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K Ramachandra

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ON THE ZEROS OF A CLASS OF GENERALISED  
DIRICHLET SERIES-XIX

BY

K. RAMACHANDRA

§ 1. **INTRODUCTION AND PRELIMINARIES.** The object of this paper is to give a simpler approach to the following theorem (this theorem namely Theorem 10 of XVIII<sup>[5]</sup> constitutes the main theorem on Balasubramanian-Ramachandra functions arising out of the works III<sup>[1]</sup>, IV<sup>[2]</sup>, V<sup>[6]</sup>, VI<sup>[3]</sup> due to R. Balasubramanian and K. Ramachandra).

**THEOREM 1.** (R. BALASUBRAMANIAN AND K. RAMACHANDRA)

(i) Let  $\lambda_n (n = 1, 2, 3, \dots)$  be an increasing sequence of positive real numbers such that  $\lambda_{n+1} - \lambda_n$  is bounded both above and below. This sequence will be further restricted by the condition (vii) or (viii) as the case may be.  $\theta$  will denote a real constant.

Let  $f(x)$  and  $g(x)$  be positive real valued functions defined in  $x \geq 0$ , satisfying

(ii)  $f(x)x^\eta$  is monotonic increasing and  $f(x)x^{-\eta}$  is monotonic decreasing for every fixed  $\eta > 0$  and all  $x \geq x_0(\eta)$ .

(iii)  $\lim_{x \rightarrow \infty} (g(x)x^{-1}) = 1$ .

(iv)  $g(x)$  is differentiable once for  $x \geq 0$  and  $g'(x)$  lies between two positive constants. Also  $g(x)$  is twice differentiable for  $x \geq x_0$  and  $(g'(x))^2 - g(x)g''(x)$  lies between two positive constants for  $x \geq x_0$ .

Let  $\{a_n\}$  and  $\{b_n\}$  ( $n = 1, 2, 3, \dots$ ) be two sequences of complex numbers having the following properties

(v)  $|b_n| (f(n))^{-1}$  lies between two positive constants (for all integers  $n \geq n_0$ ) and  $(\sum_{n \leq x} |a_n|^2) x^{-1}$  does not exceed a positive constant for all  $x \geq 1$ .

(vi) For all  $x \geq 1$ ,  $\sum_{x \leq n \leq 2x} |b_{n+1} - b_n| \ll f(x)$ .

We next assume that  $\{a_n\}$  and  $\{b_n\}$  satisfy at least one of the following conditions. We set  $\lambda_n = g(n)$ .

(vii) **MONOTONICITY CONDITION.** There exists an arithmetic progression  $\mathcal{A}$  (of natural numbers) such that

$$\lim_{x \rightarrow \infty} \left( x^{-1} \sum'_{n \leq x} a_n \right) = h \quad (0 < |h| < \infty),$$

where the accent denotes the restriction of  $n$  to  $\mathcal{A}$ . Also for every positive constant  $\nu$  we have  $|b_n| \lambda_n^{-\nu}$  is monotonic decreasing for all  $n (\geq n_0)$  in  $\mathcal{A}$ .

(viii) **REAL PART CONDITION.** There exists an arithmetic progression  $\mathcal{A}$  (of natural numbers) such that

$$\liminf_{x \rightarrow \infty} \left( \frac{1}{x} \sum'_{x \leq \lambda_n \leq 2x, \operatorname{Re} a_n > 0} \operatorname{Re} a_n \right) > 0$$

and

$$\lim_{x \rightarrow \infty} \left( x^{-1} \sum'_{x \leq \lambda_n \leq 2x, \operatorname{Re} a_n < 0} \operatorname{Re} a_n \right) = 0$$

where the accent denotes the restriction of  $n$  to  $\mathcal{A}$ .

(ix) Finally let  $\{\alpha_n\}$  ( $n = 1, 2, 3, \dots$ ) be a sequence of real numbers such that  $|\alpha_n|$  does not exceed a small positive constant (depending on other constants). We suppose that the series

$$F(s) = \sum_{n=1}^{\infty} a_n b_n e^{2\pi i n \theta} (\lambda_n + \alpha_n)^{-s} \quad (\operatorname{Re} s \geq 2)$$

can be continued analytically in  $(\sigma \geq \frac{1}{2} - \delta, T - \log T \leq t \leq 2T + \log T)$  (where  $\delta$  is a positive constant) and there  $\log \max(|F(s)| + 100) \ll \log T$ . As usual we have written  $s = \sigma + it$ .

