathcal{M}} \quad \text{and} \quad \mathcal{N}_\infty = F_{ij}\widetilde{\mathcal{M}} / \left(F_{ij}\widetilde{\mathcal{M}} \cap \bigcup_k F_{i-1,k}\widetilde{\mathcal{M}} \right)$$

By definition 2.5, as \mathcal{M} is regular along Y , the microcharacteristic variety $Ch_{\Lambda(\infty,1)}(\mathcal{M})$ is contained in $S_\Lambda = \{(x, \tau, x^*, \tau^*) \in T^*\Lambda \mid \tau\tau^* = 0\}$. By lemma 3.1, this variety is the support of $\widetilde{\mathcal{M}}$, hence by proposition 3.7, \mathcal{N}_1 and \mathcal{N}_∞ are coherent $\varrho^{-1}\mathcal{E}_Y(0)$ -modules with the good filtrations $(F_\ell\mathcal{N}_1)$ and $(F_\ell\mathcal{N}_\infty)$ induced by $(F_{i\ell}\widetilde{\mathcal{M}})_{\ell \leq j}$.

We have

$$(3.12) \quad F_\ell\mathcal{N}_1 / F_{\ell-1}\mathcal{N}_1 = F_{i\ell}\widetilde{\mathcal{M}} / (F_{i-1,j}\widetilde{\mathcal{M}} + F_{i\ell-1}\widetilde{\mathcal{M}})$$

$$(3.13) \quad F_\ell\mathcal{N}_\infty / F_{\ell-1}\mathcal{N}_\infty = F_{i\ell}\widetilde{\mathcal{M}} / \left(F_{i,\ell-1}\widetilde{\mathcal{M}} + \bigcup_\ell F_{i-1,\ell}\widetilde{\mathcal{M}} \right) = gr_{i\ell}^W \widetilde{\mathcal{M}}$$

Remark that $gr_\ell\mathcal{N}_1 = F_\ell\mathcal{N}_1 / F_{\ell-1}\mathcal{N}_1$ is not equal a priori to $gr_{i\ell}^F \widetilde{\mathcal{M}}$. Anyway, for $\ell \leq j$, we have $gr_\ell\mathcal{N}_\infty = gr_{i\ell}^W \widetilde{\mathcal{M}}$ and a surjective morphism

$$gr_{i\ell}^F \widetilde{\mathcal{M}} = F_{i\ell}\widetilde{\mathcal{M}} / (F_{i-1,\ell}\widetilde{\mathcal{M}} + F_{i,\ell-1}\widetilde{\mathcal{M}}) \rightarrow gr_\ell\mathcal{N}_1$$

If \mathcal{M} is regular then by definition 2.5 the two cycles $\widetilde{Ch}_{\Lambda(\infty,1)}(\mathcal{M})$ and $\widetilde{Ch}_{\Lambda(\infty)}(\mathcal{M})$ are equal. Hence by lemmas 3.1 and 3.2, there exists a hypersurface L of $Ch_{\Lambda(\infty,1)}(\mathcal{M})$ such that the morphism $gr^F \widetilde{\mathcal{M}} \rightarrow gr^W \widetilde{\mathcal{M}}$ is an isomorphism on the complementary of L .

Remark that the morphism $\alpha : gr_{i\ell}^F \widetilde{\mathcal{M}} \rightarrow gr_{i\ell}^W \widetilde{\mathcal{M}}$ is composed of $\beta : gr_{i\ell}^F \widetilde{\mathcal{M}} \rightarrow gr_\ell\mathcal{N}_1$ and $\gamma : gr_\ell\mathcal{N}_1 \rightarrow gr_\ell\mathcal{N}_\infty$. As β is surjective, β and γ are isomorphisms when α is an isomorphism, that is out of L .

The map $\mathcal{N}_1 \rightarrow \mathcal{N}_\infty$ is a filtered map which induces an isomorphism of graded modules on $T^*\Lambda - L$. These filtrations are good filtrations and the filtration on $\varrho^{-1}\mathcal{E}_Y(0)$ is zariskian [16, th. 2.1.1. ch II], so by [16, prop. 1.1.3. ch II] $\mathcal{N}_1 \rightarrow \mathcal{N}_\infty$ is an isomorphism on $T^*\Lambda - L$.

