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**ON THE ZEROS OF A CLASS OF GENERALISED  
DIRICHLET SERIES-XVIII**

**(A FEW REMARKS ON LITTLEWOOD'S THEOREM AND  
TITCHMARSH POINTS)**

BY

**R. BALASUBRAMANIAN, K. RAMACHANDRA and  
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**(DEDICATED TO PROFESSOR K. CHANDRASEKHARAN  
ON HIS SEVENTY-FIFTH BIRTHDAY)**

§ 1. **INTRODUCTION AND NOTATION.** This paper was necessitated because we observed that the proofs of the results of the paper XVI<sup>[7]</sup> can be simplified and that the results therein can at the same time be generalised. (In § 1, § 2 and § 3 we prove six theorems in all. For an attractive application of these see Theorem 11 of § 4). We write  $s = \sigma + it$  as usual. We begin by stating a generalisation of Theorem 9.15 (A) (on page 230 of [9]). We need some definitions. (We fix two positive constants  $a$  and  $b$  with  $a < b$  throughout). The parameter  $T$  will be assumed to exceed a large positive constant.

**GENERALISED DIRICHLET SERIES (GDS).** Let  $\{\lambda_n\}$  be a sequence of real numbers with  $a < \lambda_1 < \lambda_2 < \dots, \lambda_1 < b$  and  $a \leq \lambda_{n+1} - \lambda_n \leq b$  for  $n \geq 1$ . Let  $\{A_n\}$  be any sequence of complex numbers such that  $A_1 \neq 0$

and

$$Z(s) = \sum_{n=1}^{\infty} A_n \lambda_n^{-s} \tag{1}$$

converges for some complex  $s = s_0$ . Then  $Z(s)$  is called a *generalised Dirichlet series* (GDS). We remark that if  $Z(s)$  converges at  $s = s_0$ , then it is absolutely convergent at  $s = s_0 + 2$ . Note that a GDS is different from zero if real part of  $s(Re s)$  exceeds a certain constant. In fact as  $Re s \rightarrow \infty, |Z(s)|$  tends to a non-zero constant. A GDS is said to be a *normalised generalised Dirichlet series* (NGDS) if  $\sum_{n \leq x} |A_n|^2 \ll_\epsilon x^{1+\epsilon}$  for every  $\epsilon > 0$ . A GDS is said to be a *Dirichlet series* if  $\{\lambda_n\}$  is a subsequence of the sequence of natural numbers.

**$\{\alpha_n\}$  TRANSFORMATION OF AN NGDS.** Let  $Z(s)$  be an NGDS. We consider only such sequences  $\{\alpha_n\}$  of real numbers which satisfy  $\sum_{n \leq x} |A_n \alpha_n|^2 \ll_\epsilon x^{1+\epsilon}$  for every  $\epsilon > 0$ .  $F(s) = \sum_{n=1}^{\infty} A_n (\lambda_n + \alpha_n)^{-s}$  is said to be an  $\{\alpha_n\}$  transformation of  $Z(s)$  if  $F(s)$  is a GDS. Note that

$$F(s) = D(s) + Z(s) \text{ where } D(s) = \sum_{n=1}^{\infty} A_n ((\lambda_n + \alpha_n)^{-s} - \lambda_n^{-s}) \tag{2}$$

and that  $D(s)$  is analytic in  $\sigma > 0$ . Moreover we have

**LEMMA 1.** For  $\sigma > 0$  and every  $\epsilon > 0$  we have  $D(s) = O_\sigma(|t| + 2)^2$  and also

$$\frac{1}{T} \int_T^{2T} |D(\sigma + it)|^2 dt \ll_\epsilon \max \left( T^{2(\frac{1}{2} - \sigma) + \epsilon}, T^\epsilon \right). \tag{3}$$

**PROOF.** See Theorems 7 and 7' of XV<sup>[5]</sup>.

**THEOREM 1** (J.E. LITTLEWOOD). Let  $Z(s)$  be a GDS which can be continued analytically in  $(\sigma \geq \frac{1}{2} - \delta_0, T - \log T \leq t \leq 2T + \log T)$ , where  $\delta_0(> 0)$  is a constant, and there  $\log(\max |Z(s)| + 100)$  is  $\ll \log T$ . Let  $Z(s) \rightarrow 1$  as  $Re s \rightarrow \infty$ . For  $\alpha \geq \frac{1}{2}$  let  $N(\alpha, T)$  denote the number of zeros of  $Z(s)$  in  $(\sigma \geq \alpha, T \leq t \leq 2T)$ . Then (for  $\sigma_0 > \frac{1}{2}$ ) we have

$$2\pi \int_{\sigma_0}^{\infty} N(\sigma, T) d\sigma = \int_T^{2T} \log |Z(\sigma_0 + it)| dt + O(\log T) \tag{4}$$

and hence

$$N\left(\frac{1}{2} + 2\delta, T\right) \leq (g\delta)^{-1} T \log \left( \frac{1}{T} \int_T^{2T} |Z\left(\frac{1}{2} + \delta + it\right)|^g dt \right) + O(\log T) \quad (5)$$

holds uniformly for all real positive constants  $g$  and  $\delta$ . If  $\delta$  is any fixed constant we may take  $\delta_0 = 0$  and then replace  $O(\log T)$  by  $O_\delta(\log T)$ .