Then on every line segment  $(\sigma = \frac{1}{2} - \delta_4, T \leq t \leq 2T)$ , ( $\delta_4$  being any constant with  $0 < \delta_4 < \delta$ ) there are  $\gg T(\log \log T)^{-1}$  well-spaced Titchmarsh points with the lower bound  $\gg T^{\delta_4} f(T)$  for  $|F(s)|$ . If further

$$\frac{1}{T} \int_{T-\sqrt{\log T}}^{2T+\sqrt{\log T}} |F(\frac{1}{2} - \delta_4 + it)|^2 dt \ll T^{2\delta_4} (f(T))^2$$

for every constant  $\delta_4$  (with  $0 < \delta_4 \leq \delta$ ), then there are  $\gg T$  well-spaced Titchmarsh points on every line segment  $(\sigma = \frac{1}{2} - \delta_4, T \leq t \leq 2T)$  with the lower bound  $\gg T^{\delta_4} f(T)$  for  $|F(s)|$ .

In other words there exist real numbers  $t_1, t_2, \dots, t_r$  (with  $r \gg T(\log \log T)^{-1}$  and  $r \gg T$  respectively in the two cases) such that  $T \leq t_j \leq 2T$  ( $j = 1, 2, \dots, r$ ), the minimum of  $|t_j - t_{j'}|$  taken over all pairs  $(j, j')$  with  $j \neq j'$  is bounded below and further

$$|F(\frac{1}{2} - \delta_4 + it_j)| \gg T^{\delta_4} f(T).$$

The proof of this theorem depends on the following two lemmas.

**LEMMA 1** (van-der-CORPUT). If  $f_1(x)$  is real and twice differentiable and  $0 < \mu_2 \leq f_1''(x) \leq h'\mu_2$  (or  $\mu_2 \leq -f_1''(x) \leq h'\mu_2$ ) throughout the interval  $[a, b]$ , and  $b \geq a + 1$ , then

$$\sum_{a < n \leq b} \text{Exp}(2\pi i f_1(n)) = O(h'(b-a)\mu_2^{\frac{1}{2}}) + O(\mu_2^{-\frac{1}{2}}).$$

**REMARK.** This result is Theorem 5.9 on page 104 of [8], with a slight change of notation.

**LEMMA 2** (H.L. MONTGOMERY AND R.C. VAUGHAN). If  $\{\lambda_n\}$  is any increasing sequence of real numbers and  $\{A_n\}$  and  $\{B_n\}$  are any two sequences of complex numbers, then

$$|\sum_{m \neq n} \sum \frac{A_m \bar{B}_n}{\log(\lambda_m \lambda_n^{-1})}| \leq K (\sum \delta_n^{-1} |A_n|^2)^{\frac{1}{2}} (\sum \delta_n^{-1} |B_n|^2)^{\frac{1}{2}},$$

where  $\delta_n = \min_{m \neq n} |\lambda_n - \lambda_m|$  and  $K$  is a numerical constant.

**REMARK.** We need only a special case of this result where  $\lambda_{n+1} - \lambda_n$  lies between two positive constants (and so the same is true of  $n\delta_n$ ). For the proof in this special case and also for a reference to the paper of Montgomery and Vaughan see [7].

## § 2. SOME MORE LEMMAS.

**LEMMA 3.** Let  $y > 0$ ,  $w = u + iv$ ,  $R(w) = \text{Exp}((\text{Sin } \frac{w}{100})^2)$ , and

$$\Delta(y) = \frac{1}{2\pi i} \int_{u=2} y^w R(w) \frac{dw}{w}.$$

Then for  $|u| \leq 3$  we have  $|R(w)| \ll (\text{Exp } \text{Exp } |\frac{v}{100}|)^{-1}$ . Consequently  $\Delta(y) = 1 + O(y^{-2})$  and also  $\Delta(y) = O(y^2)$ .

