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SPECTRAL ANALYSIS OF THE SCHRÖDINGER OPERATOR ON BINARY TREE-SHAPED NETWORKS AND APPLICATIONS

KAÏS AMMARI, DENIS MERCIER, AND VIRGINIE RÉGNIER

ABSTRACT. In this paper we analyse the spectrum of the dissipative Schrödinger operator on binary tree-shaped networks. As applications, we study the stability of the Schrödinger system using a Riesz basis as well as the transfer function associated to the system.

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1. Introduction

First of all, we introduce some notation needed to formulate the problem under consideration, which is simply that of [20]. We refer to [20] for more details.

To construct the binary tree \mathcal{T} which will be considered in the following, we need some definitions (recall that a tree is a planar connected graph without paths).

A multi-index $\bar{\alpha}$ is a k -tuple $(\alpha_1, \dots, \alpha_k)$ if k lies in $\mathbb{N} - \{0\}$ and it is empty if $k = 0$. For a fixed integer n , we choose for I the set of multi-indices $\bar{\alpha}$, with length k in $\{0, 1, \dots, n\}$, such that, if $k \neq 0$, $\alpha_j \in \{1, 2\}$, for all j in $\{1, \dots, k\}$. Then the set of vertices V of the tree \mathcal{T} is $V := (\cup_{\bar{\alpha} \in I} \mathcal{O}_{\bar{\alpha}}) \cup \{\mathcal{R}\}$ where \mathcal{R} is an additional vertex which will be the root of the tree \mathcal{T} .

The edges are denoted by $e_{\bar{\alpha}}$ with $\bar{\alpha}$ in I . Note that the number of edges is the cardinal of I and it holds: $|I| = N = 2^{n+1} - 1$.

Define, for any non-empty multi-indices $\bar{\alpha} = (\alpha_1, \dots, \alpha_k)$ and $\bar{\beta} = (\beta_1, \dots, \beta_m)$, the multi-index $\bar{\alpha} \circ \bar{\beta} := (\alpha_1, \dots, \alpha_k, \beta_1, \dots, \beta_m)$ of length $(k + m)$. Then, for a non-empty multi-index $\bar{\alpha} = (\alpha_1, \dots, \alpha_k)$, the edge $e_{\bar{\alpha}}$ is chosen to have the extremities $\mathcal{O}_{\bar{\alpha}}$ and $\mathcal{O}_{\bar{\alpha}'}$ with $\bar{\alpha} = \bar{\alpha}' \circ (\alpha_k)$ and the edge e (corresponding to the case $\bar{\alpha} = \emptyset$) has the extremities \mathcal{R} and \mathcal{O} .

See Figure 1 for a representation in the case $n = 2$.

By the multiplicity of a vertex of \mathcal{T} we mean the number of edges that branch out from that vertex. If the multiplicity is equal to one, the vertex is called exterior. Otherwise, it is said to be interior. We denote by Int the set of the interior vertices of the tree \mathcal{T} and by Dir the set of the exterior vertices, except \mathcal{R} , which has a particular status in our problem. Dir is chosen for Dirichlet (see the problem below). A dissipation law is imposed at the root \mathcal{R} which explains why it is isolated from the other exterior vertices.

Define

$$I_{Int} = \{\bar{\alpha}; \mathcal{O}_{\bar{\alpha}} \in Int\}, I_{Dir} = \{\bar{\alpha}; \mathcal{O}_{\bar{\alpha}} \in Dir\}$$

which are the sets of the indices of the interior and exterior vertices, except \mathcal{R} , respectively.

Note that the multiplicity of each interior point of the tree \mathcal{T} is equal to 3 and that the integer $(n + 1)$ represents the maximum level of the binary tree \mathcal{T} .

Furthermore, the length of the edge $e_{\bar{\alpha}}$ is equal to 1. Then, $e_{\bar{\alpha}}$ will be parametrized by its arc length by means of the functions $\pi_{\bar{\alpha}}$, defined in $[0, 1]$ such that $\pi_{\bar{\alpha}}(0) = \mathcal{O}_{\bar{\alpha}}$ and $\pi_{\bar{\alpha}}(1)$ is the other vertex of this edge. This choice seems unconventional but it is made for technical reasons.

In this paper, we study the dissipative Schrödinger operator under the tree-shaped network \mathcal{T} introduced above. The case $N \geq 3$ is the one we are interested in: it corresponds to $n \geq 1$. The case $N = 1$ is well-known. See [24] and [20] concerning the model.

More precisely, we consider the following initial and boundary value problem:

$$(1.1) \quad \frac{\partial u_{\bar{\alpha}}}{\partial t}(x, t) + i \frac{\partial^2 u_{\bar{\alpha}}}{\partial x^2}(x, t) = 0, \quad 0 < x < 1, t > 0, \bar{\alpha} \in I,$$

$$(1.2) \quad i u(1, t) + \frac{\partial u}{\partial x}(1, t) = 0, u_{\bar{\alpha}}(0, t) = 0, \bar{\alpha} \in I_{Dir}, t > 0,$$

$$(1.3) \quad u_{\bar{\alpha} \circ (\beta)}(1, t) = u_{\bar{\alpha}}(0, t), \quad t > 0, \beta = 1, 2, \bar{\alpha} \in I_{Int},$$

$$(1.4) \quad \sum_{\beta=1}^2 \frac{\partial u_{\bar{\alpha} \circ (\beta)}}{\partial x}(1, t) = \frac{\partial u_{\bar{\alpha}}}{\partial x}(0, t), \quad t > 0, \bar{\alpha} \in I_{Int},$$

$$(1.5) \quad u_{\bar{\alpha}}(x, 0) = (u_{\bar{\alpha}})_0(x), \quad 0 < x < 1, \bar{\alpha} \in I,$$

where $u_{\bar{\alpha}} : [0, 1] \times (0, +\infty) \rightarrow \mathbb{R}$, $\bar{\alpha} \in I$, is the transverse displacement of the edge $e_{\bar{\alpha}}$. These functions allow us to identify the network with its rest graph. In this sense, the vertices of \mathcal{T} are called nodes and the edges are called branches.

Note that in the problem above, (1.1) is the Schrödinger equation imposed on all the branches of the tree, (1.2) concerns the root and the other exterior nodes (recall that $u = u_{\bar{\alpha}}$ with $\bar{\alpha} = \emptyset$ and that this empty multi-index is

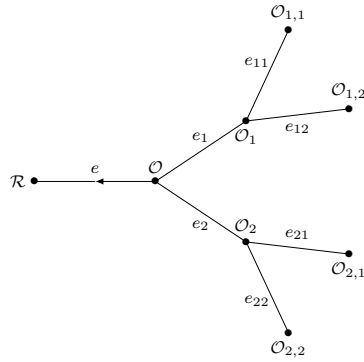


FIGURE 1. A Tree-Shaped network

chosen for the edge containing the root \mathcal{R}). A dissipation law is imposed at the root while Dirichlet conditions are put at the other exterior nodes. At last, (1.3) and (1.4) are transmission conditions.

In the last few years various physical models of multi-link flexible structures consisting of finitely many interconnected flexible elements such as strings, beams, plates, shells have been of great interest. See the references by Ali Mehmeti, von Below and Nicaise in [27] as well as [19], [20], [24] and the references therein. The spectral analysis of such structures has some applications to control or stabilization problems (cf. [24]).

For interconnected strings (corresponding to a second-order operator on each string), a lot of results have been obtained: the asymptotic behaviour of the eigenvalues (see the references by Ali Mehmeti, von Below and Nicaise in [27] as well as [2]), the relationship between the eigenvalues and algebraic theory (cf. papers by von Below, Nicaise and [24]), qualitative properties of solutions (see papers by von Below cited in [27] for example) etc...

For interconnected beams (corresponding to a fourth-order operator on each beam), some results on the asymptotic behaviour of the eigenvalues and on the relationship between the eigenvalues and algebraic theory were obtained by Nicaise and Dekoninck with different kinds of connections using the method developed by von Below in [16] to get the characteristic equation associated to the eigenvalues.

Mercier and Régnier used the same method in [26] to compute the spectrum for a hybrid system of N flexible beams connected by n vibrating point masses. This type of structure was studied by Castro and Zuazua in many papers (see [18] and the papers by the same authors cited in [27] as well as one by Castro and Hansen also cited there).

In another paper (see [27]), Mercier and Régnier used the technique of exterior matrices due to W. H. Paulsen (presented for other purposes in [29]) which D. Mercier had already used in the same type of context in [25]. The aim of these papers was to establish controllability.

Later on, they have investigated the same problem as in [19]. In that paper, Chen and al. have established the exponential stability of the problem but with an assumption on the material constants. They seem to think that the exponential stability can not hold without this assumption. Mercier and Régnier prove, using another method (that of the exterior matrices - Chen and al. had used a moment method), that in fact, the exponential stability always holds.

In this paper a feedback stabilization problem for Schrödinger equations in networks is studied. See [4]-[11], [24]. In [13], Banica and Ignat consider the Schrödinger equation on a network formed by a tree with the last generation of edges formed by infinite strips. They prove dispersive estimates which are useful for solving the linear Schrödinger equation.

Let us also cite [33], a paper in which Zhang and Xu study the stability of wave networks. They give recursive expressions that can be used to establish the stability of any given tree-shaped network. The geometry of their problem is close to ours but we give a general result of stability on binary trees with the Schrödinger equation and not the wave equation.

This paper is organized as follows:

In Section 2, the proper functional setting for system (1.1)-(1.5) is given: in particular a dissipative operator is defined. The system is proved to be well-posed. A notion of energy is defined by (2.11) and the energy of the solution of the system is proved to be a non-increasing function of the variable t .

In Section 3, the spectrum of the Schrödinger operator associated to the dissipative system (1.1)-(1.5) is studied. The

families of eigenvalues and the dimension of the corresponding eigenspaces are given for the dissipative operator as well as for the associated conservative operator. The localization and the asymptotic behaviour of the large eigenvalues are also studied. All the properties of the spectrum developed in this section are useful for the applications.

In Section 4, some of the eigenfunctions of the dissipative operator associated to system (1.1)-(1.5) are proved to form a Riesz basis of the space they span.

The rewriting of the solution in this Riesz basis allows to prove, in Section 5, that the energy of the solution decreases exponentially to a non-vanishing value depending on the initial datum. The decay rate is explicitly given in Theorem 5.1.

Section 6 is dedicated to the transfer function of the dissipative operator associated to system (1.1)-(1.5).

We study, in Section 7, the stabilization result for (7.65)-(7.70) (which is system (1.1)-(1.5) with another feedback law) by the frequency domain technique. The explicit decay rate of the energy of the solution is given.

2. Well-posedness of the system

In order to study system (1.1)-(1.5) we need a proper functional setting. We define the following space

$$H = \prod_{\bar{\alpha} \in I} L^2(0, 1)$$

equipped with the inner product

$$(2.6) \quad \langle \underline{u}, \tilde{\underline{u}} \rangle_H = \sum_{\bar{\alpha} \in I} \int_0^1 u_{\bar{\alpha}}(x) \tilde{u}_{\bar{\alpha}}(x) dx.$$

It is well-known that system (1.1)-(1.5) may be rewritten as the first order evolution equation

$$(2.7) \quad \begin{cases} \underline{u}' = \mathcal{A}_d \underline{u}, \\ \underline{u}(0) = \underline{u}_0, \end{cases}$$

where the operator $\mathcal{A}_d : \mathcal{D}(\mathcal{A}_d) \subset H \rightarrow H$ (the index d is for dissipative) is defined by

$$\mathcal{A}_d \underline{u} := (-i \partial_x^2 u_{\bar{\alpha}})_{\bar{\alpha} \in I},$$

with

$$\mathcal{D}(\mathcal{A}_d) := \left\{ \underline{u} \in \prod_{\bar{\alpha} \in I} H^2(0, 1) : \text{satisfies (2.8) to (2.10) hereafter} \right\},$$

$$(2.8) \quad i u(1) + \frac{du}{dx}(1) = 0, \quad u_{\bar{\alpha}}(0) = 0, \quad \bar{\alpha} \in I_{Dir},$$

$$(2.9) \quad u_{\bar{\alpha} \circ (\beta)}(1) = u_{\bar{\alpha}}(0), \quad \beta = 1, 2,$$

$$(2.10) \quad \sum_{\beta=1}^2 \frac{du_{\bar{\alpha} \circ (\beta)}}{dx}(1) = \frac{du_{\bar{\alpha}}}{dx}(0), \quad \bar{\alpha} \in I_{Int}.$$

The natural energy $E(t)$ of a solution $\underline{u} = (u_{\bar{\alpha}})_{\bar{\alpha} \in I}$ of (1.1)-(1.5) is defined by:

$$(2.11) \quad E(t) = \frac{1}{2} \sum_{\bar{\alpha} \in I} \int_0^1 |u_{\bar{\alpha}}(x, t)|^2 dx.$$

Proposition 2.1. (i) For an initial datum $\underline{u}_0 \in H$, there exists a unique solution $\underline{u} \in C([0, +\infty), H)$ to problem (2.7). Moreover, if $\underline{u}_0 \in \mathcal{D}(\mathcal{A}_d)$, then

$$\underline{u} \in C([0, +\infty), \mathcal{D}(\mathcal{A}_d)) \cap C^1([0, +\infty), H).$$

(ii) The solution \underline{u} of (1.1)-(1.5) with initial datum in $\mathcal{D}(\mathcal{A}_d)$ satisfies the dissipation law:

$$(2.12) \quad E'(t) = -|u(1, t)|^2 \leq 0,$$

(recall that $u = u_{\bar{\alpha}}$ with $\bar{\alpha} = \emptyset$ and that this empty multi-index is chosen for the edge containing the root \mathcal{R}).

Therefore the energy is a non-increasing function of the time variable t .

Proof. (i) By Lumer-Phillips' theorem (see [30, 32]), it suffices to show that \mathcal{A}_d is dissipative and maximal.

We first prove that \mathcal{A}_d is dissipative. Take $\underline{u} \in \mathcal{D}(\mathcal{A}_d)$. Then

$$\langle \mathcal{A}_d \underline{u}, \underline{u} \rangle_H = \sum_{\bar{\alpha} \in I} \int_0^1 -i \partial_x^2 u_{\bar{\alpha}}(x) \overline{u_{\bar{\alpha}}(x)} dx.$$

By integration by parts and using the transmission and boundary conditions, it holds:

$$(2.13) \quad \Re(\langle \mathcal{A}_d \underline{u}, \underline{u} \rangle_H) = -|u(1)|^2 \leq 0.$$

This shows the dissipativeness of \mathcal{A}_d (recall that $u(1)$ is the value of the solution at the root \mathcal{R}).

Let us now prove that \mathcal{A}_d is maximal, i.e. that $\lambda I - \mathcal{A}_d$ is surjective for some $\lambda > 0$.