**REMARK.** This theorem is essentially due to J.E. Littlewood, since the special case  $g = 2$  and  $Z(s) = \zeta(s)$  (due to J.E. Littlewood) is dealt with on pages 229 and 230 of [9]. The general case stated as Theorem 1 above follows by a trivial generalisation of Littlewood's method. If we do not assume  $Z(s) \rightarrow 1$  as  $\text{Re } s \rightarrow \infty$ , we have to replace  $O(\log T)$  by  $O(T)$  in (4) and (5). This does not matter for our purposes.

## § 2. A COROLLARY TO THEOREM 1.

**THEOREM 2.** Let  $r \geq 1$  be any integer constant and let  $\varphi_1(s), \varphi_2(s), \dots, \varphi_r(s)$  be  $r$  Dirichlet series each of which is continuable analytically in  $(\sigma \geq \frac{1}{2} - \delta_0, T - \log T \leq t \leq 2T + \log T)$  and there  $\log(\max_j |\varphi_j(s)| + 100) \ll \log T$ . Suppose further that

$$\max_j \left( \frac{1}{T} \int_{T-\log T}^{2T+\log T} |\varphi_j\left(\frac{1}{2} + it\right)|^2 dt \right) \ll_\epsilon T^\epsilon \quad (6)$$

holds for every  $\epsilon > 0$ . Let  $P(X_1, \dots, X_r)$  be any fixed polynomial (with complex coefficients) such that when we put  $X_j = \varphi_j(s)$  ( $j = 1, 2, \dots, r$ ),  $P = P(s) = P(X_1, \dots, X_r)$  is a normalised Dirichlet series. Let  $F(s)$  be any  $\{\alpha_n\}$  transformation of  $P(s)$ . Then the function  $N(\sigma, T)$  defined (as before) for  $F(s)$  satisfies

$$N(\sigma, T) \ll_\sigma T \quad \left(\sigma > \frac{1}{2}\right). \quad (7)$$

**REMARK.** We define the degree of a monomial  $X_1^{d_1} \dots X_r^{d_r}$  to be  $d_1 + \dots + d_r$  and the degree of  $P(X_1, \dots, X_r)$  to be the maximum of  $d_1 + \dots + d_r$  taken over all monomials occurring in  $P(X_1, \dots, X_r)$ . If the degree of  $P$  is 1 then we can allow each  $\varphi_j(s)$  to be a GDS. Then  $P$  has to be an NGDS.

**PROOF.** The proof follows from the fact that (6) implies

$$\max_j \left( \frac{1}{T} \int_T^{2T} |\varphi_j(\sigma + it)|^2 dt \right) \ll_{\sigma} 1 \quad \left( \sigma > \frac{1}{2} \right), \quad (8)$$

and that for a suitable small constant  $g > 0$  we have

$$|F(s)|^g \ll |D(s)|^2 + |\varphi_1(s)|^2 + \dots + |\varphi_r(s)|^2 + 1. \quad (9)$$

Note that in view of Lemma 1 it is not hard to deduce that

$$\frac{1}{T} \int_T^{2T} |D(\sigma + it)|^2 dt \ll_{\sigma} 1 \quad \left( \sigma > \frac{1}{2} \right). \quad (10)$$

From these facts Theorem 2 follows from Theorem 1.

§ 3. **TITCHMARSH POINTS.** Let  $F(s)$  be a GDS continuable analytically in  $(\sigma \geq \beta, T - \log T \leq t \leq 2T + \log T)$  and there  $\log(\max |F(s)| + 100) \ll \log T$ . A point  $s_0 = \sigma_0 + it_0$  in  $(\sigma \geq \beta + \delta_1, T \leq t \leq 2T)$ , where  $\delta_1 > 0$  is a constant, is said to be a *Titchmarsh point with the lower bound  $T^\ell$*  for  $|F(s)|$  if  $\ell(> 0)$  is bounded below independent of  $T$  and  $t_0$ .

**THEOREM 3.** *If  $s_0 = \sigma_0 + it_0$  (with  $F(s)$  as above) is a Titchmarsh point of  $F(s)$ , then the region  $(\sigma \geq \beta, |t - t_0| \leq \delta_2)$  where  $\delta_2(> 0)$  is any small constant, contains  $\gg \log T$  zeros of  $F(s)$ .*

**PROOF.** For the proof of this theorem due to R. Balasubramanian and K. Ramachandra see Theorem 3 of III<sup>[1]</sup>. It should be mentioned that this theorem is not too-trivial a generalisation of Theorem 9.14 (on page 227 of [9]) due to E.C. Titchmarsh.

