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K Ramachandra

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## ON A PROBLEM OF IVIĆ

BY K.RAMACHANDRA

TO DR. A. SANKARANARAYANAN ON

HIS FORTIETH BIRTHDAY

§1. INTRODUCTION AND STATEMENT OF RESULTS. In [A.I] A.IVIĆ has proved the following inequality. Let  $\epsilon > 0$  be any constant and let  $\gamma$  run over the ordinates of all the zeros (counted with multiplicity) of the Riemann zeta-function in the critical strip. Then

$$\sum_{T \leq \gamma \leq 2T} |\zeta(\frac{1}{2} + i\gamma)|^2 \ll_{\epsilon} T(\log T)^2 (\log \log T)^{\frac{3}{2} + \epsilon} \quad (1.1)$$

where the implied constant depends only on  $\epsilon$ . Here  $T \geq T_0$ , a large positive constant.

In the present paper it is our object to improve this inequality in two ways. First we replace the LHS by a bigger quantity and at the same time replace the RHS by a smaller quantity namely  $T(\log T)^2 \log \log T$  (where now the implied constant is absolute). Also we consider further generalisations and sketch their proof. In place of  $|\zeta(\frac{1}{2} + i\gamma)|^2$  our key function (associated with arbitrary (but fixed) constants  $A > 0$  and  $B > 0$ ) is

$$M(\gamma) = \max |\zeta(s)|^2, \quad (1.2)$$

where the maximum is taken over all  $s(= \sigma + it)$  in the rectangle

$$\frac{1}{2} - A(\log T)^{-1} \leq \sigma \leq 2, \quad |t - \gamma| \leq B(\log \log T)(\log T)^{-1}. \quad (1.3)$$

Plainly the investigations go through for  $L(s, \times)$  in place of  $\zeta(s)$ . Accordingly our Theorem is

**THEOREM 1.** *We have, for every fixed  $\epsilon > 0$ ,*

$$(i) \quad \sum_{\gamma \in I} M(\gamma) \ll_{\epsilon} H(\log T)^2 \log \log T$$

where  $I(\subset [T + 1, 2T - 1])$  is any interval of length  $H(\geq T^{\frac{2}{3}+\epsilon})$  and  $T \geq T_0(\epsilon)$ .

$$(ii) \quad \sum_{\gamma \in I} (M(\gamma))^{\frac{1}{2}} \ll_{\epsilon} H(\log T)^{\frac{1}{2}} \log \log T$$

where  $I(\subset [T + 1, 2T - 1])$  is any interval of length  $H(\geq T^{\frac{1}{2}+\epsilon})$  and  $T \geq T_0(\epsilon)$ .

**REMARK.** In (ii) we can replace  $\frac{1}{2}$  by any constant  $k > 0$  for which  $k^{-1}$  is an integer and correspondingly the number  $\frac{5}{4}$  will have to be replaced by  $1 + k^2$ . (Of course we should have  $H \geq T^{\frac{1}{2}+\epsilon}$ ). In this connection we state the following Theorem 2.

**THEOREM 2.** If  $k(> 0)$  is any constant, then

$$\sum_{\gamma \in I} (M(\gamma))^k \ll H(\log T)^{1+k^2} \log \log T$$

where  $I(\subset [T + 1, 2T - 1])$  is any interval of length  $H(> 0)$  satisfying the following two Hypotheses.

**HYPOTHESIS 1.** We should have

$$\frac{1}{H} \int_T^{T+H} |\zeta(\frac{1}{2} + it)|^{2k} dt \ll (\log T)^{k^2}.$$

**HYPOTHESIS 2.** For some constant  $k^* > k$ , we should have

$$\frac{1}{H} \int_T^{T+H} |\zeta(\frac{1}{2} + it)|^{2k^*} \ll (\log T)^C$$

where  $C(> 0)$  is some constant depending on  $k^*$ .

**REMARKS.** If  $k = 2$  we do not know whether Hypothesis 2 holds or not. However it holds if  $k < 2$  (namely we can take  $k^* = 2$  and  $C = 4$  and  $H \geq T^{\frac{2}{3}+\epsilon}$ ). Naturally we have to restrict to  $k < 2$ . Even here we do not know the truth of Hypothesis 1 unless  $k^{-1}$  is an integer and  $H \geq T^{\frac{1}{2}+\epsilon}$ . (Of course if  $k = 1$  we can take things like  $H \geq T^{\frac{1}{3}+\epsilon}$  but the corresponding Hypothesis 2 forces us to take  $H \geq T^{\frac{2}{3}+\epsilon}$ ). It should be remarked that our method is completely different from that of A.IVIĆ.