Let $\underline{f} \in H$. We look for $\underline{u} \in \mathcal{D}(\mathcal{A}_d)$ solution of

$$(2.14) \quad (\lambda I - \mathcal{A}_d) \underline{u} = \underline{f},$$

or equivalently

$$(2.15) \quad \lambda u_{\bar{\alpha}} + i \partial_x^2 u_{\bar{\alpha}} = f_{\bar{\alpha}}, \quad \forall \bar{\alpha} \in I.$$

Multiplying this identity by a test function $\underline{\phi}$, integrating in space and using integration by parts, it follows, since $\underline{u} \in \mathcal{D}(\mathcal{A}_d)$,

$$(2.16) \quad \sum_{\bar{\alpha} \in I} \int_0^1 (\lambda u_{\bar{\alpha}}(x) \overline{\phi_{\bar{\alpha}}(x)} + i \partial_x u_{\bar{\alpha}}(x) \partial_x \overline{\phi_{\bar{\alpha}}(x)}) dx + u(1) \overline{\phi(1)} = \sum_{\bar{\alpha} \in I} \int_0^1 f_{\bar{\alpha}}(x) \overline{\phi_{\bar{\alpha}}(x)} dx.$$

This problem has a unique solution

$$\underline{u} \in V := \left\{ \underline{u} \in \prod_{\bar{\alpha} \in I} H^1(0, 1) : u_{\bar{\alpha}}(0) = 0, \bar{\alpha} \in I_{Dir}, \text{ and satisfies (2.9)} \right\}$$

by Lax-Milgram's lemma, because the left-hand side of (2.16) is coercive on V . If we consider $\underline{\phi} \in \prod_{\bar{\alpha}} \mathcal{D}(0, 1) \subset V$,

then \underline{u} satisfies

$$\lambda u_{\bar{\alpha}} + i \partial_x^2 u_{\bar{\alpha}} = f_{\bar{\alpha}} \quad \text{in } \mathcal{D}'(0, 1), \quad \bar{\alpha} \in I.$$

This directly implies that $\underline{u} \in V \cap \prod_{\bar{\alpha} \in I} H^2(0, 1)$. Coming back to (2.16) and integrating by parts lead to:

$$\sum_{\bar{\alpha} \in I} (i \partial_x u_{\bar{\alpha}}(1) \overline{\phi_{\bar{\alpha}}(1)} - i \partial_x u_{\bar{\alpha}}(0) \overline{\phi_{\bar{\alpha}}(0)}) + u(1) \overline{\phi(1)} = 0.$$

Consequently, by taking particular test functions $\underline{\phi}$, we obtain

$$\begin{aligned} i u(1) + \frac{du}{dx}(1) &= 0, \\ \sum_{\beta=1}^2 \frac{du_{\bar{\alpha} \circ (\beta)}}{dx}(1) &= \frac{\partial u_{\bar{\alpha}}}{\partial x}(0), \quad \bar{\alpha} \in I_{Int}. \end{aligned}$$

In summary we have found $\underline{u} \in \mathcal{D}(\mathcal{A}_d)$ satisfying (2.14), which finishes the proof of (i).

(ii) To prove (ii), it suffices to derive the energy (2.11) for regular solutions and to use system (1.1)-(1.5). The calculations are analogous to those of the proof of the dissipativeness of \mathcal{A}_d in (i), and then, are left to the reader. \square

3. Spectral analysis

The goal of this section is to look for the eigenvalues and eigenvectors of the dissipative operator \mathcal{A}_d as well as those of the associated conservative operator \mathcal{A}_0 . To that end, the operator \mathcal{A}^ϵ is defined like \mathcal{A}_d in Section 2 except for equation (2.8) which is replaced by:

$$(3.17) \quad i\epsilon u(1) + \frac{du}{dx}(1) = 0, \quad u_{\bar{\alpha}}(0) = 0, \quad \bar{\alpha} \in I_{Dir}.$$

Thus \mathcal{A}_0 and \mathcal{A}_d are \mathcal{A}^ϵ with $\epsilon = 0$ and $\epsilon = 1$ respectively.

The main result (which is proved at the end of the section) is the following one:

Theorem 3.1. (*Spectra of \mathcal{A}_0 and \mathcal{A}_d*)

i) Let $\sigma^{(0)}$ be the spectrum of the conservative operator \mathcal{A}_0 , then

$$(3.18) \quad \sigma^{(0)} = \sigma_1^{(0)} \cup \sigma_2^{(0)},$$

where

$$\sigma_1^{(0)} = \{i(k\pi)^2 : k \in \mathbb{Z}^*\} \cup \left\{ i \left(k\pi + \frac{\pi}{2} \right)^2 : k \in \mathbb{Z} \right\},$$

and, if n is even,

$$\sigma_2^{(0)} = \left\{ i \left(k\pi + \frac{1}{2} \arg(z_{A,j}^{(n)}) \right)^2 : j = 2, \dots, n+1, k \in \mathbb{Z} \right\}$$

if n is odd,

$$\sigma_2^{(0)} = \left\{ i \left(k\pi + \frac{1}{2} \arg(z_{A,j}^{(n)}) \right)^2 : j = 1, \dots, n+1, k \in \mathbb{Z}, n \text{ odd} \right\}$$

where $z_{A,j}^{(n)}$, $j = 1, \dots, n+1$ is the family of the complex roots of the polynomial $P_{A,n}$ defined in Proposition 3.4.

ii) Let σ be the spectrum of the dissipative operator \mathcal{A}_d , then

$$(3.19) \quad \sigma = \sigma_1 \cup \sigma_2 \cup \tilde{\sigma}_2,$$

where $\sigma_1 = \sigma_1^{(0)}$, $\tilde{\sigma}_2 = \{(\lambda_k)_{k \in S} : S \text{ is finite, } \Re(\lambda_k) < 0\}$ and

$$\sigma_2 = \{i(\omega_{j,k})^2 : j = 1, \dots, n+1, k \in \mathbb{Z}, |k| \geq k_0\},$$

k_0 being an integer.

Moreover

$$\Re(i(\omega_{j,k})^2) < 0, \quad \forall i(\omega_{j,k})^2 \in \sigma_2,$$

and the following asymptotic behaviour holds:

$$(3.20) \quad i(\omega_{j,k})^2 = i \left(k^2 \pi^2 + k\pi \arg(z_{A,j}^{(n)}) + \frac{(\arg(z_{A,j}^{(n)}))^2}{4} \right) + 2\pi\gamma_j + o(1)$$

where γ_j is a real negative number (its expression is given by (3.36)). The polynomial $P_{A,n}$ defined in Proposition 3.4 admits $n+1$ distinct complex roots $z_{A,j}^{(n)} \neq 1$, $j = 1, \dots, n+1$ with modulus equal to 1.

Note that when n is even, (-1) is a root of $P_{A,n}$ which is denoted by $z_{A,1}^{(n)}$ (cf. Lemma 3.6). Since $k\pi + \frac{1}{2} \arg(z_{A,1}^{(n)}) = k\pi + \frac{\pi}{2}$, when n is even, the index j starts from 2 in the definition of $\sigma_2^{(0)}$. This ensures $\sigma_1^{(0)} \cap \sigma_2^{(0)} = \emptyset$ for any value of $n \geq 1$.

3.1. The characteristic equation. In order to prove Theorem 3.1, we establish the characteristic equation.

Proposition 3.2. (*The characteristic equation*)

Define, for $\omega \in \mathbb{C}$, $M(\omega)$ and T the matrices $M(\omega) := \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix}$ and $T := \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$.

The complex $\lambda = i\omega^2$ ($\omega \in \mathbb{C}^*$, $\omega \neq k\pi$, $k \in \mathbb{Z}$ and $\omega \neq (\pi/2) + k\pi$, $k \in \mathbb{Z}$) is an eigenvalue of the operator \mathcal{A}^ε defined at the beginning of the section on the binary tree \mathcal{T} with level $(n+1)$ (constructed in the introduction) if and only if

$$(3.21) \quad (i \in 1) e^{M(\omega)} T \dots e^{M(\omega)} T e^{M(\omega)} T e^{M(\omega)} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0$$

with $(n+1)$ matrices $e^{M(\omega)}$ in the product.

Proof. First, the vector $F_{\bar{\alpha}}(x) := \begin{pmatrix} \phi_{\bar{\alpha}}(x) \\ \partial_x \phi_{\bar{\alpha}}(x) \end{pmatrix}$ is introduced to reduce the order of the eigenvalue problem which reads: $\lambda = i\omega^2$ ($\omega \in \mathbb{C}$) is an eigenvalue of \mathcal{A}^ϵ with associated eigenvector $\underline{\phi} \in \mathcal{D}(\mathcal{A}^\epsilon)$ if and only if $\underline{\phi}$ satisfies the transmission and boundary conditions (3.17), (2.9) and (2.10) of Section 2 and

$$(EP) : F'_{\bar{\alpha}}(x) = M(\omega)F_{\bar{\alpha}}(x) \text{ on } (0,1), \quad \forall \bar{\alpha} \in I.$$

In the following, ω is supposed to be different from 0 (0 is not an eigenvalue, cf. the proof of Theorem 3.3). Thus

$$F_{\bar{\alpha}}(x) = e^{M(\omega)x}F_{\bar{\alpha}}(0) \text{ where } e^{M(\omega)x} = \begin{pmatrix} \cos(\omega x) & \frac{\sin(\omega x)}{\omega} \\ -\omega \sin(\omega x) & \cos(\omega x) \end{pmatrix}.$$

For simplicity, the computations to establish the characteristic equation are first presented in the case $I_{Dir} = \{(1,1); (1,2); (2,1); (2,2)\}$ ($n = 2$, see Figure 1. The results will then be generalized to any tree defined as in the introduction.

The Dirichlet condition at the exterior vertices (except the root, see the second part of (3.17) for $\bar{\alpha} \in I_{Dir}$) implies

$$F_{\bar{\alpha}}(0) = \begin{pmatrix} 0 \\ \partial_x \phi_{\bar{\alpha}}(0) \end{pmatrix} \text{ for } \bar{\alpha} \in I_{Dir}. \text{ Then } F_{\bar{\alpha}}(1) = e^{M(\omega)}F_{\bar{\alpha}}(0) \text{ i.e. } F_{\bar{\alpha}}(1) = \phi'_{\bar{\alpha}}(0) \begin{pmatrix} \frac{\sin(\omega)}{\omega} \\ \cos(\omega) \end{pmatrix}, \text{ for } \bar{\alpha} \in I_{Dir}$$

and $\omega \neq 0$.

Now the continuity at the interior vertices \mathcal{O}_1 and \mathcal{O}_2 (condition (2.9)) implies:

$$(3.22) \quad \phi'_{j,1}(0) \cdot \frac{\sin(\omega)}{\omega} = \phi'_{j,2}(0) \cdot \frac{\sin(\omega)}{\omega} \text{ for } j = 1 \text{ and } j = 2.$$

Either $\omega = k\pi$ with $k \in \mathbb{Z}^*$ (first family of eigenvalues which is studied in the proof of Theorem 3.3) or the following condition is imposed:

$$(3.23) \quad \phi'_{j,1}(0) = \phi'_{j,2}(0) \text{ for } j = 1 \text{ and } j = 2.$$

Then the second transmission condition at the interior vertices \mathcal{O}_1 and \mathcal{O}_2 (condition (2.10)) implies, for $j = 1$ and $j = 2$:

$$F_j(0) = \phi'_{j,1}(0) \begin{pmatrix} \frac{\sin(\omega)}{\omega} \\ 2\cos(\omega) \end{pmatrix}.$$

Once more, this vector is multiplied by $e^{M(\omega)}$ and it follows from the continuity at the vertex \mathcal{O} : either $\omega = \frac{\pi}{2} + k\pi$, $k \in \mathbb{Z}$ (second family of eigenvalues which is studied in the proof of Theorem 3.3) or

$$(3.24) \quad \phi'_{1,1}(0) = \phi'_{2,1}(0).$$

Using the diagonal matrix T , it can be written as:

$$(3.25) \quad F(1) = \phi'_{1,1}(0)e^{M(\omega)}Te^{M(\omega)}Te^{M(\omega)} \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

At last, for $\omega \in \mathbb{C}^*$, $\omega \neq k\pi$, $k \in \mathbb{Z}$ and $\omega \neq (\pi/2) + k\pi$, $k \in \mathbb{Z}$, the dissipation law (first part of condition (3.17)) imposed at the root \mathcal{R} gives the characteristic equation:

$$(3.26) \quad (i\epsilon 1)e^{M(\omega)}Te^{M(\omega)}Te^{M(\omega)} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0.$$

And for a general tree as described in the introduction, the characteristic equation is:

$$(3.27) \quad (i\epsilon 1)e^{M(\omega)}T \dots e^{M(\omega)}Te^{M(\omega)}Te^{M(\omega)} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0.$$

with $(n + 1)$ matrices $e^{M(\omega)}$ in the product where $(n + 1)$ is the maximum level of the binary tree \mathcal{T} defined in the introduction.

Note that the cases $\omega = k\pi, k \in \mathbb{Z}^*$ and $\omega = (\pi/2) + k\pi, k \in \mathbb{Z}$ are treated in next section where the dimension of the associated eigenspace is computed. For the other values of ω , the condition $\partial_x \phi_{\bar{\alpha}}(0) = \partial_x \phi_{\bar{\gamma}}(0)$ for any $\bar{\alpha}$ and $\bar{\gamma}$ in I_{Dir} is imposed. Thus the dimension of the corresponding eigenspace is one. \square

3.2. The families of eigenvalues. This section is devoted to the characterization of the different types of eigenvalues for both the conservative and the dissipative operators defined at the beginning of this section. The geometric multiplicity of each eigenvalue is also computed.

Theorem 3.3. (Families of eigenvalues of \mathcal{A}_0 and \mathcal{A}_d)

The operators \mathcal{A}_0 and \mathcal{A}_d are defined in Section 3.1 on the binary tree \mathcal{T} with level $(n + 1)$ constructed in the introduction.

- (1) The complex $\lambda = i\omega^2$ ($\omega \in \mathbb{C}$) is an eigenvalue of the operator \mathcal{A}_0 if and only if
- either $\omega = k\pi, k \in \mathbb{Z}^*$ and the dimension of the corresponding eigenspace is equal to: $2^n - 1$,
 - or $\omega = (\pi/2) + k\pi, k \in \mathbb{Z}$ and the dimension of the corresponding eigenspace is equal to: $\frac{1}{3}2^n + \frac{2}{3}$ if n is even and $\frac{1}{3}(2^n - 2)$ if n is odd,
 - or $\omega = ((\arg(z_{A,j}^{(n)})/2) + k\pi, k \in \mathbb{Z}$ where $z_{A,j}^{(n)}$ ($j = 1, \dots, n$ if n is odd, $j = 2, \dots, n$ if n is even) is the family of the roots of the polynomial $P_{A,n}$ defined in Proposition 3.4 below (if n is even, the first root $z_{A,1}^{(n)} = -1$ is excluded). The dimension of the eigenspace associated to each eigenvalue is one.
- (2) The complex $\lambda = i\omega^2$ ($\omega \in \mathbb{C}$) is an eigenvalue of the operator \mathcal{A}_d if and only if
- either $\omega = k\pi, k \in \mathbb{Z}^*$ and the dimension of the corresponding eigenspace is equal to: $2^n - 1$,
 - or $\omega = (\pi/2) + k\pi, k \in \mathbb{Z}$ and the dimension of the corresponding eigenspace is equal to: $\frac{1}{3}(2^n - 1)$ if n is even and $\frac{1}{3}(2^n - 2)$ if n is odd,
 - or ω satisfies $\omega \neq k\pi, \omega \neq (\pi/2) + k\pi, k \in \mathbb{Z}$ and

$$(3.28) \quad P_{A,n}(z) + \frac{P_{B,n}(z)}{\omega} = 0.$$

where $z = e^{2i\omega}$ and the polynomials $P_{A,n}$ and $P_{B,n}$ are defined in Proposition 3.4 below. The localization of the corresponding eigenvalues is studied in Proposition 3.1. The dimension of the eigenspace associated to each eigenvalue is one.