**WELL-SPACED POINTS.** The points  $s^{(q)} = \sigma_q + it_q$  ( $q = 1, 2, \dots$ ) in the complex plane are said to be *well-spaced* if  $|s^{(q)} - s^{(q')}|$  is bounded below for all pairs  $(q, q')$  with  $q \neq q'$ .

**THEOREM 4.** *If there are  $N_0$  well-spaced Titchmarsh points for  $F(s)$  ( $F(s)$  as in Theorem 3), then  $F(s)$  has  $\gg N_0 \log T$  zeros in  $(\sigma \geq \beta, T \leq t \leq 2T)$ .*

**PROOF.** The proof follows from the fact that  $|F(s)|$  tends to a non-zero

limit uniformly in  $t$  as  $\sigma \rightarrow \infty$ .

**THEOREM 5.** Let  $\beta (< \frac{1}{2})$  be a constant and  $r \geq 1$  any integer constant and  $\varphi_1(s), \dots, \varphi_r(s)$  be  $r$  Dirichlet series each of which is continuable analytically in  $(\sigma \geq \beta, T - \log T \leq t \leq 2T + \log T)$  and there  $\log \max_j (|\varphi_j(s)| + 100)$  is  $\ll \log T$ . Suppose further that for  $j = 1, 2, \dots, r$  and  $\sigma \geq \beta$ , we have

$$\frac{1}{T} \int_{T-\log T}^{2T+\log T} |\varphi_j(\sigma + it)|^2 dt \ll_{\epsilon} \max \left( T^{2m_j(\frac{1}{2}-\sigma)+\epsilon}, T^{\epsilon} \right) \quad (11)$$

where  $m_j > 0$  are constants. Let  $\mu > 0$  be a constant. Put  $X_0 = T^{\mu(\frac{1}{2}-\sigma)-\epsilon}$ . Let  $X_0^{d_0} X_1^{d_1} \dots X_r^{d_r}$  ( $d_j \geq 0$  integers,  $j = 0, 1, 2, \dots, r$ ) be any fixed monomial in  $X_0, X_1, \dots, X_r$ . Let the weighted  $\mu$ -degree  $d(\mu)$  of the monomial be defined as  $\mu d_0 + m_1 d_1 + \dots + m_r d_r$ . Put  $Q_0(s) = X_0^{d_0} (\varphi_1(s))^{d_1} \dots (\varphi_r(s))^{d_r}$ . Then given any well-spaced set of points  $\{s_q\}$  with  $s_q = \sigma + it_q$  ( $q = 1, 2, \dots; \sigma \geq \beta + \delta_3, T \leq t_q \leq 2T$ ) where  $\delta_3 > 0$  is a small constant we have

$$|Q_0(\sigma + it_q)| \ll \max \left( T^{d(\mu)(\frac{1}{2}-\sigma)+\epsilon}, T^{\epsilon} \right), \quad (12)$$

except for  $O(T^{1-\epsilon})$  values of  $q$ .

**REMARK.** If  $\sum_{j=1}^r d_j = 1$ , then we can allow  $\varphi_j(s)$  ( $j = 1, 2, \dots, r$ ) to be GDS.

**PROOF.** We use the fact that the value  $|\varphi_j(s_q)|$  of  $\varphi_j(s)$  is majorised by the mean value over a disc (with  $s_q$  as centre and  $\epsilon$  as radius) of  $|\varphi_j(s)|$ . We choose a small radius and sum over all the discs taking  $s_q$  to be  $\sigma + it_q$ . We obtain

$$\frac{1}{T} \sum_q |\varphi_j(s_q)|^2 \ll \max \left( T^{2m_j(\frac{1}{2}-\sigma)+\epsilon}, T^{\epsilon} \right).$$

Hence  $|\varphi_j(s_q)| > \max(T^{m_j(\frac{1}{2}-\sigma)+\epsilon}, T^{\epsilon})$  is possible for at most  $O(T^{1-\epsilon})$  values of  $q$ . We next sum over all  $j$  and obtain the result.

**THEOREM 6.** Let  $\beta (< \frac{1}{2})$  be a constant and let  $\varphi_0(s)$  be a Dirichlet series continuable analytically in  $(\sigma \geq \beta, T - \log T \leq t \leq 2T + \log T)$  and there

$\log \max(|\varphi_0(s)| + 100) \ll \log T$ . Suppose that it has  $\gg T$  (resp.  $T(\log \log T)^{-1}$ ) well-spaced Titchmarsh points  $\{\sigma + it_j\}$  (where  $\sigma$  is any constant with  $\beta < \sigma < \frac{1}{2}$ ) with the lower bound  $T^{\mu(\frac{1}{2}-\sigma)-\varepsilon}$  where  $\mu(> 0)$  is a constant. Let  $Q(X_0, X_1, \dots, X_r)$  be a fixed polynomial (with complex coefficients) such that for some positive integer  $M$ , the maximum of  $d(\mu)$  (defined in Theorem 5) taken over all the monomials  $X_0^{d_0} X_1^{d_1} \dots X_r^{d_r}$  occurring in  $Q(X_0, X_1, \dots, X_r)$  is less than  $M\mu$ . Put  $Q(s) = Q(\varphi_0(s), \varphi_1(s), \dots, \varphi_r(s))$  (where  $\varphi_j(s)$   $j = 1, 2, \dots$ , are as in Theorem 5). Assume that  $(\varphi_0(s))^M - Q(s)$  is an NGDS, and let  $F(s)$  be its  $\{\alpha_n\}$  transformation. Then  $F(s)$  has  $\gg T \log T$  (resp.  $T(\log T)(\log \log T)^{-1}$ ) zeros in  $(\sigma \geq \beta, T \leq t \leq 2T)$ .

