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| Pierre Gaillard. Higher order Peregrine breathers solutions to the NLS equation. 2015. hal-01109143

**HAL Id: hal-01109143**

**<https://hal.science/hal-01109143v1>**

Preprint submitted on 24 Jan 2015

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# Higher order Peregrine breathers solutions to the NLS equation.

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January 24, 2015

**Abstract.** The solutions to the one dimensional focusing nonlinear Schrödinger equation (NLS) can be written as a product of an exponential depending on  $t$  by a quotient of two polynomials of degree  $N(N+1)$  in  $x$  and  $t$ . These solutions depend on  $2N-2$  parameters : when all these parameters are equal to 0, we obtain the famous Peregrine breathers which we call  $P_N$  breathers. Between all quasi-rational solutions of the rank  $N$  fixed by the condition that its absolute value tends to 1 at infinity and its highest maximum is located at the point  $(x=0, t=0)$ , the  $P_N$  breather is distinguished by the fact that  $P_N(0,0) = 2N+1$ .

We construct Peregrine breathers of the rank  $N$  explicitly for  $N \leq 11$ . We give figures of these  $P_N$  breathers in the  $(x;t)$  plane; plots of the solutions  $P_N(0;t)$ ,  $P_N(x;0)$ , never given for  $6 \leq N \leq 11$  are constructed in this work. It is the first time that the Peregrine breather of order 11 is explicitly constructed.

PACS numbers :

33Q55, 37K10, 47.10A-, 47.35.Fg, 47.54.Bd

## 1. Introduction

After the first results concerning the NLS equation obtained by Zakharov and Shabat in 1972 who solved it using the inverse scattering method [1, 2] a lot of studies have been realized. The first quasi rational solutions to NLS equation were constructed in 1983 by Peregrine [3]. Akhmediev, Eleonski and Kulagin obtained in 1986 in particular the first higher order analogue of the Peregrine breather [4, 5]. Other analogues of the Peregrine breathers of order 3 and 4 were constructed in a series of articles by Akhmediev et al. [6, 7] using Darboux transformations.

Rational solutions to the NLS equation were written in 2010 as a quotient of two wronskians in [8]. Another representation of the solutions to the NLS equation in terms of a ratio of two wronskians of even order  $2N$  composed of elementary functions using truncated Riemann theta functions in 2011 has been given in [9]. In 2013 rational solutions in terms of determinants which do not involve limits were given in [10].

We recall the representation of the solutions to the NLS equation in terms quasi rational solutions depending a priori on  $2N-2$  parameters at order  $N$  as a ratio of two polynomials of degree  $N(N+1)$  of  $x$  and  $t$  multiplied by an exponential depending on  $t$ . The present paper presents Peregrine breathers as particular case of multi-parametric families of quasi rational solutions to NLS of order  $N$  depending on  $2N-2$  real parameters : Peregrine breather  $P_N$  of order  $N$  are obtained when all the parameters are equal to 0.

## 2. Families of solutions to NLS equation depending on $2N - 2$ parameters and $P_N$ breathers

We consider the focusing NLS equation

$$iv_t + v_{xx} + 2|v|^2v = 0. \quad (1)$$

Then we get the following result [9, 10] :

**Theorem 2.1** *The function  $v$  defined by*

$$v(x, t) = \exp(2it - i\varphi) \times \frac{\det((n_{jk})_{j,k \in [1, 2N]})}{\det((d_{jk})_{j,k \in [1, 2N]})} \quad (2)$$

*is a quasi-rational solution to the NLS equation (1)*

$$iv_t + v_{xx} + 2|v|^2v = 0,$$

*quotient of two polynomials  $N(x, t)$  and  $D(x, t)$  depending on  $2N - 2$  real parameters  $\tilde{a}_j$  and  $\tilde{b}_j$ ,  $1 \leq j \leq N - 1$ .*