Note that the first two families of eigenvalues of \mathcal{A}^ϵ lie on the imaginary axis. The third one also lies on the imaginary axis for the conservative operator.

The third family of eigenvalues of the dissipative operator lies on the half-plane of complex numbers with a negative real part.

Proof. Let us begin by proving that $\lambda = 0$ is not an eigenvalue. If 0 is an eigenvalue associated to the eigenvector $\underline{u} = (u_{\bar{\alpha}})_{\bar{\alpha} \in I}$, then

$$0 = \langle \mathcal{A}_d \underline{u}, \underline{u} \rangle_H = \sum_{\bar{\alpha} \in I} \int_0^1 -i \partial_x^2 u_{\bar{\alpha}}(x) \overline{u_{\bar{\alpha}}(x)} dx.$$

By integration by parts and using the transmission and boundary conditions, it holds

$$(3.29) \quad 0 = -i|u(1)|^2 - \sum_{\bar{\alpha} \in I_{Dir}} \partial_x u_{\bar{\alpha}}(0) \overline{u_{\bar{\alpha}}(0)} - \sum_{\bar{\alpha} \in I} \int_0^1 |\partial_x u_{\bar{\alpha}}(x)|^2 dx.$$

Since $\overline{u_{\bar{\alpha}}(0)} = 0$ for any $\bar{\alpha} \in I_{Dir}$, $u(1) = 0$ and $\partial_x u_{\bar{\alpha}} \equiv 0$ for any $\bar{\alpha} \in I$. It follows $u_{\bar{\alpha}} \equiv 0$ for any $\bar{\alpha} \in I$ which is a contradiction with the fact that 0 is an eigenvalue associated to the eigenvector \underline{u} .

The characteristic equation is established in the latest section and studied in Section 3.3. We are interested only in the dimension properties here.

- (1) First case $\omega = k\pi, k \in \mathbb{Z}^*$:

We use here the notation introduced in Proposition 3.2 and its proof.

The matrix $e^{M(\omega)}$ is $(-1)^k I$ where I is for the 2×2 identity matrix. The "eigenvector" $F_{\bar{\alpha}}$ for $\bar{\alpha}$ in I_{Dir} is:

$$F_{\bar{\alpha}}(x) := \begin{pmatrix} 0 \\ f_{\bar{\alpha}} \end{pmatrix} = f_{\bar{\alpha}} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \text{ with } f_{\bar{\alpha}} = \partial_x \phi_{\bar{\alpha}}(0). \text{ Recall that } I_{Dir} \text{ contains } 2^n \text{ elements. Then condition (2.9)}$$

is satisfied at all the interior nodes since the multiplication by $(-1)^k I$ does not change the first component of $F_{\bar{\alpha}}$ which is always zero. The other transmission condition (2.10) at the interior nodes is translated into a sum of the second components. At last the dissipation law at the root leads to one equation: $(-1)^{k(n+1)} \sum_{\bar{\alpha} \in I_{Dir}} f_{\bar{\alpha}} = 0$ hence the announced dimension.

- (2) Second case $\omega = (\pi/2) + k\pi, k \in \mathbb{Z}$:

There are 2^n degrees of freedom as in the first case (cardinal of I_{Dir}) but the number of constraints increases.

The "eigenvector" $F_{\bar{\alpha}}$ for $\bar{\alpha}$ in I_{Dir} is still: $F_{\bar{\alpha}}(0) := f_{\bar{\alpha}} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ with $f_{\bar{\alpha}} = \partial_x \phi_{\bar{\alpha}}(0)$. But the matrix $e^{M(\omega)}$ is not diagonal any more. It is: $e^{M(\omega)} := (-1)^k \begin{pmatrix} 0 & \omega^{-1} \\ -\omega & 0 \end{pmatrix}$.

Thus, for $\bar{\alpha}$ in I_{Dir} (i.e. for any $\bar{\alpha} \in I$ such that $|\bar{\alpha}| = n$), $F_{\bar{\alpha}}(1) = e^{M(\omega)} F_{\bar{\alpha}}(0) = (-1)^k \omega^{-1} f_{\bar{\alpha}} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

Step $n-1$: The continuity at the interior nodes $\mathcal{O}_{\bar{\alpha}}$ where the length $|\bar{\alpha}|$ is equal to $n-1$ implies the 2^{n-1} independent equations:

$$f_{\bar{\alpha}o(1)} = f_{\bar{\alpha}o(2)} \text{ for any } \bar{\alpha} \in I \text{ such that } |\bar{\alpha}| = n-1.$$

Then the value $f_{\bar{\alpha}o(1)}$ is renamed $f_{\bar{\alpha}}$ and $F_{\bar{\alpha}}(1) = e^{M(\omega)} F_{\bar{\alpha}}(0) = -f_{\bar{\alpha}} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Step $n-2$: At the next 2^{n-2} interior nodes, no continuity condition is needed and the other transmission condition is translated into a sum at each node:

$$F_{\bar{\alpha}}(0) = f_{\bar{\alpha}} \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \text{ where } -(f_{\bar{\alpha}o(1)} + f_{\bar{\alpha}o(2)}) \text{ is renamed } f_{\bar{\alpha}} \text{ and } |\bar{\alpha}| = n-2. \text{ Then } F_{\bar{\alpha}}(1) = (-1)^k \frac{1}{\omega} f_{\bar{\alpha}} \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

The process continues. Two cases are to be envisaged:

- (a) **Case of an even n .** The number of equations is: $2^1 + 2^3 + \dots + 2^{n-1} = 2(1 + 2^2 + \dots + 2^{n-2}) = 2(4^0 + 4^1 + \dots + 4^{n/2-1}) = 2 \left(\frac{4^{n/2} - 1}{3} \right) = \frac{2}{3}(2^n - 1)$.

At last, for $\bar{\alpha} \in I$, such that $|\bar{\alpha}| = 0$, $F(0) = f \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ where $-(f_{(1)} + f_{(2)})$ is renamed f and $F(1) = (-1)^k \frac{1}{\omega} f \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. This means, by definition, that $\phi(1) = -f \frac{(-1)^k}{\omega}$ and $\partial_x \phi(1) = 0$ where $\phi_{\bar{\alpha}}$ is the eigenfunction.

For the conservative operator \mathcal{A}_0 , the dissipation law (3.17) is then satisfied since $\epsilon = 0$. Thus the dimension of the corresponding eigenspace is: $2^n - \frac{2}{3}(2^n - 1) = \frac{1}{3}2^n + \frac{2}{3}$.

For the dispersive operator \mathcal{A}_d ($\epsilon = 1$), the dissipation law is satisfied if $f = 0$. Thus the dimension of the corresponding eigenspace is:

$$2^n - \left(1 + \frac{2}{3}(2^n - 1) \right) = \frac{1}{3}2^n - \frac{1}{3}.$$

- (b) **Case of an odd n .** The number of equations is: $2^0 + 2^2 + \dots + 2^{n-1} = \frac{4^{(n-1)/2+1} - 1}{3} = \frac{2^{n+1} - 1}{3}$.

At last, for $\bar{\alpha} \in I$, such that $|\bar{\alpha}| = 0$, $F(0) = (-1)^k \frac{1}{\omega} f \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ where $f_{(1)} = f_{(2)}$ is renamed f and $F(1) = -f \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. This means, by definition, that $\phi(1) = 0$ and $\partial_x \phi(1) = -f$.

For both operators \mathcal{A}_0 and \mathcal{A}_d , the dissipation law is satisfied if $f = 0$. Thus the dimension of the corresponding eigenspace is:

$$2^n - \left(1 + \frac{2^{n+1} - 1}{3}\right) = \frac{1}{3}2^n - \frac{2}{3}.$$

Note that the constraints are always independent from one another. This can be proved by iteration on p for the two separate cases $n = 2p$ and $n = 2p + 1$.

- (3) Third case: it is treated in next section. Cf. the end of the proof of Proposition 3.2 for the dimension of the corresponding eigenspace.

□

3.3. Iterative study of the characteristic equation.

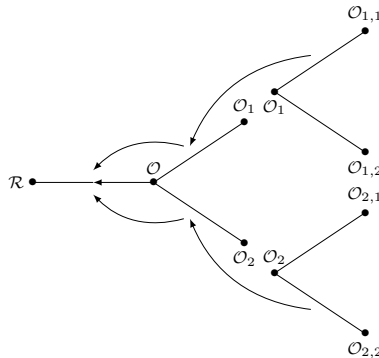


FIGURE 2. Iterative strategy for analysing the spectrum

Proposition 3.4. (*Iterative rewriting of the characteristic equation*)

The complex $\lambda = i\omega^2$ ($\omega \in \mathbb{C}^*$, $\omega \neq k\pi$, $k \in \mathbb{Z}$ and $\omega \neq (\pi/2) + k\pi$, $k \in \mathbb{Z}$) is an eigenvalue of the operator \mathcal{A}^ϵ defined in Section 3.1 on the binary tree \mathcal{T} with level $(n + 1)$ (constructed in the introduction) if and only if

$$(3.30) \quad P_{A,n}(z) + \epsilon \frac{P_{B,n}(z)}{\omega} = 0.$$

where $z = e^{2i\omega}$ and the polynomials $P_{A,n}$ and $P_{B,n}$ are defined by:

$$(3.31) \quad P_{A,m+1}(z) = 2(z + 1)P_{A,m}(z) + (z - 1)P_{B,m}(z), \forall m \in \mathbb{N},$$

$$(3.32) \quad P_{B,m+1}(z) = 2(z - 1)P_{A,m}(z) + (z + 1)P_{B,m}(y), \forall m \in \mathbb{N},$$

$$(3.33) \quad P_{A,0}(z) = z + 1, \quad P_{B,0}(z) = z - 1.$$

Proof. The iterative procedure follows from the rewriting of the product $e^{M(\omega)}T$ with $z = e^{2i\omega}$. Indeed, if $s = e^{i\omega}$, using the classical trigonometric formulae leads to:

$$e^{M(\omega)}T = \frac{1}{2} \begin{pmatrix} s + \frac{1}{s} & -\frac{2i}{\omega} \left(s - \frac{1}{s}\right) \\ i\omega \left(s - \frac{1}{s}\right) & 2 \left(s + \frac{1}{s}\right) \end{pmatrix} \text{ and } e^{M(\omega)} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} -\frac{i}{\omega} \left(s - \frac{1}{s}\right) \\ \left(s + \frac{1}{s}\right) \end{pmatrix}.$$

Since $z = s^2$, it is also:

$$e^{M(\omega)}T = \frac{1}{2s} \begin{pmatrix} z + 1 & -\frac{2i}{\omega}(z - 1) \\ i\omega(z - 1) & 2(z + 1) \end{pmatrix} \text{ and } e^{M(\omega)} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{2s} \begin{pmatrix} -\frac{i}{\omega}(z - 1) \\ z + 1 \end{pmatrix}$$

Now, for $m = 0$, the characteristic equation reads:

$$(i\epsilon 1)e^{M(\omega)} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0 \text{ or } (z+1) + \frac{\epsilon}{\omega}(z-1) = 0,$$

which is the announced result. By iteration, the rest follows. \square

The rewriting of the characteristic equation as (3.30) gives the expression of the eigenvalues of both operators as well as their asymptotic behaviour: the conservative one, \mathcal{A}_0 and the dissipative one, \mathcal{A}_d , using Rouché's Theorem. Hence Theorem 3.1 announced at the beginning of Section 3 which is proved in the following.

Proof. i) Let ω be a solution of (3.30) then $\Im(\omega)$ is bounded from below. Indeed, the characteristic equation involves

$$\text{the matrix } e^{M(\omega)} = \begin{pmatrix} \cos(\omega) & \frac{\sin(\omega)}{\omega} \\ -\omega \sin(\omega) & \cos(\omega) \end{pmatrix}.$$

If $\omega = x + iy$ with y tending to $-\infty$, it follows

$$e^{M(\omega)} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{e^{-y}}{2} \begin{pmatrix} -\frac{1}{y}e^{ix} + o(1/y) \\ e^{ix} + o(1) \end{pmatrix}$$

and

$$e^{M(\omega)} T = \frac{e^{-y}}{2} \begin{pmatrix} e^{ix} + o(1) & -\frac{2}{y}e^{ix} + o(1/y) \\ -ye^{ix} + o(y) & 2e^{ix} + o(1) \end{pmatrix}$$

and

$$e^{M(\omega)} T e^{M(\omega)} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{e^{-2y}}{4} \begin{pmatrix} e^{2ix}(-3/y) + o(1/y) \\ 3e^{2ix} + o(y) \end{pmatrix}.$$

By iteration the characteristic equation (3.27) is:

$$\frac{e^{-(n+1)y}}{2^{n+1}} (3^n \cdot e^{-i(n+1)x} + o(1)) = 0$$

when y tends to $-\infty$. This is a contradiction with the behaviour of the exponential function.

So we have proved that $\Im(\omega)$ is bounded from below.

ii) First asymptotic behaviour of the solution of (3.30).

Let us define the complex-valued functions f and g , respectively by: $f(\omega) = P_A(e^{2i\omega})$ and $g(\omega) = \frac{P_B(e^{2i\omega})}{\omega}$. From the previous step, the solutions ω of (3.30) such that $|\omega| \rightarrow \infty$ satisfy $\lim_{|\omega| \rightarrow \infty} g(\omega) = 0$. Moreover the large roots of f are $w_{j,k}^0 = k\pi + \frac{1}{2} \arg(z_{A,j}^{(n)})$, $j = 1, \dots, n$, $k \in \mathbb{Z}$, $|k| \rightarrow \infty$. Thus, we can apply Rouché's Theorem to the functions f and g as follows: we fix j and k large enough and we consider the disk Γ_k with center $w_{j,k}^0$ and radius $r_k = \frac{1}{\sqrt{|k|}}$.

