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NOTES ON THE RIEMANN ZETA-FUNCTION -V

BY

R.BALASUBRAMANIAN, K.RAMACHANDRA, A.SANKARANARAYANAN AND
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*Dedicated with deepest regards to PROFESSOR ALAN BAKER, FRS,
on his sixtieth birth day.*

§1.INTRODUCTION. This is a continuation of the paper [KR, AS] with the same title and this paper is also dedicated to him. Enough details are there in [RB.KR], [KR.AS] and [RB. KR. AS, KS]-III already and for this reason presentation in the present paper is some what sketchy. We begin by introducing a meromorphic function $F_0(s)$ as follows. Let k and l be any fixed integers subject to $0 \leq k \leq l$ and $l \geq 1$. Let $\alpha_1, \dots, \alpha_k, \beta_1, \dots, \beta_l$ be any $k + l$ complex constants. Let $P(s)$ be any Dirichlet polynomial (i.e. a terminating Dirichlet series). Put

$$F_0(s) = P(s)(\prod \zeta(s + \alpha)) (\prod \zeta(s + \beta))^{-1}$$

where the first product runs over $\alpha_1, \dots, \alpha_k$ and the second over β_1, \dots, β_l . The main theorem in [RB. KR. AS, KS]-IV runs as follows:

THEOREM 1. *Let $F_0(s)$ be not identically zero. Then $F_0(s)$ has infinitely many poles in $t \geq t_0$ (for every fixed $t_0 > 0$). There exist poles with ordinates in $[T, 2T]$. for all $T \geq T_0$ (for some $T_0 > 0$). If we arrange these ordinates in the non decreasing order then the successive gaps are majorised by*

$$d \log T$$

where $d > 0$ is a constant independent of T .

It is our object in this paper to generalise this theorem to functions of the type

$$F(s) = \sum F_0(s)$$

where the sum is a finite one and is over terms $F_0(s)$ with varying $P(s); k, l; \alpha_1, \dots, \alpha_k, \beta_1, \dots, \beta_l$. We prove

THEOREM 2. *The statement of Theorem 1 is true word for word if we replace $F_0(s)$ by $F(s)$ and assume that (in some far off right off plane) $F(s)$ has a non-terminating Dirichlet series expansion.*

REMARK 1. Note that we have completely knocked off the condition $\sum^* < 1$ of [KR, AS] which was necessary there since we have to deal with "Explicit formula" for the partial sums of the coefficients of the Dirichlet series expansions of $F(s)$ in some far off right half plane. This needed some delicate estimates for $F(s)$.

REMARK 2. Our proof of theorem 2 allows as to consider terms $F_0(s)$ (of the sum $F(s)$) with $\zeta(s + \alpha)$ being replaced by (any derivative of bounded orders of $\zeta(s + \alpha)$) plus a Dirichlet polynomial which may depend on α 's. Similarly we may as the same thing for $\zeta(s + \beta)$. But we need the condition that none of the denominators in the terms $F_0(s)$ have a zero constant term. (Here some are all the ζ can be replaced by $\sum_{n=1}^{\infty} (an + b)^{-s}$ ($a > 0, b > 0$ integer constants) or L-functions).

REMARK 3. In remark 3 we may replace $\zeta(s + \alpha)$ by $\zeta_K(s + \alpha, R)$ the zeta function of a ray class in an algebraic number field K of degree $n(K)$. Also in each of the terms $F_0(s)$ during replacements it is understood that the triplet (K, α, R) may vary. Similarly for the denominators but with the condition that none of the denominators of the terms $F_0(s)$ (of the sum) have a zero constant term. But now we need also the condition

$$\sum n(K) \leq \sum n(K')$$

for each term $F_0(s)$ (of the sum $F(s)$) $n(K')$ being the degree of the fields K' in $\zeta_{K'}(s + \beta, R')$.

REMARK 4. When $k \leq l$ and all the $\alpha_1, \alpha_2, \dots, \alpha_k$ and $\beta_1, \beta_2, \dots, \beta_l$ (involved in each term $F_0(s)$) are purely imaginary we get : The gaps between the ordinates of the successive poles of $F(s)$ (of theorem 2) in $[T, 2T]$ ($T \geq T_0$) are majorised in absolute value by the quantity $C \log \log T$. This is still true for number fields provided $\sum n(K) \leq \sum n(K')$. In [RB.KR.AS.KS]-IV, we proved the following result. If $k < l$ or ($k = l$ and $\Re(\sum(\beta - \alpha)) \leq 0$), then for $F_0(s)$ the gaps between the ordinates of the successive poles (in the sense of

ordinates being arranged in the non-decreasing order) in absolute value are bounded by a positive constant times $\log \log T$.

REMARK 5. Our method has lots of applications. We state only a few below.

