



HAL
open science

Disproof of some conjectures of K.Ramachandra

Johan Anderson

► **To cite this version:**

Johan Anderson. Disproof of some conjectures of K.Ramachandra. Hardy-Ramanujan Journal, 1999, Volume 22 - 1999 (1), pp.2-7. 10.46298/hrj.1999.137 . hal-01109508

HAL Id: hal-01109508

<https://hal.science/hal-01109508>

Submitted on 26 Jan 2015

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

DISPROOF OF SOME CONJECTURES OF K.RAMACHANDRA

BY

JOHAN ANDERSSON

1. INTRODUCTION

In a recent paper [9] K.Ramachandra states some conjectures, and gives consequences in the theory of the Riemann zeta function. In this paper I will present two different disproofs of them. The first will be an elementary application of the Szasz-Müntz theorem. The second will depend on a version of the Voronin universality theorem, and is also slightly stronger in the sense that it disprove a weaker conjecture. An elementary (but more complicated) disproof has been given by Rusza-Lazkovich [11].

2. DISPROOF OF SOME CONJECTURES

2.1 The conjectures. I will first state the three conjectures as given by Ramachandra [9], and Ramachandra-Balasubramanian [5].

Conjecture 1. For all $N \geq H \geq 1000$ and all N -tuples $a_1 = 1, a_2, \dots, a_N$ of complex numbers we have

$$\frac{1}{H} \int_0^H \left| \sum_{n=1}^N a_n n^{it} \right| dt \geq 10^{-1000}$$

*

Conjecture 2. For all $N \geq H \geq 1000$ and all N -tuples $a_1 = 1, a_2, \dots, a_N$ of complex numbers we have when $M = H(\log H)^{-2}$

$$\frac{1}{H} \int_0^H \left| \sum_{n=1}^N a_n n^{it} \right|^2 dt \geq (\log H)^{-1000} \sum_{n=1}^M |a_n|^2$$

*Recieved on 07-09-1999.

1991 *Mathematics Subject Classification.* Primary 11M41, Secondary 42A65, 11N30.

Key words and phrases. Dirichlet series, Titchmarsh series.

Conjecture 3. There exist a constant $c > 0$ such that

$$\int_0^T \left| \sum_{n=1}^N a_n n^{it} \right|^2 dt \geq c \sum_{n \leq cT} |a_n|^2$$

2.2 The Szász-Müntz theorem To disprove these conjectures, we first consider the following classic result of Szász

Lemma 1. (Szász) If we have that

$$\sum_{n=1}^{\infty} \frac{1 + 2\Re(\lambda_n)}{1 + |\lambda_n|^2} = +\infty$$

where $\Re(\lambda_n) \geq 0$ then the set of finite linear combinations of x^{λ_n} is dense in $L^2(0, 1)$

Proof See Szász [12], theorem A.

I will now state a theorem that will effectively disprove the above conjectures:

Theorem 1. For each $D \geq 0$ and $\varepsilon > 0$ there exists an $N \geq 0$ and complex numbers a_2, \dots, a_N , such that

$$\int_0^D \left| 1 + \sum_{n=2}^N a_n n^{it} \right|^2 dt \leq \varepsilon$$

Proof Since $-1 \in L^2(0, 1)$ and

$$\sum_{n=2}^{\infty} \frac{1 + 2\Re(-i \log n)}{1 + |-i \log n|^2} = +\infty$$

we have by the lemma that for each $\delta > 0$ there exists an $N > 0$ and complex numbers a_2, \dots, a_N such that

$$\int_0^1 \left| 1 + \sum_{n=2}^N a_n x^{-i \log n} \right|^2 dx < \delta$$

We obtain

$$\begin{aligned} \delta > \int_0^1 \left| 1 + \sum_{n=2}^N a_n x^{-i \log n} \right|^2 dx &\geq \int_{e^{-D}}^1 \left| 1 + \sum_{n=2}^N a_n x^{-i \log n} \right|^2 dx = \\ &= (\text{Substituting } t = -\log x) = \end{aligned}$$

$$\int_0^D e^{-t} \left| 1 + \sum_{n=2}^N a_n n^{it} \right|^2 dt \geq e^{-D} \int_0^D \left| 1 + \sum_{n=2}^N a_n n^{it} \right|^2 dt$$

By choosing $\delta = e^{-D}\varepsilon$ we obtain the theorem.

It is now an easy task to falsify the conjectures.

Proposition. *Conjectures 1, 2 and 3 are false.*

Proof. For conjecture 1, choose $\varepsilon < 10^{-2000}H$ and $D = H$ in Theorem 1 and apply the Cauchy-Schwarz inequality

$$\int_0^H \left| \sum_{n=1}^N a_n n^{it} \right| dt \leq \sqrt{H} \sqrt{\int_0^H \left| \sum_{n=1}^N a_n n^{it} \right|^2 dt}$$

For conjecture 2, choose $\varepsilon < (\log H)^{-1000}H$, and $D = H$ in Theorem. To disprove Conjecture 3, chose e.g. $a_1 = T = 1$, and $\varepsilon = c/2$.

