



HAL
open science

A bounded degree SOS hierarchy for polynomial optimization

Jean-Bernard Lasserre, Toh Kim-Chuan, Yang Shouguang

► **To cite this version:**

Jean-Bernard Lasserre, Toh Kim-Chuan, Yang Shouguang. A bounded degree SOS hierarchy for polynomial optimization . 2015. hal-01109125v1

HAL Id: hal-01109125

<https://hal.science/hal-01109125v1>

Preprint submitted on 24 Jan 2015 (v1), last revised 26 Jun 2015 (v2)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

A bounded degree SOS hierarchy for polynomial optimization*

Jean B. Lasserre[†], Kim-Chuan Toh[‡] and Shouguang Yang[§]

January 24, 2015

Abstract

We consider a new hierarchy of semidefinite relaxations for the general polynomial optimization problem $(P) : f^* = \min\{f(x) : x \in K\}$ on a compact basic semi-algebraic set $K \subset \mathbb{R}^n$. This hierarchy combines some advantages of the standard LP-relaxations associated with Krivine's positivity certificate and some advantages of the standard SOS-hierarchy. In particular it has the following attractive features: (a) In contrast to the standard SOS-hierarchy, for each relaxation in the hierarchy, the size of the matrix associated with the semidefinite constraint is the same and fixed in advance by the user. (b) In contrast to the LP-hierarchy, finite convergence occurs at the first step of the hierarchy for an important class of convex problems. Finally (c) some important techniques related to the use of point evaluations for declaring a polynomial to be zero and to the use of rank-one matrices make an efficient implementation possible. Preliminary results on a sample of non convex problems are encouraging.

1 Introduction

We consider the polynomial optimization problem:

$$(P) : \quad f^* = \min_x \{f(x) : x \in \mathbf{K}\} \quad (1)$$

where $f \in \mathbb{R}[x]$ is a polynomial and $\mathbf{K} \subset \mathbb{R}^n$ is the basic semi-algebraic set

$$\mathbf{K} = \{x \in \mathbb{R}^n : g_j(x) \geq 0, \quad j = 1, \dots, m\}, \quad (2)$$

for some polynomials $g_j \in \mathbb{R}[x]$, $j = 1, \dots, m$. In order to approximate (and sometimes solve exactly) (P) one may instead solve a hierarchy of *convex relaxations* of (P) of increasing sizes, namely for instance:

• *Semidefinite* relaxations based on Putinar's certificate of positivity on K [22], where the d -th convex relaxation of the hierarchy is a semidefinite program given by

$$\gamma_d = \max_{t, \sigma_j} \left\{ t : f - t = \sigma_0 + \sum_{j=1}^m \sigma_j g_j \right\}. \quad (3)$$

*The work of the first author is partially supported by a PGMO grant from *Fondation Mathématique Jacques Hadamard*.

[†]LAAS-CNRS and Institute of Mathematics, University of Toulouse, LAAS, 31031 Toulouse cédex 4, France (lasserre@laas.fr).

[‡]Department of Mathematics, National University of Singapore, 10 Lower Kent Ridge Road, Singapore 119076 (mattohk@nus.edu.sg).

[§]Department of Mathematics, National University of Singapore, 10 Lower Kent Ridge Road, Singapore 119076.

The unknowns σ_j are sums of squares (SOS) polynomials with the degree bound constraint, $\text{degree}(\sigma_j g_j) \leq 2d$, $j = 0, \dots, m$, and the expression in (3) is a certificate of positivity on \mathbf{K} for the polynomial $x \mapsto f(x) - t$.

- LP-relaxations based on Krivine-Stengle’s certificate of positivity on \mathbf{K} [13, 25], where the d -th convex relaxation of the hierarchy is a linear program given by

$$\theta_d = \max_{\lambda \geq 0, t} \left\{ t : f - t = \sum_{(\alpha, \beta) \in \mathbb{N}_d^{2m}} \lambda_{\alpha\beta} \prod_{j=1}^m \left(g_j^{\alpha_j} (1 - g_j)^{\beta_j} \right) \right\}, \quad (4)$$

where $\mathbb{N}_d^{2m} = \{(\alpha, \beta) \in \mathbb{N}^{2m} : \sum_j \alpha_j + \beta_j \leq d\}$. The unknown are t and the nonnegative scalars $\lambda = (\lambda_{\alpha\beta})$, and it is assumed that $0 \leq g_j \leq 1$ on \mathbf{K} (possibly after scaling) and the family $\{g_i, 1 - g_i\}$ generates the algebra $\mathbb{R}[x]$ of polynomials. Problem (4) is an LP because stating that the two polynomials in both sides of “=” are equal yields linear constraints on the $\lambda_{\alpha\beta}$ ’s. For instance, the LP-hierarchy from Sherali-Adams RLT [23] and their variants [24] are of this form.

In both cases, (γ_d) and (θ_d) , $d \in \mathbb{N}$, provide two monotone nondecreasing sequences of lower bounds on f^* and if \mathbf{K} is compact, then both converge to f^* as one lets d increases. For more details as well as a comparison of such relaxations, the reader is referred to e.g. Lasserre [18, 15] and Laurent [19], as well as Chlamtac and Tulsiani [7] for the impact of LP- and SOS-hierarchies on approximation algorithms in combinatorial optimization.

Of course, in principle, one would much prefer to solve LP-relaxations rather than semidefinite relaxations (i.e. compute θ_d rather than γ_d) because present LP-software packages can solve sparse problems with millions of variables and constraints, which is far from being the case for today’s semidefinite solvers. And so the hierarchy (3) applies to problems of modest size only unless some sparsity or symmetry is taken into account in which case specialized variants can handle problems of much larger size; see e.g. Waki et al. [28]. However, on the other hand, the LP-relaxations (4) suffer from several serious theoretical and practical drawbacks. For instance, it has been shown in [15, 18] that the LP-relaxations *cannot* be exact for most convex problems, i.e., the sequence of the associated optimal values converges to the global optimum only *asymptotically* and not in finitely many steps. Moreover, the LPs of the hierarchy are numerically ill-conditioned. This is in contrast with the semidefinite relaxations (3) for which finite convergence takes place for convex problems where $\nabla^2 f(x^*)$ is positive definite at every minimizer $x^* \in \mathbf{K}$ (see de Klerk and Laurent [9, Corollary 3.3]) and occurs at the first relaxation for SOS-convex¹ problems [17, Theorem 3.3]. In fact, as demonstrated in recent works of Marshall [20] and Nie [21], finite convergence is generic even for non convex problems.

1.1 Contribution

This paper is in the vein of recent attempts in Lasserre [16] and Ahmadi and Majumdar [1] to overcome the important computational burden associated with the standard SOS-hierarchy (3). In particular, in [16] we have suggested another hierarchy of convex relaxations which combines some of the advantages of the SOS- and LP- hierarchies (3) and (4). In the present paper we take advantage of attractive features of the SDPT3 solver [26, 27] to provide an effective implementation of this new hierarchy. First preliminary tests on a sample of non convex problems suggest that this new hierarchy might be efficient. This new hierarchy is another type of SOS-hierarchy labelled BSOS (for hierarchy with *bounded* degree SOS) with the following attractive features:

¹An SOS-convex polynomial is a convex polynomial whose Hessian factors as $L(x)L(x)^T$ for some rectangular matrix polynomial L . For instance, separable convex polynomials are SOS-convex.

- In contrast to the standard SOS-hierarchy (3), for *each* semidefinite program in the hierarchy, the size $\binom{n+k}{n}$ of the semidefinite matrix variable is now fixed, parametrized by an integer k that one fixes in advance. This integer k determines the degree of a certain SOS polynomial (for instance one may fix $k = 2$), whence the label BSOS (for “bounded”-SOS). Recall that in the standard SOS-hierarchy (3) the size of the semidefinite matrix variable is $\binom{n+d}{n}$ with rank d in the hierarchy.

- In contrast to the LP-hierarchy (4), *finite* convergence occurs at the first step in the hierarchy for a large class of convex problems; typically convex problems defined with convex quadratic polynomials or SOS-convex polynomials of degree at most k . Recall that such finite convergence is impossible for the LP-hierarchy (4).

- Just as in the standard SOS-hierarchy (3), there also exists a sufficient condition for finite convergence of the hierarchy. Namely it suffices to check whether at an optimal solution of the corresponding SDP, some associated moment matrix is rank-one.

- Last but not least, to implement this hierarchy one uses important techniques that dramatically alleviate the computational burden associated with a standard (careless) implementation. Namely, (a) to declare that two polynomials are identical one uses that their values are equal on finitely many randomly chosen points (instead of equating their coefficients), and (b) the SDP solver SDPT3 [26, 27] can be used to handle efficiently some type of matrices used in our positivity certificate.

Preliminary computational experiments First we have compared our results with those obtained with the GloptiPoly software [12] (devoted to solving the SOS-hierarchy (3)) on a sample of non convex problems with up to 20 variables. For problems with low degree (in the initial data) and/or low dimension we obtain the global optimum whereas good lower bounds are always obtained for problems with high degree or higher dimension (e.g. problems with degree 4 and up to 20 variables).

Next, we have also tested the LP-hierarchy (4) on a sample of convex problems and as expected the convergence is very poor and the resulting LPs become ill-conditioned. In addition, the LP can be expensive to solve as the LP data is typically dense. In contrast, the new hierarchy (with smallest value $k = 1$ of its parameter) converges at the first step even though some of the problems are defined with polynomials of degree larger than 2.

Finally we have considered a sample of non convex quadratic problems of the form $\inf\{x^T A x : x \in \Delta\}$ where $\Delta \subset \mathbb{R}^n$ is the canonical simplex and A is a randomly generated real symmetric matrix with r negative eigenvalues and $n - r$ positive eigenvalues. For all problems that could be solved with GloptiPoly (up to $n = 20$ variables) we obtain the optimal values. For the other problems with $n = 50, 100$ variables a lower bound is obtained in reasonable amount of time.

Of course this new hierarchy of semidefinite relaxations also has its drawbacks (at least in its present version). Namely some submatrix (of the matrix used to describe the linear equality constraints of the resulting SDP) is fully dense and many of these linear constraints are nearly dependent, which yields a lack of accuracy in the optimal solution when the order of relaxation d is increased.

