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ON THE ZEROS OF A CLASS OF GENERALISED
DIRICHLET SERIES-XIX

BY

K. RAMACHANDRA

§ 1. **INTRODUCTION AND PRELIMINARIES.** The object of this paper is to give a simpler approach to the following theorem (this theorem namely Theorem 10 of XVIII^[5] constitutes the main theorem on Balasubramanian-Ramachandra functions arising out of the works III^[1], IV^[2], V^[6], VI^[3] due to R. Balasubramanian and K. Ramachandra).

THEOREM 1. (R. BALASUBRAMANIAN AND K. RAMACHANDRA)

(i) Let $\lambda_n (n = 1, 2, 3, \dots)$ be an increasing sequence of positive real numbers such that $\lambda_{n+1} - \lambda_n$ is bounded both above and below. This sequence will be further restricted by the condition (vii) or (viii) as the case may be. θ will denote a real constant.

Let $f(x)$ and $g(x)$ be positive real valued functions defined in $x \geq 0$, satisfying

(ii) $f(x)x^\eta$ is monotonic increasing and $f(x)x^{-\eta}$ is monotonic decreasing for every fixed $\eta > 0$ and all $x \geq x_0(\eta)$.

(iii) $\lim_{x \rightarrow \infty} (g(x)x^{-1}) = 1$.

(iv) $g(x)$ is differentiable once for $x \geq 0$ and $g'(x)$ lies between two positive constants. Also $g(x)$ is twice differentiable for $x \geq x_0$ and $(g'(x))^2 - g(x)g''(x)$ lies between two positive constants for $x \geq x_0$.

Let $\{a_n\}$ and $\{b_n\}$ ($n = 1, 2, 3, \dots$) be two sequences of complex numbers having the following properties

(v) $|b_n| (f(n))^{-1}$ lies between two positive constants (for all integers $n \geq n_0$) and $(\sum_{n \leq x} |a_n|^2) x^{-1}$ does not exceed a positive constant for all $x \geq 1$.

(vi) For all $x \geq 1$, $\sum_{x \leq n < 2x} |b_{n+1} - b_n| \ll f(x)$.

We next assume that $\{a_n\}$ and $\{b_n\}$ satisfy at least one of the following conditions. We set $\lambda_n = g(n)$.

(vii) **MONOTONICITY CONDITION.** There exists an arithmetic progression \mathcal{A} (of natural numbers) such that

$$\lim_{x \rightarrow \infty} \left(x^{-1} \sum'_{n \leq x} a_n \right) = h \quad (0 < |h| < \infty),$$

where the accent denotes the restriction of n to \mathcal{A} . Also for every positive constant ν we have $|b_n| \lambda_n^{-\nu}$ is monotonic decreasing for all $n (\geq n_0)$ in \mathcal{A} .

(viii) **REAL PART CONDITION.** There exists an arithmetic progression \mathcal{A} (of natural numbers) such that

$$\liminf_{x \rightarrow \infty} \left(\frac{1}{x} \sum'_{x \leq \lambda_n \leq 2x, \operatorname{Re} a_n > 0} \operatorname{Re} a_n \right) > 0$$

and

$$\lim_{x \rightarrow \infty} \left(x^{-1} \sum'_{x \leq \lambda_n \leq 2x, \operatorname{Re} a_n < 0} \operatorname{Re} a_n \right) = 0$$

where the accent denotes the restriction of n to \mathcal{A} .

(ix) Finally let $\{\alpha_n\}$ ($n = 1, 2, 3, \dots$) be a sequence of real numbers such that $|\alpha_n|$ does not exceed a small positive constant (depending on other constants). We suppose that the series

$$F(s) = \sum_{n=1}^{\infty} a_n b_n e^{2\pi i n \theta} (\lambda_n + \alpha_n)^{-s} \quad (\operatorname{Re} s \geq 2)$$

can be continued analytically in $(\sigma \geq \frac{1}{2} - \delta, T - \log T \leq t \leq 2T + \log T)$ (where δ is a positive constant) and there $\log \max(|F(s)| + 100) \ll \log T$. As usual we have written $s = \sigma + it$.

