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On the numbers e^e, e^{e^2} and e^{π^2}

by

Michel WALDSCHMIDT

Abstract. We give two measures of simultaneous approximation by algebraic numbers, the first one for the triple (e, e^e, e^{e^2}) and the second one for (π, e, e^{π^2}). We deduce from these measures two transcendence results which had been proved in the early 70's by W.D. Brownawell and the author.

Introduction

In 1949, A.O. Gel'fond introduced a new method for algebraic independence, which enabled him to prove that the two numbers $2^{\sqrt[3]{2}}$ and $2^{\sqrt[3]{4}}$ are algebraically independent. At the same time, he proved that one at least of the three numbers e^e, e^{e^2}, e^{e^3} is transcendental (see[G] Chap. III, §4).

At the end of his book [S] on transcendental numbers, Th. Schneider suggested that one at least of the two numbers e^e, e^{e^2} is transcendental; this was the last of a list of eight problems, and the first to be solved, in 1973, by W.D. Brownawell [B] and M.Waldschmidt [W 1], independently and simultaneously. For this result they shared the Distinguished Award of the Hardy-Ramanujan Society in 1986. Another consequence of their main result is that one at least of the two following statements holds true:

(i) The numbers e and π are algebraically independent

(ii)The number e^{π^2} is transcendental

Our goal is to shed a new light on these results. It is hoped that our approach will yield further progress towards a solution of the following open problems:

(?) Two at least of the three numbers e, e^e, e^{e^2} are algebraically independent.

(?) Two at least of the three numbers π, e, e^{π^2} are algebraically independent.

Further conjectures are as follows:

(?) Each of the numbers e^e, e^{e^2} e^{π^2} is transcendental

(?) The numbers e and π are algebraically independent.

We conclude this note by showing how stronger statements are consequences of Schanuel's conjecture.

1. Heights

Let γ be a complex algebraic number. The *minimal polynomial* of γ over $\mathbb{Z}$ is the unique polynomial

$$f(X) = a_0X^d + a_1X^{d-1} + ... + a_d \epsilon \mathbb{Z}[X]$$

which vanishes at the point γ, is irreducible in the factorial ring $\mathbb{Z}[X]$ and has leading coefficient $a_0 > 0$. The integer $d = \deg f$ is the *degree* of γ, denoted by $[Q(\gamma) : Q]$. The *usual height* $H(\gamma)$ of γ is defined by

$$H(\gamma) = \max\{|a_0|, ..., |a_d|\}.$$

It will be convenient to use also the so-called *Mahler's measure* of γ, which can be defined in three equivalent ways. The first one is

$$M(\gamma) = \exp\left(\int_0^1 \log|f(e^{2i\pi t})|dt\right).$$

For the second one, let $\gamma_1, ...\gamma_d$ denote the complex roots of f, so that

$$f(X) = a_0 \prod_{i=1}^{d}(X - \gamma_i).$$

Then, according to Jensen's formula, we have

$$M(\gamma) = |a_0| \prod_{i=1}^{d} \max\{1, |\gamma_i|\}.$$

For the third one, let K be a number field (that is a subfield of $\mathbb{C}$which is a $\mathbb{Q}$-vector space of finite dimension [K:$\mathbb{Q}$]) containing γ, and let M_K be the set of (normalized) absolute

values of K. Then

$$M(\gamma) = \prod_{v \in M_K} \max\{1, |\gamma|_v\}^{[K_v:\mathbb{Q}]}$$

where K_v is the completion of K for the absolute value v and Q_v the topological closure of $\mathbb{Q}$ in K_v and $[K_v : \mathbb{Q}_v]$ the local degree.

Mahler's measure is related to the usual height by

$$2^{-d}H(\gamma) \leq M(\gamma) \leq \sqrt{d+1}.H(\gamma).$$

From this point of view it does not make too much difference to use H or M, but one should be careful that d denotes the exact degree of γ, not an upper bound. We shall deal below with algebraic numbers of degree d bounded by some parameter D.

Definition. For an algebraic number γ of degree d and Mahler's measure $M(\gamma)$, we define the *absolute logarithmic height* $h(\gamma)$ by

$$h(\gamma) = \frac{1}{d} \log M(\gamma).$$

2.Simultaneous Approximation

We state two results dealing with simultaneous Diophantine approximation. Both of them are consequences of the main result in [W 2]. Details of the proof will appear in the forthcoming book [W 3].