Now we prove that there exist $k_1 \in \mathbb{N}$ and $c_1 > 0$ such that for all $|k| \geq k_1$ then

$$(3.34) \quad \forall \omega \in \partial\Gamma_k, |f(\omega)| > c_1 r_k.$$

Indeed, for such a ω it holds

$$f(\omega) = P_{A,n}(e^{2iw_{j,k}^0 + r_k e^{i\theta}}), \theta \in [0, 2\pi].$$

But $e^{2iw_{j,k}^0 + r_k e^{i\theta}} = z_{A,j}^{(n)} e^{r_k e^{i\theta}} = z_{A,j}^{(n)} + z_{A,j}^{(n)} r_k e^{i\theta} + O\left(\frac{1}{|k|}\right)$. Consequently,

$$f(\omega) = P_{A,n}(z_{A,j}^{(n)}) + (P_{A,n})'(z_{A,j}^{(n)}) r_k e^{i\theta} + O\left(\frac{1}{|k|}\right),$$

and we get (3.34) since $P_{A,n}(z_{A,j}^{(n)}) = 0$ and $(P_{A,n})'(z_{A,j}^{(n)}) \neq 0$ (see Lemma 3.6). Finally, f and $f + g$ have the same number of zeroes on Γ , where each zero is counted as many times as its multiplicity, since there exist $c_2 > 0$ and $k_0 \in \mathbb{N}$, $k_0 \geq k_1$ such that if $|k| \geq k_0$ then

$$|g(\omega)| \leq \frac{c_2}{|k|} \leq c_1 r_k < |f(\omega)|, \forall \omega \in \Gamma_k.$$

Hence we deduce the following asymptotic behaviour of the roots of (3.30):

$$\omega_{j,k} = k\pi + \frac{1}{2} \arg(z_{A,j}^{(n)}) + o(1), j = 1, \dots, n, k \in \mathbb{Z}, |k| \geq k_0.$$

iii) Second asymptotic behaviour of the solution of (3.30).

From the previous step we can write for a fixed $j = 1, \dots, n$:

$$\omega_{j,k} = k\pi + \frac{1}{2} \arg(z_{A,j}^{(n)}) + \epsilon_k, \lim_{|k| \rightarrow \infty} \epsilon_k = 0.$$

Thus (3.30) gives

$$\begin{aligned} 0 &= P_{A,n}(e^{2ik\pi + i \arg(z_{A,j}^{(n)}) + 2i\epsilon_k}) + \frac{P_{B,n}(e^{2ik\pi + i \arg(z_{A,j}^{(n)}) + 2i\epsilon_k})}{k\pi + \frac{1}{2} \arg(z_{A,j}^{(n)}) + \epsilon_k} \\ &= P_{A,n}(z_{A,j}^{(n)} + 2iz_{A,j}^{(n)}\epsilon_k + o(\epsilon_k)) + \frac{P_{B,n}(z_{A,j}^{(n)} + o(1))}{k\pi} + o\left(\frac{1}{k}\right) \\ &= 2iz_{A,j}^{(n)}\epsilon_k(P_{A,n})'(z_{A,j}^{(n)}) + o(\epsilon_k) + \frac{P_{B,n}(z_{A,j}^{(n)})}{k\pi} + o\left(\frac{1}{k}\right), \end{aligned}$$

which leads to

$$(3.35) \quad \omega_{j,k} = k\pi + \frac{1}{2} \arg(z_{A,j}^{(n)}) - i\frac{\gamma_j}{k} + o\left(\frac{1}{k}\right),$$

where

$$(3.36) \quad \gamma_j = -\frac{P_{B,n}(z_{A,j}^{(n)})}{2\pi z_{A,j}^{(n)}(P_{A,n})'(z_{A,j}^{(n)})}.$$

Now, from Property (v) of the following Lemma, γ_j is real and non-vanishing. Note that (3.35) implies

$$i(\omega_{j,k})^2 = i\left(k^2\pi^2 + k\pi \arg(z_{A,j}^{(n)}) + \frac{(\arg(z_{A,j}^{(n)}))^2}{4}\right) + 2\pi\gamma_j + o(1).$$

Thus γ_j is negative since the problem we consider is dissipative. The proof is complete. \square

Remark 3.5. *Existence of generalized eigenfunctions.* As announced in Theorem 3.3 the dimension of the eigenspace associated to each eigenvalue of the third family of the eigenvalues of \mathcal{A}_0 and \mathcal{A}_d is one. In other words, if $\lambda \in \sigma_2^{(0)}$ or if $\lambda \in \tilde{\sigma}_2 \cup \sigma_2$ (see Theorem 3.1 for the notation) then the geometrical multiplicity of λ is one.

Obviously, if $\lambda \in \sigma_2^{(0)}$, then its algebraic multiplicity is also one since \mathcal{A}_0 is skew-adjoint.

The part ii) of the previous proof shows that the large eigenvalues of σ_2 have an algebraic multiplicity equal to one. Moreover, without loss of generality, we can assume that all the eigenvalues of σ_2 are algebraically simple (see the definition of σ_2 in Theorem 3.1: the integer k_0 may be chosen large enough in order that σ_2 does not contain any eigenvalue with an algebraic multiplicity different from one).

On the other hand, the algebraic multiplicity of each eigenvalue of the finite set $\tilde{\sigma}_2$ remains an open question.

For convenience, let us recall that if $\lambda \in \tilde{\sigma}_2$ has an algebraic multiplicity $\mu_\lambda \geq 2$ and since the geometrical multiplicity is one then there exists a Jordan chain $\phi_0, \phi_1, \dots, \phi_{\mu_\lambda-1}$

$$(\lambda I - \mathcal{A}_d)\phi_0 = 0, (\lambda I - \mathcal{A}_d)\phi_j = \phi_{j-1}, j = 1, \dots, \mu_\lambda - 1,$$

$\{\phi_0, \phi_1, \dots, \phi_{\mu_\lambda-1}\}$ providing a basis of $N[(\lambda I - \mathcal{A}_d)^{\mu_\lambda}]$. Later on, we will refer to such a family as the set of generalized eigenfunctions associated to λ .

Lemma 3.6. (Properties of the polynomials P_A and P_B)

The polynomials $P_{A,n}$ and $P_{B,n}$ defined in the latter proposition have the following properties:

- (i) $\deg P_{A,n} = \deg P_{B,n} = n + 1$.
- (ii) $P_{A,n}$ admits $n + 1$ distinct complex roots $z_{A,j}^{(n)} \neq 1$, $j = 1, \dots, n + 1$ with modulus equal to 1.
- (iii) $P_{A,n}(-1) = 0$ if n is even (this root is chosen to be $z_{A,1}^{(n)}$ in the main Theorem).
- (iv) $P_{B,n}(z_{A,j}^{(n)}) \neq 0$, $j = 1, \dots, n + 1$.
- (v) The quotient $\frac{P_{B,n}(z_{A,j}^{(n)})}{z_{A,j}^{(n)} P'_{A,n}(z_{A,j}^{(n)})}$ is a real number different from 0, for $j = 1, \dots, n + 1$.

Proof. First 1 is not a root of $P_{A,n}$. The proof is done by descending induction. Using (3.31), it is clear that if 1 is a root of $P_{A,n+1}$, it is also a root of $P_{A,n}$. But 1 is not a root of $P_{A,0}$. Hence the result.

For the other results, let us construct other sequences of polynomials which are easier to study. The first step is to use the transformation φ defined on $(i\mathbb{R})$ by $\varphi(u) = \frac{u+1}{u-1}$. Then we define the following two sequences:

$$q_{A,n}(u) = P_{A,n}(\varphi(u)) \text{ and } q_{B,n}(u) = P_{B,n}(\varphi(u)).$$

At last the sequences of polynomials $r_{A,n}$ and $r_{B,n}$ are given by:

$$(3.37) \quad \begin{cases} q_{A,n}(u) = \left(\frac{2}{u-1}\right)^{n+1} r_{A,n}(u), \\ q_{B,n}(u) = \left(\frac{2}{u-1}\right)^{n+1} r_{B,n}(u). \end{cases}$$

Calculations left to the reader lead to:

$$(3.38) \quad \begin{cases} r_{A,0}(u) = u, \quad r_{B,0}(u) = 1, \\ r_{A,n+1}(u) = 2ur_{A,n}(u) + r_{B,n}(u), \\ r_{B,n+1}(u) = 2r_{A,n}(u) + ur_{B,n}(u). \end{cases}$$

Now define the following two sequences of polynomials:

$$(3.39) \quad \begin{cases} Q_{A,n}(y) = (-i)^{n+1} r_{A,n}(iy), \\ Q_{B,n}(y) = (-i)^n r_{B,n}(iy). \end{cases}$$

They satisfy the iteration laws:

$$(3.40) \quad \begin{cases} Q_{A,0}(y) = y, \quad Q_{B,0}(y) = 1, \\ Q_{A,n+1}(y) = 2yQ_{A,n}(y) - Q_{B,n}(y), \\ Q_{B,n+1}(y) = 2Q_{A,n}(y) + yQ_{B,n}(y). \end{cases}$$

From Lemma 3.7 follow Properties (i), (ii) and (iv).

Property (iii) comes from Property (i.2) of Lemma 3.7 above: n even implies $Q_{A,n}$ is odd. Thus $Q_{A,n}(0) = 0$ which is equivalent to $P_{A,n}(-1) = 0$.

As for (v), it requires the rewriting of the quotient in terms of Q_A and Q_B .

Introduce $u_j^{(n)} := \varphi(z_{A,j}^{(n)})$, then $P_{B,n}(z_{A,j}^{(n)}) = q_{B,n}(u_j^{(n)})$ since the reciprocal function of φ is φ itself. And

$$(3.41) \quad \begin{aligned} P'_{A,n}(z) &= \frac{d}{dz}(q_{A,n})(\varphi(z)) = \varphi'(z)(q_{A,n})'(\varphi(z)) \\ &= -\frac{2}{(z-1)^2}(q_{A,n})'(\varphi(z)) = -\frac{2}{(\varphi(u)-1)^2}(q_{A,n})'(u) = -\frac{1}{2}(u-1)^2(q_{A,n})'(u). \end{aligned}$$

The quotient is:

$$(3.42) \quad \frac{P_{B,n}(z_{A,j}^{(n)})}{z_{A,j}^{(n)} P'_{A,n}(z_{A,j}^{(n)})} = \frac{q_{B,n}(u_j^{(n)})}{(-1/2)\varphi(u_j^{(n)})(u_j^n - 1)^2 (q_{A,n})'(u_j^{(n)})} = \frac{-2}{((u_j^{(n)})^2 - 1)} \frac{q_{B,n}(u_j^{(n)})}{(q_{A,n})'(u_j^{(n)})}.$$

Now, since $u_j^{(n)}$ lies in $(i\mathbb{R})$, $(u_j^{(n)})^2 - 1$ is real and non-positive. Thus the first factor is real and non-negative. It remains to be seen whether the other factor is well defined, real and non-vanishing. To this end, it is rewritten in terms of the other sequences of polynomials: $Q_{A,n}$ and $Q_{B,n}$.

Since $(q_{A,n})'(u) = \frac{2^{n+1}}{(u-1)^{n+1}} \left[-\frac{1}{(n+1)(u-1)} r_{A,n}(u) + (r_{A,n})'(u) \right]$ and since $r_{A,n}(u_j^{(n)}) = 0$ (by definition of $u_j^{(n)}$), it holds:

$$(3.43) \quad \frac{q_{B,n}(u_j^{(n)})}{(q_{A,n})'(u_j^{(n)})} = \frac{r_{B,n}(u_j^{(n)})}{(r_{A,n})'(u_j^{(n)})} = \frac{Q_{B,n}(y_j^{(n)})}{(Q_{A,n})'(y_j^{(n)})},$$

where $u_j^{(n)} = iy_j^{(n)}$.

Thus the quotient (3.42) is real since $y_j^{(n)}$ is real. It is well defined and non-vanishing due to Lemma 3.7 below: the multiplicity of the roots of $Q_{A,n}$ is one and both polynomials $Q_{A,n}$ and $Q_{B,n}$ have no common root. This ends the proof. \square

Lemma 3.7. (*Properties of the polynomials Q_A and Q_B*)

Let $\{Q_{A,n}\}_{n \in \mathbb{N}}$ and $\{Q_{B,n}\}_{n \in \mathbb{N}}$ be two sequence of polynomials defined for all $y \in \mathbb{R}$ as follows:

$$(3.44) \quad Q_{A,n+1}(y) = 2yQ_{A,n}(y) - Q_{B,n}(y), \forall n \in \mathbb{N},$$

$$(3.45) \quad Q_{B,n+1}(y) = 2Q_{A,n}(y) + yQ_{B,n}(y), \forall n \in \mathbb{N},$$

$$(3.46) \quad Q_{A,0}(y) = y, \quad Q_{B,0}(y) = 1$$

Then the following properties hold:

(i)

(i.1) $\deg Q_{A,n} = n + 1$ and $\deg Q_{B,n} = n$.

(i.2) If n is even (resp. odd) then $Q_{A,n}$ is odd (resp. even) and $Q_{B,n}$ is even (resp. odd).

(i.3) $\lim_{y \rightarrow +\infty} Q_{A,n}(y) = \lim_{y \rightarrow +\infty} Q_{B,n}(y) = +\infty$.

(ii) $Q_{A,n}$ admits $n + 1$ distinct real roots $y_{A,j}$, $j = 1, \dots, n + 1$.

(iii) $Q_{B,n}$ admits n distinct real roots $y_{B,j}$, $j = 1, \dots, n$.

(iv) The roots of $Q_{A,n}$ and $Q_{B,n}$ are ordered alternately, i.e.;

$$y_{A,1}^{(n)} < y_{B,n,1}^{(n)} < y_{A,2}^{(n)} < y_{B,2}^{(n)} < \dots < y_{B,n}^{(n)} < y_{A,n+1}^{(n)}.$$

Proof. (i) is easily checked by iteration. We will prove (ii), (iii) (iv) by iteration. For convenience we start the iteration at $n = 1$, since $n = 0$ is a particular case.

For $n = 1$, $Q_{A,1} = 2y^2 - 1$ and $Q_{B,1}(y) = 3y$ so the basis hypothesis is satisfied.

Let $n \geq 1$, be fixed and assume that we have the inductive hypothesis, i.e $Q_{A,n}$ and $Q_{B,n}$ satisfy, (ii),(iii), (iv).

From (3.44) and (iv) we see that the sign of $Q_{A,n+1}$ changes at each root of $Q_{A,n}$ and we deduce that $Q_{A,n+1}$ has at least n distinct real roots $y_{A,j}^{(n+1)}$, $j = 2, \dots, n + 1$ such that

$$y_{A,1}^{(n)} < y_{A,2}^{(n+1)} < y_{A,2}^{(n)} < y_{A,3}^{(n+1)} < \dots < y_{A,n+1}^{(n+1)} < y_{A,n+1}^{(n)}.$$

Now from (i.3) and (iv) we have $Q_{A,n+1}(y_{A,n+1}^{(n)}) = -Q_{B,n}(y_{A,n+1}^{(n)}) < 0$. Therefore again from (i.3) we see that $Q_{A,n+1}$ admits a supplementary root $y_{A,n+2}^{(n+1)}$ greater than $y_{A,n+1}^{(n)}$ and by symmetry $Q_{A,n+1}$ admits also another root $y_{A,1}^{(n+1)}$

smaller than $y_{A,1}^{(n)}$. At this step we have proved that $Q_{A,n+1}$ has exactly $n+2$ distinct real roots alternately with the roots of $Q_{A,n}$, i.e

$$(3.47) \quad y_{A,1}^{(n+1)} < y_{A,1}^{(n)} < y_{A,2}^{(n+1)} < y_{A,2}^{(n)} < y_{A,3}^{(n+1)} < \dots < y_{A,n+1}^{(n+1)} < y_{A,n+1}^{(n)} < y_{A,n+2}^{(n+1)}.$$

Now, for any root $y_{A,j}^{(n+1)}$ of $Q_{A,n+1}$, $j \in \{1, \dots, n+2\}$ we have from (3.44)

$$0 = 2 y_{A,j}^{(n+1)} Q_{A,n}(y_{A,j}^{(n+1)}) - Q_{B,n}(y_{A,j}^{(n+1)}),$$

therefore from (3.45)

$$Q_{B,n+1}(y_{A,j}^{(n+1)}) = 2 \left(1 + (y_{A,j}^{(n+1)})^2\right) Q_{A,n}(y_{A,j}^{(n+1)}).$$

From (3.47) we see that the sign of $Q_{A,n}$ changes at each root $y_{A,j}^{(n+1)}$ of $Q_{A,n+1}$, then the previous identity shows that the same fact holds also for $Q_{B,n+1}$. Consequently, between two consecutive roots $y_{A,j}^{(n+1)}$ and $y_{A,j+1}^{(n+1)}$ of $Q_{A,n+1}$, there is a root of $Q_{B,n+1}$. Finally the inductive hypothesis is true for $n+1$.