**PROOF.** Follows from  $|(\varphi_0(s))^M - Q(s)| \geq |\varphi_0(s)|^M (1 - |Q(s)|/|\varphi_0(s)|^{-M})$ .

**REMARK.** Note that none of the functions  $\varphi_0(s)$ ,  $(\varphi_0(s))^M$  and  $Q(s)$  need be normalised Dirichlet series. If  $M = 1$ , then  $Q(s)$  does not involve  $\varphi_0(s)$  (which can now be taken to be a GDS). If  $M = 1$  and  $Q(X_0, X_1, \dots, X_r)$  (now independent of  $X_0$ ) is linear in  $X_1, \dots, X_r$  (i.e.  $\sum_{j=1}^r d_j \leq 1$  for every monomial  $X_1^{d_1} \dots X_r^{d_r}$  occurring in  $Q(X_0, X_1, \dots, X_r)$  and equality holds for at least one monomial) then all of  $\varphi_1(s), \dots, \varphi_r(s)$  can be taken to be GDS. Also  $Q(X_0, X_1, \dots, X_r)$  can be a constant.

§ 4. SOME APPLICATIONS OF THEOREMS 2 TO 6. Theorems 2 to 6 are only easy formalisms. These would be completely uninteresting without examples. Finding examples is a difficult task. For example we do not know how to prove the expected result  $N(\sigma, T) \ll_\sigma T(\sigma > \frac{1}{2})$  for the abelian  $L$ -series of an algebraic number field. However we have a somewhat general theorem namely.

**THEOREM 7.** Let  $\{\lambda_n\} (n = 1, 2, \dots)$  be a sequence of real numbers as in the definition of GDS. Let  $|\sum_{n \leq x} a_n| \leq B(x)$ ,  $\sum_{n \leq x} |a_n|^2 \leq xB(x)$  and  $\sum_{m \leq x} |\sum_{n \leq m} a_n|^2 \leq xB(x)$ , where  $B(x)$  depends on  $x$ . If  $B(x) \ll_\varepsilon x^\varepsilon$  (for every

$\varepsilon > 0$ ) then  $Z(s) = \sum_{n=1}^{\infty} a_n \lambda_n^{-s}$  converges uniformly over compact subsets of  $\sigma > 0$  and hence is analytic there. We have

$$N\left(\frac{1}{2} + \delta, T\right) \ll_{\delta} T \quad (\delta > 0). \quad (13)$$

If further  $\log B(x) \ll \log \log x$  then we have

$$N\left(\frac{1}{2} + \delta, T\right) \ll \delta^{-1} T \log(\delta^{-1}) \quad (14)$$

uniformly for  $0 < \delta \leq \frac{1}{2}$ .

**REMARK.** Results like

$$\frac{1}{T} \int_{T-\log T}^{2T+\log T} |Z\left(\frac{1}{2} + it\right)|^2 dt \ll_{\varepsilon} T^{\varepsilon} \quad (15)$$

for every  $\varepsilon > 0$  and more general and powerful results have been proved in paper V<sup>[6]</sup>. Results like (15) imply (13) and (14). If  $\{Z(s)\}$  is any finite set of Dirichlet series each subject to (15) we can apply Theorem 2.

We now turn to series of the type

$$\sum_{n=1}^{\infty} a_n b_n e^{2\pi i n \theta} \lambda_n^{-s} \quad (\theta \text{ is a real constant}), \quad (16)$$

their analytic continuations and their Titchmarsh points. Investigations dealing with such series were carried out in a series of papers by R. Balasubramanian and K. Ramachandra (see III<sup>[1]</sup>, IV<sup>[2]</sup>, V<sup>[6]</sup>, VI<sup>[3]</sup>, XIV<sup>[4]</sup> and also the paper [8] by K. Ramachandra and A. Sankaranarayanan). The paper XIV<sup>[4]</sup> is nearly final. In paper XIV<sup>[4]</sup> the condition  $a_n = O(1)$  is assumed. This can be relaxed to  $\sum_{n \leq x} |a_n|^2 = O(x)$ . This last mentioned condition on  $a_n$  will be assumed in the rest of this paper.

Lest we get lost in generalities we state two special cases first.