**PROOF.** By trivial computation (and moving the line of integration to  $u = -2$  and  $u = 2$  respectively).

$A$  will denote the arithmetic progression consisting of an infinite subset of natural numbers. Let  $\lambda(0 < \lambda < 1)$  be a constant. We put  $X = T\lambda$  (later we will choose  $\lambda$  to be a small constant).  $S$  will denote the set  $A \cap [\frac{1}{2}X, X]$ . All our  $O$ -constants and the constants implied by the Vinogradov symbols  $\gg$  and  $\ll$  will be independent of  $\lambda$ .

**LEMMA 4.** For  $T \leq t \leq 2T$ , we have,

$$\left| \sum_{n \in S} \text{Exp}(-2\pi i n \theta + it \log g(n)) \right| \ll T^{\frac{1}{2}}.$$

**PROOF.** Noting that the second derivative of  $-2\pi x \theta + t \log g(x)$  is  $t((g'(x))^2 - g(x)g''(x))(g(x))^{-2}$  the lemma follows by Lemma 1.

**LEMMA 5.** For  $T \leq t \leq 2T$ , we have

$$\left| \sum_{n \in S} \bar{b}_n \text{Exp}(-2\pi i n \theta + it \log g(n)) \right| \ll T^{\frac{1}{2}} f(X).$$

**PROOF.** The proof follows by partial summation (from Lemma 4) on using

$$\sum_{x \leq n \leq 2x} |b_{n+1} - b_n| \ll f(x) \text{ for all } x \geq 1.$$

Next we put  $g(n) = \lambda_n$ . We consider the case  $\alpha_n \equiv 0$  first. Our object is to obtain a good lower bound for the LHS of (2) below.

**LEMMA 6.** For  $s = \frac{1}{2} - \delta + it, T \leq t \leq 2T$ , put

$$F_X(s) = \sum_{n=1}^{\infty} a_n b_n \text{Exp}(2\pi i n \theta) \lambda_n^{-s} \Delta\left(\frac{X}{\lambda_n}\right). \quad (1)$$

Then

$$\frac{1}{T} \int_T^{2T} |F_X(s)| dt \gg (T^{\frac{3}{2}} f(X))^{-1} |I|, \quad (2)$$

where

$$I = \int_T^{2T} F_X(s) \sum_{n \in S} \bar{b}_n \text{Exp}(-2\pi i n \theta + it \log \lambda_n) dt. \quad (3)$$

**PROOF.** Follows from Lemma 5.

**LEMMA 7.** We have,

$$I = T \sum_{n \in S} a_n |b_n|^2 \lambda_n^{-\frac{1}{2} + \delta} + O(J) \quad (4)$$

where

$$J = (J_1 J_2)^{\frac{1}{2}}, J_1 = \sum_{n=1}^{\infty} |a_n b_n|^2 n^{2\delta} \left(\Delta\left(\frac{X}{\lambda_n}\right)\right)^2 \quad (5)$$

and

$$J_2 = \sum_{n \in [\frac{1}{2}X, X]} n |b_n|^2. \quad (6)$$

**PROOF.** Follows from Lemma 2.

**LEMMA 8.** We have

$$J_1 = O(X^{1+2\delta} (f(X))^2) \quad (7)$$

and

$$J_2 = O(X^2 (f(X))^2). \quad (8)$$

**PROOF.** Follows from  $\Delta(y) = 1 + O(y^{-2}) = O(y^2)$  and also from (ii) of Theorem 1.

**LEMMA 9.** *Let*

$$\sum_0 = \sum_{n \in S} a_n |b_n|^2 \lambda_n^{-\frac{1}{2} + \delta} \Delta\left(\frac{X}{\lambda_n}\right). \quad (9)$$

*Then*

$$I = T \sum_0 + O(X^{\frac{3}{2} + \delta} (f(X))^2) \quad (10)$$

**PROOF.** Follows from Lemmas 7 and 8.

**LEMMA 10.** *Under monotonicity condition, we have,*

$$|\sum_0| \gg |h| X^{\frac{1}{2} + \delta} (f(X))^2. \quad (11)$$

**PROOF.** We write

$$\frac{1}{x} \sum_{n \in \mathcal{A}, n \leq x} a_n = h + \varepsilon_x$$

where  $\varepsilon_x \rightarrow 0$  as  $x \rightarrow \infty$ . We obtain the result by the monotonicity of  $|b_n|^2 \lambda_n^{-\frac{1}{4} + \frac{1}{2}\delta}$ .