Remark that \mathcal{N}_1 and \mathcal{N}_∞ are coherent $\varrho^{-1}\mathcal{E}_Y(0)$ -modules hence are locally constant on the fibers of ϱ , so L is equal to $\varrho^{-1}(L_0)$ where L_0 is a subvariety of T^*Y of dimension strictly lower than the dimension of Z . If X is of dimension n , $Ch_{\Lambda(\infty,1)}(\mathcal{M})$ is of dimension n (because \mathcal{M} is holonomic), hence L is of dimension at most $n-1$ and L_0 at most $n-2$. Hence by lemma 3.11 applied to its kernel, the morphism $\mathcal{N}_1 \rightarrow \mathcal{N}_\infty$ is an isomorphism on $T^*\Lambda$ (outside of the zero section of T^*X).

Second step: We still remain outside of the zero section of T^*X and we assume now that Λ is any homogeneous lagrangian submanifold of T^*X . The problem being local on T^*X , there is a homogeneous symplectic transformation which sends Λ to the conormal

bundle to a smooth hypersurface Y of X . Using a quantized canonical transformation associated to this symplectic transformation, we get the result by the first step.

Third step: We assume now that \mathcal{M} is defined on the zero section of T^*X . Hence \mathcal{M} is a \mathcal{D}_X -module and $T^*\Lambda = T_Y^*X$ for some submanifold Y of X (not necessarily a hypersurface). We define \mathcal{N}_1 and \mathcal{N}_∞ as in (3.11) and \mathcal{K} as the kernel of $\mathcal{N}_1 \rightarrow \mathcal{N}_\infty$. The three modules \mathcal{K} , \mathcal{N}_1 and \mathcal{N}_∞ are coherent $\mathcal{O}_{[\Lambda]}$ -modules.

By the first step \mathcal{K} is a coherent $\mathcal{O}_{[\Lambda]}$ -module such that $\mathcal{D}_\Lambda \otimes \mathcal{K}$ is a coherent \mathcal{D}_Λ -module whose characteristic variety is contained in the zero section. So \mathcal{K} is a locally free $\mathcal{O}_{[\Lambda]}$ -module [1, th. 7.1., ch.5] and its support is equal to Λ .

But the cycles $\widetilde{Ch}_{\Lambda(\infty,1)}(\mathcal{M})$ and $\widetilde{Ch}_{\Lambda(\infty)}(\mathcal{M})$ are equal hence $gr\mathcal{N}_1 \rightarrow gr\mathcal{N}_\infty$ is generically an isomorphism and \mathcal{K} must be supported by a strict subvariety of Λ . So we must have $\mathcal{K} = 0$ and the proposition is proved. \square

To end this section we remark that theorem 2.6 is a direct consequence of theorem 3.10. Indeed, a b -function is a polynomial such that $b(\theta + i)$ annihilates $F_{ij}\mathcal{M} / \bigcup_k F_{i-1,k}\mathcal{M}$ while a regular b -function is a polynomial such that $b(\theta + i)$ annihilates $F_{ij}\mathcal{M} / F_{i-1,j}\mathcal{M}$.

4. APPLICATIONS AND EXTENSIONS

4.1. Regular meromorphic b -functions.

Corollary 4.1. *Assume that a \mathcal{D}_X -module is regular along a submanifold Y outside a hypersurface Z . Let φ be an equation of Z . Then any section u of \mathcal{M} satisfy an equation $(\varphi^N b(\vartheta) + Q)u = 0$ where b is a polynomial and Q is a differential operator in $V_{-1}\mathcal{D}_X$ which is of order at most the degree of b .*

Proof. By the hypothesis, the kernel of the morphism defined in theorem 2.6 is contained in Z . As its restriction to the zero section X of T^*X is a coherent \mathcal{O}_X -module there is some $N \in \mathbb{N}$ such that φ^N annihilates this kernel. \square

4.2. b -function and Newton polygon. When a holonomic module is not regular, its irregularity is characterized by its Newton Polygon [12]. In particular, the first slope of this Newton polygon determines the growth of formal solutions (which are not convergent if the module is not regular). We will see in this section that this slope determines also the order of the operator Q in the definition 1.1.

Let us briefly redefine the first slope of a coherent \mathcal{D}_X or \mathcal{E}_X -module, details may be found in [8].