**THEOREM 8.** Let  $\theta_0 (0 < \theta_0 < \frac{1}{2})$  be a constant and let  $\{a_n\}$  be a sequence of complex numbers satisfying the inequality  $|\sum_{m=1}^N a_m - N| \leq (\frac{1}{2} - \theta_0)^{-1}$  for



$N = 1, 2, 3, \dots$ . Also for  $n = 1, 2, 3, \dots$ , let  $\alpha_n$  be real and  $|\alpha_n| \leq C(\theta_0)$  where  $C(\theta_0)$  is a certain (small) constant depending only on  $\theta_0$ . Then the number of zeros of the function

$$\sum_{n=1}^{\infty} a_n (n + \alpha_n)^{-s} = \zeta(s) + \sum_{n=1}^{\infty} (a_n (n + \alpha_n)^{-s} - n^{-s})$$

in the rectangle ( $|\sigma - \frac{1}{2}| \leq \delta, T \leq t \leq 2T$ ) is  $\geq C(\theta_0, \delta)T \log T$  where  $C(\theta_0, \delta)$  is a positive constant depending only on  $\theta_0$  and  $\delta$ , and  $T \geq T_0(\theta_0, \delta)$  a large positive constant.

**PROOF.** Theorem 10 (below) gives  $\gg T$  well spaced Titchmarsh points on every line segment ( $\sigma = \frac{1}{2} - \delta, T \leq t \leq 2T$ ) with the lower bound  $\gg T^\delta$ , while actually (14) gives

$$N\left(\frac{1}{2} + \frac{C \log \log T}{\log T}, T\right) \ll C^{-1}T \log T$$

for every fixed  $C(> 0)$ . (It is not hard to prove the required mean-square upper bound for the function).

**THEOREM 9.** In the above theorem we can relax the condition on  $a_n$  to

$$\left| \sum_{m=1}^N a_m - N \right| \leq \left(\frac{1}{2} - \theta_0\right)^{-1} N^{\theta_0} \quad \text{and} \quad \sum_{n \leq x} |a_n|^2 \leq \left(\frac{1}{2} - \theta_0\right)^{-1} x.$$

Then the lower bound for the number of zeros in ( $\sigma \geq \frac{1}{2} - \delta, T \leq t \leq 2T$ ) ( $\delta$  being any constant with  $\frac{1}{2} - \delta > \theta_0$ ) is  $\geq C(\theta_0, \delta)T(\log T)(\log \log T)^{-1}$ . But only when  $\sum_{n \leq x} a_n = x + O_\epsilon(x^\epsilon)$  we can prove that  $N(\frac{1}{2} + \delta, T) \ll_\delta T$ . Also if

$$\sum_{n \leq x} a_n = x + O((\log x)^{C_1}) \quad (C_1 > 0 \text{ being a constant we can prove})$$

$$N\left(\frac{1}{2} + \frac{C(\log \log T)^2}{\log T}, T\right) \ll C^{-1}T(\log T)(\log \log T)^{-1}$$

for every fixed  $C > 0$ .

**PROOF.** Theorem 10 (below) gives  $\gg T(\log \log T)^{-1}$  well-spaced Titchmarsh points on every line segment ( $\sigma = \frac{1}{2} - \delta, T \leq t \leq 2T$ ) with the lower

bound  $\gg T^\delta$  ( $\delta$  being a constant subject to  $\frac{1}{2} - \delta > \theta_0$ ).

**THEOREM 10.** (i) Let  $\{\lambda_n\}$  be as in the definition of GDS. This sequence will be further restricted by the condition (vii) or (viii) as the case may be.  $\theta$  will denote a real constant.

Let  $f(x)$  and  $g(x)$  be positive real valued functions defined in  $x \geq 0$  satisfying

(ii)  $f(x)x^\eta$  is monotonic increasing and  $f(x)x^{-\eta}$  is monotonic decreasing for every fixed  $\eta > 0$  and all  $x \geq x_0(\eta)$ .

(iii)  $\lim_{x \rightarrow \infty} (g(x)x^{-1}) = 1$ .

(iv) For all  $x \geq 0$ ,  $g'(x)$  lies between two positive constants and  $(g'(x))^2 - g(x)g''(x)$  lies between two positive constants (it being assumed that  $g(x)$  is twice continuously differentiable for  $x \geq x_0$ ).

Let  $\{a_n\}$  and  $\{b_n\}$  be two sequences of complex numbers having the following properties.

(v)  $|b_n| (f(n))^{-1}$  lies between two positive constants (for all integers  $n \geq n_0$ ) and  $(\sum_{n \leq x} |a_n|^2)x^{-1}$  does not exceed a positive constant for all  $x \geq 1$ .

(vi) For all  $X \geq 1$ ,  $\sum_{X \leq n \leq 2X} |b_{n+1} - b_n| \ll f(X)$ .

We next assume that  $\{a_n\}$  and  $\{b_n\}$  satisfy at least one of the two following conditions.