*$N$  and  $D$  are polynomials of degrees  $N(N + 1)$  in  $x$  and  $t$ , where*

$$\begin{aligned} n_{j1} &= \varphi_{j,1}(x, t, 0), \quad 1 \leq j \leq 2N & n_{jk} &= \frac{\partial^{2k-2} \varphi_{j,1}}{\partial \epsilon^{2k-2}}(x, t, 0), \\ n_{jN+1} &= \varphi_{j,N+1}(x, t, 0), \quad 1 \leq j \leq 2N & n_{jN+k} &= \frac{\partial^{2k-2} \varphi_{j,N+1}}{\partial \epsilon^{2k-2}}(x, t, 0), \\ d_{j1} &= \psi_{j,1}(x, t, 0), \quad 1 \leq j \leq 2N & d_{jk} &= \frac{\partial^{2k-2} \psi_{j,1}}{\partial \epsilon^{2k-2}}(x, t, 0), \\ d_{jN+1} &= \psi_{j,N+1}(x, t, 0), \quad 1 \leq j \leq 2N & d_{jN+k} &= \frac{\partial^{2k-2} \psi_{j,N+1}}{\partial \epsilon^{2k-2}}(x, t, 0), \\ & & & 2 \leq k \leq N, \quad 1 \leq j \leq 2N \end{aligned}$$

*The functions  $\varphi$  and  $\psi$  are defined in (3), (4), (5), (6).*

$$\begin{aligned} \varphi_{4j+1,k} &= \gamma_k^{4j-1} \sin X_k, & \varphi_{4j+2,k} &= \gamma_k^{4j} \cos X_k, \\ \varphi_{4j+3,k} &= -\gamma_k^{4j+1} \sin X_k, & \varphi_{4j+4,k} &= -\gamma_k^{4j+2} \cos X_k, \end{aligned} \quad (3)$$

for  $1 \leq k \leq N$ , and

$$\begin{aligned} \varphi_{4j+1,N+k} &= \gamma_k^{2N-4j-2} \cos X_{N+k}, & \varphi_{4j+2,N+k} &= -\gamma_k^{2N-4j-3} \sin X_{N+k}, \\ \varphi_{4j+3,N+k} &= -\gamma_k^{2N-4j-4} \cos X_{N+k}, & \varphi_{4j+4,N+k} &= \gamma_k^{2N-4j-5} \sin X_{N+k}, \end{aligned} \quad (4)$$

for  $1 \leq k \leq N$ .

$$\begin{aligned} \psi_{4j+1,k} &= \gamma_k^{4j-1} \sin Y_k, & \psi_{4j+2,k} &= \gamma_k^{4j} \cos Y_k, \\ \psi_{4j+3,k} &= -\gamma_k^{4j+1} \sin Y_k, & \psi_{4j+4,k} &= -\gamma_k^{4j+2} \cos Y_k, \end{aligned} \quad (5)$$

for  $1 \leq k \leq N$ , and

$$\begin{aligned} \psi_{4j+1,N+k} &= \gamma_k^{2N-4j-2} \cos Y_{N+k}, & \psi_{4j+2,N+k} &= -\gamma_k^{2N-4j-3} \sin Y_{N+k}, \\ \psi_{4j+3,N+k} &= -\gamma_k^{2N-4j-4} \cos Y_{N+k}, & \psi_{4j+4,N+k} &= \gamma_k^{2N-4j-5} \sin Y_{N+k}, \end{aligned} \quad (6)$$

for  $1 \leq k \leq N$ .

The arguments  $X_k$  and  $Y_k$  are defined by

$$\begin{aligned} X_\nu &= \kappa_\nu x/2 + i\delta_\nu t - ix_{3,\nu}/2 - ie_\nu/2, \\ Y_\nu &= \kappa_\nu x/2 + i\delta_\nu t - ix_{1,\nu}/2 - ie_\nu/2, \end{aligned}$$

for  $1 \leq \nu \leq 2N$ .