2 Main result

2.1 Notation and definitions

Let $\mathbb{R}[x]$ be the ring of polynomials in the variables $x = (x_1, \dots, x_n)$. Denote by $\mathbb{R}[x]_d \subset \mathbb{R}[x]$ the vector space of polynomials of degree at most d , which forms a vector space of dimension $s(d) = \binom{n+d}{d}$, with e.g., the usual canonical basis (x^α) of monomials. Also, denote by $\Sigma[x] \subset \mathbb{R}[x]$

(resp. $\Sigma[x]_d \subset \mathbb{R}[x]_{2d}$) the space of sums of squares (s.o.s.) polynomials (resp. s.o.s. polynomials of degree at most $2d$). If $f \in \mathbb{R}[x]_d$, we write $f(x) = \sum_{\alpha \in \mathbb{N}_d^n} f_\alpha x^\alpha$ in the canonical basis and denote by $\mathbf{f} = (f_\alpha) \in \mathbb{R}^{s(d)}$ its vector of coefficients. Finally, let \mathcal{S}^n denote the space of $n \times n$ real symmetric matrices, with inner product $\langle \mathbf{A}, \mathbf{B} \rangle = \text{trace } \mathbf{A}\mathbf{B}$. We use the notation $\mathbf{A} \succeq 0$ (resp. $\mathbf{A} \succ 0$) to denote that \mathbf{A} is positive semidefinite (definite). With $g_0 := 1$, the quadratic module $Q(g_1, \dots, g_m) \subset \mathbb{R}[x]$ generated by polynomials g_1, \dots, g_m , is defined by

$$Q(g_1, \dots, g_m) := \left\{ \sum_{j=0}^m \sigma_j g_j : \sigma_j \in \Sigma[x] \right\}.$$

We briefly recall two important theorems by Putinar [22] and Krivine-Stengle [13, 25] respectively, on the representation of polynomials that are positive on \mathbf{K} .

Theorem 1. *Let $g_0 = 1$ and \mathbf{K} in (2) be compact.*

(a) *If the quadratic polynomial $x \mapsto M - \|x\|^2$ belongs to $Q(g_1, \dots, g_m)$ and if $f \in \mathbb{R}[x]$ is strictly positive on \mathbf{K} then $f \in Q(g_1, \dots, g_m)$.*

(b) *Assume that $0 \leq g_j \leq 1$ on \mathbf{K} for every j , and the family $\{1, g_j\}$ generates $\mathbb{R}[x]$. If f is strictly positive on \mathbf{K} then*

$$f = \sum_{\alpha, \beta \in \mathbb{N}^m} c_{\alpha\beta} \prod_j \left(g_j^{\alpha_j} (1 - g_j)^{\beta_j} \right),$$

for some (finitely many) nonnegative scalars $(c_{\alpha\beta})$.

2.2 The Bounded-SOS-hierarchy (BSOS)

Consider the problem

$$(P) : \quad f^* = \min\{f(x) \mid x \in \mathbf{K}\}$$

where $K \subset \mathbb{R}^n$ is the basic semi-algebraic set defined in (2), assumed to be compact. Moreover we also assume that $g_j(x) \leq 1$ for all $x \in \mathbf{K}$ and $j = 1, \dots, m$, and $\{g_j, 1 - g_j\}$ generates the ring of polynomials $\mathbb{R}[x]$. For a fixed $d \geq 1$, the problem (P) is easily seen to be equivalent to the following by adding redundant constraints:

$$(\tilde{P}) : \quad f^* = \min\{f(x) \mid h_{\alpha\beta}(x) \geq 0 \quad \forall (\alpha, \beta) \in \mathbb{N}_d^{2m}\}$$

where $\mathbb{N}_d^{2m} = \{(\alpha, \beta) \in \mathbb{N}^{2m} \mid |\alpha| + |\beta| \leq d\}$ and $h_{\alpha\beta} \in \mathbb{R}[x]$ is the polynomial

$$x \mapsto h_{\alpha\beta}(x) := \prod_{j=1}^m g_j(x)^{\alpha_j} (1 - g_j(x))^{\beta_j}, \quad x \in \mathbb{R}^n.$$

Given $\lambda = (\lambda_{\alpha\beta})$, $(\alpha, \beta) \in \mathbb{N}_d^{2m}$, consider the Lagrangian function:

$$x \mapsto L_d(x, \lambda) = f(x) - \sum_{(\alpha, \beta) \in \mathbb{N}_d^{2m}} \lambda_{\alpha\beta} h_{\alpha\beta}(x), \quad x \in \mathbb{R}^n.$$

The Lagrangian dual of (\tilde{P}) is given by

$$(\tilde{P}_d^*) : \quad \sup_{\lambda} \{G_d(\lambda) : \lambda \geq 0\}$$

where the function $G_d(\cdot)$ is given by:

$$\lambda \mapsto G_d(\lambda) := \inf_{x \in \mathbb{R}^n} \{L_d(x, \lambda)\}, \quad \lambda \geq 0.$$

Now for a fixed λ , the evaluation of $G_d(\lambda)$ is computational intractable. However, let $k \in \mathbb{N}$ be fixed and observe that

$$\begin{aligned} G_d(\lambda) &= \inf_{x \in \mathbb{R}^n} L_d(x, \lambda) = \sup_t \{t \mid L_d(x, \lambda) - t \geq 0, \quad \forall x\} \\ &\geq \sup_t \{t \mid L_d(x, \lambda) - t \in \Sigma[x]_k\} \end{aligned}$$

where $\Sigma[x]_k$ is the space of sum-of-squares polynomials up to degree $2k$. Hence, with $k \in \mathbb{N}$ fixed, consider the family of optimization problems indexed by $d \in \mathbb{N}$:

$$q_d^k := \sup_{\lambda, t} \{t \mid L_d(x, \lambda) - t \in \Sigma[x]_k, \lambda \geq 0\}, \quad d \in \mathbb{N}. \quad (5)$$

Observe that when k is fixed, then for each $d \in \mathbb{N}$:

- Computing q_d^k in (5) reduces to solving a semidefinite program and so (5) defines a hierarchy of semidefinite programs because $q_{d+1}^k \geq q_d^k$ for all $d \in \mathbb{N}$.
- The semidefinite constraint is associated with the constraint $L_d(x, \lambda) - t \in \Sigma[x]_k$ and the associated matrix has fixed size $\binom{n+k}{n}$, independent of $d \in \mathbb{N}$, a crucial feature for computational efficiency of the approach.

Moreover,

$$f^* \geq \sup_{\lambda \geq 0} G_d(\lambda) \geq q_d^k, \quad \forall d \in \mathbb{N},$$

and in fact we even have the more precise and interesting result.

Theorem 2 ([16]). *Let $\mathbf{K} \subset \mathbb{R}^n$ in (2) be compact with nonempty interior and $g_j(x) \leq 1$ for $x \in \mathbf{K}$ and $j = 1, \dots, m$. Assume further that the family $\{1, g_j\}$ generates the algebra $\mathbb{R}[x]$. Let $k \in \mathbb{N}$ be fixed. Then the sequence (q_d^k) , $d \in \mathbb{N}$, defined in (5) is monotone nondecreasing and $q_d^k \rightarrow f^*$ as $d \rightarrow \infty$.*

Moreover, if f and $-g_j$, $j = 1, \dots, m$, are SOS-convex polynomials² of degree at most $2k$ then $q_1^k = f^$, i.e., finite convergence takes places at the first relaxation in the hierarchy! In particular when $f, -g_j$ are convex quadratic polynomials then $q_1^1 = f^*$.*

2.3 The SDP formulation of (5)

To formulate (5) as a semidefinite program one has at least two possibilities depending on how we state that two polynomials $p, q \in \mathbb{R}[x]_d$ are identical. Either by *equating their coefficients* (e.g. in the monomial basis), i.e., $p_\alpha = q_\alpha$ for all $\alpha \in \mathbb{N}_d^n$, or by *equating their values* on $\binom{n+d}{n}$ generic points (e.g. randomly generated on the box $[-1, 1]^n$). In the present context of (5) we prefer the latter option since expanding the polynomial $h_{\alpha\beta}(x)$ symbolically to get the coefficients with respect to the monomial basis can be expensive and memory intensive.

²A polynomial $f \in \mathbb{R}[x]$ is SOS-convex if its Hessian $\nabla^2 f$ is an SOS matrix, i.e., $\nabla^2 f(x) = L(x)L(x)^T$ for some matrix polynomial $L \in \mathbb{R}[x]^{n \times p}$ and some $p \in \mathbb{N}$.

Let $\tau = \max\{\deg(f), 2k, d \max_j \{\deg(g_j)\}\}$. Then for k fixed and for each d , we get

$$\begin{aligned} q_d^k &= \max \left\{ t \mid f(x) - t - \sum_{(\alpha, \beta) \in \mathbb{N}_d^{2m}} \lambda_{\alpha\beta} h_{\alpha\beta}(x) = \langle Q, v_k(x) v_k(x)^T \rangle, Q \in \mathcal{S}_+^{s(k)}, \lambda \geq 0 \right\} \\ &= \max \left\{ t \mid \begin{array}{l} f(x^{(p)}) = t + \sum_{(\alpha, \beta) \in \mathbb{N}_d^{2m}} \lambda_{\alpha\beta} h_{\alpha\beta}(x^{(p)}) + \langle Q, v_k(x^{(p)}) v_k(x^{(p)})^T \rangle, \\ p = 1, \dots, L, Q \in \mathcal{S}_+^{s(k)}, \lambda \geq 0, t \in \mathbb{R} \end{array} \right\} \end{aligned} \quad (6)$$

where $L := |\mathbb{N}_\tau^n| = \binom{n+\tau}{n}$ and $\{x^{(p)} \in \mathbb{R}^n \mid p = 1, \dots, L\}$ are randomly selected points in $[-1, 1]^n$; $s(k) = \binom{n+k}{k}$, and $v_k(x)$ is a vector of polynomial basis for $\mathbb{R}[x]_k$, the space of polynomials of degree at most k .

2.4 Sufficient condition for finite convergence

By looking at the dual of the semidefinite program (6) one obtains a sufficient condition for finite convergence. To describe the dual of the semidefinite program (6) we need to introduce some notation.

For every $p = 1, \dots, L$, denote by $\delta_{x^{(p)}}$ the Dirac measure at the point $x^{(p)} \in \mathbb{R}^n$ and let $\langle q, \delta_{x^{(p)}} \rangle = q(x^{(p)})$ for all $p = 1, \dots, L$, and all $q \in \mathbb{R}[x]$.