Next we state Theorem 3 which gives lower bounds for

$$\sum_{\gamma \in I} (M(\gamma))^k.$$

**THEOREM 3.** We have, for any  $k > 0$

$$\sum_{\gamma \in I} (M(\gamma))^k \gg \left( \int_I |\zeta(\frac{1}{2} + it)|^{2k} dt \right) (\log T) (\log \log T)^{-1}$$

where  $I(\subset [T, 2T])$  is any interval of length  $H$  satisfying

$$H \geq 4B(\log T)^{-1} \log \log T$$

provided the intervals

$$|t - \gamma| \leq B(\log T)^{-1} \log \log T$$

associated with two successive  $\gamma$ 's have at least one point in common.

Since the proof of theorem 3 is very simple we give it here itself.

**PROOF OF THEOREM 3.** Contribution (to the integral on the right) from each interval

$$|t - \gamma| \leq B(\log T)^{-1} \log \log T$$

is clearly  $\ll (M_0(\gamma))^k (\log T)^{-1} \log \log T$ , where  $M_0(\gamma)$  is the maximum of  $|\zeta(\frac{1}{2} + it)|^2$  in the interval. Also these intervals cover  $I$  completely so that each  $\gamma$  is counted in  $\sum_{\gamma \in I} (M_0(\gamma))^k$ . Clearly  $M_0(\gamma) \leq M(\gamma)$  and this completes the proof of Theorem 3.

Next we state a Conjecture.

**CONJECTURE:**  $\sum_{T \leq \gamma \leq 2T} (M(\gamma))^2 \ll T(\log T)^5 \log \log T$ .

**FURTHER REMARKS.** Lower bounds (and sometimes upper bounds) for

$$\int_I |\zeta(\frac{1}{2} + it)|^{2k} dt$$

have been studied first by K.RAMACHANDRA and then by D.R.HEATH-BROWN and then by M.JUTILA (for these results see the book [K.R]<sub>1</sub> by K.RAMACHANDRA and also the famous classic [E.C.T, D.R.H-B] by E.C.TITCHMARSH and D.R.HEATH-BROWN). There have been some further work on this integral by K.RAMACHANDRA [K.R]<sub>2</sub>. Another typical result of K.RAMACHANDRA [K.R]<sub>2</sub> is

$$\frac{1}{H} \int_T^{T+H} |\zeta(\frac{1}{2} + it)|^{2\sqrt{2}} dt \gg (\log H)^2 (\log \log H)^{-1},$$

where  $H$  exceeds some positive constant times  $\log \log T$ . In particular

$$\frac{1}{T} \int_T^{2T} |\zeta(\frac{1}{2} + it)|^{2\sqrt{2}} dt \gg (\log T)^2 (\log \log T)^{-1}.$$

It seems very difficult to knock off the factor  $(\log \log T)^{-1}$ .

**§2. NOTATION.** We adopt standard notation. The Vinogradov symbols  $\ll$  and  $\gg$  mean "less than a positive constant times" and "greater than a positive constant times"

respectively. Some times we use  $\ll \dots$  and  $\gg \dots$  to denote that these constants depend on  $\dots$  (for example  $\ll_\varepsilon$ ). We use the Landau symbol  $O(\dots)$  to denote "less than a positive constant times  $\dots$ ". In §3 the letters A, B, C, D, E, F and G will denote positive constants. A and B need not necessarily be the same as in §1.  $T$  will exceed a large positive constant.  $\varepsilon (> 0)$  will be an arbitrary constant. Some times we write  $T_0(\varepsilon)$  to denote that the positive constant  $T_0$  depends on  $\varepsilon$ .