These terms are defined by means of  $\lambda_\nu$  such that  $-1 < \lambda_\nu < 1$ ,  $\nu = 1, \dots, 2N$ ,

$$\begin{aligned} -1 < \lambda_{N+1} < \lambda_{N+2} < \dots < \lambda_{2N} < 0 < \lambda_N < \lambda_{N-1} < \dots < \lambda_1 < 1 \\ \lambda_{N+j} &= -\lambda_j, \quad j = 1, \dots, N. \end{aligned} \quad (7)$$

$\kappa_\nu$ ,  $\delta_\nu$  and  $\gamma_\nu$  are defined by

$$\begin{aligned} \kappa_j &= 2\sqrt{1 - \lambda_j^2}, \quad \delta_j = \kappa_j \lambda_j, \quad \gamma_j = \sqrt{\frac{1 - \lambda_j}{1 + \lambda_j}}, \\ \kappa_{N+j} &= \kappa_j, \quad \delta_{N+j} = -\delta_j, \quad \gamma_{N+j} = 1/\gamma_j, \quad j = 1 \dots N. \end{aligned} \quad (8)$$

The parameters  $a_j$  and  $b_j$  in the form

$$a_j = \sum_{k=1}^{N-1} \tilde{a}_k j^{2k+1} \epsilon^{2k+1}, \quad b_j = \sum_{k=1}^{N-1} \tilde{b}_k j^{2k+1} \epsilon^{2k+1}, \quad 1 \leq j \leq N. \quad (9)$$

Complex numbers  $e_\nu$   $1 \leq \nu \leq 2N$  are defined by

$$e_j = ia_j - b_j, \quad e_{N+j} = ia_j + b_j, \quad 1 \leq j \leq N, \quad a, b \in \mathbf{R}. \quad (10)$$

The terms  $x_{r,\nu}$  ( $r = 3, 1$ ) are defined by

$$x_{r,\nu} = (r - 1) \ln \frac{\gamma_\nu - i}{\gamma_\nu + i}, \quad 1 \leq j \leq 2N. \quad (11)$$

In particular we have the following structure for the  $P_N$  breathers

**Theorem 2.2** *The function  $v_0$  defined by*

$$v_0(x, t) = \exp(2it - i\varphi) \times \left( \frac{\det((n_{jk})_{j,k \in [1, 2N]})}{\det((d_{jk})_{j,k \in [1, 2N]})} \right)_{(\tilde{a}_j = \tilde{b}_j = 0, 1 \leq j \leq N-1)} \quad (12)$$

is the Peregrine breather of order  $N$  solution to the NLS equation (1) whose highest amplitude in module is equal to  $2N + 1$ .

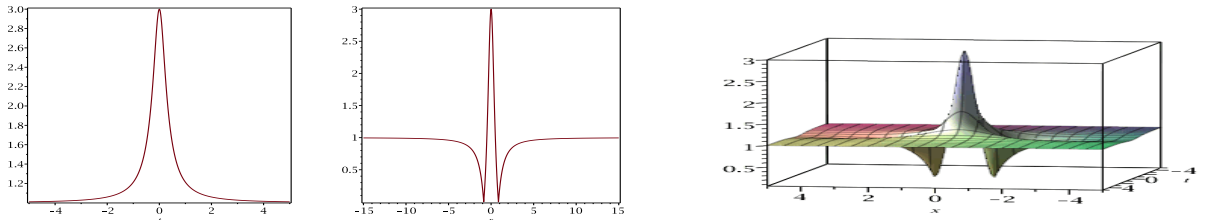
### 3. Peregrine breathers $P_N$ of order $N$

We have already constructed in [9, 11, 12, 13, 14, 15, 16, 17] solutions for the cases  $N = 1$  until  $N = 10$ .

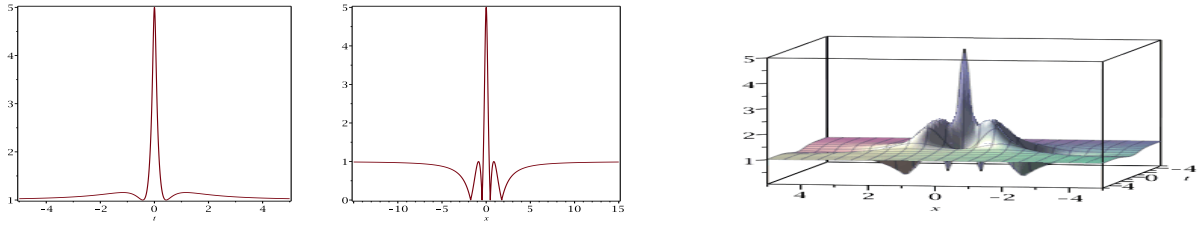
Because of the length of the expressions of polynomials  $N$  and  $D$  of the solutions  $v$  to the NLS equation defined by