With a real sequence $\mathbf{y} = (y_\alpha)$, $\alpha \in \mathbb{N}_{2\ell}^n$, denote by $\mathbf{M}_\ell(\mathbf{y})$ the moment matrix associated with \mathbf{y} . It is a real symmetric matrix with rows and columns indexed in the basis of monomials (x^α) , $\alpha \in \mathbb{N}_\ell^n$, and with entries

$$\mathbf{M}_\ell(\mathbf{y})(\alpha, \beta) = y_{\alpha+\beta}, \quad \forall \alpha, \beta \in \mathbb{N}_\ell^n.$$

The dual of the semidefinite program (6) reads:

$$\begin{aligned} \tilde{q}_d^k &:= \inf_{\theta \in \mathbb{R}^L} \sum_{p=1}^L \theta_p \langle f, \delta_{x^{(p)}} \rangle \\ \text{s.t.} \quad &\sum_{p=1}^L \theta_p (v_k(x^{(p)}) v_k(x^{(p)})^T) \succeq 0 \\ &\sum_{p=1}^L \theta_p \langle h_{\alpha\beta}, \delta_{x^{(p)}} \rangle \geq 0, \quad (\alpha, \beta) \in \mathbb{N}_d^{2m} \\ &\sum_{p=1}^L \theta_p = 1. \end{aligned} \quad (7)$$

(Notice that the weights θ_p are not required to be nonnegative.) By standard weak duality in convex optimization, and for every fixed $k \in \mathbb{N}$, one has

$$f^* \geq \tilde{q}_d^k \geq q_d^k, \quad \forall d \in \mathbb{N}.$$

Let $s \in \mathbb{N}$ be the smallest integer such that $2s \geq \max[\deg(f); \deg(g_j)]$. We have the following lemma.

Lemma 1. *Let $\theta^* \in \mathbb{R}^L$ be an optimal solution of (7) and let $\mathbf{y}^* = (y_\alpha^*)$, $\alpha \in \mathbb{N}_{2s}^n$, with*

$$y_\alpha^* := \sum_{p=1}^L \theta_p^* (x^{(p)})^\alpha, \quad \alpha \in \mathbb{N}_{2s}^n.$$

If $\text{rank } \mathbf{M}_s(\mathbf{y}^) = 1$ then $\tilde{q}_d^k = f^*$ and $x^* = (y_\alpha^*)$, $|\alpha| = 1$, i.e., $x^* = \sum_{p=1}^L \theta_p^* x^{(p)}$, is an optimal solution of problem (P).*

Proof. If $\text{rank } \mathbf{M}_s(\mathbf{y}^*) = 1$ then (\mathbf{y}_α^*) , $\alpha \in \mathbb{N}_{2s}^n$, is the vector of moments (up to order $2s$) of the Dirac measure δ_{x^*} at the point $x^* = \sum_{p=1}^L \theta_p^* x^{(p)} \in \mathbb{R}^n$. That is,

$$\langle q, \delta_{x^*} \rangle = \langle q, \sum_{p=1}^L \theta_p^* \delta_{x^{(p)}} \rangle, \quad \forall q \in \mathbb{R}[x]_{2s}.$$

In other words, up to moments of order $2s$, one cannot distinguish the Dirac measure at x^* from the signed measure $\mu = \sum_p \theta_p^* \delta_{x^{(p)}}$. Next, as θ^* is feasible for (7) and $2s \geq \max[\deg(f); \deg(g_j)]$,

$$0 \leq \sum_{p=1}^L \theta_p^* \langle h_{\alpha\beta}, \delta_{x^{(p)}} \rangle = \left\langle h_{\alpha\beta}, \sum_{p=1}^L \theta_p^* \delta_{x^{(p)}} \right\rangle = h_{\alpha\beta}(x^*), \quad \forall (\alpha, \beta) \in \mathbb{N}_{2s}^{2m}.$$

In particular $g_j(x^*) \geq 0$, $j = 1, \dots, m$, which shows that $x^* \in \mathbf{K}$. In addition,

$$f^* \geq \tilde{q}_d^k = \sum_{p=1}^L \theta_p^* \langle f, \delta_{x^{(p)}} \rangle = \left\langle f, \sum_{p=1}^L \theta_p^* \delta_{x^{(p)}} \right\rangle = f(x^*),$$

which proves that x^* is an optimal solution of problem (P). \square

2.5 On the rank-one matrices of (5) and SDPT3

Note that in the SDP (6), the constraint matrices associated with Q are all dense rank-1 matrices of the form $A_p = v_k(x^{(p)})v_k(x^{(p)})^T$. Thus if we let $\mathbf{v}_p = v_k(x^{(p)})$, then the linear maps involved in the equality constraints of the SDP can be evaluated cheaply based on the following formulas:

$$\mathcal{A}(X) := \left[\langle A_p, X \rangle \right]_{p=1}^L = \left[\langle \mathbf{v}_p, X \mathbf{v}_p \rangle \right]_{p=1}^L, \quad \mathcal{A}^* y := \sum_{p=1}^L y_p A_p = V \text{Diag}(y) V^T$$

where $X \in \mathcal{S}^{s(k)}$, $y \in \mathbb{R}^L$, $V = [\mathbf{v}_1, \dots, \mathbf{v}_L] \in \mathbb{R}^{s(k) \times L}$. Moreover, one need not store the dense constraint matrices $\{A_p \mid p = 1, \dots, L\}$ but only the vectors $\{\mathbf{v}_p \mid p = 1, \dots, L\}$. *To solve the SDP (6) efficiently, we need to exploit the rank-1 structure of the constraint matrices during the iterations.* Fortunately, the SDPT3 solver [26, 27] based on interior point methods has already been designed to exploit such a rank-1 structure to minimize the memory needed to store the constraint matrices, as well as to minimize the computational cost required to compute the Schur complement matrix arising in each interior-point iteration. More precisely, in each iteration where a positive definite matrix $W \in \mathcal{S}^{s(k)}$ is given, one needs to compute the Schur complement matrix \mathbf{S} whose (p, q) element is given by

$$\mathbf{S}_{pq} = \langle A_p, W A_q W \rangle = \langle \mathbf{v}_p \mathbf{v}_p^T, W \mathbf{v}_q \mathbf{v}_q^T W \rangle = \langle \mathbf{v}_p, W \mathbf{v}_q \rangle^2, \quad p, q = 1, \dots, L.$$

It is the combination of these two implementation techniques (point evaluation in the formulation and exploiting rank-one structure in the interior point algorithm) that makes our implementation of the SOS-hierarchy (5) efficient.

3 Computational issues

Given $f \in \mathbb{R}[x]_d$, in order to efficiently evaluate the vector $f(x^{(p)})$, $p = 1, \dots, L$, we need a convenient representation of the polynomial $f(x)$. In our implementation of BSOS, we use the following data format to input a polynomial:

$$\mathbf{F}(i, 1 : n + 1) = [\alpha^T, f_\alpha]$$

where f_α is the i th coefficient corresponding to the monomial x^α . Note that the enumeration of the coefficients of $f(x)$ is not important. For a given point $z \in \mathbb{R}^n$ such that $z_i \neq 0$ for all $i = 1, \dots, n$, we evaluate $f(z)$ via the following procedure written in MATLAB syntax:

Step 1. Set $\mathbf{P} = \mathbf{F}(:, 1 : n)$, $\mathbf{f} = \mathbf{F}(:, n + 1)$, and $s = (s_1, \dots, s_n)^T$, where $s_i = 1$ if $z_i < 0$, and $s_i = 0$ if $z_i \geq 0$.

Step 2. Compute $\bar{s} = \text{rem}(\mathbf{P}s, 2)$ and $\mathbf{z} = \exp(\mathbf{P} \log |z|)$.

Step 3. Compute $f(z) = \langle \mathbf{f}^{(a)}, \mathbf{z}^{(a)} \rangle - \langle \mathbf{f}^{(b)}, \mathbf{z}^{(b)} \rangle$, where $\mathbf{f}^{(a)} = \mathbf{f}(\text{find}(\bar{s} == 0))$ and $\mathbf{f}^{(b)} = \mathbf{f}(\text{find}(\bar{s} == 1))$.

(The above procedure can be modified slightly to handle the case when z has some zero components.) Note that in the above procedure, $\langle \mathbf{f}^{(a)}, \mathbf{z}^{(a)} \rangle$ and $\langle \mathbf{f}^{(b)}, \mathbf{z}^{(b)} \rangle$ correspond to the sum of positive terms and sum of negative terms in the evaluation of $f(z)$. By separating the summation of the positive and negative terms in the evaluation of $f(z)$, it is hoped that cancellation errors can be minimized.

We should mention that some of the equality constraints in (6) may be redundant. For the sake of reducing the computational cost and improve the numerical stability, we remove these redundant constraints before solving the SDP. However, as d increases, the linear constraints would become more and more nearly dependent, and typically the SDP problem cannot be solved accurately by either SDPT3 or SEDUMI.

Another numerical issue which we should point out is that the constraint matrix

$$\begin{bmatrix} (h_{\alpha\beta}(x^{(1)}))_{(\alpha,\beta) \in \mathbb{N}_d^{2m}} \\ \vdots \\ (h_{\alpha\beta}(x^{(L)}))_{(\alpha,\beta) \in \mathbb{N}_d^{2m}} \end{bmatrix}$$

associated with the nonnegative vector $(\lambda_{\alpha\beta})$ is typically fully dense. Such a matrix would consume too much memory and also computational cost when d increases or when m is large.

4 Numerical experiments

We call our approach BSOS (for hierarchy with bounded degree SOS). As mentioned in the Introduction, we conduct experiments on three classes of problems which will be described in the ensuing subsections.

4.1 Comparison of BSOS with Gloptiploy

We construct a set of test functions with 5 constraints. The test functions are mainly generated based on the following two problems:

$$\begin{aligned} (P_1) \quad & f = \quad x_1^2 \quad -x_2^2 \quad \quad \quad +x_3^2 \quad -x_4^2 \quad \quad +x_1 \quad -x_2 \\ \text{s.t.} \quad & 0 \leq g_1 = \quad 2x_1^2 \quad +3x_2^2 \quad +2x_1x_2 \quad +2x_3^2 \quad +3x_4^2 \quad +2x_3x_4 \leq 1 \\ & 0 \leq g_2 = \quad 3x_1^2 \quad +2x_2^2 \quad -4x_1x_2 \quad +3x_3^2 \quad +2x_4^2 \quad -4x_3x_4 \leq 1 \\ & 0 \leq g_3 = \quad x_1^2 \quad +6x_2^2 \quad -4x_1x_2 \quad +x_3^2 \quad +6x_4^2 \quad -4x_3x_4 \leq 1 \\ & 0 \leq g_4 = \quad x_1^2 \quad +4x_2^2 \quad -3x_1x_2 \quad +x_3^2 \quad +4x_4^2 \quad -3x_3x_4 \leq 1 \\ & 0 \leq g_5 = \quad 2x_1^2 \quad +5x_2^2 \quad +3x_1x_2 \quad +2x_3^2 \quad +5x_4^2 \quad +3x_3x_4 \leq 1 \\ & 0 \leq x. \end{aligned}$$

The optimal value of (P_1) is $f(x^*) = -0.57491$, as computed by Gloptipoly3. For BSOS, we get the result $q_{d=1}^{k=1} = -0.57491$, which is the exact result.