Then on every line segment $(\sigma = \frac{1}{2} - \delta_4, T \leq t \leq 2T)$, (δ_4 being any constant with $0 < \delta_4 < \delta$) there are $\gg T(\log \log T)^{-1}$ well-spaced Titchmarsh points with the lower bound $\gg T^{\delta_4} f(T)$ for $|F(s)|$. If further

$$\frac{1}{T} \int_{T-\sqrt{\log T}}^{2T+\sqrt{\log T}} |F(\frac{1}{2} - \delta_4 + it)|^2 dt \ll T^{2\delta_4} (f(T))^2$$

for every constant δ_4 (with $0 < \delta_4 \leq \delta$), then there are $\gg T$ well-spaced Titchmarsh points on every line segment $(\sigma = \frac{1}{2} - \delta_4, T \leq t \leq 2T)$ with the lower bound $\gg T^{\delta_4} f(T)$ for $|F(s)|$.

In other words there exist real numbers t_1, t_2, \dots, t_r (with $r \gg T(\log \log T)^{-1}$ and $r \gg T$ respectively in the two cases) such that $T \leq t_j \leq 2T$ ($j = 1, 2, \dots, r$), the minimum of $|t_j - t_{j'}|$ taken over all pairs (j, j') with $j \neq j'$ is bounded below and further

$$|F(\frac{1}{2} - \delta_4 + it_j)| \gg T^{\delta_4} f(T).$$

The proof of this theorem depends on the following two lemmas.

LEMMA 1 (van-der-CORPUT). *If $f_1(x)$ is real and twice differentiable and $0 < \mu_2 \leq f_1''(x) \leq h'\mu_2$ (or $\mu_2 \leq -f_1''(x) \leq h'\mu_2$) throughout the interval $[a, b]$, and $b \geq a + 1$, then*

$$\sum_{a < n \leq b} \text{Exp}(2\pi i f_1(n)) = O(h'(b-a)\mu_2^{\frac{1}{2}}) + O(\mu_2^{-\frac{1}{2}}).$$

REMARK. This result is Theorem 5.9 on page 104 of [8], with a slight change of notation.

LEMMA 2 (H.L. MONTGOMERY AND R.C. VAUGHAN). *If $\{\lambda_n\}$ is any increasing sequence of real numbers and $\{A_n\}$ and $\{B_n\}$ are any two sequences of complex numbers, then*

$$|\sum_{m \neq n} \sum \frac{A_m \bar{B}_n}{\log(\lambda_m \lambda_n^{-1})}| \leq K (\sum \delta_n^{-1} |A_n|^2)^{\frac{1}{2}} (\sum \delta_n^{-1} |B_n|^2)^{\frac{1}{2}},$$

where $\delta_n = \min_{m \neq n} |\lambda_n - \lambda_m|$ and K is a numerical constant.

REMARK. We need only a special case of this result where $\lambda_{n+1} - \lambda_n$ lies between two positive constants (and so the same is true of $n\delta_n$). For the proof in this special case and also for a reference to the paper of Montgomery and Vaughan see [7].

§ 2. SOME MORE LEMMAS.

LEMMA 3. Let $y > 0$, $w = u + iv$, $R(w) = \text{Exp}((\text{Sin } \frac{w}{100})^2)$, and

$$\Delta(y) = \frac{1}{2\pi i} \int_{u=2} y^w R(w) \frac{dw}{w}.$$

Then for $|u| \leq 3$ we have $|R(w)| \ll (\text{Exp } \text{Exp } |\frac{v}{100}|)^{-1}$. Consequently $\Delta(y) = 1 + O(y^{-2})$ and also $\Delta(y) = O(y^2)$.

PROOF. By trivial computation (and moving the line of integration to $u = -2$ and $u = 2$ respectively).