Our method enables us to prove

THEOREM 3. Let $P(s)$ be any Dirichlet polynomial ($\neq 0$) for which $F(s)$ is defined for complex constants a and b (with $a \neq 1$ and $b \neq 1$) by

$$F(s) = \frac{P(s)(\zeta'(s))^2}{(\zeta(s) - a)(\zeta(s) - b)}.$$

Then $F(s)$ has infinitely many poles in $t \geq t_0$ and the gaps between the ordinates in $[T, 2T], (T \geq T_0)$, of the successive poles are majorised in absolute value by

$$d \log \log T$$

where $d > 0$ is a constant independent of T .

REMARK 1. Note that the poles of $F(s)$ are precisely simple zeros of $(P(s))^{-1} (\zeta(s) - a)(\zeta(s) - b)$ whenever $a \neq b$.

REMARK 2. Theorem 3 is true when $\zeta(s)$ is replaced by many other functions such as the zeta function of a number field.

REMARK 3. When $a = b = 0$, in fact one can say much more: The gaps between the ordinates in $[T, 2T] (T \geq T_0)$ of these poles are majorised in absolute value by $d_1 (\log \log \log T)^{-1}$ which follows from a result of J.E.Littlewood (see [ECT]). When $a = 1$ or $b = 1$, $F(s)$ can not be expanded as a Dirichlet series and hence it is an exclusion.

We state two more theorems.

THEOREM 4. Let $P(s)$ and $Q(s)$ be any two Dirichlet polynomials of which $Q(s)$ has non-zero constant term. Let $F(s)$ defined by $F(s) = P(s)(Q(s))^{-1}$ have a non-terminating Dirichlet series expansion in some far off right half plane. Then $F(s)$ has infinitely many poles and the gaps between successive poles are bounded.

REMARK . Successive in the sense of poles with ordinates being arranged in the non-decreasing order.

THEOREM 5. Let $P(s)$ and $Q(s)$ be two Dirichlet polynomials of which $Q(s)$ has a non zero constant term. Let $F(s)$ defined by

$$F(s) = P(s)(Q'(s))^2 (Q(s))^{-1}$$

have a non-terminating Dirichlet series expansion in some far off right half plane. Then $F(s)$ has infinitely many poles and the gaps between successive poles are bounded.

REMARK Note that the poles of $F(s)$ are precisely the simple zeros of $Q(s)(P(s))^{-1}$.

§2. SOME REMARKS ABOUT THE METHOD OF PROOFS.

In [KR, AS] we dealt with 'Explicit formula' for $\sum_{n \leq x} a_n$ where $F(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ in some far off right half plane. In our new method we consider "Explicit formula" for E defined by $E = \sum_{n \leq x} a_n (x-n)^q$, where q is a large positive integer constant. This enables us to cope up with "large growth of $F(s)$ " and to knock off the condition $\sum^* < 1$ completely. Note that

$$E = E(x) = \frac{1}{2\pi i} \int_{L_1} \frac{F(s) x^{s+q}}{s(s+1)\dots(s+q)} ds$$

over a suitable vertical line L_1 (far off to the right). It is not hard to prove that $E(x)$ is $q-1$ times continuously differentiable and that

$$\frac{d^{q-1}}{dx^{q-1}} E(x) = \sum_{n \leq x} a_n (x-n).$$

This helps us to observe that $E(x)$ does not have a continuous derivative of order q , provided that $a_n \neq 0$ for infinitely many n . On the other hand the explicit formula for $E(x)$ (which is easy to derive), shows that if there are no poles of $F(s)$ in $|t| \geq t_0 (> 0)$, a large constant then $E(x)$ is q times continuously differentiable. To verify this all that we have to verify is that each of the functions $E_1(x)$ defined by

$$E_1(x) = \frac{1}{2\pi i} \int_L \frac{F(s)x^{s+q} ds}{s(s+1)\dots(s+q)}$$

(where L is the (anti-clockwise) boundary of the rectangles

$$\{|t| = 2t_0, -N \leq \sigma \leq -q_1\}$$

for suitable sequence $N = N_\nu (\nu = 1, 2, 3, \dots)$ even in the limit as N tends to ∞) are q times continuously differentiable provided x exceeds a large positive constant. Here q_1 is a large

positive constant. This contradiction proves that $F(s)$ has infinitely many poles in $|t| \geq t_0$, since the poles in $\sigma > -q_1$ contribute a C^∞ function to the sum $E(x)$. Once we have the infinitude of poles in $|t| \geq t_0$ we can apply the results of [RB. KR] (see remark 1 on page 194). We take this opportunity to correct a misprint on page 193, last line; $(F(s)s)$ should read $(F(s))$. According to this result it follows that $F(s)$ is a good Dirichlet series to be defined presently.