The second problem is :

$$\begin{aligned}
(P_2) \quad & f = x_1^4 x_2^2 + x_1^2 x_2^4 - x_1^2 x_2^2 \\
\text{s.t.} \quad & 0 \leq g_1 = x_1^2 + x_2^2 \leq 1 \\
& 0 \leq g_2 = 3x_1^2 + 2x_2^2 - 4x_1 x_2 \leq 1 \\
& 0 \leq g_3 = x_1^2 + 6x_2^4 - 8x_1 x_2 + 2.5 \leq 1 \\
& 0 \leq g_4 = x_1^4 + 3x_2^4 \leq 1 \\
& 0 \leq g_5 = x_1^2 + x_2^3 \leq 1 \\
& 0 \leq x_1, \quad 0 \leq x_2.
\end{aligned}$$

The optimal value of (P_2) is $f(x^*) = -0.037037$, as computed by Gloptipoly3. The results obtained by BSOS are

$$\begin{aligned}
q_{d=1}^{k=3} &= -0.041855, & q_{d=2}^{k=3} &= -0.037139, & q_{d=3}^{k=3} &= -0.037087 \\
q_{d=4}^{k=3} &= -0.037073, & q_{d=5}^{k=3} &= -0.037046 \\
q_{d=1}^{k=4} &= -0.038596, & q_{d=2}^{k=4} &= -0.037046, & q_{d=3}^{k=4} &= -0.037040 \\
q_{d=4}^{k=4} &= -0.037038, & q_{d=5}^{k=4} &= -0.037037.
\end{aligned}$$

Based on the above two problems, we increase the degree of the objective function and constraint functions to generate other test instances which are given explicitly in the Appendix.

Table 4.1 compares the results obtained by BSOS and Gloptipoly3 for the tested instances. We observe that BSOS can give the exact result for those problems with either low degree or low dimension, while also providing a good lower bound for high degree and high dimensional problems. In addition, BSOS can usually get a better bound for most of the test problems when d increases, and in most cases, the bound is good enough for $d = 2, 3$.

In Table 4.1, we also use the sufficient condition stated in Lemma 1 to check whether the generated lower bound is indeed optimal. For quite a number of instances, the moment matrix $\mathbf{M}_\ell(\mathbf{y}^*)$ associated with the optimal solution θ^* of (7) indeed has numerical rank equal to one (we declare that the matrix has numerical rank equal to one if the largest eigenvalue is at least 10^4 times larger than the second largest eigenvalue), which certifies that the lower bound is actually the optimal value. We should note that for some of the instances, although the lower bound is actually the optimal value (as declared by Gloptipoly), but the rank of the moment matrix $\mathbf{M}_\ell(\mathbf{y}^*)$ is larger than one.

4.2 Comparison of BSOS with the LP relaxations of Krivine-Stengle on convex problems

Here we compare the performance of BSOS with the LP relaxations of Krivine-Stengle on convex problems where each test problem has 5 constraint functions in addition to the nonnegative constraint $x \geq 0$. Note that the LP relaxation problem has exactly the same form as in (6), except that the positive semidefinite matrix variable Q is set to 0. We should mention that even though the Krivine-Stengle scheme generate LP problems instead of SDP problems, the sizes of the LP problems also increase rapidly as d increases. In particular, the dimension of the nonnegative variable λ is $\binom{2m+d}{d}$, and the constraint matrix is fully dense.

Table 1: Comparison of BSOS and Gloptipoly3. An entry marked with “*” means that the correspond SDP was not solved to high accuracy.

Problem	BSOS				Gloptipoly3		
	(d, k)	Result	Time(s)	rank(M(y*))	Result	Time(s)	Order, Optimal
P4_2	1,1	-6.7747e-001	0.3	1	-6.7747e-001	0.2	1,yes
	2,1	-6.7747e-001	0.5	1			
P4_4	1,2	-2.9812e-001	0.5	7	-3.3539e-002	0.3	2,yes
	2,2	-3.3539e-002	0.6	4			
P4_6	1,3	-6.2500e-002	0.8	31	-6.0693e-002	0.5	3,yes
	2,3	-6.0937e-002	0.9	7			
	3,3	-6.0693e-002	1.8	4			
P4_8	1,4	-9.3354e-002*	3.2	> 10	-8.5813e-002	2.6	4,yes
	2,4	-8.5813e-002	3.7	9			
	3,4	-8.5813e-002	5.1	4			
P6_2	1,1	-5.7491e-001	0.3	1	-5.7491e-001	0.2	1,yes
	2,1	-5.7491e-001	0.8	1			
P6_4	1,2	-5.7716e-001	1.1	10	-5.7696e-001	0.3	2,yes
	2,2	-5.7696e-001	1.1	4			
	3,2	-5.7696e-001	4.3	1			
P6_6	1,3	-6.5972e-001	7.1	> 10	-4.1288e-001	6.4	3,yes
	2,3	-6.5972e-001	10.2	> 10			
	3,3	-4.1288e-001	32.0	1			
P6_8	1,4	-6.5973e-001	74.2	> 10	-4.0902e-001	207.2	4,yes
	2,4	-6.5973e-001	168.6	> 10			
	3,4	-6.5973e-001	264.1	> 10			
	4,4	-4.0928e-001*	1656.0	1*			
8 var, deg 2	1,1	-5.7491e-001	0.5	1	-5.7491e-001	0.3	1,yes
	2,1	-5.7491e-001	0.9	1			
8 var, deg 4	1,2	-6.5946e-001	2.8	> 10	-4.3603e-001	1.5	2,yes
	2,2	-4.3603e-001	4.8	1			
8 var, deg 6	1,3	-6.5973e-001	127.1	> 10	-4.1288e-001	161.3	3,yes
	2,3	-6.5973e-001	126.6	> 10			
	3,3	-4.1322e-001*	258.7	1*			
10 var, deg 2	1,1	-5.7491e-001	0.4	1	-5.7491e-001	0.2	1,yes
	2,1	-5.7491e-001	1.0	1			
10 var, deg 4	1,2	-6.5951e-001	7.8	1	-4.3603e-001	5.3	2,yes
	2,2	-4.3603e-001	20.0	1			
	3,2	-4.3603e-001*	66.7	1*			
20 var, deg 2	1,1	-5.7491e-001	1.2	1	-5.7491e-001	0.4	1,yes
	2,1	-5.7491e-001	3.0	1			
20 var, deg 4	1,2	infeasible	302.1	-	-4.3603e-001	5600.8	2,yes
	2,2	-4.3602e-001*	1942.2	1*			

The following example illustrates the performance of LP relaxation method:

$$\begin{aligned}
(C_1) \quad \min f &= x_1^4 + x_2^4 + 2x_1^2x_2^2 - x_1 - x_2 \\
\text{s.t.} \quad 0 \leq g_1 &= -x_1^4 - 2x_2^4 + 1 \\
0 \leq g_2 &= -2x_1^4 - x_2^4 + 1 \\
0 \leq g_3 &= -x_1^4 - 4x_2^2 + 1.25 \\
0 \leq g_4 &= -4x_1^4 - x_2^4 + 1.25 \\
0 \leq g_5 &= -2x_1^4 - 3x_2^2 + 1.1 \\
0 \leq x_1 & \\
0 \leq x_2 &.
\end{aligned}$$

For this problem, the functions f and $-g_i$'s are all convex. The optimal value for this problem is $f(x^*) = -0.7500$, as computed by Gloptipoly3. For BSOS, we get $q_{d=1}^{k=2} = -0.7500$, and we obtained the exact result by just choosing $d = 1$. This observation is consistent with Theorem 4.1 in [16]. For the LP relaxation method, we get the following values for various choices of d :

$$q_{d=1}^{LP} = \text{infeasible}, \quad q_{d=2}^{LP} = -1.2200, \quad q_{d=3}^{LP} = -1.0944, \quad q_{d=4}^{LP} = -0.9696, \quad q_{d=5}^{LP} = \text{fail}.$$

Observe that when d increases, we could get a better lower bound for the exact optimal value. However, as d increases, the LP relaxation problem would become increasing ill-posed and the solver has difficulty in solving LP problem accurately. In particular, for $d = 5$, both the solvers SeDuMi and SDPT3 fail to compute an accurate enough solution for the LP to generate a sensible lower bound for $f(x^*)$.

In Table 2, we observe that BSOS can achieve the exact result with $d = 1$ for all the test instances. In contrast, the LP relaxation method of Krivine-Stengle does not perform very well even though the test instances are convex problems. In particular, observe that for the last instance C20.2, the LP relaxation method cannot produce a good lower bound even when we choose $d = 3$, and the time taken to solve the correspond LP is about 40 minutes.

Table 2: Comparison of BSOS with LP relaxations of Krivine-Stengle on convex problems.

	LP			BSOS			Gloptipoly3		
	d	Result	Time(s)	d, k	Result	Time(s)	Result	Time(s)	Order, Optimal
C4.2	1	infeasible		1, 1	-2.5000e-001	0.4	-2.5000e-001	0.2	1,yes
	2	-9.0000e-001	0.1						
	3	-5.8852e-001	0.3						
	4	-4.2500e-001	5.6						
	5	-3.4975e-001	98.4						
	6	-3.1001e-001	4074.1						
C4.4	≤ 3	infeasible		1, 2	-6.9574e-001	0.6	-6.9574e-001	0.2	2,yes
	4	-1.1094e+000	16.9						
	5	-8.8542e-001	788.4						
C4.6	≤ 5	infeasible		1, 3	-1.1933e+000	1.5	-1.1933e+000	0.5	3,unknown
	6	fail							
C6.2	1	infeasible		1, 1	-2.5000e-001	0.3	-2.5000e-001	0.2	1,yes
	2	-9.0000e-001	0.1						
	3	-5.8852e-001	0.6						
	4	-4.2500e-001	66.3						
	5	-3.4975e-001	4069.3						
C6.4	≤ 3	infeasible		1, 2	-6.9574e-001	1.3	-6.9574e-001	0.4	2,yes
	4	-1.1094e+000	177.5						
C6.6	≤ 5	infeasible		1, 3	-1.1933e+000	1.3	-1.1933e+000	0.4	3,unknown
	6	out of memory							
C8.2	1	infeasible		1, 1	-2.5000e-001	0.4	-2.5000e-001	0.2	1,yes
	2	-9.0000e-001	0.1						
	3	-5.8852e-001	3.5						
	4	-4.2500e-001	508.8						

Table 2: Comparison of BSOS with LP relaxations of Krivine-Stengle on convex problems.