(vii) **MONOTONICITY CONDITION.** There exists an arithmetic progression  $\mathcal{A}$  (of integers) such that

$$\lim_{x \rightarrow \infty} \left( x^{-1} \sum'_{n \leq x} a_n \right) = h \quad (h \neq 0),$$

where the accent denotes the restriction of  $n$  to  $\mathcal{A}$ . Also for every positive constant  $\nu$  we have that  $|b_n| \lambda_n^{-\nu}$  is monotonic decreasing for all  $n (\geq n_0)$  in  $\mathcal{A}$ .

(viii) **REAL PART CONDITION.** There exists an arithmetic pro-

gression  $\mathcal{A}$  (of integers) such that

$$\liminf_{x \rightarrow \infty} \left( \frac{1}{x} \sum'_{x \leq \lambda_n \leq 2x, \operatorname{Re} a_n > 0} \operatorname{Re} a_n \right) > 0$$

and

$$\lim_{x \rightarrow \infty} \left( \frac{1}{x} \sum'_{x \leq \lambda_n \leq 2x, \operatorname{Re} a_n < 0} \operatorname{Re} a_n \right) = 0$$

where the accent denotes the restriction of  $n$  to  $\mathcal{A}$ .

(ix) Finally we set  $\lambda_n = g(n)$  and let  $\{\alpha_n\}$  be a sequence of real numbers such that  $|\alpha_n|$  does not exceed a small positive constant (depending on other constants). We suppose that the GDS

$$F(s) = \sum_{n=1}^{\infty} a_n b_n e^{2\pi i n \theta} (\lambda_n + \alpha_n)^{-s}$$

can be continued analytically in  $(\sigma \geq \frac{1}{2} - \delta, T - \log T \leq t \leq 2T + \log T)$  and there  $\log \max(|F(s)| + 100) \ll \log T$ .

Then on every line segment  $(\sigma = \frac{1}{2} - \delta_4, T \leq t \leq 2T)$  ( $\delta_4$  being a constant with  $0 < \delta_4 \leq \delta$ ) there are  $\gg T(\log \log T)^{-1}$  well-spaced Titchmarsh points with the lower bound  $\gg T^{\delta_4} f(T)$ . If further we have

$$\frac{1}{T} \int_T^{2T} |F(\frac{1}{2} - \delta_4 + it)|^2 dt \ll T^{2\delta_4} (f(T))^2$$

for every constant  $\delta_4$  (with  $0 < \delta_4 \leq \delta$ ), then the number of well-spaced Titchmarsh points on the line segment  $(\sigma = \frac{1}{2} - \delta_4, T \leq t \leq 2T)$  (with the lower bound  $\gg T^{\delta_4} f(T)$ ) is  $\gg T$ .

**REMARK.** This theorem is proved by R. Balasubramanian and K. Ramachandra in this form in the paper XIV<sup>[4]</sup> except that we have now to use  $\sum_{n \leq x} |a_n|^2 \ll x$  in place of  $a_n = O(1)$  and also except that we have to involve  $\theta$ . Lemmas necessary (see Lemma 6 of IV<sup>[2]</sup>) for these generalities and also the method have been developed in previous papers mentioned before by R. Balasubramanian and K. Ramachandra.

Finally we would like to mention paper XV<sup>[5]</sup> of this series of papers. Here we assume a functional equation of a very general type for a GDS and prove that a large class of  $\{\alpha_n\}$  transformations of it have  $\gg T$  well-spaced Titchmarsh points on every line segment ( $\sigma = \frac{1}{2} - \delta, T \leq t \leq 2T$ ) with a lower bound of the type  $\gg_\epsilon T^{m\delta - \epsilon}$  where  $m > 0$  is a real constant and  $\epsilon (> 0)$  is an arbitrary constant (for example for the zeta function of a ray class in an algebraic number field of degree  $m$ . If  $m \geq 2$  we can allow  $\sum_{n \leq x} |\alpha_n|^2 \ll x^{1+\epsilon}$  in place of  $\sum_{n \leq x} |A_n \alpha_n|^2 \ll x^{1+\epsilon}$ , see the definition in § 1 for the meaning of  $A_n$ ). Note that if  $A_n = O_\epsilon(n^\epsilon)$  then the condition on  $\alpha_n$  is simply  $\sum_{n \leq x} |\alpha_n|^2 \ll x^{1+\epsilon}$ . These results are very general. But out of these GDS only in very special cases (but still a somewhat large class of GDS) we can prove that

$$\frac{1}{T} \int_T^{2T} |F(\frac{1}{2} + \delta + it)|^2 dt \ll_{\epsilon, \delta} T^\epsilon$$

for all  $\delta > 0$  and  $\epsilon > 0$ . Some examples (not already covered by Theorem 7) are (i) zeta function of any ray class of a quadratic field (ii) zeta function of a positive definite quadratic form  $Q(X_1, \dots, X_\ell)$  (in  $\ell \geq 2$  variables and with integer coefficients) namely  $\sum_{n=1}^{\infty} (a_n n^{-\frac{\ell}{2}+1}) n^{-s}$ , where  $a_n$  is the number of  $\ell$ -tuples  $(m_1, \dots, m_\ell)$  of integers with  $Q(m_1, \dots, m_\ell) = n$ . In this case  $m = 1$  and the lower bound is  $\gg T^\delta$  (resp.  $\gg T^{2\delta - \epsilon}$ ) according as  $\ell > 2$  or  $\ell = 2$  see [8].