$$v(x, t) = \frac{N(x, t)}{D(x, t)} \exp(2it - i\varphi),$$

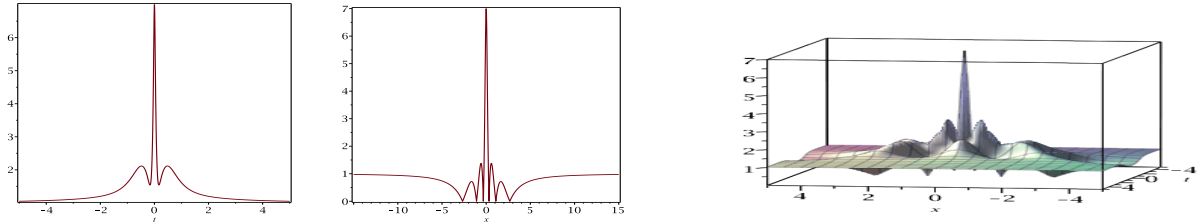
we cannot give it in this paper. We only give figure in the  $(x; t)$  plane and plots of the solutions  $v(0; t)$ ,  $v(x; 0)$ .



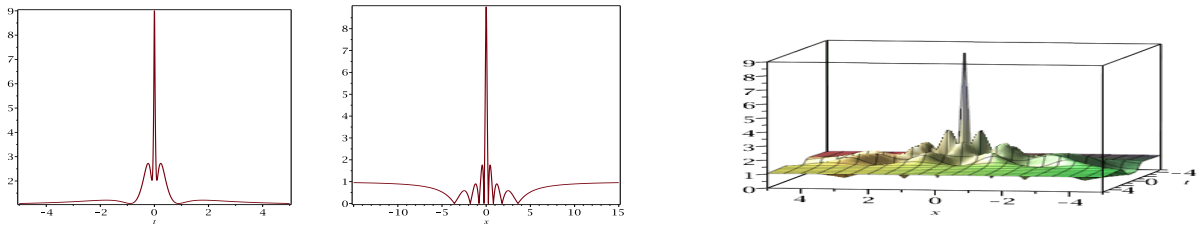
**Figure 1.** Solution to NLS,  $N=1$ , to the left  $v(x, 0)$ ; in the center  $v(0, t)$ ; to the right  $v(x, t)$ .



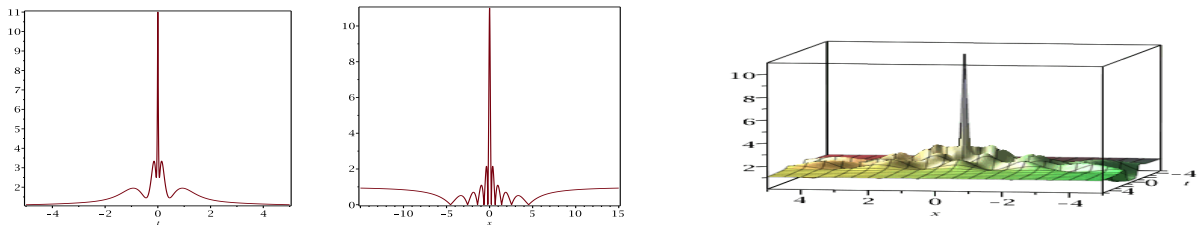
**Figure 2.** Solution to NLS,  $N=2$ , to the left  $v(x, 0)$ ; in the center  $v(0, t)$ ; to the right  $v(x, t)$ .