	LP			BSOS			Gloptipoly3		
	d	Result	Time(s)	d, k	Result	Time(s)	Result	Time(s)	Order, Optimal
C8_4	≤ 3	infeasible		1, 2	-6.9574e-001	3.3	-6.9574e-001	1.2	2,yes
	4	-1.1094e+000	1167.3						
C10_2	1	infeasible		1, 1	-2.5000e-001	0.8	-2.5000e-001	0.3	1,yes
	2	-9.0000e-001	0.1						
	3	-5.8852e-001	15.8						
	4	-4.2500e-001	9993.0						
C10_4	≤ 3	infeasible		1, 2	-6.9574e-001	11.4	-6.9574e-001	5.5	2,yes
	4	-1.1094e+000	5544.2						
C20_2	1	infeasible		1, 1	-2.5000e-001	1.7	-2.5000e-001	0.4	1,yes
	2	-9.0000e-001	0.9						
	3	-5.8852e-001	2398.0						

4.3 Performance of BSOS on quadratic problems with polyhedral constraints

Here consider the following problem:

$$\begin{aligned}
 \min \quad & x^T A x \\
 \text{s.t.} \quad & e^T x \leq 1, \quad x \geq 0, \quad x \in \mathbb{R}^n,
 \end{aligned} \tag{8}$$

where A is a given $n \times n$ symmetric matrix. In our numerical experiments, we generate random instances such as Qn10_r2 for which $n = 10$ and A is randomly generated so that it has $r = 2$ negative eigenvalues and $n - r$ positive eigenvalues as follows:

```

rng('default')
A1 = randn(n); A2 = A1*A1'; perm=randperm(n);
[V,D] = eig(A); eigval=diag(D); idx1=perm(1:r); idx2=perm(r+1:n);
V1=V(:,idx1); V2=V(:,idx2); d1=eigval(idx1); d2=eigval(idx2);
A = V2*diag(d2)*V2' - V1*diag(d1)*V1';

```

Table 4.3 compares the performance of BSOS and Gloptipoly3. From the numerical results, we can see that BSOS is far more efficient than Gloptipoly3 in solving the problems (8). For example, for the problem Qn20_r2 with $n = 20$, BSOS took only 1.9 seconds to generate the lower bound $-2.0356e3$ for the problem, but Gloptipoly3 took more than 1 hour to generate the same bound. The disparity in the efficiency between BSOS and Gloptipoly3 is expected to become even wider for other instances with n larger than 20.

In Table 4.3, we again use the sufficient condition stated in Lemma 1 to check whether the generated lower bound is indeed optimal. For each of the first eight instances, the moment matrix $M_\ell(\mathbf{y}^*)$ associated with the optimal solution θ^* of (7) has numerical rank equal to one (we declare that the matrix has numerical rank equal to one if the largest eigenvalue is at least 10^4 times larger than the second largest eigenvalue), which certifies that the lower bound is actually the optimal value.

5 Conclusion

We have described and tested a new hierarchy of semidefinite relaxations for global polynomial optimization. It tries to combine some advantages of previously defined LP- and SOS-hierarchies. Essentially, it uses a positivity certificate already used in the LP-hierarchy but with an additional semidefinite constraint which thus makes it an SOS-hierarchy. However the main and crucial

Table 3: Comparison of BSOS and Gloptipoly3 on quadratic problems with polyhedral constraints.

Problem	BSOS				Gloptipoly3			
	(d, k)	Result	Time(s)	rank($M_\ell(\mathbf{y}^*)$)	Order	Result	Time(s)	Optimal
Qn10_r2 $n = 10, r = 2$	1,1	infeasible	0.8		1	infeasible	0.1	
	2,1	-2.8023e+000	0.5	1	2	-2.8023e+000	2.8	yes
Qn10_r5 $n = 10, r = 5$	1,1	infeasible	0.7		1	infeasible	0.1	
	2,1	-1.9685e+001	0.4	1	2	-1.9685e+001	2.3	yes
Qn20_r2 $n = 20, r = 2$	1,1	infeasible	1.5		1	infeasible	0.1	
	2,1	-2.0356e-003	1.9	1	2	-2.0356e-003	4057.0	yes
Qn20_r5 $n = 20, r = 5$	1,1	infeasible	1.7		1	infeasible	0.1	
	2,1	-1.7900e+001	1.0	1	2	-1.7900e+001	3587.4	yes
Qn40_r4 $n = 40, r = 4$	1,1	infeasible	10.9		1	infeasible	1.2	
	2,1	-7.0062e+000	10.9	1				
Qn50_r5 $n = 50, r = 5$	1,1	infeasible	24.9		1	infeasible	1.5	
	2,1	-5.9870e+000	34.8	1				
Qn100_r10 $n = 100, r = 10$	1,1	infeasible	385.8		1	infeasible	108.7	
	2,1	-8.8502e+000	1617.4	1				
Problem 2.9 in [8] $n = 10, r = 6$	1,1	infeasible	0.9		1	infeasible	0.1	
	2,1	3.7500e-001	0.5	1	2	3.7500e-001	2.7	yes
A=-toeplitz([0,1,1,1,1,zeros(1,10)]) $n = 15, r = 3$	1,1	infeasible	1.2		1	infeasible	0.1	
	2,1	-8.0000e-001	0.6	11	2	-8.0087e-001	233.7	unknown
A=-toeplitz([0,1,1,zeros(1,17)]) $n = 20, r = 7$	1,1	infeasible	1.9		1	infeasible	0.2	
	2,1	-6.6667e-001	1.3	18	2	-6.6905e-001	4753.3	unknown

point is that the size of this additional semidefinite constraint is fixed in advance and decided by the user (in contrast to the standard SOS-hierarchy in which the size of the semidefinite constraint increases in the hierarchy). Preliminary results are encouraging especially for non convex problems on convex polytopes where problems with up to 100 variables have been solved in a reasonable amount of time (whereas the standard SOS-hierarchy cannot be implemented).

For problems of larger size one needs to consider some serious numerical issues due to the presence of some fully dense submatrix and some nearly dependent linear constraints. In addition, to be able to handle large-scale problems one also needs to provide a “sparse version” of this hierarchy, an analogue of the sparse version of the SOS-hierarchy defined in [28]. Both issues (a topic of further investigation) are certainly non trivial, in particular the latter issue because the positivity certificate used in this new hierarchy involves products of initial polynomial constraints, which destroys the sparsity pattern considered in [28].

References

- [1] Ahmadi A.A., Majumdar A. *DSOS and SDSOS Optimization: LP and SOCP-Based Alternatives to SOS Optimization*, Proceedings of the 48th Annual Conference on Information Sciences and Systems, Princeton, NJ, pp. 1–5, March 2014.
- [2] Ben-Tal A., Nemirovski A. *Lectures on Modern Convex Optimization*, SIAM, Philadelphia, 2001.
- [3] Belousov E.G., Klatte D. A Frank-Wolfe type theorem for convex polynomial programs, *Comp. Optim. Appl.* **22**, pp. 37–48, 2002.
- [4] Benabbas S., Georgiou K., Magen A., Tulsiani M. SDP gaps from pairwise independence, *Theory of Computing* **8**, pp. 269–289, 2012.
- [5] Benabbas S., Magen A. Extending SDP integrality gaps to Sherali-Adams with applications to Quadratic Programming and MaxCutGain, in *Integer Programming and Combinatorial Optimization*, Lecture Notes in Computer Science, Springer 2010, pp. 299–312.

- [6] Bertsekas D.P., Nedić A., Ozdaglar E. *Convex Analysis and Optimization*, Athena Scientific, Belmont, Massachusetts, 2003.
- [7] Chlamtac E., Tulsiani M. Convex relaxations and integrality gaps, in *Handbook of Semidefinite, Conic and Polynomial Optimization*, M. Anjos and J.B. Lasserre Eds., Springer, New York, 2012, pp. 139–170.
- [8] C.A. Floudas and P.M. Pardalos, *A Collection of Test Problems for Constrained Global Optimization Algorithms*, Lecture Notes in Comput. Sci. 455, Springer-Verlag, Berlin, 1990.
- [9] de Klerk E., Laurent M. On the Lasserre hierarchy of semidefinite programming relaxations of convex polynomial optimization problems, *SIAM J. Optim.* **21**, pp. 824–832, 2011.
- [10] Handelman D. Representing polynomials by positive linear functions on compact convex polyhedra, *Pac. J. Math.* **132**, pp. 35–62, 1988.
- [11] Helton J.W., Nie J. Semidefinite representation of convex sets and convex hulls, in *Handbook on Semidefinite, Conic and Polynomial Optimization*, M. Anjos and J.B. Lasserre Eds., Springer, New York, 2012, pp. 77–112.
- [12] Henrion D., Lasserre J.B., Lofberg J. GloptiPoly 3: moments, optimization and semidefinite programming, *Optim. Methods and Softwares***24**, pp. 761–779, 2009.
- [13] Krivine J.L. Anneaux préordonnés, *J. Anal. Math.* **12**, pp. 307–326, 1964.
- [14] Lasserre J.B. Global optimization with polynomials and the problem of moments, *SIAM J. Optim.* **11**, pp. 796–817, 2001.
- [15] Lasserre J.B. Semidefinite programming vs. LP relaxations for polynomial programming, *Math. Oper. Res.* **27**, pp. 347–360, 2002.
- [16] Lasserre J.B. A Lagrangian relaxation view of linear and semidefinite hierarchies, *SIAM J. Optim.* **23**, pp. 1742–1756, 2013
- [17] Lasserre J.B. Convexity in semi-algebraic geometry and polynomial optimization, *SIAM J. Optim.* **19**, pp. 1995–2014, 2009.
- [18] Lasserre J.B. *Moments, Positive Polynomials and Their Applications*, Imperial College Press, London, 2009.
- [19] Laurent M. A comparison of the Sherali-Adams, Lovász-Schrijver and Lasserre relaxations for 0-1 programming, *Math. Oper. Res.* **28**, pp. 470–496, 2003.
- [20] Marshall M. Representation of non-negative polynomials, degree bounds and applications to optimization, *Canad. J. Math.* **61**, pp. 205–221, 2009.
- [21] Nie J. Optimality conditions and finite convergence of Lasserre’s hierarchy, *Math. Program.* **146**, pp. 97–121, 2014.
- [22] Putinar M. Positive polynomials on compact semi-algebraic sets, *Ind. Univ. Math. J.* **42**, pp. 969–984, 1993.
- [23] Sherali H.D., Adams W.P. A hierarchy of relaxations between the continuous and convex hull representations for zero-one programming problems, *SIAM J. Discr. Math.* **3**, pp. 411–430, 1990.