A will denote the arithmetic progression consisting of an infinite subset of natural numbers. Let $\lambda(0 < \lambda < 1)$ be a constant. We put $X = T\lambda$ (later we will choose λ to be a small constant). S will denote the set $A \cap [\frac{1}{2}X, X]$. All our O -constants and the constants implied by the Vinogradov symbols \gg and \ll will be independent of λ .

LEMMA 4. For $T \leq t \leq 2T$, we have,

$$\left| \sum_{n \in S} \text{Exp}(-2\pi i n \theta + it \log g(n)) \right| \ll T^{\frac{1}{2}}.$$

PROOF. Noting that the second derivative of $-2\pi x \theta + t \log g(x)$ is $t((g'(x))^2 - g(x)g''(x))(g(x))^{-2}$ the lemma follows by Lemma 1.

LEMMA 5. For $T \leq t \leq 2T$, we have

$$\left| \sum_{n \in S} \bar{b}_n \text{Exp}(-2\pi i n \theta + it \log g(n)) \right| \ll T^{\frac{1}{2}} f(X).$$

PROOF. The proof follows by partial summation (from Lemma 4) on using

$$\sum_{x \leq n \leq 2x} |b_{n+1} - b_n| \ll f(x) \text{ for all } x \geq 1.$$

Next we put $g(n) = \lambda_n$. We consider the case $\alpha_n \equiv 0$ first. Our object is to obtain a good lower bound for the LHS of (2) below.

LEMMA 6. For $s = \frac{1}{2} - \delta + it, T \leq t \leq 2T$, put

$$F_X(s) = \sum_{n=1}^{\infty} a_n b_n \text{Exp}(2\pi i n \theta) \lambda_n^{-s} \Delta\left(\frac{X}{\lambda_n}\right). \quad (1)$$

Then

$$\frac{1}{T} \int_T^{2T} |F_X(s)| dt \gg (T^{\frac{3}{2}} f(X))^{-1} |I|, \quad (2)$$

where

$$I = \int_T^{2T} F_X(s) \sum_{n \in S} \bar{b}_n \text{Exp}(-2\pi i n \theta + it \log \lambda_n) dt. \quad (3)$$

PROOF. Follows from Lemma 5.

LEMMA 7. We have,

$$I = T \sum_{n \in S} a_n |b_n|^2 \lambda_n^{-\frac{1}{2} + \delta} + O(J) \quad (4)$$

where

$$J = (J_1 J_2)^{\frac{1}{2}}, J_1 = \sum_{n=1}^{\infty} |a_n b_n|^2 n^{2\delta} \left(\Delta\left(\frac{X}{\lambda_n}\right)\right)^2 \quad (5)$$

and

$$J_2 = \sum_{n \in [\frac{1}{2}X, X]} n |b_n|^2. \quad (6)$$

PROOF. Follows from Lemma 2.

LEMMA 8. We have

$$J_1 = O(X^{1+2\delta} (f(X))^2) \quad (7)$$

and

$$J_2 = O(X^2 (f(X))^2). \quad (8)$$

PROOF. Follows from $\Delta(y) = 1 + O(y^{-2}) = O(y^2)$ and also from (ii) of Theorem 1.

LEMMA 9. *Let*

$$\sum_0 = \sum_{n \in S} a_n |b_n|^2 \lambda_n^{-\frac{1}{2} + \delta} \Delta\left(\frac{X}{\lambda_n}\right). \quad (9)$$

Then

$$I = T \sum_0 + O(X^{\frac{3}{2} + \delta} (f(X))^2) \quad (10)$$

PROOF. Follows from Lemmas 7 and 8.

LEMMA 10. *Under monotonicity condition, we have,*

$$|\sum_0| \gg |h| X^{\frac{1}{2} + \delta} (f(X))^2. \quad (11)$$

PROOF. We write

$$\frac{1}{x} \sum_{n \in \mathcal{A}, n \leq x} a_n = h + \varepsilon_x$$

where $\varepsilon_x \rightarrow 0$ as $x \rightarrow \infty$. We obtain the result by the monotonicity of $|b_n|^2 \lambda_n^{-\frac{1}{4} + \frac{1}{2}\delta}$.