DEFINITION. $F(s)$ is said to be good if there exist positive constants C_1, C_2, C_3 such that for all $X \geq C_1$ there holds

$$\max_{X \leq n \leq X^{C_2}} |a_n| > X^{-C_3}.$$

Next the paper [RB, KR, AS,KS]-III comes to our rescue to prove all that we want. For the convenience of readers we will give a complete statement of the main result of [RB.KR, AS,KS]-III in §3. Apart from goodness the other conditions needed therein are easy to verify.

§3. MAIN RESULT OF [RB, KR, AS, KS]III. In what follows T will exceed a large positive constant. The letters ψ , φ and H will denote positive functions of T bounded below by large positive constants. They are assumed to satisfy

$$H = o(T), \log \log \varphi = O(H)$$

and

$$\log \psi = O(H)$$

where the two O -constants are assumed to be sufficiently small.

(A) Let $F_1(s)$ and $F_2(s)(s = \sigma + it)$ be two Dirichlet series (which may depend on a parameter T and we consider only the interval $T-H \leq t \leq T+H$) convergent absolutely in $\sigma \geq C_0 (\geq 100)$ and bounded there. The letter $g > 0$ will denote a large absolute constant and we assume that $F_1(s)$ and $F_2(s)$ can be continued analytically in

$$\sigma \geq -g(\log \psi)(\log \log \psi)^{-1}.$$

Also we assume that $F_2(\sigma) \rightarrow 1$ as $\sigma \rightarrow \infty$ and that in $\sigma \geq C_0$ the function $\log F_2(s)$ is bounded

(B) Let

$$|F_2(s)| < \varphi^g \text{ in } \sigma \geq -20g.$$

(C) Let $F_2(s) \neq 0$ in $-g \geq \sigma \geq -g(\log H)(\log \log H)^{-1}$ and also in the same region we have

$$|F_1(s)(F_2(s))^{-1}| \leq C_4^\sigma \psi(C_4 > 0),$$

where C_4 is a constant. For convenience we assume that the constant C_4 is bounded below and also above.

(D) Let $|F_1(s)| \leq \exp((g^2 \log \varphi)^3)$ in $\sigma \geq -g$. Under these conditions we have the following main theorem.

THEOREM. *We have a pole $p_1 + ip_2$ with*

$$T - H \leq p_2 \leq T + H,$$

provided $F(s)$ defined as $F_1(s)(F_2(s))^{-1}$ is good.

REMARK 1. For theorem 3, it is not difficult to show that $F(s)$ is a non-terminating Dirichlet series whenever $P(s) \neq 0$. If we let $P(s) = \sum_{\nu \leq M} a_\nu \nu^{-s}$ and ν_0 is the least ν for which $a_\nu \neq 0$, then we find that for all large p , the coefficient of $p^{-2s} \nu_0^{-s}$ in $F(s)$ is $\frac{a_{\nu_0}(\log p)^2}{(1-a)(1-b)}$ which in fact gives the "goodness" of $F(s)$. These remarks go through for number fields case etc with the additional condition $\sum n(K) \leq \sum n(K')$. For theorem 3, we find that $\varphi \ll T^A$ and $\psi \ll (\log T)^A$ and hence the proof of theorem 3 follows from the above theorem of [RB, KR, AS, KS]III. Of course we have to use the functional equation of $\zeta(s)$ appropriately here.

REMARK 2. For theorems 4 and 5, we find that φ and ψ of the above theorem satisfy $\varphi \ll B$ and $\psi \ll B$ for a large positive constant B . Hence the proofs follow from the above theorem of the paper [RB, KR, AS, KS]III.

R E F E R E N C E S

- [RB, KR], R.Balasubramanian and K.Ramachandra, *Some Problems of Analytic Number Theory - II*, *Studia Scientiarum Mathematicarum Hungarica*, 14 (1979), 193-202.
- [RB, KR, AS, KS], R.Balasubramanian, K.Ramachandra, A.Sankranarayanan and K.Srinivas. *Notes on the Riemann zeta-function -III*, *Hardy-Ramanujan J.*, Vol.22 (1999), 23-33.
- [RB, KR, AS, KS], R.Balasubramanian, K.Ramachandra, A.Sankaranarayanan and K.Srinivas, *Notes on the Riemann Zeta-function - IV*, *Hardy-Ramanujan J.*, Vol.22 (1999), 34-41.
- [KR, AS], K.Ramachandra and A.Sankaranarayanan, *Notes on the Riemann zeta-function - II*, *Acta Arith.*, Vol.XCI, 4(1999), 351-365.
- [ECT], E.C. Titchmarsh, *The theory of the Riemann zeta-function*, second edition, Edited by D.R.Heath-Brown, Clarendon Press (Oxford) (1986).

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