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ON A PROBLEM OF IVIĆ

BY K.RAMACHANDRA

TO DR. A. SANKARANARAYANAN ON

HIS FORTIETH BIRTHDAY

§1. INTRODUCTION AND STATEMENT OF RESULTS. In [A.I] A.IVIĆ has proved the following inequality. Let $\epsilon > 0$ be any constant and let γ run over the ordinates of all the zeros (counted with multiplicity) of the Riemann zeta-function in the critical strip. Then

$$\sum_{T \leq \gamma \leq 2T} |\zeta(\frac{1}{2} + i\gamma)|^2 \ll_{\epsilon} T(\log T)^2 (\log \log T)^{\frac{3}{2} + \epsilon} \quad (1.1)$$

where the implied constant depends only on ϵ . Here $T \geq T_0$, a large positive constant.

In the present paper it is our object to improve this inequality in two ways. First we replace the LHS by a bigger quantity and at the same time replace the RHS by a smaller quantity namely $T(\log T)^2 \log \log T$ (where now the implied constant is absolute). Also we consider further generalisations and sketch their proof. In place of $|\zeta(\frac{1}{2} + i\gamma)|^2$ our key function (associated with arbitrary (but fixed) constants $A > 0$ and $B > 0$) is

$$M(\gamma) = \max |\zeta(s)|^2, \quad (1.2)$$

where the maximum is taken over all $s(= \sigma + it)$ in the rectangle

$$\frac{1}{2} - A(\log T)^{-1} \leq \sigma \leq 2, \quad |t - \gamma| \leq B(\log \log T)(\log T)^{-1}. \quad (1.3)$$

Plainly the investigations go through for $L(s, \chi)$ in place of $\zeta(s)$. Accordingly our Theorem is

THEOREM 1. *We have, for every fixed $\epsilon > 0$,*

$$(i) \quad \sum_{\gamma \in I} M(\gamma) \ll_{\epsilon} H(\log T)^2 \log \log T$$

where $I(\subset [T + 1, 2T - 1])$ is any interval of length $H(\geq T^{\frac{2}{3}+\epsilon})$ and $T \geq T_0(\epsilon)$.

$$(ii) \quad \sum_{\gamma \in I} (M(\gamma))^{\frac{1}{2}} \ll_{\epsilon} H(\log T)^{\frac{1}{2}} \log \log T$$

where $I(\subset [T + 1, 2T - 1])$ is any interval of length $H(\geq T^{\frac{1}{2}+\epsilon})$ and $T \geq T_0(\epsilon)$.

REMARK. In (ii) we can replace $\frac{1}{2}$ by any constant $k > 0$ for which k^{-1} is an integer and correspondingly the number $\frac{5}{4}$ will have to be replaced by $1 + k^2$. (Of course we should have $H \geq T^{\frac{1}{2}+\epsilon}$). In this connection we state the following Theorem 2.

THEOREM 2. If $k(> 0)$ is any constant, then

$$\sum_{\gamma \in I} (M(\gamma))^k \ll H(\log T)^{1+k^2} \log \log T$$

where $I(\subset [T + 1, 2T - 1])$ is any interval of length $H(> 0)$ satisfying the following two Hypotheses.

HYPOTHESIS 1. We should have

$$\frac{1}{H} \int_T^{T+H} |\zeta(\frac{1}{2} + it)|^{2k} dt \ll (\log T)^{k^2}.$$

HYPOTHESIS 2. For some constant $k^* > k$, we should have

$$\frac{1}{H} \int_T^{T+H} |\zeta(\frac{1}{2} + it)|^{2k^*} \ll (\log T)^C$$

where $C(> 0)$ is some constant depending on k^* .