LEMMA 11. *Under the real part condition, we have,*

$$Re \sum_0 \gg X^{\frac{1}{2} + \delta} (f(X))^2 \quad (12)$$

PROOF. Follows since the contribution from those a_n with $Re a_n < 0$ is of a smaller order.

LEMMA 12. *We have*

$$|I| > C_1 T (f(X))^2 X^{\frac{1}{2} + \delta} - C_2 (f(X))^2 X^{\frac{3}{2} + \delta} \quad (13)$$

where C_1 and C_2 are positive constants independent of λ .

PROOF. Follows from Lemma 7 to 11.

LEMMA 13. *We have, with $s = \frac{1}{2} - \delta + it$, $X = T\lambda$, where $\lambda(> 0)$ is some fixed small constant, the inequality*

$$\frac{1}{T} \int_T^{2T} |F_X(s)| dt \gg T^\delta f(T). \quad (14)$$

PROOF. RHS of (13) is

$$\left(C_1 T^{\frac{3}{2}+\delta} \lambda^{\frac{1}{2}+\delta} - C_2 T^{\frac{3}{2}} \lambda^{\frac{3}{2}+\delta}\right) (f(X))^2.$$

Using (ii) it follows that $f(X)X^{-1} \geq f(T)T^{-1}$ and so $f(X) \geq \lambda f(T)$. Lemma 13 follows on fixing λ to be a small positive constant.

LEMMA 14. *Let now $Y = T\lambda'$ where λ' ($0 < \lambda' < \lambda$) is a small constant. We have*

$$\frac{1}{T} \int_T^{2T} |F_Y(s)| dt \leq \eta_0 T^\delta f(T), \tag{15}$$

where η_0 depends on λ' and is small enough if λ' is small.

PROOF. Note that

$$\frac{1}{T} \int_T^{2T} |F_Y(s)|^2 dt \ll Y^{2\delta} (f(Y))^2$$

and that here RHS is $\leq Y^\delta (Y^{\frac{1}{2}\delta} f(Y))^2 \leq (\lambda')^\delta T^{2\delta} (f(T))^2$. Lemma 14 follows from this on using Hölder's inequality.

LEMMA 15. *We have, with $X = T\lambda, Y = T\lambda'$ where λ is as before and λ' ($0 < \lambda' < \lambda$) is fixed to be a sufficiently small constant, the inequality*

$$\frac{1}{T} \int_T^{2T} |F_X(s) - F_Y(s)| dt \gg T^\delta f(T). \tag{16}$$

PROOF. Follows from Lemma 13 and 14.

From now on we fix the positive constants λ and λ' so that (16) is satisfied.

LEMMA 16. *Now let α_n be real and let $|\alpha_n|$ be bounded above by a small positive constant. Then with $s = \frac{1}{2} - \delta + it, X = T\lambda, Y = T\lambda'$ we have*

$$\frac{1}{T} \int_T^{2T} \left| \sum_{n=1}^{\infty} a_n b_n \text{Exp}(2\pi in\theta) (\lambda_n + \alpha_n)^{-s} \right. \\ \left. \left(\Delta \left(\frac{X}{\lambda_n + \alpha_n} \right) - \Delta \left(\frac{Y}{\lambda_n + \alpha_n} \right) \right) \right| dt \gg T^\delta f(T). \tag{17}$$

