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## DISPROOF OF SOME CONJECTURES OF K.RAMACHANDRA

BY

JOHAN ANDERSSON

## 1. INTRODUCTION

In a recent paper [9] K.Ramachandra states some conjectures, and gives consequences in the theory of the Riemann zeta function. In this paper I will present two different disproofs of them. The first will be an elementary application of the Szasz-Müntz theorem. The second will depend on a version of the Voronin universality theorem, and is also slightly stronger in the sense that it disprove a weaker conjecture. An elementary (but more complicated) disproof has been given by Rusza-Lazkovich [11].

## 2. DISPROOF OF SOME CONJECTURES

**2.1 The conjectures.** I will first state the three conjectures as given by Ramachandra [9], and Ramachandra-Balasubramanian [5].

**Conjecture 1.** For all  $N \geq H \geq 1000$  and all  $N$ -tuples  $a_1 = 1, a_2, \dots, a_N$  of complex numbers we have

$$\frac{1}{H} \int_0^H \left| \sum_{n=1}^N a_n n^{it} \right| dt \geq 10^{-1000}$$

\*

**Conjecture 2.** For all  $N \geq H \geq 1000$  and all  $N$ -tuples  $a_1 = 1, a_2, \dots, a_N$  of complex numbers we have when  $M = H(\log H)^{-2}$

$$\frac{1}{H} \int_0^H \left| \sum_{n=1}^N a_n n^{it} \right|^2 dt \geq (\log H)^{-1000} \sum_{n=1}^M |a_n|^2$$

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**Conjecture 3.** There exist a constant  $c > 0$  such that

$$\int_0^T \left| \sum_{n=1}^N a_n n^{it} \right|^2 dt \geq c \sum_{n \leq cT} |a_n|^2$$

**2.2 The Szász-Müntz theorem** To disprove these conjectures, we first consider the following classic result of Szász

**Lemma 1.** (Szász) If we have that

$$\sum_{n=1}^{\infty} \frac{1 + 2\Re(\lambda_n)}{1 + |\lambda_n|^2} = +\infty$$

where  $\Re(\lambda_n) \geq 0$  then the set of finite linear combinations of  $x^{\lambda_n}$  is dense in  $L^2(0, 1)$

*Proof* See Szász [12], theorem A.

I will now state a theorem that will effectively disprove the above conjectures:

**Theorem 1.** For each  $D \geq 0$  and  $\varepsilon > 0$  there exists an  $N \geq 0$  and complex numbers  $a_2, \dots, a_N$ , such that

$$\int_0^D \left| 1 + \sum_{n=2}^N a_n n^{it} \right|^2 dt \leq \varepsilon$$

**Proof** Since  $-1 \in L^2(0, 1)$  and

$$\sum_{n=2}^{\infty} \frac{1 + 2\Re(-i \log n)}{1 + |-i \log n|^2} = +\infty$$

we have by the lemma that for each  $\delta > 0$  there exists an  $N > 0$  and complex numbers  $a_2, \dots, a_N$  such that

$$\int_0^1 \left| 1 + \sum_{n=2}^N a_n x^{-i \log n} \right|^2 dx < \delta$$

We obtain

$$\begin{aligned} \delta &> \int_0^1 \left| 1 + \sum_{n=2}^N a_n x^{-i \log n} \right|^2 dx \geq \int_{e^{-D}}^1 \left| 1 + \sum_{n=2}^N a_n x^{-i \log n} \right|^2 dx = \\ &= (\text{Substituting } t = -\log x) = \end{aligned}$$

$$\int_0^D e^{-t} \left| 1 + \sum_{n=2}^N a_n n^{it} \right|^2 dt \geq e^{-D} \int_0^D \left| 1 + \sum_{n=2}^N a_n n^{it} \right|^2 dt$$

By choosing  $\delta = e^{-D}\varepsilon$  we obtain the theorem.

It is now an easy task to falsify the conjectures.

**Proposition.** *Conjectures 1, 2 and 3 are false.*

*Proof.* For conjecture 1, choose  $\varepsilon < 10^{-2000}H$  and  $D = H$  in Theorem 1 and apply the Cauchy-Schwarz inequality

$$\int_0^H \left| \sum_{n=1}^N a_n n^{it} \right| dt \leq \sqrt{H} \sqrt{\int_0^H \left| \sum_{n=1}^N a_n n^{it} \right|^2 dt}$$

For conjecture 2, choose  $\varepsilon < (\log H)^{-1000}H$ , and  $D = H$  in Theorem. To disprove Conjecture 3, chose e.g.  $a_1 = T = 1$ , and  $\varepsilon = c/2$ .