□

Figure 3 represents the polynomials $Q_{A,3}$, $Q_{B,3}$ and $Q_{A,4}$:

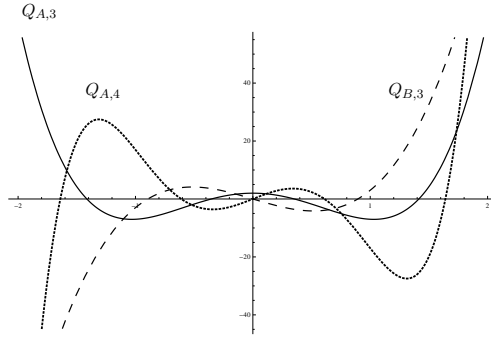


FIGURE 3. $Q_{A,3}(y) = 8y^4 - 17y^2 + 2$, $Q_{B,3}(y) = 15y^3 - 12y$, $Q_{A,4}(y) = 16y^5 - 49y^3 + 16y$.

4. Riesz basis

In this section, it is proved that the generalized eigenfunctions of the dissipative operator \mathcal{A}_d associated to the eigenvalues in $\sigma_2 \cup \tilde{\sigma}_2$ (introduced in Section 2 and Theorem 3.1 for the spectrum) form a Riesz basis of the subspace of H which they span (denoted by H_2 in next section). To this end, we use Theorem 1.2.10 of [1] which is a rewriting of Guo's version of Bari Theorem with another proof (see [22]).

The resolvent of the conservative operator \mathcal{A}_0 restricted to H_2 is compact since the domain of \mathcal{A}_0 is compactly embedded in H . Thus, it is enough to show that the eigenfunctions of \mathcal{A}_0 associated to the eigenvalues in $\sigma_2^{(0)}$ and those of the dissipative operator \mathcal{A}_d associated to the eigenvalues in $\sigma_2 \cup \tilde{\sigma}_2$ are quadratically close to one another, except from a finite number of eigenfunctions.

The parity of n plays a role in the proof: if n is even, one eigenfunction has a different form. It is, for any fixed value of k in \mathbb{Z} , the eigenfunction $\varphi^0(\omega_{1,k}^0, \cdot)$. It is associated to $\omega_{1,k}^0 = k\pi + \frac{\pi}{2}$ ($i(\omega_{1,k}^0)^2$ is an eigenvalue in $\sigma_1^{(0)}$ and not in $\sigma_2^{(0)}$) and this function has the same form as the eigenfunctions associated to the eigenvalues of $\sigma_1^{(0)}$. See Theorem 3.1 for the notation of the spectra and the proof of Theorem 3.3 for the construction of the eigenfunctions.

Theorem 4.1. (*Riesz basis for the operator \mathcal{A}_d*)

The notation of Theorem 3.1 for the spectra of both operators \mathcal{A}_0 and \mathcal{A}_d is kept here.

Denote by $\omega_{j,k}^0$ the number $\omega_{j,k}^0 := k\pi + \frac{1}{2} \arg(z_{A,j}^{(n)})$ which is such that $i(\omega_{j,k}^0)^2 \in \sigma_2^{(0)}$ (except from the case $n = 2$ and $j = 1$: $i(\omega_{1,k}^0)^2$ is an eigenvalue in $\sigma_1^{(0)}$).

Denote by $\omega := \omega_{j,k}$ the value which is such that $i(\omega_{j,k})^2 \in \sigma_2 \cup \tilde{\sigma}_2$. The indices j and k are dropped for simplicity since they are fixed here.

Denote by $\varphi^0(\omega^0, \cdot)$ (resp. $\varphi(\omega, \cdot)$) the eigenfunction of \mathcal{A}_0 (resp. \mathcal{A}_d) associated to the eigenvalue $i(\omega^0)^2$ (resp. $i\omega^2$).

For an odd n and any $i = 1, \dots, n$, there exists an integer k_0 such that, for any $j = 1, \dots, n$:

$$\sum_{|k| > k_0} \|\varphi_i(\omega_{j,k}, \cdot) - \varphi_i^0(\omega_{j,k}^0, \cdot)\|_2^2 < \infty.$$

The index i comes from an indexation based on the level of the vertices of the tree which is read from the leaves to the root here and not the other way round as before (see the proof).

For an even n and any $i = 1, \dots, n$, there exists an integer k_0 such that, for any $j = 2, \dots, n$:

$$\sum_{|k| > k_0} \|\varphi_i(\omega_{j,k}, \cdot) - \varphi_i^0(\omega_{j,k}^0, \cdot)\|_2^2 < \infty.$$

For an even n and any $\bar{\alpha} \in I$, there exists an integer k_0 such that:

$$\sum_{|k| > k_0} \|\varphi_{n-|\bar{\alpha}|+1}(\omega_{1,k}, \cdot) - \varphi_{\bar{\alpha}}^0(\omega_{1,k}^0, \cdot)\|_2^2 < \infty.$$

For an even n , the eigenfunction $\varphi^0(\omega_{1,k}^0, \cdot)$ associated to $\omega_{1,k}^0 = k\pi + \frac{\pi}{2}$ in $\sigma_1^{(0)}$ has the same form as the eigenfunctions associated to the eigenvalues of $\sigma_1^{(0)}$ (cf. Proof of Theorem 3.3 and the proof of this Theorem). The index is $\bar{\alpha}$ as in the preceding sections.

Proof. For both operators, the "eigenvector" $F_{\bar{\alpha}}$ for $\bar{\alpha}$ in I_{Dir} is: $F_{\bar{\alpha}}(0) := f \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ with $f = \partial_x \phi_{\bar{\alpha}}(0)$ for any $\bar{\alpha} \in I_{Dir}$ (cf. the beginning of the proof of Proposition 3.2 for the definition of $F_{\bar{\alpha}}$ and the end for its form in this case). Let us rename $F_{\bar{\alpha}}(0)$ the vector $\omega(1/f) \times F_{\bar{\alpha}}(0)$ which is denoted by F_0 , for simplicity.

For $\bar{\alpha}$ in I_{Dir} (i.e. for any $\bar{\alpha} \in I$ such that $|\bar{\alpha}| = n$), $F_{\bar{\alpha}}(x) = e^{M(\omega)x} F_0$ which can be denoted by $F_1(x)$ since it does not depend on $\bar{\alpha}$. Likewise $F_1^0(x) := e^{M(\omega^0)x} F_0$.

Then $F_2(x) = e^{M(\omega)x} T F_1(1)$ with $F_2(x) := F_{\bar{\alpha}}(x)$ for any $\bar{\alpha} \in I$ such that $|\bar{\alpha}| = n - 1$ and T defined in Proposition 3.2. Idem for $F_2^0(x)$ with ω^0 instead of ω . And so on...

The first component of $F_i(x)$ (resp. $F_i^0(x)$) gives the expression of the eigenfunction $\psi_{\bar{\alpha}}(x)$ of the dissipative operator \mathcal{A}_d on the adequate branch (resp. $\phi_{\bar{\alpha}}(x)$, eigenfunction of the conservative operator \mathcal{A}_0 on the appropriate branch) with the new indexation. Let call them respectively $\varphi_i(x)$ and $\varphi_i^0(x)$.

Since $e^{M(\omega)x} = \begin{pmatrix} \cos(\omega x) & \frac{\sin(\omega x)}{\omega} \\ -\omega \sin(\omega x) & \cos(\omega x) \end{pmatrix}$, it holds:

$$(4.48) \quad \varphi_i(\omega, x) = \cos(\omega x) q_i(\omega) + \frac{\sin(\omega x)}{\omega} h_i(\omega)$$

$$(4.49) \quad \varphi_i^0(\omega^0, x) = \cos(\omega^0 x) q_i(\omega^0) + \frac{\sin(\omega^0 x)}{\omega^0} h_i(\omega^0)$$

and the construction of φ_i has been done in such a way that the sequences q_i and h_i are defined iteratively by:

$$(4.50) \quad q_{i+1}(\omega) = \cos(\omega) q_i(\omega) + \frac{\sin(\omega)}{\omega} h_i(\omega), \forall i \in \mathbb{N},$$

$$(4.51) \quad h_{i+1}(\omega) = -2\omega \sin(\omega) q_i(\omega) + 2 \cos(\omega) h_i(\omega), \forall i \in \mathbb{N},$$

$$(4.52) \quad q_1(\omega) = 0, h_1(\omega) = \omega.$$

Denoting by $Q_i(\omega) := \begin{pmatrix} q_i(\omega) \\ \frac{h_i(\omega)}{\omega} \end{pmatrix}$, equations (4.50), (4.51) and (4.52) are equivalent to:

$$(4.53) \quad Q_{i+1}(\omega) = T R(\omega) Q_i(\omega), \forall i \in \mathbb{N},$$

$$(4.54) \quad Q_1(\omega) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

with T defined as in Proposition 3.2 and $R(\omega)$ the rotation matrix: $R(\omega) := \begin{pmatrix} \cos(\omega) & \sin(\omega) \\ -\sin(\omega) & \cos(\omega) \end{pmatrix}$.

Denote by $Q_i^0(\omega) := Q_i(\omega^0)$. Using Lemma 4.2 below ends the proof except for the particular case n even and $j = 1$. Indeed, with the notation introduced in the Theorem

$$\begin{aligned} \|\varphi_i(\omega_{j,k}, \cdot) - \varphi_i^0(\omega_{j,k}^0, \cdot)\|_2 &\leq \|\cos(\omega \cdot)(q_i(\omega) - q_i(\omega^0))\|_2 + \|(\cos(\omega \cdot) - \cos(\omega^0 \cdot))q_i(\omega_0)\|_2 \\ &\quad + \|\sin(\omega \cdot)(h_i(\omega)/\omega - h_i(\omega^0)/\omega^0)\|_2 + \|h_i(\omega^0)/\omega^0(\sin(\omega \cdot) - \sin(\omega^0 \cdot))\|_2. \end{aligned}$$

Note that both the norms of $\varphi_i(\omega_{j,k}, \cdot)$ and $\varphi_i^0(\omega_{j,k}^0, \cdot)$ in H are bounded from above and below by a constant number independent of k .

Now, when n is even, the dimension of the eigenspace for the conservative operator is different from that of the eigenspace for the dispersive operator. Let us construct an eigenfunction for \mathcal{A}_0 which is not an eigenfunction for \mathcal{A}_d . We start as in the beginning of this proof. The "eigenvector" $F_{\bar{\alpha}}^0$ for $\bar{\alpha}$ in I_{Dir} is: $F_{\bar{\alpha}}^0(0) := f_{\bar{\alpha}} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ with $f_{\bar{\alpha}} = \partial_x \phi_{\bar{\alpha}}(0)$. Let us rename $F_{\bar{\alpha}}^0(0)$ the vector $\omega^0 \times F_{\bar{\alpha}}(0)$. Thus, for any $\bar{\alpha} \in I_{Dir}$, $F_{\bar{\alpha}}^0(x) = e^{M(\omega^0)x} f_{\bar{\alpha}} \begin{pmatrix} 0 \\ \omega^0 \end{pmatrix}$ i.e.

$$(4.55) \quad \varphi_{\bar{\alpha}}^0(\omega^0, x) = \frac{h_{\bar{\alpha}}(\omega^0)}{\omega^0} \sin(\omega^0 x) \text{ where } \frac{h_{\bar{\alpha}}(\omega^0)}{\omega^0} = 1, \text{ if } |\bar{\alpha}| = n.$$

Using the transmission conditions and denoting by $q_{\bar{\alpha}}$ and $h_{\bar{\alpha}}$ the new resulting coefficients as we have done in the proof of Theorem 3.3 lead to:

$$\varphi_{\bar{\alpha}}^0(\omega^0, x) = q_{\bar{\alpha}}(\omega^0) \cos(\omega^0 x) + \frac{h_{\bar{\alpha}}(\omega^0)}{\omega^0} \sin(\omega^0 x)$$

with $q_{\bar{\alpha}}(\omega^0) = 0$ when $|\bar{\alpha}|$ is even and $h_{\bar{\alpha}}(\omega^0) = 0$ when $|\bar{\alpha}|$ is odd. Thus the particular case n even and $j = 1$ is treated as soon as we have proved Lemma 4.3 below. \square

Lemma 4.2. *Let A and B be real numbers. The notation $A \lesssim B$ means the existence of a positive constant C , which is independent of A and B such that $A \leq CB$. Then it holds:*

- (1) $|q_i(\omega) - q_i(\omega^0)|$, $|h_i(\omega)/\omega - h_i(\omega^0)/\omega^0|$, $\|\cos(\omega \cdot) - \cos(\omega^0 \cdot)\|_2$ and $\|\sin(\omega \cdot) - \sin(\omega^0 \cdot)\|_2$ have the same asymptotic behaviour when k tends to infinity: $|q_i(\omega) - q_i(\omega^0)| \lesssim \frac{1}{k}$ (idem for the other expressions).
- (2) $|q_i(\omega^0)|$ and $|h_i(\omega^0)/\omega^0|$ have the same asymptotic behaviour when k tends to infinity: $|q_i(\omega^0)| \lesssim 1$ (idem for the second expression).
- (3) $|\varphi_{n+1}^0(\omega^0, 1)| \lesssim 1$.

Proof. It is equivalent to prove that $|Q_i^0| \lesssim 1$ and $|Q_i - Q_i^0| \lesssim \frac{1}{k}$ where Q_i and Q_i^0 have been introduced in the latest proof (all these expressions depend on ω that we drop for simplicity). The proof is done by induction.

- Base case: Since $|Q_1 - Q_1^0| = 0$ and since $|Q_1^0|$ does not depend on ω (and thus on k), both assertions hold for $i = 1$.
- Inductive step: Suppose the statement holds for some natural number i , then for $i + 1$: $|Q_{i+1}^0| = |TR^0 Q_i^0| \lesssim 1$ since T does not depend on k and the terms of $R^0 := R(\omega^0)$ are bounded with respect to k . Now $|Q_{i+1} - Q_{i+1}^0| = |TR(\omega)Q_i - TR(\omega^0)Q_i^0| = |T(R(\omega) - R(\omega^0))Q_i + TR(\omega^0)(Q_i - Q_i^0)|$. Since $|Q_i| \lesssim 1$, $|R(\omega^0)| \lesssim 1$ and $|Q_i - Q_i^0| \lesssim \frac{1}{k}$, it remains to prove: $|R(\omega) - R(\omega^0)| \lesssim \frac{1}{k}$.