REMARKS. If $k = 2$ we do not know whether Hypothesis 2 holds or not. However it holds if $k < 2$ (namely we can take $k^* = 2$ and $C = 4$ and $H \geq T^{\frac{2}{3}+\epsilon}$). Naturally we have to restrict to $k < 2$. Even here we do not know the truth of Hypothesis 1 unless k^{-1} is an integer and $H \geq T^{\frac{1}{2}+\epsilon}$. (Of course if $k = 1$ we can take things like $H \geq T^{\frac{1}{3}+\epsilon}$ but the corresponding Hypothesis 2 forces us to take $H \geq T^{\frac{2}{3}+\epsilon}$). It should be remarked that our method is completely different from that of A.IVIĆ.

Next we state Theorem 3 which gives lower bounds for

$$\sum_{\gamma \in I} (M(\gamma))^k.$$

THEOREM 3. We have, for any $k > 0$

$$\sum_{\gamma \in I} (M(\gamma))^k \gg \left(\int_I |\zeta(\frac{1}{2} + it)|^{2k} dt \right) (\log T) (\log \log T)^{-1}$$

where $I(\subset [T, 2T])$ is any interval of length H satisfying

$$H \geq 4B(\log T)^{-1} \log \log T$$

provided the intervals

$$|t - \gamma| \leq B(\log T)^{-1} \log \log T$$

associated with two successive γ 's have at least one point in common.

Since the proof of theorem 3 is very simple we give it here itself.

PROOF OF THEOREM 3. Contribution (to the integral on the right) from each interval

$$|t - \gamma| \leq B(\log T)^{-1} \log \log T$$

is clearly $\ll (M_0(\gamma))^k (\log T)^{-1} \log \log T$, where $M_0(\gamma)$ is the maximum of $|\zeta(\frac{1}{2} + it)|^2$ in the interval. Also these intervals cover I completely so that each γ is counted in $\sum_{\gamma \in I} (M_0(\gamma))^k$. Clearly $M_0(\gamma) \leq M(\gamma)$ and this completes the proof of Theorem 3.

Next we state a Conjecture.

CONJECTURE: $\sum_{T \leq \gamma \leq 2T} (M(\gamma))^2 \ll T(\log T)^5 \log \log T$.

FURTHER REMARKS. Lower bounds (and sometimes upper bounds) for

$$\int_I |\zeta(\frac{1}{2} + it)|^{2k} dt$$

have been studied first by K.RAMACHANDRA and then by D.R.HEATH-BROWN and then by M.JUTILA (for these results see the book [K.R]₁ by K.RAMACHANDRA and also the famous classic [E.C.T, D.R.H-B] by E.C.TITCHMARSH and D.R.HEATH-BROWN). There have been some further work on this integral by K.RAMACHANDRA [K.R]₂. Another typical result of K.RAMACHANDRA [K.R]₂ is

$$\frac{1}{H} \int_T^{T+H} |\zeta(\frac{1}{2} + it)|^{2\sqrt{2}} dt \gg (\log H)^2 (\log \log H)^{-1},$$

where H exceeds some positive constant times $\log \log T$. In particular

$$\frac{1}{T} \int_T^{2T} |\zeta(\frac{1}{2} + it)|^{2\sqrt{2}} dt \gg (\log T)^2 (\log \log T)^{-1}.$$

It seems very difficult to knock off the factor $(\log \log T)^{-1}$.

§2. NOTATION. We adopt standard notation. The Vinogradov symbols \ll and \gg mean "less than a positive constant times" and "greater than a positive constant times"

respectively. Some times we use $\ll \dots$ and $\gg \dots$ to denote that these constants depend on \dots (for example \ll_ε). We use the Landau symbol $O(\dots)$ to denote "less than a positive constant times \dots ". In §3 the letters A, B, C, D, E, F and G will denote positive constants. A and B need not necessarily be the same as in §1. T will exceed a large positive constant. $\varepsilon (> 0)$ will be an arbitrary constant. Some times we write $T_0(\varepsilon)$ to denote that the positive constant T_0 depends on ε .