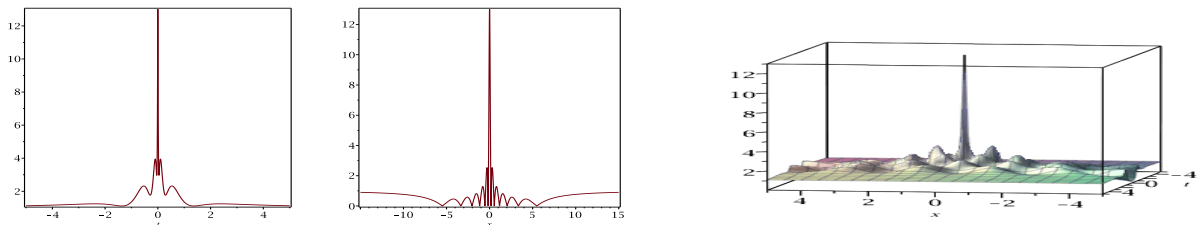
**Figure 3.** Solution to NLS,  $N=3$ , to the left  $v(x, 0)$ ; in the center  $v(0, t)$ ; to the right  $v(x, t)$ .



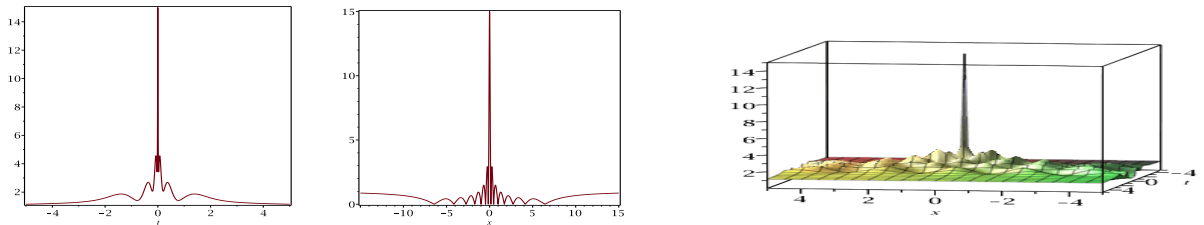
**Figure 4.** Solution to NLS,  $N=4$ , to the left  $v(x, 0)$ ; in the center  $v(0, t)$ ; to the right  $v(x, t)$ .



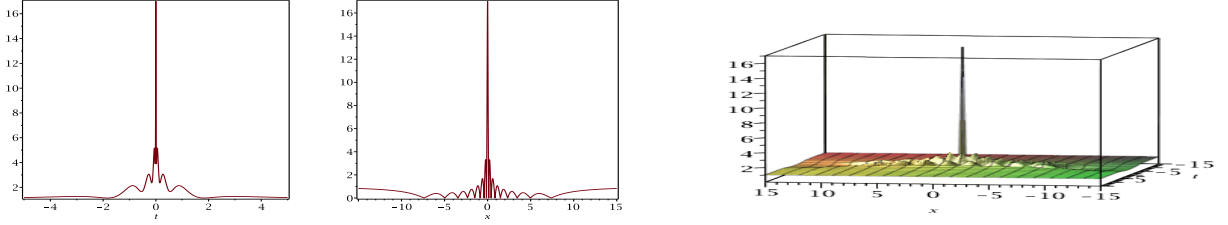
**Figure 5.** Solution to NLS,  $N=5$ , to the left  $v(x, 0)$ ; in the center  $v(0, t)$ ; to the right  $v(x, t)$ .



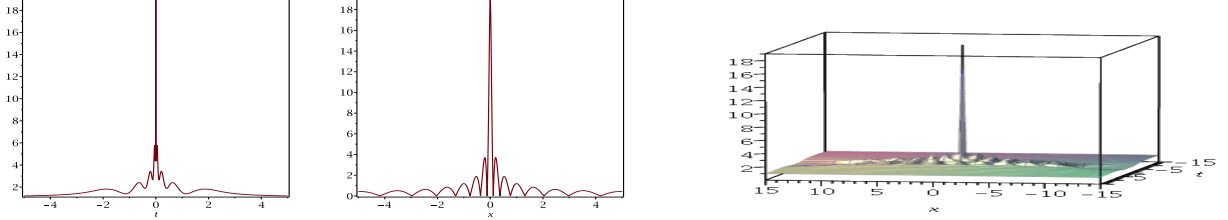
**Figure 6.** Solution to NLS,  $N=6$ , to the left  $v(x, 0)$ ; in the center  $v(0, t)$ ; to the right  $v(x, t)$ .