2.1. Simultaneous Approximation to e, e^e and e^{e^2}

Theorem 1. *There exists a positive absolute constant c_1 such that, if $\gamma_0, \gamma_1, \gamma_2$ are algebraic numbers in a field of degree D, then*

$$|e-\gamma_0| + |e^e - \gamma_1| + |e^{e^2} - \gamma_2| > \exp\{-c_1 D^2 (h_0+h_1+h_2)^{1/2}(h_1+h_2)^{1/2}(h_0+\log D)(\log D)^{-1}\}$$

where $h_i = \max\{e, h(\gamma_i)\}$ $(i = 0, 1, 2)$.

2.2. Simultaneous Approximation to π, e and e^{π^2}

Theorem 2. *There exists a positive absolute constant c_2 such that, if $\gamma_0, \gamma_1, \gamma_2$ are algebraic numbers in a field of degree D, then*

$$|\pi - \gamma_0| + |e^e - \gamma_1| + e^{e^2} - \gamma_2| > \exp\{-c_2 D^2 (h_0 + \log(Dh_1\ h_2))^{\frac{1}{2}} h_1^{\frac{1}{2}} h_2^{\frac{1}{2}} (\log\ D)^{-1}$$

where $h_i = \max\{e, h(\gamma_i)\}$ $(i = 0, 1, 2)$.

3. Transcendence Criterion

3.1 Algebraic Approximations to a Given Transcendental Number

The following result is Théoréme 3.2 of [R-W 1]; see also Theorem 1.1 of [R-W 2].

Theorem 3. *Let $\theta \in C$ be a complex number. The two following conditions are equivalent:*

(i) the number θ is transcendental.

(ii) For any real number $h \geq 10^7$, there are infinitely many integers $d \geq 1$ for which there exists an algebraic number γ of degree d and absolute logarithmic height $h(\gamma) \leq h$ which satisfies

$$0 < |\theta - \gamma| \leq \exp(-10^{-7} h d^2).$$

Notice that the proof of $(ii) \Rightarrow (i)$ is an easy consequence of Liouville's inequality.

3.2. Application to e^e and e^{e^2}

Corollary to Theorem 1. *One at least of the two numbers e^e, e^{e^2} is transcendental.*

Proof of the corollary. Assume that the two numbers e^e e^{e^2} are algebraic, say γ_1 and γ_2. Then, according to Theorem 1, there exists a constant $c_3 > 1$ such that, for any algebraic number γ of degree $\leq D$ and height $h(\gamma) \leq h$ with $h \geq e$,

$$|e - \gamma| > \exp\{-c_3 D^2 h^{1/2} (h + \log D)(\log D)^{-1}\}.$$

We now use Theorem 3 for $\theta = e$ with $h = 10^{15} c_3^2$ and derive a contradiction.

4. Algebraic Independence

4.1 Simultaneous Approximation

The proof of the following result is given in [R-W 1], Théoréme 3.1, as a consequence of Theorem 3 (see also [R-W 2] Corollary 1.2).

Corollary to Theorem 3. *Let $\theta_1, ..., \theta_m$ be complex numbers such that the field $Q(\theta_1, .., \theta_m)$ has tyranscendence degree 1 over Q. There exists a constant $c > 0$ such that, for any real number $h \geq c$, there are infinitely many integers D for which there exists a tuple $(\gamma_1, ..., \gamma_m)$ of algebraic numbers satisfying*

$$[Q(\gamma_1, ..., \gamma_m) : Q] \leq D, \quad \max_{1 \leq i \leq m} h(\gamma_i) \leq h$$

and

$$\max_{1 \leq i \leq m} \{|\theta_i - \gamma_i|\} \leq \exp(-c^{-1} h D^2).$$

4.2. Application to π, e and e^{π^2}

Corollary to Theorem 2. *One at least of the two following statements is true:*
(i) The numbers e and π are algebraiclly independent.
(ii) The number e^{π^2} is transcendental.

Remark. This corollary can be stated in an equivalent way as follows:

For any non constant polynomial $P \epsilon Z[X]$, the complex number

$$e^{\pi^2} + iP(e, \pi)$$

is transcendental.

The idea behind this remark originates in [R].

Proof of the Corollary. Assume that the number e^{π^2} is algebraic. Theorem 2 with $\gamma_2 = e^{\pi^2}$ shows that there exists a constant $c_4 > 0$ such that, for any pair (γ_0, γ_1) of algebraic numbers, if we set

$$D = [Q(\gamma_0, \gamma_1) : Q] \quad \textit{and} \quad h = \max\{e, h(\gamma_0), h(\gamma_1)\},$$

then

$$|\pi - \gamma_0| + |e - \gamma_1| > \exp\{-c_4 D^2 (h + \log D)^{1/2} h^{1/2} (\log D)^{-1}\}$$

Therefore we dedeuce from the Corollary to Theorem 3 that the field $Q(\pi, e)$ has transcendence degree 2.