§3. PROOF OF THEOREM 1. We divide the proof into seven steps for convenience. To illustrate our method we take $H = T - 2$. The general case in (i) follows from the deep result

$$\frac{1}{H} \int_T^{T+H} |\zeta(\frac{1}{2} + it)|^4 dt \ll_\varepsilon (\log T)^4$$

where $T \geq H \geq T^{\frac{2}{3} + \varepsilon}$ due to N.ZAVOROTNYI [N.Z] (see also [D.R.H-B], [H.I], [M.J] and [Y.M] for the history to date) and (ii) from the crucial result

$$\frac{1}{H} \int_T^{T+H} |\zeta(\frac{1}{2} + it)| dt \ll_\varepsilon (\log T)^{\frac{1}{2}}$$

where $T \geq H \geq T^{\frac{1}{2} + \varepsilon}$ due to K.RAMACHANDRA (see (4.3.2) of [K.R.]₁).

STEP I. Consider the various rectangles

$$\left[\frac{1}{2} - \frac{D \log \log T}{\log T} \leq \sigma \leq 2 \right] \times \left[T + (\log T)^E \leq t \leq T + 2(\log T)^E, T + 2(\log T)^E \leq t \leq T + 3(\log T)^E, \dots \right]$$

where the right extremity of the last t -interval does not exceed $2T - (\log T)^E$. Consider the maximum

$$\max |\zeta(s)|^4$$

over a typical rectangle. We will prove that

$$\sum \max |\zeta(s)|^4 \ll T(\log T)^{6+8D}. \tag{3.1}$$

(We can improve RHS to $T(\log T)^{5+8D}(\log \log T)^{-2}$; but we do not need this). Let s_1, s_2, \dots denote points where the maxima are attained. Then the required quantity is (see Theorem (1.7.1) of [K.R.]₁)

$$\leq |\zeta(s_1)|^4 + |\zeta(s_2)|^4 + \dots \leq \left(\frac{\log T}{D \log \log T} \right)^2 \pi^{-1} \left\{ \int_{D_1} \int |\zeta(s)|^4 da + \int_{D_2} \int |\zeta(s)|^4 da + \dots \right\}$$

(where D_j is the disc of radius $D(\log \log T)(\log T)^{-1}$ with centre s_j and da the element of area)

$$\leq 10\pi^{-1}(\log T)^2(D \log \log T)^{-2} \iint |\zeta(s)|^4 d\sigma dt$$

where the last integration is over the rectangle

$$\left[\frac{1}{2} - \frac{2D \log \log T}{\log T} \leq \sigma \leq 3 \right] \times [T \leq t \leq 2T].$$

The last integral is easily seen to be $\ll T(\log T)^{4+8D}$

STEP II. We now record a Corollary to step I. We have

$$\max |\zeta(s)| \leq (\log T)^C$$

for all rectangles except N of them where

$$N(\log T)^{4C} \ll T(\log T)^{6+8D}$$

and so

$$N \ll T(\log T)^{6-4C+8D}. \quad (3.2)$$

STEP III. We now write

$$M(\gamma) = \max |\zeta(s)|^2$$

where the maximum is taken over the rectangle

$$\left[\frac{1}{2} - \frac{A}{\log T} \leq \sigma \leq 2 \right] \times \left[|t - \gamma| \leq \frac{B \log \log T}{\log T} \right].$$

Let the asterisk denote the sum over those γ for which $M(\gamma) \leq (\log T)^{-F}$. Then

$$\sum_{T+1 \leq \gamma \leq 2T-1}^* (M(\gamma)) \ll T(\log T)^{1-F} \quad (3.3)$$

since the total number of γ 's is $\ll T \log T$ (we may fix F to be 2).

STEP IV. Denote the rectangle in step III by $R(\gamma)$ and by s_γ the point at which $|\zeta(s)|$ attains its maximum in $R(\gamma)$. Consider those γ 's for which $M(\gamma) > (\log T)^{-F}$. Then the number of zeros of $\zeta(s)$ in the disc with centre s_γ and radius $G(\log \log T)(\log T)^{-1}$ is $O(\log \log T)$ uniformly provided that this disc is inside any of the rectangles of step II. (Note that the number of exceptional rectangles is $N \leq T(\log T)^{6-4C+8D}$). This follows by Jensen's Theorem (see page 150 of [K.R]₁).

