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On the numbers $e^e$, $e^{e^2}$ and $e^{\pi^2}$

by

Michel WALDSCHMIDT

Abstract. We give two measures of simultaneous approximation by algebraic numbers, the first one for the triple $(e, e^e, e^{e^2})$ and the second one for $(\pi, e, e^{\pi^2})$. We deduce from these measures two transcendence results which had been proved in the early 70's by W.D. Brownawell and the author.

Introduction

In 1949, A.O. Gel'fond introduced a new method for algebraic independence, which enabled him to prove that the two numbers $2^{\sqrt{2}}$ and $2^{\sqrt{3}}$ are algebraically independent. At the same time, he proved that one at least of the three numbers $e^e$, $e^{e^2}$, $e^{e^3}$ is transcendental (see [G] Chap. III, §4).

At the end of his book [S] on transcendental numbers, Th. Schneider suggested that one at least of the two numbers $e^e$, $e^{e^2}$ is transcendental; this was the last of a list of eight problems, and the first to be solved, in 1973, by W.D. Brownawell [B] and M. Waldschmidt [W 1], independently and simultaneously. For this result they shared the Distinguished Award of the Hardy-Ramanujan Society in 1986. Another consequence of their main result is that one at least of the two following statements holds true:

(i) The numbers $e$ and $\pi$ are algebraically independent
(ii) The number $e^{\pi^2}$ is transcendental

Our goal is to shed a new light on these results. It is hoped that our approach will yield further progress towards a solution of the following open problems:
Two at least of the three numbers $e, e^e, e^{e^2}$ are algebraically independent.

Two at least of the three numbers $\pi, e, e^{e^2}$ are algebraically independent.

Further conjectures are as follows:

1. Each of the numbers $e^e, e^{e^2} e^{e^2}$ is transcendental
2. The numbers $e$ and $\pi$ are algebraically independent.

We conclude this note by showing how stronger statements are consequences of Schanuel's conjecture.

1. Heights

Let $\gamma$ be a complex algebraic number. The minimal polynomial of $\gamma$ over $\mathbb{Z}$ is the unique polynomial

$$f(X) = a_0 X^d + a_1 X^{d-1} + \ldots + a_d \in \mathbb{Z}[X]$$

which vanishes at the point $\gamma$, is irreducible in the factorial ring $\mathbb{Z}[X]$ and has leading coefficient $a_0 > 0$. The integer $d = \deg f$ is the degree of $\gamma$, denoted by $[\mathbb{Q}(\gamma) : \mathbb{Q}]$. The usual height $H(\gamma)$ of $\gamma$ is defined by

$$H(\gamma) = \max\{|a_0|, \ldots, |a_d|\}.$$ 

It will be convenient to use also the so-called Mahler's measure of $\gamma$, which can be defined in three equivalent ways. The first one is

$$M(\gamma) = \exp \left( \int_0^1 \log |f(e^{2\pi it})| dt \right).$$

For the second one, let $\gamma_1, \ldots, \gamma_d$ denote the complex roots of $f$, so that

$$f(X) = a_0 \prod_{i=1}^d (X - \gamma_i).$$

Then, according to Jensen's formula, we have

$$M(\gamma) = |a_0| \prod_{i=1}^d \max\{1, |\gamma_i|\}.$$ 

For the third one, let $K$ be a number field (that is a subfield of $\mathbb{C}$ which is a $\mathbb{Q}$-vector space of finite dimension $[K: \mathbb{Q}]$) containing $\gamma$, and let $M_K$ be the set of (normalized) absolute
values of \( K \). Then

\[
M(\gamma) = \prod_{v \in M_K} \max\{1, |\gamma_v|^{[K_v:Q_v]}\}
\]

where \( K_v \) is the completion of \( K \) for the absolute value \( v \) and \( Q_v \) the topological closure of \( Q \) in \( K_v \) and \( [K_v:Q_v] \) the local degree.

Mahler's measure is related to the usual height by

\[
2^{-d}H(\gamma) \leq M(\gamma) \leq \sqrt{d+1}.H(\gamma).
\]

From this point of view it does not make too much difference to use \( H \) or \( M \), but one should be careful that \( d \) denotes the exact degree of \( \gamma \), not an upper bound. We shall deal below with algebraic numbers of degree \( d \) bounded by some parameter \( D \).