**§3. PROOF OF THEOREM 1.** We divide the proof into seven steps for convenience. To illustrate our method we take  $H = T - 2$ . The general case in (i) follows from the deep result

$$\frac{1}{H} \int_T^{T+H} |\zeta(\frac{1}{2} + it)|^4 dt \ll_\varepsilon (\log T)^4$$

where  $T \geq H \geq T^{\frac{2}{3} + \varepsilon}$  due to N.ZAVOROTNYI [N.Z] (see also [D.R.H-B], [H.I], [M.J] and [Y.M] for the history to date) and (ii) from the crucial result

$$\frac{1}{H} \int_T^{T+H} |\zeta(\frac{1}{2} + it)| dt \ll_\varepsilon (\log T)^{\frac{1}{2}}$$

where  $T \geq H \geq T^{\frac{1}{2} + \varepsilon}$  due to K.RAMACHANDRA (see (4.3.2) of [K.R.]<sub>1</sub>).

**STEP I.** Consider the various rectangles

$$\left[ \frac{1}{2} - \frac{D \log \log T}{\log T} \leq \sigma \leq 2 \right] \times \left[ T + (\log T)^E \leq t \leq T + 2(\log T)^E, T + 2(\log T)^E \leq t \leq T + 3(\log T)^E, \dots \right]$$

where the right extremity of the last  $t$ -interval does not exceed  $2T - (\log T)^E$ . Consider the maximum

$$\max |\zeta(s)|^4$$

over a typical rectangle. We will prove that

$$\sum \max |\zeta(s)|^4 \ll T(\log T)^{6+8D}. \tag{3.1}$$

(We can improve RHS to  $T(\log T)^{5+8D}(\log \log T)^{-2}$ ; but we do not need this). Let  $s_1, s_2, \dots$  denote points where the maxima are attained. Then the required quantity is (see Theorem (1.7.1) of [K.R.]<sub>1</sub>)

$$\leq |\zeta(s_1)|^4 + |\zeta(s_2)|^4 + \dots \leq \left( \frac{\log T}{D \log \log T} \right)^2 \pi^{-1} \left\{ \int_{D_1} \int |\zeta(s)|^4 da + \int_{D_2} \int |\zeta(s)|^4 da + \dots \right\}$$

(where  $D_j$  is the disc of radius  $D(\log \log T)(\log T)^{-1}$  with centre  $s_j$  and  $da$  the element of area)

$$\leq 10\pi^{-1}(\log T)^2(D \log \log T)^{-2} \iint |\zeta(s)|^4 d\sigma dt$$

where the last integration is over the rectangle

$$\left[ \frac{1}{2} - \frac{2D \log \log T}{\log T} \leq \sigma \leq 3 \right] \times [T \leq t \leq 2T].$$

The last integral is easily seen to be  $\ll T(\log T)^{4+8D}$

**STEP II.** We now record a Corollary to step I. We have

$$\max |\zeta(s)| \leq (\log T)^C$$

for all rectangles except  $N$  of them where

$$N(\log T)^{4C} \ll T(\log T)^{6+8D}$$

and so

$$N \ll T(\log T)^{6-4C+8D}. \quad (3.2)$$

**STEP III.** We now write

$$M(\gamma) = \max |\zeta(s)|^2$$

where the maximum is taken over the rectangle

$$\left[ \frac{1}{2} - \frac{A}{\log T} \leq \sigma \leq 2 \right] \times \left[ |t - \gamma| \leq \frac{B \log \log T}{\log T} \right].$$

Let the asterisk denote the sum over those  $\gamma$  for which  $M(\gamma) \leq (\log T)^{-F}$ . Then

$$\sum_{T+1 \leq \gamma \leq 2T-1}^* (M(\gamma)) \ll T(\log T)^{1-F} \quad (3.3)$$

since the total number of  $\gamma$ 's is  $\ll T \log T$  (we may fix  $F$  to be 2).

**STEP IV.** Denote the rectangle in step III by  $R(\gamma)$  and by  $s_\gamma$  the point at which  $|\zeta(s)|$  attains its maximum in  $R(\gamma)$ . Consider those  $\gamma$ 's for which  $M(\gamma) > (\log T)^{-F}$ . Then the number of zeros of  $\zeta(s)$  in the disc with centre  $s_\gamma$  and radius  $G(\log \log T)(\log T)^{-1}$  is  $O(\log \log T)$  uniformly provided that this disc is inside any of the rectangles of step II. (Note that the number of exceptional rectangles is  $N \leq T(\log T)^{6-4C+8D}$ ). This follows by Jensen's Theorem (see page 150 of [K.R]<sub>1</sub>).