STEP V. Now consider all the $M(\gamma)$ with $T+1 \leq \gamma \leq 2T-1$. We claim

$$\sum M(\gamma) \ll T(\log T)^2 \log \log T. \quad (3.4)$$

(this proves part (i) of Theorem 1). To see this we split up the sum on the LHS into $\sum_1 + \sum_2 + \sum_3$, where \sum_3 is the sum in (3.3) and \sum_2 is over all γ lying in all the exceptional N rectangles occurring in step II. We have

$$\sum_2 \leq (\sum (M(\gamma))^2)^{\frac{1}{2}} N^{\frac{1}{2}}.$$

Note that $\sum (M(\gamma))^2 \ll T(\log T)^{6+8D+E+40} \leq T(\log T)^{50+8D+E}$ since each of the $M(\gamma)$'s may be replaced by the maxima in step I. Also

$$N \ll T(\log T)^{6-4C+8D}.$$

Thus

$$\sum_2 \ll T(\log T)^{25+4D+E+3-2C+4D} \leq T(\log T)^{-1}$$

if $C = 4D + E + 20$.

We now look at the sum \sum_1 . It consists of those γ 's for which

$$(\log T)^{-F} \leq M(\gamma) \leq (\log T)^C \quad (3.5)$$

and so by step IV the number of zeros of $\zeta(s)$ in

$$|s_\gamma - s| \leq G(\log \log T)(\log T)^{-1} \text{ is } O(\log \log T).$$

Here G is any constant such that this disc lies in

$$\left[\frac{1}{2} - \frac{D \log \log T}{\log T} \leq \sigma \leq 3 \right] \times J$$

where J is any of the non exceptional intervals of step II. This is so if $D = 10G$.

STEP VI. We are now in a position to complete the estimation of \sum_1 . Divide the σ -range $\left[\frac{1}{2} - \frac{A}{\log T} \leq \sigma \leq 2 \right]$ into intervals

$$I_0 = \left[\frac{1}{2} - \frac{A}{\log T} \leq \sigma \leq \frac{1}{2} + \frac{A}{\log T} \right],$$

$$I_1 = \left[\frac{1}{2} + \frac{A}{\log T} \leq \sigma \leq \frac{1}{2} + \frac{2A}{\log T} \right],$$

$$I_2 = \left[\frac{1}{2} + \frac{2A}{\log T} \leq \sigma \leq \frac{1}{2} + \frac{3A}{\log T} \right], \dots$$

$$I_n = \left[\frac{1}{2} + \frac{nA}{\log T} \leq \sigma \leq \frac{1}{2} + \frac{(n+1)A}{\log T} \right], \dots$$

the last interval projecting a little beyond 2. First consider I_0 . Look at the s_γ of the sum Σ_1 for which $\operatorname{Re} s_\gamma \in I_0$. We have (see page 34 of [K.R]₁)

$$|\zeta(s_\gamma)|^2 \leq (\pi R_0^2)^{-1} \int \int_{|s-s_\gamma| \leq R_0} |\zeta(s)|^2 da$$

where da is the element of area and $R_0 > 0$. Choose $R_0 = \frac{20A}{\log T}$. We find

$$\begin{aligned} \sum_{\gamma, \operatorname{Re} s_\gamma \in I_0} |\zeta(s_\gamma)|^2 &\ll (\log \log T)(\log T)^2 \int \int_{T \leq t \leq 2T, |\sigma - \frac{1}{2}| \leq \frac{100A}{\log T}} |\zeta(s)|^2 d\sigma dt \\ &\ll (\log \log T)(\log T)^2 (T \log T)(\log T)^{-1} \\ &= T(\log T)^2 \log \log T \end{aligned}$$