Recall the dependence of ω with respect to k :

$$(4.56) \quad \omega := \omega_{j,k} = k\pi + \frac{1}{2} \arg(z_{A,j}^{(n)}) - i \frac{\gamma_j}{k} + o\left(\frac{1}{k}\right) = \omega^0 - i \frac{\gamma_j}{k} + o\left(\frac{1}{k}\right).$$

Using the classical trigonometric difference identities for the sine and cosine functions, it holds, for any $x \in [0; 1]$:

$$\|\cos(\omega \cdot) - \cos(\omega^0 \cdot)\|_2 \lesssim \frac{1}{k} \text{ and } \|\sin(\omega \cdot) - \sin(\omega^0 \cdot)\|_2 \lesssim \frac{1}{k}.$$

Hence the result. \square

Lemma 4.3. *Suppose that n is even and choose the particular value $\omega^0 = \omega_{1,k}^0 = k\pi + \frac{\pi}{2}$. Then, with the notation of Lemma 4.2:*

- (1) $|q_i(\omega)|$ for an odd i , $|h_i(\omega)/\omega|$ for an even i , $|h_i(\omega)/\omega - h_{\bar{\alpha}}(\omega^0)/\omega^0|$ for an odd i and $|\bar{\alpha}| = n - i + 1$ and $|q_i(\omega^0) - q_{\bar{\alpha}}(\omega^0)|$ for an even i and $|\bar{\alpha}| = n - i + 1$, have the same asymptotic behaviour when k tends to infinity: $|q_i(\omega)| \lesssim \frac{1}{k}$ (idem for the other expressions).
- (2) $|q_i(\omega)|$ for an even i and $|h_{\bar{\alpha}}(\omega^0)/\omega^0|$ for an odd $|\bar{\alpha}|$ have the same asymptotic behaviour when k tends to infinity: $|q_i(\omega)| \lesssim 1$ (idem for the second expression).

Proof. (1) First the fact that $|q_i(\omega)|$ for an odd i and $|h_i(\omega)/\omega|$ for an even i are both bounded from above by a constant times $(1/k)$ follows from Lemma 4.2 and $q_i(\omega^0) = 0$ for an odd i and $h_i(\omega^0)/\omega^0 = 0$ for an even i and $\omega^0 = k\pi + \frac{\pi}{2}$ (the latest property is proved by induction, using $\cos(\omega^0) = 0$).

Now, to prove that $|h_i(\omega)/\omega - h_{\bar{\alpha}}(\omega^0)/\omega^0| \lesssim (1/k)$ for an odd i and $|q_i(\omega^0) - q_{\bar{\alpha}}(\omega^0)| \lesssim (1/k)$ for an even i (where $|\bar{\alpha}| = n - i + 1$), we proceed by induction on i . The base case is $i = 1$ for h and $i = 2$ for q . Both differences are equal to $|1 - f_{\bar{\alpha}}|$.

It is enough to choose the $f_{\bar{\alpha}}$'s such that $|1 - f_{\bar{\alpha}}| \lesssim (1/k)$, for any $\bar{\alpha}$, such that $|\bar{\alpha}| = n$. Indeed the construction procedure explained in the proof of Theorem 4.1 for an even n guarantees that it is sufficient to get the desired estimate.

To explicit this iterative procedure, let us detail the example $n = 4$: there are 6 independent eigenfunctions (cf. Theorem 3.3 for the dimension). Choose $f_{1,1,1}$, $f_{1,1,2}$, $f_{1,2,1}$, $f_{2,1,1}$, $f_{2,1,2}$ and $f_{2,2,1}$ all satisfying $|1 - f_{\bar{\alpha}}| \lesssim (1/k)$, then if $|\bar{\alpha}| = 4$, there exists $\bar{\beta}$ and j in $\{1; 2\}$ such that $|\bar{\beta}| = 3$ and $\bar{\alpha} = \bar{\beta} \circ (j)$. Then choose $f_{\bar{\alpha}} := f_{\bar{\beta}}$ and $f_{1,2,2} := f_{1,1,1} + f_{1,1,2} - f_{1,2,1}$ and $f_{2,2,2} := f_{2,1,1} + f_{2,1,2} - f_{2,2,1}$. At last $f_1 := f_{1,1,1} + f_{1,1,2}$ and $f_2 := f_{2,1,1} + f_{2,1,2}$.

Moreover $q_{\bar{\alpha}}(\omega^0) = 0$ when $|\bar{\alpha}|$ is even and $q_{\bar{\alpha}}(\omega^0) = (-1)^k |f_{\bar{\alpha}}|$ when $|\bar{\alpha}|$ is odd whereas $h_{\bar{\alpha}}(\omega^0) = 0$ when $|\bar{\alpha}|$ is odd. When $|\bar{\alpha}| = 4$, $h_{\bar{\alpha}}(\omega^0) = f_{\bar{\beta}}\omega^0$ where $\bar{\alpha} = \bar{\beta} \circ (j)$, $j = 1; 2$ and when $|\bar{\alpha}| = 2$, $h_{\bar{\alpha}}(\omega^0) = -(f_{\bar{\alpha} \circ (1)} + f_{\bar{\alpha} \circ (2)})\omega^0$. Now $q_1(\omega^0) = 0$, $h_1(\omega^0) = \omega^0$, $q_2(\omega^0) = \sin(\omega^0) = (-1)^k$, $h_2(\omega^0) = 2\omega^0 \cos(\omega^0) = 0$, $q_3(\omega^0) = 0$, $h_3(\omega^0) = -2\omega^0$, $q_4(\omega^0) = -2(-1)^k$, $h_4(\omega^0) = 0$.

Thus, for $i = 2$ and so $|\bar{\alpha}| = 3$, $|q_2(\omega^0) - q_{\bar{\alpha}}(\omega^0)| = |1 - f_{\bar{\alpha}}| \lesssim (1/k)$ and for $i = 4$ and so $|\bar{\alpha}| = 1$, $|q_4(\omega^0) - q_{\bar{\alpha}}(\omega^0)| = |-2 - f_j| \lesssim (1/k)$, since $f_j := f_{j,1,1} + f_{j,1,2}$, $j = 1; 2$.

And, for $i = 1$ and so $|\bar{\alpha}| = 4$, $|h_1(\omega)/\omega - h_{\bar{\alpha}}(\omega^0)/\omega^0| \leq |h_1(\omega)/\omega - h_1(\omega^0)/\omega^0| + |h_1(\omega^0)/\omega^0 - h_{\bar{\alpha}}(\omega^0)/\omega^0| = |h_1(\omega)/\omega - h_1(\omega^0)/\omega^0| + |1 - f_{\bar{\beta}}| \lesssim (1/k)$, due to Lemma 4.2. At last, for $i = 3$ and so $|\bar{\alpha}| = 2$, $|h_3(\omega)/\omega - h_{\bar{\alpha}}(\omega^0)/\omega^0| \leq |h_3(\omega)/\omega - h_3(\omega^0)/\omega^0| + |h_3(\omega^0)/\omega^0 - h_{\bar{\alpha}}(\omega^0)/\omega^0| = |h_3(\omega)/\omega - h_3(\omega^0)/\omega^0| + |-2 + (f_{\bar{\alpha} \circ (1)} + f_{\bar{\alpha} \circ (2)})| \lesssim (1/k)$, due to Lemma 4.2.

- (2) Since $h_{\bar{\alpha}}(\omega^0)/\omega^0 = 0$ for an odd $|\bar{\alpha}|$ (cf. Proof of Theorem 4.1), it is bounded from above by a constant independent of k .

For an even i , $|q_i(\omega)|$ is proved to satisfy the same property by induction on m where $i = 2m$. It is enough to assume that both $|q_{2m}(\omega)|$ and $|h_{2m}(\omega)/\omega|$ are bounded from above by a constant independent of k and to use the recursive definition of q_i and h_i twice to compute the same expressions with indices $2m + 1$ and $2m + 2$ ((4.50) and (4.51)).

□

5. Energy decreasing

Using the Riesz basis constructed in the latest section, the energy is proved to decrease exponentially to a non-vanishing value depending on the initial datum. The decay rate is explicitly given at the end of Theorem 5.1 below since the ω 's satisfying $i\omega^2 \in (\sigma_2 \cup \tilde{\sigma}_2)$ are the solutions of (3.28). For any fixed value of n , the constant C is computable numerically. Its value is given for $n = 2$.

5.1. Energy decreasing using the Riesz basis.

Theorem 5.1. (*Energy decreasing of the solution*)

Let $E(t)$ be the energy defined by (2.11) in the introduction and H the Hilbert space introduced at the beginning of Section 2. Keeping the notation of Theorem 3.1, let H_1 (respectively H_2) be the subspace of H spanned by the $\underline{\psi}^1(\omega, \cdot)$'s (resp. $\underline{\psi}^2(\omega, \cdot)$'s), which are the normalized (in H) eigenfunctions of \mathcal{A}_d associated to the eigenvalues $i\omega^2$ in σ_1 (resp. $\sigma_2 \cup \tilde{\sigma}_2$).

- (1) H_1 is orthogonal to H_2 .

(2) Let \underline{u}_0 in H be the initial condition of the boundary value problem given in the introduction and \underline{u}_0^1 its orthogonal projection onto H_1 .

Then $E(t)$ decreases exponentially to $E_1(0) := \|\underline{u}_0^1\|_H^2$ when t tends to $+\infty$. More precisely

$$(5.57) \quad E(t) \leq E_1(0) + e^{-2Ct} E_2(0)$$

where $-C := \sup_{i\omega^2 \in (\sigma_2 \cup \tilde{\sigma}_2)} \Re(i\omega^2) < 0$.

Proof. First part: Above all, it is easy to see that the operator \mathcal{A}_d^* is obtained by changing i by $-i$ in \mathcal{A}_d . Thus, if $\mathcal{A}_d \underline{\psi}^1(\omega, \cdot) = \lambda \underline{\psi}^1(\omega, \cdot)$, with $\lambda = i\omega^2$ then $\mathcal{A}_d^* \underline{\psi}^1(\omega, \cdot) = -\lambda \underline{\psi}^1(\omega, \cdot)$.

Now, to prove that H_1 is orthogonal to H_2 , it suffices to check that any generalized eigenfunction $\underline{\psi}^2(\omega', \cdot)$ of H_2 is orthogonal to any eigenfunction $\underline{\psi}^1(\omega, \cdot)$ of H_1 .

First we assume that $\underline{\psi}^2(\omega', \cdot)$ is an eigenfunction, i.e

$$\mathcal{A}_d \underline{\psi}^2(\omega', \cdot) = \lambda' \underline{\psi}^2(\omega', \cdot)$$

with $\lambda' = i\omega'^2$. Therefore, since λ is purely imaginary,

$$\begin{aligned} \lambda' \langle \underline{\psi}^2(\omega', \cdot), \underline{\psi}^1(\omega, \cdot) \rangle_H &= \langle \mathcal{A}_d \underline{\psi}^2(\omega', \cdot), \underline{\psi}^1(\omega, \cdot) \rangle_H \\ &= \langle \underline{\psi}^2(\omega', \cdot), \mathcal{A}_d^* \underline{\psi}^1(\omega, \cdot) \rangle_H \\ &= -\langle \underline{\psi}^2(\omega', \cdot), \mathcal{A}_d \underline{\psi}^1(\omega, \cdot) \rangle_H \\ &= -\bar{\lambda} \langle \underline{\psi}^2(\omega', \cdot), \underline{\psi}^1(\omega, \cdot) \rangle_H \\ &= \lambda \langle \underline{\psi}^2(\omega', \cdot), \underline{\psi}^1(\omega, \cdot) \rangle_H. \end{aligned}$$

Consequently $\langle \underline{\psi}^2(\omega', \cdot), \underline{\psi}^1(\omega, \cdot) \rangle_H = 0$.

Secondly, we assume that λ' is not simple. Let $\underline{\psi}^2(\omega', \cdot)$ be an associated generalized eigenfunction of order $p \geq 2$, in the sense that

$$(\mathcal{A}_d - \lambda')^p \underline{\psi}^2(\omega', \cdot) = 0, \quad (\mathcal{A}_d - \lambda')^{p-1} \underline{\psi}^2(\omega', \cdot) \neq 0.$$

Setting $\psi = (\mathcal{A}_d - \lambda') \underline{\psi}^2(\omega', \cdot)$, then ψ is a generalized eigenfunction associated to λ' of order $p - 1$, so arguing by iteration with respect to the order p we can assume that $\langle \psi, \underline{\psi}^1(\omega, \cdot) \rangle_H = 0$.

Therefore

$$\begin{aligned} \lambda' \langle \underline{\psi}^2(\omega', \cdot), \underline{\psi}^1(\omega, \cdot) \rangle_H &= \langle \mathcal{A}_d \underline{\psi}^2(\omega', \cdot) + \psi, \underline{\psi}^1(\omega, \cdot) \rangle_H \\ &= \langle \mathcal{A}_d \underline{\psi}^2(\omega', \cdot), \underline{\psi}^1(\omega, \cdot) \rangle_H \\ &= \lambda \langle \underline{\psi}^2(\omega', \cdot), \underline{\psi}^1(\omega, \cdot) \rangle_H, \end{aligned}$$

as previously. Consequently $\langle \underline{\psi}^2(\omega', \cdot), \underline{\psi}^1(\omega, \cdot) \rangle_H = 0$.

Second part: the spectrum of the dissipative operator \mathcal{A}_d is the union of σ_1 - set of the purely imaginary eigenvalues and $\sigma_2 \cup \tilde{\sigma}_2$ - set of the other eigenvalues (see Theorem 3.1).

The initial condition $\underline{u}_0 := ((u_{\bar{\alpha}})_{\bar{\alpha} \in I})_0$ is written as a sum of two terms:

$$\underline{u}_0 := \sum_{i\omega^2 \in \sigma_1} \underline{u}_0^1(\omega, \cdot) \underline{\psi}^1(\omega, \cdot) + \sum_{i\omega^2 \in (\sigma_2 \cup \tilde{\sigma}_2)} \underline{u}_0^2(\omega, \cdot) \underline{\psi}^2(\omega, \cdot)$$

where $\underline{\psi}^1(\omega, \cdot)$ (respectively $\underline{\psi}^2(\omega, \cdot)$) is a normalized (in H) eigenfunction of \mathcal{A}_d associated to the eigenvalue $i\omega^2$ in σ_1 (resp. σ_2). Note that the sum takes into account the multiplicities of the eigenvalues here.