5. Schanuel's Conjecture

The following conjecture is stated in [L] Chap. III p. 30: (The resulsts of this section are based on the conjecture to be stated).

Schanuel's Conjecture. *Let $x_1, ..., x_n$ be Q-linearly independent complex numbers. Then, among the $2n$ numbers*

$$x_1, ..., x_n, e^{x_1}, ..., e^{x_n},$$

at least n are algebraically independent.

Let us deduce from Schanuel's Conjecture the following statement (which is an open problem):

(?) The 7 numbers

$$e,\ \pi,\ e^e,\ e^{e^2},\ e^{\pi^2},\ 2^{\sqrt[3]{2}},\ 2^{\sqrt[3]{2}}$$

are algebraically independent.

We shall use Schanuel's cinjecture twice. We start with the numbers $1, \log 2$, and $i\pi$ which are linearly independent over Q because $\log 2$ is irrational. Therefore, according to Schanuel's conjecture, three at least of the numbers

$$1,\ \log\ 2,\ i\pi,\ \ e,\ 2,\ -1$$

are algebraically independent. This means that the three numbers $\log 2, \pi$ and e are algebraically independent. Therefore the 8 numbers

$$1,\ i\pi,\ \pi^2,\ e,\ e^2,\ \log 2,\ 2^{\frac{1}{3}} \log\ 2,\ 4^{\frac{1}{3}} \log\ 2$$

are Q-linearly independent. Again, Schanuel's conjecture implies that 8 at least of the numbers

$$1,\ i\pi,\ \pi^2,\ e,\ e^2,\ \log^2,\ 2^{\frac{1}{3}} \log\ 2,\ 4^{\frac{1}{3}} \log\ 2\ e,\ -1,\ e^{\pi^2},\ e^e,\ e^{e^2},\ 2, 2^{\sqrt[3]{2}},\ 2^{\sqrt[3]{4}}$$

are algebraically independent, and this means that the 8 numbers

$$e,\ \pi,\ e^e,\ e^{e^2},\ e^{\pi^2},\ 2^{\sqrt[3]{2}},\ 2^{\sqrt[3]{4}},\ \log 2$$

are algebraically independent.

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REFERENCES

[B] Brownawell, W.Dale. - *The algebraic independence of certain numbers related to the exponential function.* J.Number Th. 6 (1974), 22-31.

[G] Gel'fond, Aleksandr O, - *Transcendental and algebraic numbers.* Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1952. 224 pp. *Transcendental and algebraic numbers.* Translated from the first Russian edition by Leo F. boron Dover Publications, Inc., New York 1960 vii+190 pp.

[L] Lang, Serge. - *Introduction to transcendental numbers.* Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont. 1966 vi+105 pp.

[R] Ramachandra, K. - *Contributions to the theory of transcendental numbers.*I, II. Acta Arith. **14** (1967/68), 65-72; ibid., **14** (1967/1968), 73-88.

[R-W 1] Roy, Damien; Waldschmidt, Michel. - *Approximation diophantienne et indépendance algébrique de logarithmes.* Ann. Scient. Ec. Norm. Sup., **30** (1997), no 6, 753-796.

[R-W 2] Roy, Damien; Waldschmidt, Michel. - *Simultaneous approximation and algebraic independence.* The Ramanujan Journal, **1** Fasc. 4(1997), 379-430.

[S] Schneider, Theodor,- *Einführung in die transzendenten Zahlen.* Springer-Verlag, berlin-Gottingen-Heidelberg, 1957, v+150 pp. *Introduction aux nombres transcendants.* Traduit del'allemand par P. Eymard. Gauthier-Villars, paris 1959 viii+151 pp.

[W 1] Waldschmidt, Michel. - *Solution du huitiéme probléme de Schneider.* J.Number Theory, **5** (1973), 191-202.

[W 2] Waldschmidt, Michel. - *Approximation diophantienne dans les groupes algébriques commutatifs - (I) : Une version effective du theoreme du sous-groupe algébrique.* J. reine angew. Math. **493** (1997), 61-113.

[W 3] Waldschmidt, Michel. - *Diophantine Approximation on Linear Algebraic Groups. Transcendence Properties of the Exponential Function in Several Variables.* In preparation.

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