Let $r > 1$ be a rational number written as an irreducible quotient p/q of two integers with $p > q \geq 1$. From the two filtrations $(\mathcal{E}_{Xj})_{j \in \mathbb{Z}}$ and $(V_k \mathcal{E}_X)_{k \in \mathbb{Z}}$ of section 2.2 we define a new filtration:

$$(4.1) \quad \forall k \in \mathbb{Z} \quad F_k^r \mathcal{E}_X = \sum_{(p-q)m+qn=k} \mathcal{E}_{X,n} \cap V_m \mathcal{E}_X$$

and a bifiltration

$$(4.2) \quad \forall (k, j) \in \mathbb{Z} \times \mathbb{Z} \quad F_{k,j}^r \mathcal{E}_X = F_j^r \mathcal{E}_X \cap V_k \mathcal{E}_X$$

If \mathcal{M} is a coherent \mathcal{E}_X -module, a good $F^r \mathcal{E}_X$ -bifiltration is defined as in section 2.2 as well as the associated bigraded module $gr_{F^r} \mathcal{M}$. To this module is associated the microcharacteristic variety $Ch_{\Lambda(\infty,r)}(\mathcal{M})$ and the microcharacteristic cycle $\widetilde{Ch}_{\Lambda(\infty,r)}(\mathcal{M})$.

We may now adapt the definition of regularity 2.5:

Definition 4.2. Let Λ be a lagrangian conic submanifold of T^*X and \mathcal{M} be a holonomic \mathcal{E}_X -module. The first slope of the module \mathcal{M} is the lowest rational number such that \mathcal{M} satisfies the following equivalent properties:

- i) The microcharacteristic variety $Ch_{\Lambda(\infty,r)}(\mathcal{M})$ is contained in S_{Λ} .
- ii) The microcharacteristic variety $Ch_{\Lambda(\infty,r)}(\mathcal{M})$ is lagrangian.
- iii) The microcharacteristic variety $Ch_{\Lambda(\infty,r)}(\mathcal{M})$ is equal to $Ch_{\Lambda(\infty)}(\mathcal{M})$
- iv) The microcharacteristic cycle $\widetilde{Ch}_{\Lambda(\infty,r)}(\mathcal{M})$ is equal to $\widetilde{Ch}_{\Lambda(\infty)}(\mathcal{M})$
- v) $\forall j \geq 0, \quad \text{Ext}_{\mathcal{E}_X}^j(\mathcal{M}, \mathcal{M}_{\Lambda}) = \text{Ext}_{\mathcal{E}_X}^j(\mathcal{M}, \mathcal{E}_{X(\infty,r)} \otimes_{\mathcal{E}_X} \mathcal{M}_{\Lambda})$

We may also define the slope of a b -function of the module \mathcal{M} by adapting definition 2.4:

Definition 4.3. A polynomial b of degree n is a b -function with slope r for \mathcal{M} along Y if \mathcal{M} admits a good V -filtration $V\mathcal{M}$ and a good F^r -filtration $F^r\mathcal{M}$ such that:

$$\forall (i, j) \in \mathbb{Z} \times \mathbb{Z}, \quad b(\Theta)F_j^r\mathcal{M} \cap V_i\mathcal{M} \subset F_{j+qn}^r\mathcal{M} \cap V_{i-1}\mathcal{M}$$

where qn is the order of $b(\Theta)$ for the $F^r\mathcal{E}_X$ -filtration.

Given a b -function, the slope of this b -function is the lowest rational number such that this is true.

By the definitions, the first slope is 1 if and only if \mathcal{M} is regular along Λ . This definition takes a more explicit form if we restrict to a section of \mathcal{M} :

Definition 4.4. Let \mathcal{M} be a coherent \mathcal{E}_X -module and u a section of \mathcal{M} .

A polynomial b is b -function with slope r for u along Y if there exists a differential operator Q in $V_{-1}\mathcal{E}_X$ which is of order less or equal to the order of $b(\vartheta)$ for the $F^r\mathcal{E}_X$ -filtration and such that $(b(\vartheta) + Q)u = 0$.

In the case of \mathcal{D}_X -modules this means that Q may be written as a sum $\sum_{j \geq 1} Q_j(x, D_x, tD_t)t^j$ where Q_j is of order at most $m + (r - 1)j$ where m is the degree of b .

Theorem 4.5. *If \mathcal{M} is a holonomic \mathcal{E}_X -module, the first slope of \mathcal{M} is equal to the slope of its b -function.*

The proof of this theorem is similar to the proof of theorem 2.6 replacing the bifiltrations by the F^r bifiltrations and the sheaf \mathcal{E}_{Λ}^2 by the sheaf $\mathcal{E}_{\Lambda(\infty,r)}^2$ also defined in [7].

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