Now for any fixed $n (n = 1, 2, 3, \dots)$ consider the interval I_n for $\operatorname{Re} s_\gamma$. We have (see page 34 of [K.R]₁)

$$|\zeta(s_\gamma)|^2 \leq (\pi R_n^2)^{-1} \int \int_{|s-s_\gamma| \leq R_n} |\zeta(s)|^2 da$$

where da is the element of area and $R_n > 0$. We choose $R_n = \frac{nA}{2 \log T}$. We find

$$\begin{aligned} \sum_{\gamma, \operatorname{Re} s_\gamma \in I_n} |\zeta(s_\gamma)|^2 &\ll (\log \log T)(n^{-1} \log T)^2 \int \int_{T \leq t \leq 2T, \frac{1}{2} + \frac{nA}{2 \log T} \leq \sigma \leq \frac{1}{2} + \frac{2nA}{\log T}} |\zeta(s)|^2 d\sigma dt \\ &\ll (\log \log T)(\log T)^2 n^{-2} \int_{\sigma \in I_n} (T \sum_{m=1}^{\infty} m^{-2\sigma}) d\sigma \\ &\ll T(\log \log T)(\log T)^2 n^{-2} (n^{-1} \log T) n (\log T)^{-1} \\ &\ll T(\log \log T)(\log T)^2 n^{-2}. \end{aligned}$$

Summing up over $n = 1, 2, \dots, [\lambda \log T]$, where $\lambda > 0$ is a suitable constant (and square bracket denotes the integer part), we have

$$\sum_1 \ll T(\log T)^2 (\log \log T).$$

This proves part (i) of Theorem 1 in the case $H = T - 2$, (the case $H \geq T^{\frac{2}{3}+\varepsilon}$ is similar).

STEP VII. To prove part (ii) we have to use

$$T^{-1} \int_T^{2T} |\zeta(\frac{1}{2} + it)| dt \ll (\log T)^{\frac{1}{4}}. \quad (3.6)$$

From (3.6) the following two Corollaries can be drawn (by the use of convexity principles (see [K.R]₂)).

COROLLARY 1. For $|\sigma - \frac{1}{2}| \ll (\log T)^{-1}$, we have

$$\frac{1}{T} \int_T^{2T} |\zeta(\sigma + it)| dt \ll (\log T)^{\frac{1}{4}}. \quad (3.7)$$

COROLLARY 2. For $\sigma = \frac{1}{2} + \lambda$ with $0 < \lambda \leq 2$, $\lambda \gg (\log T)^{-1}$, We have

$$\frac{1}{T} \int_T^{2T} |\zeta(\sigma + it)| dt \ll \lambda^{-\frac{1}{4}}. \quad (3.8)$$

The rest of the proof of part (ii) of Theorem 1 is similar to that of part(i). (In this case also the proof with $H \geq T^{\frac{1}{2}+\varepsilon}$ is similar).

§4. PROOF OF THEOREM 2. The proof is similar to that of Theorem 1 (part (i)). We have simply to note that Hypothesis 1 implies (by convexity principles referred to already) the following two Corollaries.

COROLLARY 1. For $|\sigma - \frac{1}{2}| \ll (\log T)^{-1}$, we have,

$$\frac{1}{H} \int_T^{T+H} |\zeta(\sigma + it)|^{2k} dt \ll (\log T)^{k^2}. \quad (4.1)$$

COROLLARY 2. For $\sigma = \frac{1}{2} + \lambda$, with $0 < \lambda \leq 2$ and $\lambda \gg (\log T)^{-1}$, we have,

$$\frac{1}{H} \int_T^{T+H} |\zeta(\sigma + it)|^{2k} dt \ll \lambda^{-k^2}. \quad (4.2)$$

This completes the proof of all our assertions.

§5.REMARK. We have not computed the constants in Theorems 1 and 2. We will take it up on a different occasion if there are some important applications.

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