- [24] Sherali H.D., Adams W.P. *A Reformulation-Linearization Technique for Solving Discrete and Continuous Nonconvex Problems*, Kluwer, Dordrecht, MA, 1999.
- [25] Stengle G. A Nullstellensatz and a Positivstellensatz in semialgebraic geometry, *Math. Ann.* **207**, pp. 87–97, 1974.
- [26] Toh K.C., Todd M.J., Tutuncu R.H. SDPT3 — a Matlab software package for semidefinite programming, *Optimization Methods and Software* **11**, pp. 545–581, 1999.
- [27] Toh K.C., Todd M.J., Tutuncu R.H. Solving semidefinite-quadratic-linear programs using SDPT3, *Mathematical Programming* **95**, pp. 189–217, 2003.
- [28] Waki S., Kim S., Kojima M., Maramatsu M. Sums of squares and semidefinite programming relaxations for polynomial optimization problems with structured sparsity, *SIAM J. Optim.* **17**, pp. 218–242, 2006.
- [29] Zheng X.J., Sun X.L., Li D., Xu Y.F. On zero duality gap in nonconvex quadratic programming problems, *J. Glob. Optim.* **52**, pp. 229–242, 2012.

Appendix

Test functions for BSOS and Gloptipoly in Table 4.1

Example P4.2 (4 variables, degree 2):

$$\begin{aligned}
 f &= x_1^2 - x_2^2 + x_3^2 - x_4^2 + x_1 - x_2; & g_1 &= 2x_1^2 + 3x_2^2 + 2x_1x_2 + 2x_3^2 + 3x_4^2 + 2x_3x_4; \\
 g_2 &= 3x_1^2 + 2x_2^2 - 4x_1x_2 + 3x_3^2 + 2x_4^2 - 4x_3x_4; & g_3 &= x_1^2 + 6x_2^2 - 4x_1x_2 + x_3^2 + 6x_4^2 - 4x_3x_4; \\
 g_4 &= x_1^2 + 4x_2^2 - 3x_1x_2 + x_3^2 + 4x_4^2 - 3x_3x_4; & g_5 &= 2x_1^2 + 5x_2^2 + 3x_1x_2 + 2x_3^2 + 5x_4^2 + 3x_3x_4; \quad x \geq 0.
 \end{aligned}$$

Example P4.4 (4 variables, degree 4):

$$\begin{aligned}
 f &= x_1^4 - x_2^4 + x_3^4 - x_4^4; & g_1 &= 2x_1^4 + 3x_2^2 + 2x_1x_2 + 2x_3^4 + 3x_4^2 + 2x_3x_4; \\
 g_2 &= 3x_1^2 + 2x_2^2 - 4x_1x_2 + 3x_3^2 + 2x_4^2 - 4x_3x_4; & g_3 &= x_1^2 + 6x_2^2 - 4x_1x_2 + x_3^2 + 6x_4^2 - 4x_3x_4; \\
 g_4 &= x_1^2 + 4x_2^2 - 3x_1x_2 + x_3^2 + 4x_4^2 - 3x_3x_4; & g_5 &= 2x_1^2 + 5x_2^2 + 3x_1x_2 + 2x_3^2 + 5x_4^2 + 3x_3x_4; \quad x \geq 0.
 \end{aligned}$$

Example P4.6 (4 variables, degree 6):

$$\begin{aligned}
 f &= x_1^4x_2^2 + x_1^2x_2^4 - x_1^2x_2^2 + x_3^4x_4^2 + x_3^2x_4^4 - x_3^2x_4^2; & g_1 &= x_1^2 + x_2^2 + x_3^2 + x_4^2; \\
 g_2 &= 3x_1^2 + 2x_2^2 - 4x_1x_2 + 3x_3^2 + 2x_4^2 - 4x_3x_4; & g_3 &= x_1^2 + 6x_2^4 - 8x_1x_2 + x_3^2 + 6x_4^4 - 8x_3x_4 + 2.5; \\
 g_4 &= x_1^4 + 3x_2^4 + x_3^4 + 3x_4^4; & g_5 &= x_1^2 + x_2^3 + x_3^2 + x_4^3; \quad x \geq 0.
 \end{aligned}$$

Example P4.8 (4 variables, degree 8):

$$\begin{aligned}
 f &= x_1^4x_2^2 + x_1^2x_2^6 - x_1^2x_2^2 + x_3^4x_4^2 + x_3^2x_4^6 - x_3^2x_4^2; & g_1 &= x_1^2 + x_2^2 + x_3^2 + x_4^2; \\
 g_2 &= 3x_1^2 + 2x_2^2 - 4x_1x_2 + 3x_3^2 + 2x_4^2 - 4x_3x_4; & g_3 &= x_1^2 + 6x_2^4 - 8x_1x_2 + x_3^2 + 6x_4^4 - 8x_3x_4 + 2.5; \\
 g_4 &= x_1^4 + 3x_2^4 + x_3^4 + 3x_4^4; & g_5 &= x_1^2 + x_2^3 + x_3^2 + x_4^3; \quad x \geq 0.
 \end{aligned}$$

Example P6.2 (6 variables, degree 2):

$$\begin{aligned}
 f &= x_1^2 - x_2^2 + x_3^2 - x_4^2 + x_5^2 - x_6^2 + x_1 - x_2; \\
 g_1 &= 2x_1^2 + 3x_2^2 + 2x_1x_2 + 2x_3^2 + 3x_4^2 + 2x_3x_4 + 2x_5^2 + 3x_6^2 + 2x_5x_6; \\
 g_2 &= 3x_1^2 + 2x_2^2 - 4x_1x_2 + 3x_3^2 + 2x_4^2 - 4x_3x_4 + 3x_5^2 + 2x_6^2 - 4x_5x_6; \\
 g_3 &= x_1^2 + 6x_2^2 - 4x_1x_2 + x_3^2 + 6x_4^2 - 4x_3x_4 + x_5^2 + 6x_6^2 - 4x_5x_6; \\
 g_4 &= x_1^2 + 4x_2^2 - 3x_1x_2 + x_3^2 + 4x_4^2 - 3x_3x_4 + x_5^2 + 4x_6^2 - 3x_5x_6; \\
 g_5 &= 2x_1^2 + 5x_2^2 + 3x_1x_2 + 2x_3^2 + 5x_4^2 + 3x_3x_4 + 2x_5^2 + 5x_6^2 + 3x_5x_6; \quad x \geq 0.
 \end{aligned}$$

Example P6.4 (6 variables, degree 4):

$$\begin{aligned}
 f &= x_1^4 - x_2^2 + x_3^4 - x_4^2 + x_5^4 - x_6^2 + x_1 - x_2; \\
 g_1 &= 2x_1^4 + x_2^2 + 2x_1x_2 + 2x_3^4 + x_4^2 + 2x_3x_4 + 2x_5^4 + x_6^2 + 2x_5x_6; \\
 g_2 &= 3x_1^2 + x_2^2 - 4x_1x_2 + 3x_3^2 + x_4^2 - 4x_3x_4 + 3x_5^2 + x_6^2 - 4x_5x_6; \\
 g_3 &= x_1^2 + 6x_2^2 - 4x_1x_2 + x_3^2 + 6x_4^2 - 4x_3x_4 + x_5^2 + 6x_6^2 - 4x_5x_6; \\
 g_4 &= x_1^2 + 3x_2^4 - 3x_1x_2 + x_3^2 + 3x_4^4 - 3x_3x_4 + x_5^2 + 3x_6^4 - 3x_5x_6; \\
 g_5 &= 2x_1^2 + 5x_2^2 + 3x_1x_2 + 2x_3^2 + 5x_4^2 + 3x_3x_4 + 2x_5^2 + 5x_6^2 + 3x_5x_6, \quad x \geq 0.
 \end{aligned}$$

Example P6.6 (6 variables, degree 6):

$$\begin{aligned}
 f &= x_1^6 - x_2^6 + x_3^6 - x_4^6 + x_5^6 - x_6^6 + x_1 - x_2; \\
 g_1 &= 2x_1^6 + 3x_2^2 + 2x_1x_2 + 2x_3^6 + 3x_4^2 + 2x_3x_4 + 2x_5^6 + 3x_6^2 + 2x_5x_6; \\
 g_2 &= 3x_1^2 + 2x_2^2 - 4x_1x_2 + 3x_3^2 + 2x_4^2 - 4x_3x_4 + 3x_5^2 + 2x_6^2 - 4x_5x_6; \\
 g_3 &= x_1^2 + 6x_2^2 - 4x_1x_2 + x_3^2 + 6x_4^2 - 4x_3x_4 + x_5^2 + 6x_6^2 - 4x_5x_6; \\
 g_4 &= x_1^2 + 4x_2^6 - 3x_1x_2 + x_3^2 + 4x_4^6 - 3x_3x_4 + x_5^2 + 4x_6^6 - 3x_5x_6; \\
 g_5 &= 2x_1^2 + 5x_2^2 + 3x_1x_2 + 2x_3^2 + 5x_4^2 + 3x_3x_4 + 2x_5^2 + 5x_6^2 + 3x_5x_6, \quad x \geq 0.
 \end{aligned}$$

Example P6.8 (6 variables, degree 8):

$$\begin{aligned}
 f &= x_1^8 - x_2^8 + x_3^8 - x_4^8 + x_5^8 - x_6^8 + x_1 - x_2; \\
 g_1 &= 2x_1^8 + 3x_2^2 + 2x_1x_2 + 2x_3^8 + 3x_4^2 + 2x_3x_4 + 2x_5^8 + 3x_6^2 + 2x_5x_6; \\
 g_2 &= 3x_1^2 + 2x_2^2 - 4x_1x_2 + 3x_3^2 + 2x_4^2 - 4x_3x_4 + 3x_5^2 + 2x_6^2 - 4x_5x_6; \\
 g_3 &= x_1^2 + 6x_2^2 - 4x_1x_2 + x_3^2 + 6x_4^2 - 4x_3x_4 + x_5^2 + 6x_6^2 - 4x_5x_6; \\
 g_4 &= x_1^2 + 4x_2^8 - 3x_1x_2 + x_3^2 + 4x_4^8 - 3x_3x_4 + x_5^2 + 4x_6^8 - 3x_5x_6; \\
 g_5 &= 2x_1^2 + 5x_2^2 + 3x_1x_2 + 2x_3^2 + 5x_4^2 + 3x_3x_4 + 2x_5^2 + 5x_6^2 + 3x_5x_6, \quad x \geq 0.
 \end{aligned}$$