2.3. The Voronin Theorem. In private correspondence, Ramachandra asked whether the conjectures hold under the additional growth assumption $|a_k| \ll (Hk)^{100}$. Ramachandra and Balasubramanian have proved conjectures 1 and 2 under the this and the additional further assumption that $N < \exp(\exp(cH))$. This shows that N must be very large compared to H for the conjectures to be false. However, they are in fact still false, although their proof requires a deeper result. A version of the Voronin universality theorem for the Riemann zeta-function. We will state the theorem that shows that the conjectures are still false below:

Theorem 2. *Suppose that $H, \varepsilon > 0$, $0 < \delta < \frac{1}{2}$. Then there exists $|a_k| \leq k^{\delta-1}$ such that*

$$\max_{t \in [0, H]} \left| 1 + \sum_{n=2}^N a_n n^{it} \right| < \varepsilon$$

The idea goes as follows: We use the following version of the Voronin universality theorem

Theorem 3. *(Voronin-Bagchi) For any compact subset K of the complex numbers such that $x \in K \implies \frac{1}{2} < \operatorname{Re}(x) < 1$, non vanishing analytic function f on K and $\varepsilon > 0$, we have a real t such that $|\zeta(z + it) - f(z)| \leq \varepsilon$, for all $z \in K$*

Proof. See Bagchi [1].

and the following version of the approximate functional equation for the Riemann zeta-function

$$\left| \zeta(\sigma + it) - \sum_{k=1}^N k^{-\sigma-it} \right| \leq CN^{\sigma-1}, \text{ for e.g. } t < N < 2t, \sigma \geq \frac{1}{2}$$

Proof (of Theorem 2) Choose T_0 so that when $T > T_0$ then

$$\left| \zeta(1 - \delta + it) - \sum_{k < 2T} k^{\delta-1-it} \right| < \frac{\varepsilon}{3}$$

when $T \leq t \leq T + H$. Now chose $T > T_0$ so that

$$|\zeta(1 - \delta + it)| < \frac{\varepsilon}{3}$$

for $T \leq t \leq T + H$ (This is possible by applying Theorem 3 to $f(z) = \varepsilon/3$, and $K = [1 - \delta, 1 - \delta + iH]$). Using the triangle inequality we get

$$\left| \sum_{k < 2T} k^{\delta-1-it} \right| < \varepsilon$$

for all $T < t < T + H$. Now we can chose $N = T$ and $a_n = n^{\delta-1-iT}$ in the theorem.

3. Summary

In [8] K. Ramachandra stated a similar conjecture to conjectures 1 and 2. Although it was more general in the sense that it considered Dirichlet series of form $A(s) = 1 + \sum a_n \lambda_n^{-s}$, for certain λ_n generalizing $\lambda_n = n$ it was much weaker as it had three further conditions. H depended on N , $|a_n|$ were bounded from above and A was bounded in certain regions in the complex plane. Under these additional assumptions, $A(s)$ is called a Titchmarsh-series and for these, analogues of conjectures 1 and 2, and similar conjectures were proved in Ramachandra [7], and Balasubramanian and Ramachandra [2], [4] and [3]. It would certainly be interesting to see if all these additional assumptions are needed, or if a certain subset of

them implies the truth of conjectures 1 and 2. For a disproof of the $L^p(0, D)$, $p > 2$ version of conjectures 1 and 2, or for considering much more sparse sequences than $\log n$ in (1), we need a stronger version of the Lemma, the reader is referred to the literature on the theory of completeness of complex exponentials, see e.g. the classic by Levinson [6] or Redheffer [10] for a more recent survey. It should be noticed that problems similar to conjectures 1 and 2 are also studied in Turán power sum theory (although there it is essentially the λ_n 's which vary, instead of the a_n 's), see Turán [13] for a thorough treatment.

R E F E R E N C E S

1. Bhaskar Bagchi, *A joint universality theorem for Dirichlet l -functions. (English)*, Math. Z. **181** (1982), no. 3, 319–334.
2. R. Balasubramanian and K. Ramachandra, *Proof of some conjectures on the mean-value of Titchmarsh series. I*, Hardy-Ramanujan J. **13** (1990), 1–20.
3. —, *Proof of some conjectures on the mean-value of Titchmarsh series. II* Hardy-Ramanujan J. **14** (1991), 1–20.
4. —, *Proof of some conjectures on the mean-value of Titchmarsh series. III*, Proc. Indian Acad. Sci. Math. Sci. **102**(1992), no.2, 83–91.
5. —, *On Riemann zeta-function and allied questions. II*, Hardy-Ramanujan J. **18** (1995), 10–22.
6. Norman Levinson, *Gap and Density Theorems*, American Mathematical Society, New York, 1940, American Mathematical Society Colloquium Publications, v. 26.
7. K. Ramachandra, *Progress towards a conjecture on the mean-value of Titchmarsh series-1*, Recent progress in Analytic number theory (H. Halberstam and C. Hooley, eds.), vol.1, Academic Press, London, New York, Toronto, Sydney, San Francisco, 1981, pp. 303–318.

8. K.Ramachandra, *Proof of some conjectures on the mean-value of Titchmarsh series with applications to Titchmarsh's phenomenon*, Hardy-Ramanujan J. **13** (1990), 21–27.
- 9 ———, *On Riemann zeta-function and allied questions*, Astérisque (1992), no.209, 57–72, Journées Arithmétiques, 1991 (Geneva).
10. Raymond M. Redheffer, *Completeness of sets of complex exponentials*, Advances in Math. **24** (1977), no.1, 1–62.
11. M.Rusza, I.Lazkovich, *Sums of periodic functions and a problem of Ramachandra*, Tech. report, Mathematical institute of the Hungarian Academy of Sciences, 1996.
12. O.Szász, *Über die Approximation stetiger Funktionen durch lineare Aggregate von Potenzen*, Math Ann. **77** (1916), 482–496.
13. Paul Turán, *On a new method of analysis and its applications*, Pure and Applied Mathematics, John Wiley & Sons Inc., New York, 1984, With the assistance of G. Halász and J. Pintz, With a foreword by Vera T.Sós, A Wiley-Interscience Publication.

Address of the Author:

Prof.Johan Andersson
Department of Mathematics
Stockholm University
S-10691, Stockholm
SWEDEN.

E-mail:- johana@matematik.su.se