Thus the solution of the boundary value problem given in the introduction is:

$$\underline{u}(t) := \sum_{i\omega^2 \in \sigma_1} \underline{u}_0^1(\omega, \cdot) e^{i\omega^2 t} \underline{\psi}^1(\omega, \cdot) + \sum_{i\omega^2 \in (\sigma_2 \cup \tilde{\sigma}_2)} \underline{u}_0^2(\omega, \cdot) e^{i\omega^2 t} \underline{\psi}^2(\omega, \cdot).$$

The energy, defined in the introduction, by (2.11) is: $E(t) = E_1(t) + E_2(t)$ with

$$E_1(t) := \sum_{i\omega^2 \in \sigma_1} \|\underline{u}_0^1(\omega, \cdot)\|_H^2 |e^{i\omega^2 t}|^2, \quad E_2(t) := \sum_{i\omega^2 \in (\sigma_2 \cup \tilde{\sigma}_2)} \|\underline{u}_0^2(\omega, \cdot)\|_H^2 |e^{i\omega^2 t}|^2.$$

Now, since σ_1 contains only purely imaginary eigenvalues (see Theorem 3.1), $|e^{2i\omega^2 t}| = 1$, for any ω such that $i\omega^2 \in \sigma_1$ and any $t > 0$. Thus $E_1(t) = E_1(0)$ for any $t > 0$.

The real part of $i\omega^2$ is a non-positive real number if ω is such that $i\omega^2 \in (\sigma_2 \cup \tilde{\sigma}_2)$ (see Theorem 3.1). In the proof of Theorem 3.1, this real part is proved to be equal to $2\gamma_j\pi$ with γ_j independent of k if $i\omega^2 \in \sigma_2$. Since $\tilde{\sigma}_2$ is a finite set and since j belongs to a finite set, there exists $C > 0$ such that, for any ω such that $i\omega^2 \in (\sigma_2 \cup \tilde{\sigma}_2)$: $-\Re(i\omega^2) \geq C > 0$. It holds $|E_2(t)| \leq e^{-2Ct} E_2(0)$. Thus $E_2(t)$ decreases exponentially to 0 when t tends to $+\infty$ and the total energy $E(t)$ decreases exponentially to $E_1(0)$ when t tends to $+\infty$. \square

5.2. Energy decreasing: a numerical example. Let us explicit the case $n = 2$ which corresponds to Figure 1. The polynomials $P_{A,2}$ and $P_{B,2}$ defined in Proposition 3.4 are:

$$P_{A,2}(z) = 9z^3 + 7z^2 + 7z + 9, \quad P_{B,2}(z) = 9z^3 + z^2 - z - 9.$$

The polynomial $P_{A,2}$ has 3 roots which are:

$$z_{A,1}^{(2)} = -1 = e^{i\pi}, \quad z_{A,2}^{(2)} = \frac{1}{9}(1 - 4i\sqrt{5}) = e^{-i \arctan(4\sqrt{5})}, \quad z_{A,3}^{(2)} = \frac{1}{9}(1 + 4i\sqrt{5}) = e^{i \arctan(4\sqrt{5})}.$$

Thus the spectrum of the conservative operator \mathcal{A}_0 introduced at the beginning of Section 3 is given by (3.18) with

$$\sigma_2^{(0)} = \{i(k\pi \pm \arctan(4\sqrt{5}))^2 : k \in \mathbb{Z}\}.$$

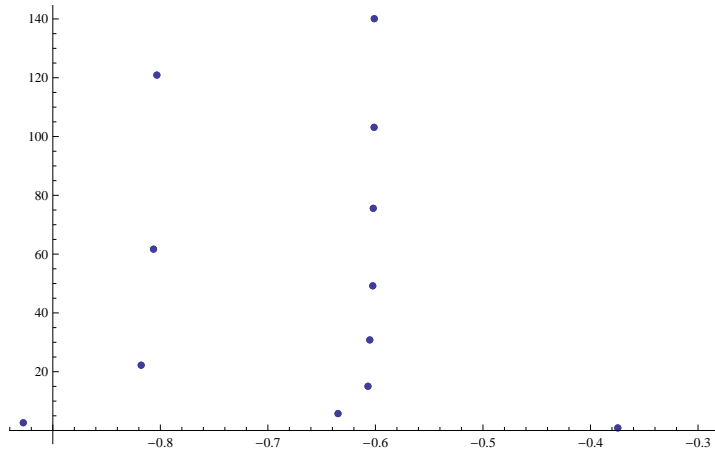


FIGURE 4. $n = 2$: the spectrum $\sigma_2 \cup \tilde{\sigma}_2$.

The values of γ_j defined by (3.36) are: $\gamma_1 = -\frac{2}{5\pi}, \gamma_2 = \gamma_3 = -\frac{3}{10\pi}$.

Then the set σ_2 which is a part of the spectrum of the dissipative operator \mathcal{A}_d (see the beginning of Section 3 and Theorem 3.1) has two vertical asymptots:

$$\Re(\lambda) = 2\pi\gamma_1 = -\frac{4}{5}, \quad \Re(\lambda) = 2\pi\gamma_2 = 2\pi\gamma_3 = -\frac{3}{5},$$

which is consistent with the numerical computation of the spectrum (see Figure 4).

At last, numerically the eigenvalue of \mathcal{A}_d with the largest real part is $\lambda \approx -0.37459 + 0.873125i$. Hence the approximate value for the decay rate: $C \approx 0.37459$.

6. Transfer function analysis

The transfer function of the conservative operator \mathcal{A}_0 introduced at the beginning of Section 3 is proved to be bounded from above. Let us recall the definition of this operator: $\mathcal{A}_0 : \mathcal{D}(\mathcal{A}_0) \subset H \rightarrow H$ is defined by

$$\mathcal{A}_0 \underline{u} := (-i \partial_x^2 u_{\bar{\alpha}})_{\bar{\alpha} \in I},$$

with

$$\mathcal{D}(\mathcal{A}_0) := \left\{ \underline{u} \in \prod_{\bar{\alpha} \in I} H^2(0, 1) : \text{satisfies (6.58) to (6.60) hereafter} \right\},$$

$$(6.58) \quad \frac{du}{dx}(1) = 0, \quad u_{\bar{\alpha}}(0) = 0, \quad \bar{\alpha} \in I_{Dir},$$

$$(6.59) \quad u_{\bar{\alpha} \circ (\beta)}(1) = u_{\bar{\alpha}}(0), \quad \beta = 1, 2, \quad \bar{\alpha} \in I_{Int},$$

$$(6.60) \quad \sum_{\beta=1}^2 \frac{du_{\bar{\alpha} \circ (\beta)}}{dx}(1) = \frac{du_{\bar{\alpha}}}{dx}(0), \quad \bar{\alpha} \in I_{Int}$$

and $B = \begin{pmatrix} \delta_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in \mathcal{L}(\mathbb{C}^N, \mathcal{D}(\mathcal{A}_0)'),$ where the duality is obtained by means of the inner product in H .

Theorem 6.1. (*Estimate of the transfer function*)

Let $\mathcal{A}_0 : \mathcal{D}(\mathcal{A}_0) \subset H \rightarrow H$ and B be the operators defined above. The transfer function is given by:

$$H(\lambda) = B^*(\lambda I + \mathcal{A}_0)^{-1} B \in \mathcal{L}(\mathbb{C}^N), \quad \lambda \in \mathbb{C}_+ = \{\lambda \in \mathbb{C}; \Re \lambda > 0\}.$$

It satisfies $\sup_{\Re \lambda = \gamma} |H(\lambda)| < \infty, \quad \gamma > 0.$

Proof. In order to compute the transfer function and to prove that it satisfies $\sup_{\Re \lambda = \gamma} |H(\lambda)| < \infty, \quad \gamma > 0,$ we solve the following problem: $z(1) = H(\lambda) \underline{g}, \quad \underline{g} \in M_{N,1}(\mathbb{C}),$ i.e.,

$$(6.61) \quad \begin{cases} \lambda z_{\bar{\alpha}} + i \frac{d^2 z_{\bar{\alpha}}}{dx^2} = 0, \quad (0, 1), \quad \bar{\alpha} \in I, \\ \frac{dz}{dx}(1) = \underline{g}, \quad z_{\bar{\alpha}}(0) = 0, \quad \bar{\alpha} \in I_{Dir}, \\ z_{\bar{\alpha} \circ (\beta)}(1) = z_{\bar{\alpha}}(0), \quad \beta = 1, 2, \quad \bar{\alpha} \in I_{Int}, \\ \sum_{\beta=1}^2 \frac{dz_{\bar{\alpha} \circ (\beta)}}{dx}(1) = \frac{dz_{\bar{\alpha}}}{dx}(0), \quad \bar{\alpha} \in I_{Int}. \end{cases}$$

We denote

$$\underline{\psi}_\lambda = (\lambda I + \mathcal{A}_0)^{-1} B \underline{g}.$$

We consider the decomposition, for an even n (an odd value for n leads to a similar expression and this case is left to the reader):

$$\begin{aligned} \underline{\psi}_\lambda &= \sum_{k \in \mathbb{Z}^*} \sum_{l=1}^{2^n-1} a_{k,l}(\lambda) \underline{\phi}_{k,l}^0(k\pi, \cdot) + \sum_{k \in \mathbb{Z}} \sum_{l=1}^{(2^n+2)/3} b_{k,l}(\lambda) \underline{\phi}_{k,l}^0(k\pi + \frac{\pi}{2}, \cdot) \\ &+ \sum_{k \in \mathbb{Z}} \sum_{j=2}^n c_{j,k}(\lambda) \underline{\phi}_k^0(\omega_{j,k}, \cdot) \end{aligned}$$

where the $\underline{\phi}_{k,l}^0(k\pi, \cdot)$'s (respectively $\underline{\phi}_{k,l}^0(k\pi + \frac{\pi}{2}, \cdot)$'s and the $\underline{\phi}_k^0(\omega_{j,k}, \cdot)$'s) are the normalized (in H) eigenfunctions of \mathcal{A}_0 with the corresponding eigenvalues $\lambda_k = i(\omega_k)^2$ with $\omega_k \in \sigma_1^0$ (resp. $\lambda_{j,k} = i(\omega_{j,k})^2 \in \sigma_2^0$). See Theorems 3.1 and 3.3 for the details and note that, for simplicity, the same notation is used for eigenfunctions which do not have the same form.

Now the above coefficients in the decomposition of $\underline{\psi}_\lambda$ are equal to:

$$\begin{aligned}
 a_{k,l}(\lambda) &= \frac{\langle B\underline{g}, \underline{\phi}_{k,l}^0(k\pi, \cdot) \rangle_{\mathcal{D}(\mathcal{A}_0)', \mathcal{D}(\mathcal{A}_0)}}{\lambda + \lambda_k} = \frac{\langle \underline{g}, B^* \underline{\phi}_{k,l}^0(k\pi, \cdot) \rangle_{\mathbb{C}^N}}{\lambda + \lambda_k}, \forall k \in \mathbb{Z}^* \\
 b_{k,l}(\lambda) &= \frac{\langle B\underline{g}, \underline{\phi}_{k,l}^0(k\pi + \frac{\pi}{2}, \cdot) \rangle_{\mathcal{D}(\mathcal{A}_0)', \mathcal{D}(\mathcal{A}_0)}}{\lambda + \lambda_k} = \frac{\langle \underline{g}, B^* \underline{\phi}_{k,l}^0(k\pi + \frac{\pi}{2}, \cdot) \rangle_{\mathbb{C}^N}}{\lambda + \lambda_k}, \forall k \in \mathbb{Z} \\
 c_{j,k}(\lambda) &= \frac{\langle B\underline{g}, \underline{\phi}_k^0(\omega_{j,k}, \cdot) \rangle_{\mathcal{D}(\mathcal{A}_0)', \mathcal{D}(\mathcal{A}_0)}}{\lambda + \lambda_{j,k}} = \frac{\langle \underline{g}, B^* \underline{\phi}_k^0(\omega_{j,k}, \cdot) \rangle_{\mathbb{C}^N}}{\lambda + \lambda_{j,k}}, \forall k \in \mathbb{Z}, j = 2, \dots, n
 \end{aligned}$$

Which implies that

$$H(\lambda)\underline{g} = \sum_{i(\omega^0)^2 \in \sigma(0)} \frac{\varphi_{n+1}^0(\omega^0, 1)}{\lambda + i(\omega^0)^2} \underline{g},$$

where $\varphi_{n+1}^0(\omega^0, \cdot)$ is defined by (4.49). It holds

$$H(\lambda) = \left(\sum_{k \in \mathbb{Z}} \frac{\varphi_{n+1}^0(\omega^0, 1)}{\lambda + \lambda_k} \quad 0 \quad \dots \quad 0 \right), \forall \lambda \text{ such that } \Re \lambda = \gamma.$$

$$\sup_{\Re \lambda = \gamma} |H(\lambda)| = \sup_{\Re \lambda = \gamma} \left| \sum_{i(\omega^0)^2 \in \sigma(0)} \frac{\varphi_{n+1}^0(\omega^0, 1)}{\lambda + i(\omega^0)^2} \right| \leq \sqrt{2} \sup_{\Re \lambda = \gamma} \sum_{i(\omega^0)^2 \in \sigma(0)} \frac{|\varphi_{n+1}^0(\omega^0, 1)|}{\gamma + |(\omega^0)^2 + \Im(\lambda)|}.$$

It is known that $|\varphi_{n+1}^0(\omega^0, 1)| \lesssim 1$ (cf. Lemma 4.2). The spectrum of \mathcal{A}_0 can be split into a finite union of sets of the form $\{i(k\pi + \theta)^2\}_{k \in \mathbb{Z}^* \text{ or } k \in \mathbb{Z}}$, where θ is a real number, (cf. Theorem 3.1) and the multiplicity of the eigenvalues is uniformly bounded. Thus, if we set $\lambda = \gamma + iy$, it suffices to show

$$(6.62) \quad \sup_{y \in \mathbb{R}} \sum_{k \in \mathbb{N}} \left| \frac{1}{\gamma + |(k\pi + \theta)^2 + y|} \right| < \infty.$$

First we consider the case $y \geq 0$ and we set $\Sigma(y) = \sum_{k \in \mathbb{N}} \left| \frac{1}{\gamma + |(k\pi + \theta)^2 + y|} \right|$. Then

$$(6.63) \quad \Sigma(y) \leq \sum_{k \in \mathbb{N}} \left| \frac{1}{(k\pi + \theta)^2} \right| < \infty, \quad y \geq 0.$$

Now, we assume that $y < 0$ and we set $y = -Y^2, Y > 0$. If $Y \geq |\theta|$, we have

$$\Sigma(y) = \Sigma_1(Y) + \left| \frac{1}{\gamma + (E[\frac{Y - \theta}{\pi}]\pi + \theta)^2 - Y^2} \right| + \Sigma_2(Y),$$

where

$$\Sigma_1(Y) = \sum_{k \leq E[\frac{Y - \theta}{\pi}] - 1} \frac{1}{\gamma + Y^2 - (k\pi + \theta)^2}, \quad \Sigma_2(Y) = \sum_{k \geq E[\frac{Y - \theta}{\pi}] + 1} \frac{1}{\gamma + (k\pi + \theta)^2 - Y^2}.$$

First, we have

$$(6.64) \quad \Sigma(y) \leq \Sigma_1(Y) + \frac{1}{\gamma} + \Sigma_2(Y),$$

and the following estimates for $\Sigma_i(Y), i = 1, 2$:

$$\begin{aligned}
\Sigma_1(Y) &\leq \sum_{k \leq E[\frac{Y-\theta}{\pi}]-1} \frac{1}{Y^2 - (k\pi + \theta)^2}, \\
&= \sum_{k \leq E[\frac{Y-\theta}{\pi}]-1} \frac{1}{\pi(\frac{Y-\theta}{\pi} - k)(Y + k\pi + \theta)}, \\
&\leq \frac{1}{\pi} \sum_{k \leq E[\frac{Y-\theta}{\pi}]-1} \frac{1}{Y}, \\
&\lesssim 1, \\
\Sigma_2(Y) &\leq \sum_{k \geq E[\frac{Y-\theta}{\pi}]+1} \frac{1}{(k\pi + \theta)^2 - Y^2} \\
&= \sum_{k \in \mathbb{N}} \frac{1}{((k + E[\frac{Y-\theta}{\pi}] + 1)\pi + \theta)^2 - Y^2} \\
&= \sum_{k \in \mathbb{N}} \frac{1}{\left((k + E[\frac{Y-\theta}{\pi}] + 1)\pi + \theta - Y \right) \left((k + E[\frac{Y-\theta}{\pi}] + 1)\pi + \theta + Y \right)} \\
&\leq \sum_{k \in \mathbb{N}} \frac{1}{\left((k + \frac{Y-\theta}{\pi})\pi + \theta - Y \right) \times k\pi} \\
&= \frac{\pi}{6}.
\end{aligned}$$

So, we have proved that $\Sigma(y)$ is uniformly bounded for large values of $|y|$, consequently (6.62) holds with a continuity argument. □

7. Application to feedback stabilization with another feedback law

This section is dedicated to some stability properties of the solutions of system (7.65)-(7.70). We obtain an exponential stability result in the energy space.