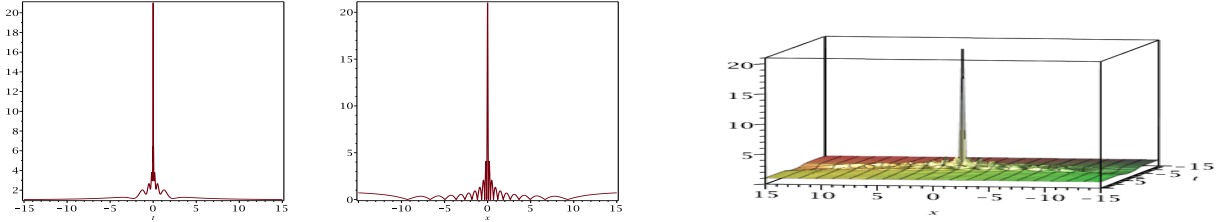
**Figure 7.** Solution to NLS,  $N=7$ , to the left  $v(x, 0)$ ; in the center  $v(0, t)$ ; to the right  $v(x, t)$ .



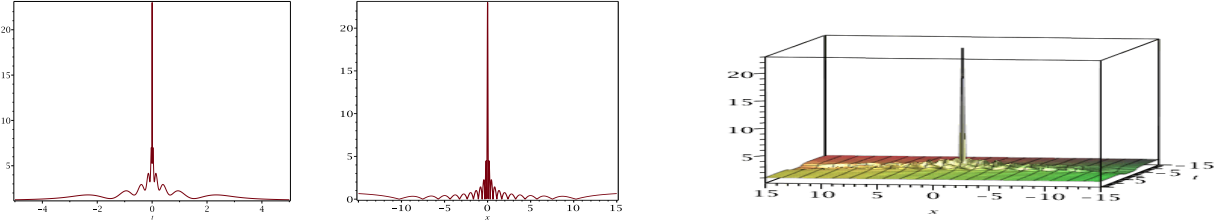
**Figure 8.** Solution to NLS,  $N=8$ , to the left  $v(x, 0)$ ; in the center  $v(0, t)$ ; to the right  $v(x, t)$ .



**Figure 9.** Solution to NLS,  $N=9$ , to the left  $v(x, 0)$ ; in the center  $v(0, t)$ ; to the right  $v(x, t)$ .



**Figure 10.** Solution to NLS,  $N=10$ , to the left  $v(x, 0)$ ; in the center  $v(0, t)$ ; to the right  $v(x, t)$ .



**Figure 11.** Solution to NLS,  $N=11$ , to the left  $v(x, 0)$ ; in the center  $v(0, t)$ ; to the right  $v(x, t)$ .

It is important to say that, contrary to the  $P_1$  breather, all higher ranks  $P_N$  breathers can be deformed thus generating the multi-rogue-waves (MRW) solutions.

Actually  $N = 11$  is a greatest rank for which this work is completed and this is one of the results of this article. We postpone to give a more precise study of this eleventh Peregrine breather to another publication.

#### 4. Conclusion

The method described in the present paper provides a powerful tool to get explicit solutions to the NLS equation and to understand the behavior of rogue waves. It is the first time that the Peregrine breather order 11 is explicitly constructed; the complete expression as a quotient of polynomials of order 132 in  $x$  and  $t$  is too long to be presented here. Moreover, the studies for the initial conditions  $x = 0$  and  $t = 0$ , in the cases  $N = 6$  until  $N = 10$  are completely new, having never yet been presented.

We hope that several promising applications may be a result of this theoretical study. Recently, the notion of the rogue wave has been transferred into the realm of nonlinear optics. Experimental studies have shown that continuous-wave laser radiation in optical fibers splits into separate

pulses and those pulses can reach very high amplitudes [18]. This kind of solution has been observed in plasma [19], Bose-Einstein condensates [20], fiber optics [21] or on a water surface [22, 23]. In the case of nonlinear optics and hydrodynamics, the results were checked until order 5.

This study leads to a better understanding of the phenomenon of rogue waves, and it would be relevant to go on with more higher orders.

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