Example P8_2 (8 variables, degree 2):

$$\begin{aligned}
f &= x_1^2 - x_2^2 + x_3^2 - x_4^2 + x_5^2 - x_6^2 + x_7^2 - x_8^2 + x_1 - x_2; \\
g_1 &= 2x_1^2 + 3x_2^2 + 2x_1x_2 + 2x_3^2 + 3x_4^2 + 2x_3x_4 + 2x_5^2 + 3x_6^2 + 2x_5x_6 + 2x_7^2 + 3x_8^2 + 2x_7x_8; \\
g_2 &= 3x_1^2 + 2x_2^2 - 4x_1x_2 + 3x_3^2 + 2x_4^2 - 4x_3x_4 + 3x_5^2 + 2x_6^2 - 4x_5x_6 + 3x_7^2 + 2x_8^2 - 4x_7x_8; \\
g_3 &= x_1^2 + 6x_2^2 - 4x_1x_2 + x_3^2 + 6x_4^2 - 4x_3x_4 + x_5^2 + 6x_6^2 - 4x_5x_6 + x_7^2 + 6x_8^2 - 4x_7x_8; \\
g_4 &= x_1^2 + 4x_2^2 - 3x_1x_2 + x_3^2 + 4x_4^2 - 3x_3x_4 + x_5^2 + 4x_6^2 - 3x_5x_6 + x_7^2 + 4x_8^2 - 3x_7x_8; \\
g_5 &= 2x_1^2 + 5x_2^2 + 3x_1x_2 + 2x_3^2 + 5x_4^2 + 3x_3x_4 + 2x_5^2 + 5x_6^2 + 3x_5x_6 + 2x_7^2 + 5x_8^2 + 3x_7x_8; \quad x \geq 0.
\end{aligned}$$

Example P8_4 (8 variables, degree 4):

$$\begin{aligned}
f &= x_1^4 - x_2^4 + x_3^4 - x_4^4 + x_5^4 - x_6^4 + x_7^4 - x_8^4 + x_1 - x_2; \\
g_1 &= 2x_1^4 + 3x_2^4 + 2x_1x_2 + 2x_3^4 + 3x_4^4 + 2x_3x_4 + 2x_5^4 + 3x_6^4 + 2x_5x_6 + 2x_7^4 + 3x_8^4 + 2x_7x_8; \\
g_2 &= 3x_1^4 + 2x_2^4 - 4x_1x_2 + 3x_3^4 + 2x_4^4 - 4x_3x_4 + 3x_5^4 + 2x_6^4 - 4x_5x_6 + 3x_7^4 + 2x_8^4 - 4x_7x_8; \\
g_3 &= x_1^4 + 6x_2^4 - 4x_1x_2 + x_3^4 + 6x_4^4 - 4x_3x_4 + x_5^4 + 6x_6^4 - 4x_5x_6 + x_7^4 + 6x_8^4 - 4x_7x_8; \\
g_4 &= x_1^4 + 4x_2^4 - 3x_1x_2 + x_3^4 + 4x_4^4 - 3x_3x_4 + x_5^4 + 4x_6^4 - 3x_5x_6 + x_7^4 + 4x_8^4 - 3x_7x_8; \\
g_5 &= 2x_1^4 + 5x_2^4 + 3x_1x_2 + 2x_3^4 + 5x_4^4 + 3x_3x_4 + 2x_5^4 + 5x_6^4 + 3x_5x_6 + 2x_7^4 + 5x_8^4 + 3x_7x_8, \quad x \geq 0.
\end{aligned}$$

Example P8_6 (8 variables, degree 6):

$$\begin{aligned}
f &= x_1^6 - x_2^6 + x_3^6 - x_4^6 + x_5^6 - x_6^6 + x_7^6 - x_8^6 + x_1 - x_2; \\
g_1 &= 2x_1^6 + 3x_2^6 + 2x_1x_2 + 2x_3^6 + 3x_4^6 + 2x_3x_4 + 2x_5^6 + 3x_6^6 + 2x_5x_6 + 2x_7^6 + 3x_8^6 + 2x_7x_8; \\
g_2 &= 3x_1^6 + 2x_2^6 - 4x_1x_2 + 3x_3^6 + 2x_4^6 - 4x_3x_4 + 3x_5^6 + 2x_6^6 - 4x_5x_6 + 3x_7^6 + 2x_8^6 - 4x_7x_8; \\
g_3 &= x_1^6 + 6x_2^6 - 4x_1x_2 + x_3^6 + 6x_4^6 - 4x_3x_4 + x_5^6 + 6x_6^6 - 4x_5x_6 + x_7^6 + 6x_8^6 - 4x_7x_8; \\
g_4 &= x_1^6 + 4x_2^6 - 3x_1x_2 + x_3^6 + 4x_4^6 - 3x_3x_4 + x_5^6 + 4x_6^6 - 3x_5x_6 + x_7^6 + 4x_8^6 - 3x_7x_8; \\
g_5 &= 2x_1^6 + 5x_2^6 + 3x_1x_2 + 2x_3^6 + 5x_4^6 + 3x_3x_4 + 2x_5^6 + 5x_6^6 + 3x_5x_6 + 2x_7^6 + 5x_8^6 + 3x_7x_8, \quad x \geq 0.
\end{aligned}$$

Example P10_2 (10 variables, degree 2):

$$\begin{aligned}
f &= x_1^2 - x_2^2 + x_3^2 - x_4^2 + x_5^2 - x_6^2 + x_7^2 - x_8^2 + x_9^2 - x_{10}^2 + x_1 - x_2; \\
g_1 &= 2x_1^2 + 3x_2^2 + 2x_1x_2 + 2x_3^2 + 3x_4^2 + 2x_3x_4 + 2x_5^2 + 3x_6^2 + 2x_5x_6 + 2x_7^2 + 3x_8^2 + 2x_7x_8 + 2x_9^2 + 3x_{10}^2 + 2x_9x_{10}; \\
g_2 &= 3x_1^2 + 2x_2^2 - 4x_1x_2 + 3x_3^2 + 2x_4^2 - 4x_3x_4 + 3x_5^2 + 2x_6^2 - 4x_5x_6 + 3x_7^2 + 2x_8^2 - 4x_7x_8 + 3x_9^2 + 2x_{10}^2 - 4x_9x_{10}; \\
g_3 &= x_1^2 + 6x_2^2 - 4x_1x_2 + x_3^2 + 6x_4^2 - 4x_3x_4 + x_5^2 + 6x_6^2 - 4x_5x_6 + x_7^2 + 6x_8^2 - 4x_7x_8 + x_9^2 + 6x_{10}^2 - 4x_9x_{10}; \\
g_4 &= x_1^2 + 4x_2^2 - 3x_1x_2 + x_3^2 + 4x_4^2 - 3x_3x_4 + x_5^2 + 4x_6^2 - 3x_5x_6 + x_7^2 + 4x_8^2 - 3x_7x_8 + x_9^2 + 4x_{10}^2 - 3x_9x_{10}; \\
g_5 &= 2x_1^2 + 5x_2^2 + 3x_1x_2 + 2x_3^2 + 5x_4^2 + 3x_3x_4 + 2x_5^2 + 5x_6^2 + 3x_5x_6 + 2x_7^2 + 5x_8^2 + 3x_7x_8 + 2x_9^2 + 5x_{10}^2 + 3x_9x_{10}; \\
& x \geq 0.
\end{aligned}$$

Example P10_4 (10 variables, degree 4):

$$\begin{aligned}
f &= x_1^4 - x_2^4 + x_3^4 - x_4^4 + x_5^4 - x_6^4 + x_7^4 - x_8^4 + x_9^4 - x_{10}^4 + x_1 - x_2; \\
g_1 &= 2x_1^4 + 3x_2^4 + 2x_1x_2 + 2x_3^4 + 3x_4^4 + 2x_3x_4 + 2x_5^4 + 3x_6^4 + 2x_5x_6 + 2x_7^4 + 3x_8^4 + 2x_7x_8 + 2x_9^4 + 3x_{11}^4 + 2x_9x_{10}; \\
g_2 &= 3x_1^4 + 2x_2^4 - 4x_1x_2 + 3x_3^4 + 2x_4^4 - 4x_3x_4 + 3x_5^4 + 2x_6^4 - 4x_5x_6 + 3x_7^4 + 2x_8^4 - 4x_7x_8 + 3x_9^4 + 2x_{10}^4 - 4x_9x_{10}; \\
g_3 &= x_1^4 + 6x_2^4 - 4x_1x_2 + x_3^4 + 6x_4^4 - 4x_3x_4 + x_5^4 + 6x_6^4 - 4x_5x_6 + x_7^4 + 6x_8^4 - 4x_7x_8 + x_9^4 + 6x_{10}^4 - 4x_9x_{10}; \\
g_4 &= x_1^4 + 4x_2^4 - 3x_1x_2 + x_3^4 + 4x_4^4 - 3x_3x_4 + x_5^4 + 4x_6^4 - 3x_5x_6 + x_7^4 + 4x_8^4 - 3x_7x_8 + x_9^4 + 4x_{10}^4 - 3x_9x_{10}; \\
g_5 &= 2x_1^4 + 5x_2^4 + 3x_1x_2 + 2x_3^4 + 5x_4^4 + 3x_3x_4 + 2x_5^4 + 5x_6^4 + 3x_5x_6 + 2x_7^4 + 5x_8^4 + 3x_7x_8 + 2x_9^4 + 5x_{10}^4 + 3x_9x_{10}; \\
& x \geq 0.
\end{aligned}$$