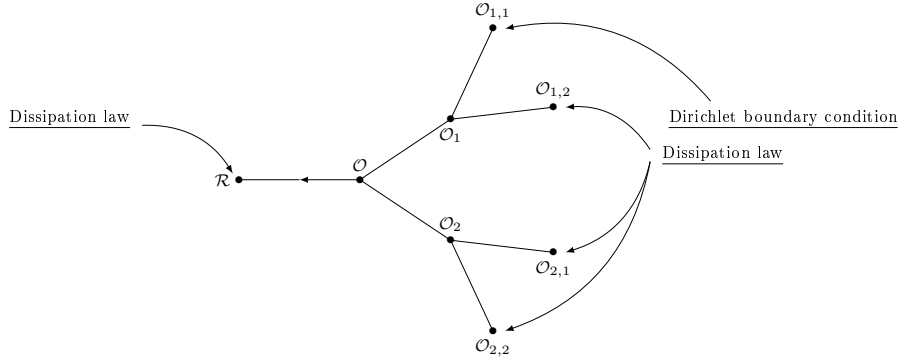


FIGURE 5. A Tree-Shaped network with exponential stabilizing feedback

Let $\bar{\alpha}^*$ be an arbitrary element of I_{Dir} . We consider the following initial and boundary value problem :

$$(7.65) \quad \frac{\partial u_{\bar{\alpha}}}{\partial t}(x, t) + i \frac{\partial^2 u_{\bar{\alpha}}}{\partial x^2}(x, t) = 0, \quad 0 < x < 1, \quad t > 0, \quad \bar{\alpha} \in I,$$

$$(7.66) \quad i u(1, t) + \frac{\partial u}{\partial x}(1, t) = 0, \quad i u_{\bar{\alpha}}(0, t) - \frac{\partial u_{\bar{\alpha}}}{\partial x}(0, t) = 0, \quad \bar{\alpha} \in I_{Dir}, \quad \bar{\alpha} \neq \bar{\alpha}^*, \quad t > 0,$$

$$(7.67) \quad u_{\bar{\alpha}^*}(0, t) = 0, \quad t > 0,$$

$$(7.68) \quad u_{\bar{\alpha} \circ \beta}(1, t) = u_{\bar{\alpha}}(0, t), \quad t > 0, \beta = 1, 2, \bar{\alpha} \in I_{Int},$$

$$(7.69) \quad \sum_{\beta=1}^2 \frac{\partial u_{\bar{\alpha} \circ \beta}}{\partial x}(1, t) = \frac{\partial u_{\bar{\alpha}}}{\partial x}(0, t), \quad t > 0, \bar{\alpha} \in I_{Int},$$

$$(7.70) \quad u_{\bar{\alpha}}(x, 0) = (u_{\bar{\alpha}})_0(x), \quad 0 < x < 1, \bar{\alpha} \in I.$$

It is well-known that system (7.65)-(7.70) may be rewritten as the first order evolution equation

$$(7.71) \quad \begin{cases} \underline{u}' = A_d \underline{u}, \\ \underline{u}(0) = \underline{u}_0, \end{cases}$$

where the operator $A_d : \mathcal{D}(A_d) \subset H \rightarrow H$ is defined by

$$A_d \underline{u} := (-i \partial_x^2 u_{\bar{\alpha}})_{\bar{\alpha} \in I},$$

with

$$\mathcal{D}(A_d) := \left\{ \underline{u} \in \prod_{\bar{\alpha} \in I} H^2(0, 1) : \text{satisfies (7.72) to (7.75) hereafter} \right\},$$

$$(7.72) \quad i u(1) + \frac{du}{dx}(1) = 0, \quad i u_{\bar{\alpha}}(0) - \frac{du_{\bar{\alpha}}}{dx}(0) = 0, \quad \bar{\alpha} \in I_{Dir}, \bar{\alpha} \neq \bar{\alpha}^*,$$

$$(7.73) \quad u_{\bar{\alpha}^*}(0) = 0,$$

$$(7.74) \quad u_{\bar{\alpha} \circ \beta}(1) = u_{\bar{\alpha}}(0), \quad \beta = 1, 2, \bar{\alpha} \in I_{Int},$$

$$(7.75) \quad \sum_{\beta=1}^2 \frac{du_{\bar{\alpha} \circ \beta}}{dx}(1) = \frac{du_{\bar{\alpha}}}{dx}(0), \quad \bar{\alpha} \in I_{Int}.$$

As in Section 2, the operator A_d generates a C_0 semigroup of contractions on H .

In this section, we show that the semigroup e^{tA_d} decays to the null steady state with an exponential decay rate. To obtain this, our technique is based on a frequency domain approach method and combines a contradiction argument with the multiplier technique to carry out a special analysis for the resolvent.

Theorem 7.1. *There exist constants $C, \tau > 0$ such that the semigroup e^{tA_d} satisfies the following estimate*

$$(7.76) \quad \|e^{tA_d}\|_{\mathcal{L}(H)} \leq C e^{-\tau t}, \quad \forall t > 0.$$

We will employ from [31, 23] the following frequency domain theorem for uniform stability of a C_0 semigroup of contractions on a Hilbert space:

Lemma 7.2. *A C_0 semigroup $e^{t\mathcal{L}}$ on a Hilbert space \mathcal{H} satisfies*

$$\|e^{t\mathcal{L}}\|_{\mathcal{L}(\mathcal{H})} \leq C e^{-\tau t},$$

for some constant $C > 0$ and for $\tau > 0$ if and only if

$$(7.77) \quad \rho(\mathcal{L}) \supset \{i\beta \mid \beta \in \mathbb{R}\} \equiv i\mathbb{R},$$

and

$$(7.78) \quad \limsup_{|\beta| \rightarrow \infty} \|(i\beta I - \mathcal{L})^{-1}\|_{\mathcal{L}(\mathcal{H})} < \infty,$$

where $\rho(\mathcal{L})$ denotes the resolvent set of the operator \mathcal{L} .

Lemma 7.3. *The spectrum of A_d contains no point on the imaginary axis.*

Proof. Since the resolvent of A_d is compact, its spectrum $\sigma(A_d)$ only consists of eigenvalues of A_d . We will show that the equation

$$(7.79) \quad A_d \underline{Z} = i\beta \underline{Z}$$

with $\underline{Z} \in \mathcal{D}(A_d)$ and $\beta \neq 0$ has only the trivial solution.

By taking the inner product of (7.79) with \underline{Z} and using

$$(7.80) \quad \Re(\langle A_d \underline{Z}, \underline{Z} \rangle_H) = -|z(1)|^2 - \sum_{\bar{\alpha}^* \neq \bar{\alpha} \in I_{Dir}} |z_{\bar{\alpha}}(0)|^2,$$

we obtain that $z(1) = 0$, $z_{\bar{\alpha}}(0) = 0$, $\bar{\alpha}^* \neq \bar{\alpha} \in I_{Dir}$. Next, we get the following ordinary differential equation:

$$(7.81) \quad \begin{cases} i\beta z_{\bar{\alpha}}(x) + i \frac{d^2 z_{\bar{\alpha}}}{dx^2}(x) = 0, & x \in (0, 1), \bar{\alpha} \in I, \\ z(1) = \frac{dz}{dx}(1) = 0, & z_{\bar{\alpha}}(1) = \frac{dz_{\bar{\alpha}}}{dx}(1) = 0, \bar{\alpha}^* \neq \bar{\alpha} \in I_{Dir}, \\ z_{\bar{\alpha}^*}(0) = 0, \\ z_{\bar{\alpha} \circ \beta}(1) = z_{\bar{\alpha}}(0), & \beta = 1, 2, \bar{\alpha} \in I_{Int}, \\ \sum_{\beta=1}^2 \frac{dz_{\bar{\alpha} \circ \beta}}{dx}(1) = \frac{dz_{\bar{\alpha}}}{dx}(0), & \bar{\alpha} \in I_{Int}. \end{cases}$$

And the above system has only the trivial solution. □

Proof of Theorem 7.1. By Lemma 7.2, it suffices to show that A_d satisfies the following two conditions:

$$(7.82) \quad \rho(A_d) \supset \{i\beta \mid \beta \in \mathbb{R}\} \equiv i\mathbb{R},$$

and

$$(7.83) \quad \limsup_{|\beta| \rightarrow \infty} \|(i\beta - A_d)^{-1}\|_{\mathcal{L}(H)} < \infty,$$

where $\rho(A_d)$ denotes the resolvent set of the operator A_d .

By Lemma 7.3 the condition (7.82) is satisfied. Suppose that condition (7.83) is false. By the Banach-Steinhaus Theorem (see [17]), there exist a sequence of real numbers $\beta_n \rightarrow \infty$ and a sequence of vectors $\underline{Z}_n \in \mathcal{D}(A_d)$ with $\|\underline{Z}_n\|_H = 1$ such that

$$(7.84) \quad \|(i\beta_n I - A_d)\underline{Z}_n\|_H \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

i.e.,

$$(7.85) \quad i\beta_n z_{\bar{\alpha},n} + i \frac{d^2 z_{\bar{\alpha},n}}{dx^2} \equiv f_{\bar{\alpha},n} \rightarrow 0 \quad \text{in } L^2(0, 1), \bar{\alpha} \in I.$$

Our goal is to derive from (7.84) that $\|\underline{Z}_{\bar{\alpha},n}\|_H$ converges to zero. Thus there will be a contradiction.

We notice that from (7.80), it holds:

$$(7.86) \quad \|(i\beta_n I - A_d)\underline{Z}_n\|_H \geq |\Re(\langle (i\beta_n I - A_d)\underline{Z}_n, \underline{Z}_n \rangle_H)| = |z_n(1)|^2 + \sum_{\bar{\alpha}^* \neq \bar{\alpha} \in I_{Dir}} |z_{\bar{\alpha},n}(0)|^2.$$

Then, by (7.84)

$$(7.87) \quad z_n(1) \rightarrow 0, \quad z_{\bar{\alpha},n}(0) \rightarrow 0, \quad \bar{\alpha}^* \neq \bar{\alpha} \in I_{Dir}$$

and

$$(7.88) \quad \frac{dz_n}{dx}(1), \quad \frac{dz_{\bar{\alpha},n}}{dx}(0) \rightarrow 0, \quad \bar{\alpha}^* \neq \bar{\alpha} \in I_{Dir}.$$

First we assume that the sequence (or a subsequence) of β_n converges towards $+\infty$. According to (7.85):

$$(7.89) \quad \begin{aligned} z_{\bar{\alpha},n} &= A_{\bar{\alpha},n} \sin \sqrt{\beta_n} x + B_{\bar{\alpha},n} \cos \sqrt{\beta_n} x - \\ & i \int_0^x \frac{\sin \sqrt{\beta_n}(x-y)}{\sqrt{\beta_n}} f_{\bar{\alpha},n}(y) dx, \quad x \in (0, 1), \bar{\alpha} \in I, \end{aligned}$$

where $A_{\bar{\alpha},n}$ and $B_{\bar{\alpha},n}$ are constants.

Moreover, for $\bar{\alpha} = \bar{\alpha}^*$,

$$(7.90) \quad z_{\bar{\alpha}^*,n} = A_{\bar{\alpha}^*,n} \sin \sqrt{\beta_n} x - i \int_0^x \frac{\sin \sqrt{\beta_n}(x-y)}{\sqrt{\beta_n}} f_{\bar{\alpha}^*,n}(y) dx, \quad x \in (0, 1).$$

According to (7.87)-(7.88), it holds for $\bar{\alpha}^* \neq \bar{\alpha} \in I_{Dir}$,

$$(7.91) \quad \begin{cases} A_{\bar{\alpha}} \sin \sqrt{\beta_n} + B_{\bar{\alpha}} \cos \sqrt{\beta_n} \rightarrow 0 \\ A_{\bar{\alpha}} \sqrt{\beta_n} \cos \sqrt{\beta_n} - \sqrt{\beta_n} B_{\bar{\alpha}} \sin \sqrt{\beta_n} \rightarrow 0, \end{cases}$$

which implies that, for $\bar{\alpha}^* \neq \bar{\alpha} \in I_{Dir}$,

$$(7.92) \quad A_{\bar{\alpha},n}, B_{\bar{\alpha},n} \rightarrow 0$$

and that

$$(7.93) \quad z_{\bar{\alpha},n} \rightarrow 0, \text{ in } L^2(0, 1), \bar{\alpha}^* \neq \bar{\alpha} \in I_{Dir}.$$

By using the continuity and transmission conditions, we deduce by iteration and by the same way as above:

$$z_{\bar{\alpha},n} \rightarrow 0, \text{ in } L^2(0, 1), \bar{\alpha} \in I.$$

Which contradicts the fact that $\|\underline{Z}_n\|_H = 1$.

Now, we assume that no subsequence of β_n converges towards $+\infty$, (i.e $\lim_{n \rightarrow \infty} \beta_n = -\infty$). Since $z_{\bar{\alpha}}$ satisfies the first equality of (7.81), we multiply it by $z_{\bar{\alpha}}$, and then integrating by parts, summing over all $\bar{\alpha} \in I$, using the boundary and transmission conditions as well as (7.86) and (7.88), we get

$$i\beta_n \|\underline{Z}_n\|_H^2 - i \sum_{\bar{\alpha} \in I} \int_0^1 \left| \frac{dz_{\bar{\alpha}}}{dx} \right|^2 dx + o(1) = 0,$$

which also contradicts the fact that $\|\underline{Z}_n\|_H = 1$.

□

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