Definition. For an algebraic number \( \gamma \) of degree \( d \) and Mahler's measure \( M(\gamma) \), we define the absolute logarithmic height \( h(\gamma) \) by

\[
h(\gamma) = \frac{1}{d} \log M(\gamma).
\]

\section*{2. Simultaneous Approximation}

We state two results dealing with simultaneous Diophantine approximation. Both of them are consequences of the main result in [W 2]. Details of the proof will appear in the forthcoming book [W 3].

\subsection*{2.1. Simultaneous Approximation to \( e, e^e \) and \( e^{e^2} \)}

\textbf{Theorem 1.} There exists a positive absolute constant \( c_1 \) such that, if \( \gamma_0, \gamma_1, \gamma_2 \) are algebraic numbers in a field of degree \( D \), then

\[
|e - \gamma_0| + |e^e - \gamma_1| + |e^{e^2} - \gamma_2| > \exp\{-c_1 D^2(h_0 + h_1 + h_2)^{1/2}(h_1 + h_2)^{1/2}(h_0 + \log D)(\log D)^{-1}\}
\]

where \( h_i = \max\{e, h(\gamma_i)\} \) (\( i = 0, 1, 2 \)).
2.2. Simultaneous Approximation to \( \pi, e \) and \( e^{x^2} \)

**Theorem 2.** There exists a positive absolute constant \( c_2 \) such that, if \( \gamma_0, \gamma_1, \gamma_2 \) are algebraic numbers in a field of degree \( D \), then

\[
|\pi - \gamma_0| + |e^x - \gamma_1| + e^{x^2} - \gamma_2| > \exp\{-c_2D^2(h_0 + \log(Dh_1h_2))^{\frac{1}{2}}h_1^{\frac{3}{4}}h_2^{\frac{1}{4}}(\log D)^{-1}\}
\]

where \( h_i = \max\{e, h(\gamma_i)\} \) (\( i = 0, 1, 2 \)).

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3. Transcendence Criterion

3.1 Algebraic Approximations to a Given Transcendental Number

The following result is Théorème 3.2 of [R-W 1]; see also Theorem 1.1 of [R-W 2].

**Theorem 3.** Let \( \theta \in \mathbb{C} \) be a complex number. The two following conditions are equivalent:

(i) the number \( \theta \) is transcendental.

(ii) For any real number \( h \geq 10^7 \), there are infinitely many integers \( d \geq 1 \) for which there exists an algebraic number \( \gamma \) of degree \( d \) and absolute logarithmic height \( h(\gamma) \leq h \) which satisfies

\[
0 < |\theta - \gamma| \leq \exp(-10^{-7}hd^2).
\]

Notice that the proof of (ii) \( \Rightarrow \) (i) is an easy consequence of Liouville's inequality.

3.2. Application to \( e^x \) and \( e^{x^2} \)

**Corollary to Theorem 1.** One at least of the two numbers \( e^x, e^{x^2} \) is transcendental.

**Proof of the corollary.** Assume that the two numbers \( e^x e^{x^2} \) are algebraic, say \( \gamma_1 \) and \( \gamma_2 \). Then, according to Theorem 1, there exists a constant \( c_3 > 1 \) such that, for any algebraic number \( \gamma \) of degree \( \leq D \) and height \( h(\gamma) \leq h \) with \( h \geq e \),

\[
|e - \gamma| > \exp\{-c_3D^2h^{1/2}(h + \log D)(\log D)^{-1}\}.
\]

We now use Theorem 3 for \( \theta = e \) with \( h = 10^{15}c_3^2 \) and derive a contradiction.
4. Algebraic Independence

4.1 Simultaneous Approximation

The proof of the following result is given in [R-W 1], Théoréme 3.1, as a consequence of Theorem 3 (see also [R-W 2] Corollary 1.2).