PROOF. We split the infinite series on the LHS of (17) to be \sum_1 with $n \leq T\lambda''$ (where $\lambda''(> 0)$ is a small constant) and \sum_2 the rest. Clearly (by Lemma 2)

$$\frac{1}{T} \int_T^{2T} |\sum_1| dt \ll (\lambda'')^{\frac{1}{2}} T^\delta f(T)$$

and also in \sum_2 using

$$\begin{aligned} & (\lambda_n + \alpha_n)^{-s} \left(\Delta \left(\frac{X}{\lambda_n + \alpha_n} \right) - \Delta \left(\frac{Y}{\lambda_n + \alpha_n} \right) \right) - \lambda_n^{-s} \left(\Delta \left(\frac{X}{\lambda_n} \right) - \Delta \left(\frac{Y}{\lambda_n} \right) \right) \\ &= \int_0^{\alpha_n} \frac{d}{dk} \left((\lambda_n + k)^{-s} \left(\Delta \left(\frac{X}{\lambda_n + k} \right) - \Delta \left(\frac{Y}{\lambda_n + k} \right) \right) \right) dk \end{aligned}$$

and Lemma 2 we are led to Lemma 16. (For details see page 173 of XIV^[4]).

LEMMA 17. *We have, with α_n as in Lemma 16,*

$$\begin{aligned} & \frac{1}{T} \int_T^{2T} \left| \sum_{n=1}^{\infty} a_n b_n \text{Exp}(2\pi i n \theta) (\lambda_n + \alpha_n)^{-s} \right. \\ & \left. \left(\Delta \left(\frac{X}{\lambda_n + \alpha_n} \right) - \Delta \left(\frac{Y}{\lambda_n + \alpha_n} \right) \right) \right|^2 dt \ll T^{2\delta} (f(T))^2. \end{aligned} \quad (18)$$

PROOF. Follows from Lemma 2.

THEOREM 2. *Denote by $G(s)$ the infinite series in the LHS of (18). Then there are real numbers t_1, t_2, \dots, t_r as in Theorem 1 with $r \gg T$ and*

$$\left| G\left(\frac{1}{2} - \delta + it_j\right) \right| \gg T^\delta f(T). \quad (19)$$

PROOF. Divide the interval $[T, 2T]$ (of integration) on the LHS of (17) into abutting intervals of length 1, ignoring a bit at one end. Ignore the integrals over intervals of length 1 which do not exceed a small (positive) constant times $T^\delta f(T)$. Now apply Hölder's inequality for the rest and apply Lemma 17. We obtain (19).

THEOREM 3. *We have*

$$\frac{1}{T} \int_{T-\sqrt{\log T}}^{2T+\sqrt{\log T}} \left| F\left(\frac{1}{2} - \delta + it\right) \right| dt \gg T^\delta f(T). \quad (20)$$

PROOF. Follows from Lemma 16 on writing $G(s)$ as a line integral over $u = 2$ and moving the line of integration to $u = 0$ using suitable horizontal connecting lines.

§ 3. **COMPLETION OF THE PROOF OF THEOREM 1.** Using the mean square upper bound for $|F(s)|$ and also Theorem 3, we can obtain (as in the proof of Theorem 2) real numbers t_1, t_2, \dots, t_r as in Theorem 1 with $r \gg T$ and

$$|F(\frac{1}{2} - \delta + it_j)| \gg T^\delta f(T).$$

Next we use Theorem 2. Out of the numbers t_1, \dots, t_r we can omit a minimal number of them and obtain numbers $\tau_1, \dots, \tau_{r'}$ such that $r' \gg T(\log \log T)^{-1}$, $|\tau_j - \tau_{j'}| \gg \log \log T$ for all pairs (j, j') with $j \neq j'$ and

$$|G(\frac{1}{2} - \delta + i\tau_j)| \gg T^\delta f(T).$$

Now writing $G(s)$ as a line integral over $u = 2$ and moving the line of integration to $u = 0$ using suitable connecting horizontal lines. We thus obtain points $\tau'_1, \dots, \tau'_{r'}$ with

$$|F(\frac{1}{2} - \delta + i\tau'_j)| \gg T^\delta f(T) \quad (j = 1, 2, \dots, r').$$

This proves Theorem 1 completely.

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