**2.3. The Voronin Theorem.** In private correspondence, Ramachandra asked whether the conjectures hold under the additional growth assumption  $|a_k| \ll (Hk)^{100}$ . Ramachandra and Balasubramanian have proved conjectures 1 and 2 under the this and the additional further assumption that  $N < \exp(\exp(cH))$ . This shows that  $N$  must be very large compared to  $H$  for the conjectures to be false. However, they are in fact still false, although their proof requires a deeper result. A version of the Voronin universality theorem for the Riemann zeta-function. We will state the theorem that shows that the conjectures are still false below:

**Theorem 2.** *Suppose that  $H, \varepsilon > 0$ ,  $0 < \delta < \frac{1}{2}$ . Then there exists  $|a_k| \leq k^{\delta-1}$  such that*

$$\max_{t \in [0, H]} \left| 1 + \sum_{n=2}^N a_n n^{it} \right| < \varepsilon$$

The idea goes as follows: We use the following version of the Voronin universality theorem

**Theorem 3.** *(Voronin-Bagchi) For any compact subset  $K$  of the complex numbers such that  $x \in K \implies \frac{1}{2} < \operatorname{Re}(x) < 1$ , non vanishing analytic function  $f$  on  $K$  and  $\varepsilon > 0$ , we have a real  $t$  such that  $|\zeta(z + it) - f(z)| \leq \varepsilon$ , for all  $z \in K$*

*Proof.* See Bagchi [1].

and the following version of the approximate functional equation for the Riemann zeta-function

$$\left| \zeta(\sigma + it) - \sum_{k=1}^N k^{-\sigma-it} \right| \leq CN^{\sigma-1}, \text{ for e.g. } t < N < 2t, \sigma \geq \frac{1}{2}$$

*Proof* (of Theorem 2) Choose  $T_0$  so that when  $T > T_0$  then

$$\left| \zeta(1 - \delta + it) - \sum_{k < 2T} k^{\delta-1-it} \right| < \frac{\varepsilon}{3}$$

when  $T \leq t \leq T + H$ . Now chose  $T > T_0$  so that

$$|\zeta(1 - \delta + it)| < \frac{\varepsilon}{3}$$

for  $T \leq t \leq T + H$  (This is possible by applying Theorem 3 to  $f(z) = \varepsilon/3$ , and  $K = [1 - \delta, 1 - \delta + iH]$ ). Using the triangle inequality we get

$$\left| \sum_{k < 2T} k^{\delta-1-it} \right| < \varepsilon$$

for all  $T < t < T + H$ . Now we can chose  $N = T$  and  $a_n = n^{\delta-1-iT}$  in the theorem.

### 3. Summary

In [8] K. Ramachandra stated a similar conjecture to conjectures 1 and 2. Although it was more general in the sense that it considered Dirichlet series of form  $A(s) = 1 + \sum a_n \lambda_n^{-s}$ , for certain  $\lambda_n$  generalizing  $\lambda_n = n$  it was much weaker as it had three further conditions.  $H$  depended on  $N$ ,  $|a_n|$  were bounded from above and  $A$  was bounded in certain regions in the complex plane. Under these additional assumptions,  $A(s)$  is called a Titchmarsh-series and for these, analogues of conjectures 1 and 2, and similar conjectures were proved in Ramachandra [7], and Balasubramanian and Ramachandra [2], [4] and [3]. It would certainly be interesting to see if all these additional assumptions are needed, or if a certain subset of

them implies the truth of conjectures 1 and 2. For a disproof of the  $L^p(0, D)$ ,  $p > 2$  version of conjectures 1 and 2, or for considering much more sparse sequences than  $\log n$  in (1), we need a stronger version of the Lemma, the reader is referred to the literature on the theory of completeness of complex exponentials, see e.g. the classic by Levinson [6] or Redheffer [10] for a more recent survey. It should be noticed that problems similar to conjectures 1 and 2 are also studied in Turán power sum theory (although there it is essentially the  $\lambda_n$ 's which vary, instead of the  $a_n$ 's), see Turán [13] for a thorough treatment.

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