Instead of enumerating all the applications of this theory we state a beautiful theorem (namely Theorem 11 below). Many other theorems can be deduced in a similar manner by the interested readers from the results of papers mentioned above and the results of § 1, § 2 and § 3, (see also the post-script at the end of this paper).

**THEOREM 11.** Let  $\mathcal{F}$  denote the class of Dirichlet series of the form  $\zeta(s) + \sum_{n=1}^{\infty} a_n n^{-s}$  with complex number sequence  $\{a_n\}$  satisfying  $\sum_{n \leq x} a_n = O(1)$ . Let  $\varphi_j = \varphi_j(s)$  ( $j = 0, 1, 2, \dots, r$ ) be any  $r+1$  Dirichlet series (may

not be distinct) of the class  $\mathcal{F}$ . Let  $P(X_0, X_1, \dots, X_r)$  be any fixed polynomial (with complex coefficients) of degree  $d$  (being the maximum of  $d_0 + d_1 + \dots + d_r$  taken over all monomials  $X_0^{d_0} X_1^{d_1} \dots X_r^{d_r}$  occurring in  $P(X_0, \dots, X_r)$ ). Let  $Q$  be defined by

$$Q = (\varphi_0(s))^{d+1} - P(\varphi_0, \varphi_1, \dots, \varphi_r) = \sum_{n=1}^{\infty} B_n n^{-s}, \quad (\sigma > 1).$$

Then first we have  $B_n \neq 0$  for at least one  $n$  (also  $Q$  is analytic in  $\sigma > 0$ ,  $t \geq 1$ ). Next put

$$F(s) = \sum B_n ((n + \alpha_n)^{-s} - n^{-s}) + Q$$

where  $\{\alpha_n\}$  is any sequence of real numbers with  $|\alpha_n| \leq \frac{1}{3}$ . Then in  $(\sigma \geq \frac{1}{2} - \delta, T \leq t \leq 2T)$ ,  $F(s)$  has  $\gg T \log T$  zeros and in  $(\sigma \geq \frac{1}{2} + \frac{C \log \log T}{\log T}, T \leq t \leq 2T)$  only  $\ll C^{-1} T \log T$  zeros ( $C \geq 1$  being any constant).

**REMARK 1.** If  $d \geq 1$  we can allow  $\sum_{n \leq x} |\alpha_n|^2 \ll x^{1+\epsilon}$  in place of  $|\alpha_n| \leq \frac{1}{3}$ . But then we have to stipulate that  $F(s)$  should be a GDS.

**REMARK 2.** That  $B_n \neq 0$  for at least one  $n$  of course follows since  $Q$  has a pole of order  $(d + 1)$  at  $s = 1$ . But then we mention that the conclusion of Theorem 11 are valid for  $\varphi_0(s) = (1 - 2^{1-s})\zeta(s)$  and  $\varphi_j(s) = \sum_{n=1}^{\infty} a_n^{(j)} n^{-s}$  ( $j = 1, 2, \dots, r$ ) where  $\max |\sum_{n \leq x} a_n^{(j)}| = O(1)$ .

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## POST-SCRIPT

1. In view of Theorem 2, it is important to find Dirichlet series which satisfy (6). This will enable us to prove  $N(\frac{1}{2} + \delta, T) \ll_{\delta} T(\delta > 0)$  for larger and larger class of GDS.

2. In view of Theorem 4, it is important to find  $N_0$  (as large as possible) well-spaced Titchmarsh points with the lower bound  $\geq T^{k\delta - \epsilon}$  (for some  $k > 0$  and every  $\epsilon > 0$ ) on the line segment ( $\sigma = \frac{1}{2} - \delta, T \leq t \leq 2T$ ) for a large class of Dirichlet series. In this direction we have Balasubramanian-Ramachandra functions given by Theorem 10 (Theorems 8 and 9 are special cases of these functions). Also we have the  $\ell$ th derivative ( $\ell \geq 0$  integer) of a class of GDS which satisfy a very general functional equation (see equation (5) of XV<sup>[5]</sup>). The case  $\ell = 0$  is treated in XV<sup>[5]</sup> and it is proved that  $N_0 \gg T$ . We can cover all integers  $\ell$  as follows. We make use of the following lemma.