Corollary to Theorem 3. Let \( \theta_1, \ldots, \theta_m \) be complex numbers such that the field \( \mathbb{Q}(\theta_1, \ldots, \theta_m) \) has transcendence degree 1 over \( \mathbb{Q} \). There exists a constant \( c > 0 \) such that, for any real number \( h \geq c \), there are infinitely many integers \( D \) for which there exists a tuple \( (\gamma_1, \ldots, \gamma_m) \) of algebraic numbers satisfying

\[
[Q(\gamma_1, \ldots, \gamma_m) : \mathbb{Q}] \leq D, \quad \max_{1 \leq i \leq m} h(\gamma_i) \leq h
\]

and

\[
\max_{1 \leq i \leq m} \{|\theta_i - \gamma_i|\} \leq \exp(-c^{-1}hD^2).
\]

4.2. Application to \( \pi, e \) and \( e^{\pi^2} \)

Corollary to Theorem 2. One at least of the two following statements is true:

(i) The numbers \( e \) and \( \pi \) are algebraically independent.

(ii) The number \( e^{\pi^2} \) is transcendental.

Remark. This corollary can be stated in an equivalent way as follows:

For any non constant polynomial \( P \in \mathbb{C}[X] \), the complex number

\[
e^{\pi^2} + iP(e, \pi)
\]

is transcendental.

The idea behind this remark originates in [R].

Proof of the Corollary. Assume that the number \( e^{\pi^2} \) is algebraic. Theorem 2 with \( \gamma_2 = e^{\pi^2} \) shows that there exists a constant \( c_4 > 0 \) such that, for any pair \( (\gamma_0, \gamma_1) \) of algebraic numbers, if we set

\[
D = [Q(\gamma_0, \gamma_1) : \mathbb{Q}] \quad \text{and} \quad h = \max\{e, h(\gamma_0), h(\gamma_1)\},
\]

then

\[
|\pi - \gamma_0| + |e - \gamma_1| > \exp\{-c_4D^2(h + \log D)^{1/2}h^{1/2}(\log D)^{-1}\}
\]
Therefore we deduce from the Corollary to Theorem 3 that the field \( \mathbb{Q}(\pi, e) \) has transcendence degree 2.

## 5. Schanuel's Conjecture

The following conjecture is stated in [L] Chap. III p. 30: (The results of this section are based on the conjecture to be stated).

**Schanuel's Conjecture.** Let \( x_1, \ldots, x_n \) be \( \mathbb{Q} \)-linearly independent complex numbers. Then, among the \( 2n \) numbers

\[
x_1, \ldots, x_n, e^{x_1}, \ldots, e^{x_n},
\]

at least \( n \) are algebraically independent.

Let us deduce from Schanuel's Conjecture the following statement (which is an open problem):

(?) The 7 numbers

\[
e, \pi, e^e, e^{e^e}, 2^{\sqrt{2}}, 2^{\sqrt[3]{2}}
\]

are algebraically independent.

We shall use Schanuel’s conjecture twice. We start with the numbers 1, \( \log 2 \), and \( i\pi \) which are linearly independent over \( \mathbb{Q} \) because \( \log 2 \) is irrational. Therefore, according to Schanuel’s conjecture, three at least of the numbers

\[
1, \log 2, i\pi, e, 2, -1
\]

are algebraically independent. This means that the three numbers \( \log 2, \pi \) and \( e \) are algebraically independent. Therefore the 8 numbers

\[
1, i\pi, \pi^2, e, e^2, \log 2, 2^{\frac{1}{2}} \log 2, 4^{\frac{1}{2}} \log 2
\]

are \( \mathbb{Q} \)-linearly independent. Again, Schanuel’s conjecture implies that 8 at least of the numbers

\[
1, i\pi, \pi^2, e, e^2, \log 2, 2^{\frac{1}{2}} \log 2, 4^{\frac{1}{2}} \log 2, -1, e^{e^2}, e^{e^e}, 2, 2^{\sqrt{2}}, 2^{\sqrt[3]{2}}
\]
are algebraically independent, and this means that the 8 numbers
\[ e, \pi, e^e, e^{e^e}, 2^{\sqrt{2}}, 2^{\sqrt{3}}, \log 2 \]
are algebraically independent.

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