**STEP V.** Now consider all the  $M(\gamma)$  with  $T+1 \leq \gamma \leq 2T-1$ . We claim

$$\sum M(\gamma) \ll T(\log T)^2 \log \log T. \quad (3.4)$$

(this proves part (i) of Theorem 1). To see this we split up the sum on the LHS into  $\sum_1 + \sum_2 + \sum_3$ , where  $\sum_3$  is the sum in (3.3) and  $\sum_2$  is over all  $\gamma$  lying in all the exceptional  $N$  rectangles occurring in step II. We have

$$\sum_2 \leq (\sum (M(\gamma))^2)^{\frac{1}{2}} N^{\frac{1}{2}}.$$

Note that  $\sum (M(\gamma))^2 \ll T(\log T)^{6+8D+E+40} \leq T(\log T)^{50+8D+E}$  since each of the  $M(\gamma)$ 's may be replaced by the maxima in step I. Also

$$N \ll T(\log T)^{6-4C+8D}.$$

Thus

$$\sum_2 \ll T(\log T)^{25+4D+E+3-2C+4D} \leq T(\log T)^{-1}$$

if  $C = 4D + E + 20$ .

We now look at the sum  $\sum_1$ . It consists of those  $\gamma$ 's for which

$$(\log T)^{-F} \leq M(\gamma) \leq (\log T)^C \quad (3.5)$$

and so by step IV the number of zeros of  $\zeta(s)$  in

$$|s_\gamma - s| \leq G(\log \log T)(\log T)^{-1} \text{ is } O(\log \log T).$$

Here  $G$  is any constant such that this disc lies in

$$\left[ \frac{1}{2} - \frac{D \log \log T}{\log T} \leq \sigma \leq 3 \right] \times J$$

where  $J$  is any of the non exceptional intervals of step II. This is so if  $D = 10G$ .

**STEP VI.** We are now in a position to complete the estimation of  $\sum_1$ . Divide the  $\sigma$ -range  $\left[ \frac{1}{2} - \frac{A}{\log T} \leq \sigma \leq 2 \right]$  into intervals

$$I_0 = \left[ \frac{1}{2} - \frac{A}{\log T} \leq \sigma \leq \frac{1}{2} + \frac{A}{\log T} \right],$$

$$I_1 = \left[ \frac{1}{2} + \frac{A}{\log T} \leq \sigma \leq \frac{1}{2} + \frac{2A}{\log T} \right],$$

$$I_2 = \left[ \frac{1}{2} + \frac{2A}{\log T} \leq \sigma \leq \frac{1}{2} + \frac{3A}{\log T} \right], \dots$$

$$I_n = \left[ \frac{1}{2} + \frac{nA}{\log T} \leq \sigma \leq \frac{1}{2} + \frac{(n+1)A}{\log T} \right], \dots$$

the last interval projecting a little beyond 2. First consider  $I_0$ . Look at the  $s_\gamma$  of the sum  $\Sigma_1$  for which  $\text{Re } s_\gamma \in I_0$ . We have (see page 34 of [K.R]<sub>1</sub>)

$$|\zeta(s_\gamma)|^2 \leq (\pi R_0^2)^{-1} \int \int_{|s-s_\gamma| \leq R_0} |\zeta(s)|^2 da$$

where  $da$  is the element of area and  $R_0 > 0$ . Choose  $R_0 = \frac{20A}{\log T}$ . We find

$$\begin{aligned} \sum_{\gamma, \text{Re } s_\gamma \in I_0} |\zeta(s_\gamma)|^2 &\ll (\log \log T)(\log T)^2 \int \int_{T \leq t \leq 2T, |\sigma - \frac{1}{2}| \leq \frac{100A}{\log T}} |\zeta(s)|^2 d\sigma dt \\ &\ll (\log \log T)(\log T)^2 (T \log T)(\log T)^{-1} \\ &= T(\log T)^2 \log \log T \end{aligned}$$