**LEMMA 2.** *Let  $h(x)$  be an  $n$ -times continuously differentiable function defined in  $a_0 \leq x \leq a_0 + nd_0$ , where  $a_0 > 0, d_0 > 0$  are constants and  $n$  is any fixed integer  $\geq 1$ . Then*

$$\sum_{r=0}^n (-1)^{n-r} \binom{n}{r} h(a_0 + rd_0) = \int_0^{d_0} \cdots \int_0^{d_0} h(a_0 + u_1 + \cdots + u_n) du_1 du_2 \cdots du_n.$$

**PROOF.** Follows by trivial induction.

We apply this lemma to  $h(\sigma) = h(\sigma, t) = \chi(\sigma + it)$  of equation (5) of XV<sup>[5]</sup> and obtain  $|\chi^{(\ell)}(s_0)| \gg T^{k(\frac{1}{2} - \sigma)} (\log T)^{\ell}$  for any fixed  $t (T \leq t \leq 2T)$  and a suitable  $s_0 = \sigma_0 + it$  (with  $\sigma_0$  at a distance of  $O((\log T)^{-1})$  from any arbitrarily given  $\sigma$ ). At the same time for all  $s$  and  $\ell$  we have (by Cauchy's theorem),  $|\chi^{(\ell)}(s)| \ll T^{k(\frac{1}{2} - \sigma)} (\log T)^{\ell}$ .

Next we apply local convexity (see for example the references [PS-1] and [PS-2] below, see especially Theorem 6-C of [PS-2] for a correction in [PS-1]) to the zeta-function like analytic function  $Z^{(\ell)}(s)(\chi(s))^{-1}(\log T)^{-\ell}$  to prove that the integral of its absolute value taken over  $|t - t_0| \leq C(\epsilon)$  on  $\sigma = \frac{1}{2} + \delta$  exceeds  $t_0^{-\epsilon} (T \leq t_0 \leq 2T)$ , where  $C(\epsilon)$  depends only on  $\epsilon$ . From this it follows that for  $Z^{(\ell)}(s)$  we have  $N_0 \gg T$  and the lower bound

is  $\geq T^{k\delta - \epsilon}$ .

3. Next given (arbitrarily)  $N_0$  well-spaced points on  $(\sigma = \frac{1}{2} - \delta, T \leq t \leq 2T)$  we can sometimes obtain a subset (of these points) of cardinality  $\gg N_0$  ( $= T$ , sometimes  $T(\log \log T)^{-1}$ ) Titchmarsh points for a class of Dirichlet series or GDS. But this class of Dirichlet series is a very restricted one. Let  $Z(s)$  be a Dirichlet series (see equation (5) of XV<sup>[5]</sup>) which have

(a) Euler product for  $Z_1(s)$ .

(b) Functional equation with  $1 \leq k \leq 2$ .

(c) Mean-square on the critical line (see equation (6) of the present paper)  
 $\sigma = \frac{1}{2}$ .

(We have to mention that (c) follows from (b))

(d)  $|\chi^{(\ell)}(s)| \asymp t^{k(\frac{1}{2} - \sigma)} (\log t)^{\ell}$  for all integers  $\ell \geq 0$ .

From these we can deduce.

**LEMMA 3.** *Let  $\{t_j\}$  ( $T \leq t_j \leq 2T$ ) be a well-spaced set of points with cardinality  $\gg T$ . Then out of these points we can select a subset of points  $t'_j$  (with cardinality  $\gg T$ ) satisfying*

$$|Z_1(\frac{1}{2} + \delta + it'_j)| \gg 1 \quad \text{and} \quad |Z_1^{(\ell)}(\frac{1}{2} + \delta + it'_j)| \ll 1$$

( $\ell = 1, 2, \dots, \ell_0$ ) where  $\ell_0 \geq 1$  is any integer.

**PROOF.** This lemma is contained implicitly in the proof of Theorem 1 of [PS-3].

From these we can formulate a general principle.

**GENERAL PRINCIPLE.** *In Theorem 11 we can replace  $(\varphi_0(s))^{d+1}$  by  $Q_1 \equiv (F_1(s))^{M_1} (\varphi^{(\ell_1)}(s))^{M_2}$  with integers  $\ell_1 \geq 0, M_1 \geq 0, M_2 \geq 0, M_1 + M_2 \geq 1$ , where  $F_1(s)$  is a power product (with non-negative integral exponents) of derivatives of functions like  $Z(s)$  satisfying (a), (b), (c) and (d) above and  $\varphi(s)$  is either a Balasubramanian-Ramachandra function or a function which has a functional equation such as (5) of XV<sup>[5]</sup>. In place of*



$P(\varphi_0, \varphi_1, \dots, \varphi_r)$  of Theorem 11, we can have a suitable modification say  $Q_2$  such that  $Q_1 - Q_2$  has  $\gg T$  (resp.  $T(\log \log T)^{-1}$ ) well-spaced Titchmarsh points on  $\sigma = \frac{1}{2} - \delta$ . Accordingly we have lower bounds for the number of zeros of  $Q_1 - Q_2$  in  $(\sigma \geq \frac{1}{2} - 2\delta, T \leq t \leq 2T)$  (and upper bounds for  $N(\frac{1}{2} + \delta, T)$  only sometimes). We can say similar things about the  $\{\alpha_n\}$  transformations of  $Q_1 - Q_2$ .

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