**LEMMA 11.** *Under the real part condition, we have,*

$$Re \sum_0 \gg X^{\frac{1}{2} + \delta} (f(X))^2 \quad (12)$$

**PROOF.** Follows since the contribution from those  $a_n$  with  $Re a_n < 0$  is of a smaller order.

**LEMMA 12.** *We have*

$$|I| > C_1 T (f(X))^2 X^{\frac{1}{2} + \delta} - C_2 (f(X))^2 X^{\frac{3}{2} + \delta} \quad (13)$$

where  $C_1$  and  $C_2$  are positive constants independent of  $\lambda$ .

**PROOF.** Follows from Lemma 7 to 11.

**LEMMA 13.** *We have, with  $s = \frac{1}{2} - \delta + it$ ,  $X = T\lambda$ , where  $\lambda(> 0)$  is some fixed small constant, the inequality*

$$\frac{1}{T} \int_T^{2T} |F_X(s)| dt \gg T^\delta f(T). \quad (14)$$

**PROOF.** RHS of (13) is

$$\left(C_1 T^{\frac{3}{2}+\delta} \lambda^{\frac{1}{2}+\delta} - C_2 T^{\frac{3}{2}} \lambda^{\frac{3}{2}+\delta}\right) (f(X))^2.$$

Using (ii) it follows that  $f(X)X^{-1} \geq f(T)T^{-1}$  and so  $f(X) \geq \lambda f(T)$ . Lemma 13 follows on fixing  $\lambda$  to be a small positive constant.

**LEMMA 14.** *Let now  $Y = T\lambda'$  where  $\lambda'$  ( $0 < \lambda' < \lambda$ ) is a small constant. We have*

$$\frac{1}{T} \int_T^{2T} |F_Y(s)| dt \leq \eta_0 T^\delta f(T), \tag{15}$$

where  $\eta_0$  depends on  $\lambda'$  and is small enough if  $\lambda'$  is small.

**PROOF.** Note that

$$\frac{1}{T} \int_T^{2T} |F_Y(s)|^2 dt \ll Y^{2\delta} (f(Y))^2$$

and that here RHS is  $\leq Y^\delta (Y^{\frac{1}{2}\delta} f(Y))^2 \leq (\lambda')^\delta T^{2\delta} (f(T))^2$ . Lemma 14 follows from this on using Hölder's inequality.

**LEMMA 15.** *We have, with  $X = T\lambda, Y = T\lambda'$  where  $\lambda$  is as before and  $\lambda'$  ( $0 < \lambda' < \lambda$ ) is fixed to be a sufficiently small constant, the inequality*

$$\frac{1}{T} \int_T^{2T} |F_X(s) - F_Y(s)| dt \gg T^\delta f(T). \tag{16}$$

**PROOF.** Follows from Lemma 13 and 14.

From now on we fix the positive constants  $\lambda$  and  $\lambda'$  so that (16) is satisfied.

**LEMMA 16.** *Now let  $\alpha_n$  be real and let  $|\alpha_n|$  be bounded above by a small positive constant. Then with  $s = \frac{1}{2} - \delta + it, X = T\lambda, Y = T\lambda'$  we have*

$$\frac{1}{T} \int_T^{2T} \left| \sum_{n=1}^{\infty} a_n b_n \text{Exp}(2\pi in\theta) (\lambda_n + \alpha_n)^{-s} \right. \\ \left. \left( \Delta \left( \frac{X}{\lambda_n + \alpha_n} \right) - \Delta \left( \frac{Y}{\lambda_n + \alpha_n} \right) \right) \right| dt \gg T^\delta f(T). \tag{17}$$