Now for any fixed  $n (n = 1, 2, 3, \dots)$  consider the interval  $I_n$  for  $\text{Re } s_\gamma$ . We have (see page 34 of [K.R]<sub>1</sub>)

$$|\zeta(s_\gamma)|^2 \leq (\pi R_n^2)^{-1} \int \int_{|s-s_\gamma| \leq R_n} |\zeta(s)|^2 da$$

where  $da$  is the element of area and  $R_n > 0$ . We choose  $R_n = \frac{nA}{2 \log T}$ . We find

$$\begin{aligned} \sum_{\gamma, \text{Re } s_\gamma \in I_n} |\zeta(s_\gamma)|^2 &\ll (\log \log T)(n^{-1} \log T)^2 \int \int_{T \leq t \leq 2T, \frac{1}{2} + \frac{nA}{2 \log T} \leq \sigma \leq \frac{1}{2} + \frac{2nA}{\log T}} |\zeta(s)|^2 d\sigma dt \\ &\ll (\log \log T)(\log T)^2 n^{-2} \int_{\sigma \in I_n} (T \sum_{m=1}^{\infty} m^{-2\sigma}) d\sigma \\ &\ll T(\log \log T)(\log T)^2 n^{-2} (n^{-1} \log T) n (\log T)^{-1} \\ &\ll T(\log \log T)(\log T)^2 n^{-2}. \end{aligned}$$

Summing up over  $n = 1, 2, \dots, [\lambda \log T]$ , where  $\lambda > 0$  is a suitable constant (and square bracket denotes the integer part), we have

$$\sum_1 \ll T(\log T)^2 (\log \log T).$$

This proves part (i) of Theorem 1 in the case  $H = T - 2$ , (the case  $H \geq T^{\frac{2}{3}+\varepsilon}$  is similar).

**STEP VII.** To prove part (ii) we have to use

$$T^{-1} \int_T^{2T} |\zeta(\frac{1}{2} + it)| dt \ll (\log T)^{\frac{1}{4}}. \quad (3.6)$$



From (3.6) the following two Corollaries can be drawn (by the use of convexity principles (see [K.R]<sub>2</sub>)).

**COROLLARY 1.** For  $|\sigma - \frac{1}{2}| \ll (\log T)^{-1}$ , we have

$$\frac{1}{T} \int_T^{2T} |\zeta(\sigma + it)| dt \ll (\log T)^{\frac{1}{4}}. \quad (3.7)$$

**COROLLARY 2.** For  $\sigma = \frac{1}{2} + \lambda$  with  $0 < \lambda \leq 2$ ,  $\lambda \gg (\log T)^{-1}$ , We have

$$\frac{1}{T} \int_T^{2T} |\zeta(\sigma + it)| dt \ll \lambda^{-\frac{1}{4}}. \quad (3.8)$$

The rest of the proof of part (ii) of Theorem 1 is similar to that of part(i). (In this case also the proof with  $H \geq T^{\frac{1}{2}+\varepsilon}$  is similar).

**§4. PROOF OF THEOREM 2.** The proof is similar to that of Theorem 1 (part (i)). We have simply to note that Hypothesis 1 implies (by convexity principles referred to already) the following two Corollaries.

**COROLLARY 1.** For  $|\sigma - \frac{1}{2}| \ll (\log T)^{-1}$ , we have,

$$\frac{1}{H} \int_T^{T+H} |\zeta(\sigma + it)|^{2k} dt \ll (\log T)^{k^2}. \quad (4.1)$$

**COROLLARY 2.** For  $\sigma = \frac{1}{2} + \lambda$ , with  $0 < \lambda \leq 2$  and  $\lambda \gg (\log T)^{-1}$ , we have,

$$\frac{1}{H} \int_T^{T+H} |\zeta(\sigma + it)|^{2k} dt \ll \lambda^{-k^2}. \quad (4.2)$$

This completes the proof of all our assertions.

**§5.REMARK.** We have not computed the constants in Theorems 1 and 2. We will take it up on a different occasion if there are some important applications.

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**Author's addresses**

K. Ramachandra  
Hon. Vis. Professor  
Nat. Inst. of Adv. Studies  
I. I. Sc. Campus  
Bangalore-560012  
India

K. Ramachandra  
Retd. Professor  
TIFR Centre  
P.O.Box 1234  
I. I. Sc. Campus  
Bangalore-560012  
India

*e-mail:* [kram@math.tifrbng.res.in](mailto:kram@math.tifrbng.res.in)

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