**PROOF.** We split the infinite series on the LHS of (17) to be  $\sum_1$  with  $n \leq T\lambda''$  (where  $\lambda''(> 0)$  is a small constant) and  $\sum_2$  the rest. Clearly (by Lemma 2)

$$\frac{1}{T} \int_T^{2T} |\sum_1| dt \ll (\lambda'')^{\frac{1}{2}} T^\delta f(T)$$

and also in  $\sum_2$  using

$$\begin{aligned} & (\lambda_n + \alpha_n)^{-s} \left( \Delta \left( \frac{X}{\lambda_n + \alpha_n} \right) - \Delta \left( \frac{Y}{\lambda_n + \alpha_n} \right) \right) - \lambda_n^{-s} \left( \Delta \left( \frac{X}{\lambda_n} \right) - \Delta \left( \frac{Y}{\lambda_n} \right) \right) \\ &= \int_0^{\alpha_n} \frac{d}{dk} \left( (\lambda_n + k)^{-s} \left( \Delta \left( \frac{X}{\lambda_n + k} \right) - \Delta \left( \frac{Y}{\lambda_n + k} \right) \right) \right) dk \end{aligned}$$

and Lemma 2 we are led to Lemma 16. (For details see page 173 of XIV<sup>[4]</sup>).

**LEMMA 17.** *We have, with  $\alpha_n$  as in Lemma 16,*

$$\begin{aligned} & \frac{1}{T} \int_T^{2T} \left| \sum_{n=1}^{\infty} a_n b_n \text{Exp}(2\pi i n \theta) (\lambda_n + \alpha_n)^{-s} \right. \\ & \left. \left( \Delta \left( \frac{X}{\lambda_n + \alpha_n} \right) - \Delta \left( \frac{Y}{\lambda_n + \alpha_n} \right) \right) \right|^2 dt \ll T^{2\delta} (f(T))^2. \end{aligned} \quad (18)$$

**PROOF.** Follows from Lemma 2.

**THEOREM 2.** *Denote by  $G(s)$  the infinite series in the LHS of (18). Then there are real numbers  $t_1, t_2, \dots, t_r$  as in Theorem 1 with  $r \gg T$  and*

$$\left| G\left(\frac{1}{2} - \delta + it_j\right) \right| \gg T^\delta f(T). \quad (19)$$

**PROOF.** Divide the interval  $[T, 2T]$  (of integration) on the LHS of (17) into abutting intervals of length 1, ignoring a bit at one end. Ignore the integrals over intervals of length 1 which do not exceed a small (positive) constant times  $T^\delta f(T)$ . Now apply Hölder's inequality for the rest and apply Lemma 17. We obtain (19).

**THEOREM 3.** *We have*

$$\frac{1}{T} \int_{T-\sqrt{\log T}}^{2T+\sqrt{\log T}} \left| F\left(\frac{1}{2} - \delta + it\right) \right| dt \gg T^\delta f(T). \quad (20)$$

**PROOF.** Follows from Lemma 16 on writing  $G(s)$  as a line integral over  $u = 2$  and moving the line of integration to  $u = 0$  using suitable horizontal connecting lines.

§ 3. **COMPLETION OF THE PROOF OF THEOREM 1.** Using the mean square upper bound for  $|F(s)|$  and also Theorem 3, we can obtain (as in the proof of Theorem 2) real numbers  $t_1, t_2, \dots, t_r$  as in Theorem 1 with  $r \gg T$  and

$$|F(\frac{1}{2} - \delta + it_j)| \gg T^\delta f(T).$$

Next we use Theorem 2. Out of the numbers  $t_1, \dots, t_r$  we can omit a minimal number of them and obtain numbers  $\tau_1, \dots, \tau_{r'}$  such that  $r' \gg T(\log \log T)^{-1}$ ,  $|\tau_j - \tau_{j'}| \gg \log \log T$  for all pairs  $(j, j')$  with  $j \neq j'$  and

$$|G(\frac{1}{2} - \delta + i\tau_j)| \gg T^\delta f(T).$$

Now writing  $G(s)$  as a line integral over  $u = 2$  and moving the line of integration to  $u = 0$  using suitable connecting horizontal lines. We thus obtain points  $\tau'_1, \dots, \tau'_{r'}$  with

$$|F(\frac{1}{2} - \delta + i\tau'_j)| \gg T^\delta f(T) \quad (j = 1, 2, \dots, r').$$

This proves Theorem 1 completely.

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**ADDRESS OF THE AUTHOR**

K. RAMACHANDRA  
SENIOR PROFESSOR  
SCHOOL OF MATHEMATICS  
TATA INSTITUTE OF FUNDAMENTAL RESEARCH  
HOMI BHABHA ROAD  
BOMBAY 400 005, INDIA

**e-mail** : KRAM@TIFRVAX.TIFR.RES.IN

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