Example P20_2 (20 variables, degree 2):

$$\begin{aligned}
f &= x_1^2 - x_2^2 + x_3^2 - x_4^2 + x_5^2 - x_6^2 + x_7^2 - x_8^2 + x_9^2 - x_{10}^2 + x_{11}^2 - x_{12}^2 + x_1 - x_2 \\
&\quad + x_{13}^2 - x_{14}^2 + x_{15}^2 - x_{16}^2 + x_{17}^2 - x_{18}^2 + x_{19}^2 - x_{20}^2; \\
g_1 &= 2x_1^2 + 3x_2^2 + 2x_1x_2 + 2x_3^2 + 3x_4^2 + 2x_3x_4 + 2x_5^2 + 3x_6^2 + 2x_5x_6 + 2x_7^2 + 3x_8^2 + 2x_7x_8 \\
&\quad + 2x_9^2 + 3x_{10}^2 + 2x_9x_{10} + 2x_{11}^2 + 3x_{12}^2 + 2x_{11}x_{12} + 2x_{13}^2 + 3x_{14}^2 + 2x_{13}x_{14} + 2x_{15}^2 + 3x_{16}^2 \\
&\quad + 2x_{15}x_{16} + 2x_{17}^2 + 3x_{18}^2 + 2x_{17}x_{18} + 2x_{19}^2 + 3x_{20}^2 + 2x_{20}x_{20}; \\
g_2 &= 3x_1^2 + 2x_2^2 - 4x_1x_2 + 3x_3^2 + 2x_4^2 - 4x_3x_4 + 3x_5^2 + 2x_6^2 - 4x_5x_6 + 3x_7^2 + 2x_8^2 - 4x_7x_8 \\
&\quad + 3x_9^2 + 2x_{10}^2 - 4x_9x_{10} + 3x_{11}^2 + 2x_{12}^2 - 4x_{11}x_{12} + 3x_{13}^2 + 2x_{14}^2 - 4x_{13}x_{14} \\
&\quad + 3x_{15}^2 + 2x_{16}^2 - 4x_{15}x_{16} + 3x_{17}^2 + 2x_{18}^2 - 4x_{17}x_{18} + 3x_{19}^2 + 2x_{20}^2 - 4x_{19}x_{20}; \\
g_3 &= x_1^2 + 6x_2^2 - 4x_1x_2 + x_3^2 + 6x_4^2 - 4x_3x_4 + x_5^2 + 6x_6^2 - 4x_5x_6 + x_7^2 + 6x_8^2 - 4x_7x_8 \\
&\quad + x_9^2 + 6x_{10}^2 - 4x_9x_{10} + x_{11}^2 + 6x_{12}^2 - 4x_{11}x_{12} + x_{13}^2 + 6x_{14}^2 - 4x_{13}x_{14} \\
&\quad + x_{15}^2 + 6x_{16}^2 - 4x_{15}x_{16} + x_{17}^2 + 6x_{18}^2 - 4x_{17}x_{18} + x_{19}^2 + 6x_{20}^2 - 4x_{19}x_{20}; \\
g_4 &= x_1^2 + 4x_2^2 - 3x_1x_2 + x_3^2 + 4x_4^2 - 3x_3x_4 + x_5^2 + 4x_6^2 - 3x_5x_6 + x_7^2 + 4x_8^2 - 3x_7x_8 \\
&\quad + x_9^2 + 4x_{10}^2 - 3x_9x_{10} + x_{11}^2 + 4x_{12}^2 - 3x_{11}x_{12} + x_{13}^2 + 4x_{14}^2 - 3x_{13}x_{14} \\
&\quad + x_{15}^2 + 4x_{16}^2 - 3x_{15}x_{16} + x_{17}^2 + 4x_{18}^2 - 3x_{17}x_{18} + x_{19}^2 + 4x_{20}^2 - 3x_{19}x_{20}; \\
g_5 &= 2x_1^2 + 5x_2^2 + 3x_1x_2 + 2x_3^2 + 5x_4^2 + 3x_3x_4 + 2x_5^2 + 5x_6^2 + 3x_5x_6 + 2x_7^2 + 5x_8^2 + 3x_7x_8 \\
&\quad + 2x_9^2 + 5x_{10}^2 + 3x_9x_{10} + 2x_{11}^2 + 5x_{12}^2 + 3x_{11}x_{12} + 2x_{13}^2 + 5x_{14}^2 + 3x_{13}x_{14} \\
&\quad + 2x_{15}^2 + 5x_{16}^2 + 3x_{15}x_{16} + 2x_{17}^2 + 5x_{18}^2 + 3x_{17}x_{18} + 2x_{19}^2 + 5x_{20}^2 + 3x_{19}x_{20}; \\
x &\geq 0.
\end{aligned}$$

Example P20_4 (20 variables, degree 4): Same as P20_2 except that f is replaced by

$$\begin{aligned}
f &= x_1^4 - x_2^4 + x_3^2 - x_4^2 + x_5^2 - x_6^2 + x_7^2 - x_8^2 + x_9^2 - x_{10}^2 + x_{11}^2 - x_{12}^2 + x_1 - x_2 \\
&\quad + x_{13}^2 - x_{14}^2 + x_{15}^2 - x_{16}^2 + x_{17}^2 - x_{18}^2 + x_{19}^2 - x_{20}^2;
\end{aligned}$$

5.1 Test functions for BSOS versus LP relaxations of Krivine-Stengle on convex problems in Table 2

Example C4_2 (4 variables, degree 2):

$$\begin{aligned}
f &= x_1^2 + x_2^2 + x_3^2 + x_4^2 + 2x_1x_2 - x_1 - x_2; & g_1 &= -x_1^2 - 2x_2^2 - x_3^2 - 2x_4^2 + 1; \\
g_2 &= -2x_1^2 - x_2^2 - 2x_3^2 - x_4^2 + 1; & g_3 &= -x_1^2 - 4x_2^2 - x_3^2 - 4x_4^2 + 1.25; \\
g_4 &= -4x_1^2 - x_2^2 - 4x_3^2 - x_4^2 + 1.25; & g_5 &= -2x_1^2 - 3x_2^2 - 2x_3^2 - 3x_4^2 + 1.1; \quad x \geq 0.
\end{aligned}$$

Example C4_4 (4 variables, degree 4):

$$\begin{aligned}
f &= x_1^4 + x_2^4 + x_3^4 + x_4^4 + 3x_1^2x_2^2 - x_1 - x_2; & g_1 &= -x_1^4 - 2x_2^4 - x_3^4 - 2x_4^4 + 1; \\
g_2 &= -2x_1^4 - x_2^4 - 2x_3^4 - x_4^4 + 1; & g_3 &= -x_1^4 - 4x_2^4 - x_3^4 - 4x_4^4 + 1.25; \\
g_4 &= -4x_1^4 - x_2^4 - 4x_3^4 - x_4^4 + 1.25; & g_5 &= -2x_1^4 - 3x_2^4 - 2x_3^4 - 3x_4^4 + 1.1; \quad x \geq 0.
\end{aligned}$$

Example C4_6 (4 variables, degree 6):

$$\begin{aligned}
f &= x_1^6 + x_2^6 + x_3^6 + x_4^6 + \frac{10}{3}x_1^3x_2^3 - x_1 - x_2; & g_1 &= -x_1^6 - 2x_2^6 - x_3^6 - 2x_4^6 + 1; \\
g_2 &= -2x_1^6 - x_2^6 - 2x_3^6 - x_4^6 + 1; & g_3 &= -x_1^6 - 4x_2^6 - x_3^6 - 4x_4^6 + 1.25; \\
g_4 &= -4x_1^6 - x_2^6 - 4x_3^6 - x_4^6 + 1.25; & g_5 &= -2x_1^6 - 3x_2^6 - 2x_3^6 - 3x_4^6 + 1.1; \quad x \geq 0.
\end{aligned}$$

Example C6_2 (6 variables, degree 2):

$$\begin{aligned}
f &= x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2 + 2x_1x_2 - x_1 - x_2; & g_1 &= -x_1^2 - 2x_2^2 - x_3^2 - 2x_4^2 - x_5^2 - 2x_6^2 + 1; \\
g_2 &= -2x_1^2 - x_2^2 - 2x_3^2 - x_4^2 - 2x_5^2 - x_6^2 + 1; & g_3 &= -x_1^2 - 4x_2^2 - x_3^2 - 4x_4^2 - x_5^2 - 4x_6^2 + 1.25; \\
g_4 &= -4x_1^2 - x_2^2 - 4x_3^2 - x_4^2 - 4x_5^2 - x_6^2 + 1.25; & g_5 &= -2x_1^2 - 3x_2^2 - 2x_3^2 - 3x_4^2 - 2x_5^2 - 3x_6^2 + 1.1; \quad x \geq 0.
\end{aligned}$$

Example C6_4 (6 variables, degree 4):

$$\begin{aligned}
f &= x_1^4 + x_2^4 + x_3^4 + x_4^4 + x_5^4 + x_6^4 + 3x_1^2x_2^2 - x_1 - x_2; & g_1 &= -x_1^4 - 2x_2^4 - x_3^4 - 2x_4^4 - x_5^4 - 2x_6^4 + 1; \\
g_2 &= -2x_1^4 - x_2^4 - 2x_3^4 - x_4^4 - 2x_5^4 - x_6^4 + 1; & g_3 &= -x_1^4 - 4x_2^4 - x_3^4 - 4x_4^4 - x_5^4 - 4x_6^4 + 1.25; \\
g_4 &= -4x_1^4 - x_2^4 - 4x_3^4 - x_4^4 - 4x_5^4 - x_6^4 + 1.25; & g_5 &= -2x_1^4 - 3x_2^4 - 2x_3^4 - 3x_4^4 - 2x_5^4 - 3x_6^4 + 1.1; \quad x \geq 0.
\end{aligned}$$

Example C6_6 (6 variables, degree 6):

$$\begin{aligned}
f &= x_1^6 + x_2^6 + x_3^6 + x_4^6 + x_5^6 + x_6^6 + \frac{10}{3}x_1^2x_2^3 - x_1 - x_2; & g_1 &= -x_1^6 - 2x_2^6 - x_3^6 - 2x_4^6 - x_5^6 - 2x_6^6 + 1; \\
g_2 &= -2x_1^6 - x_2^6 - 2x_3^6 - x_4^6 - 2x_5^6 - x_6^6 + 1; & g_3 &= -x_1^6 - 4x_2^6 - x_3^6 - 4x_4^6 - x_5^6 - 4x_6^6 + 1.25; \\
g_4 &= -4x_1^6 - x_2^6 - 4x_3^6 - x_4^6 - 4x_5^6 - x_6^6 + 1.25; & g_5 &= -2x_1^6 - 3x_2^6 - 2x_3^6 - 3x_4^6 - 2x_5^6 - 3x_6^6 + 1.1; \quad x \geq 0.
\end{aligned}$$

Example C8_2 (8 variables, degree 2):

$$\begin{aligned}
f &= x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2 + x_7^2 + x_8^2 + 2x_1x_2 - x_1 - x_2; & g_1 &= -x_1^2 - 2x_2^2 - x_3^2 - 2x_4^2 - x_5^2 - 2x_6^2 - x_7^2 - 2x_8^2 + 1; \\
g_2 &= -2x_1^2 - x_2^2 - 2x_3^2 - x_4^2 - 2x_5^2 - x_6^2 - 2x_7^2 - x_8^2 + 1; & g_3 &= -x_1^2 - 4x_2^2 - x_3^2 - 4x_4^2 - x_5^2 - 4x_6^2 - x_7^2 - 4x_8^2 + 1.25; \\
g_4 &= -4x_1^2 - x_2^2 - 4x_3^2 - x_4^2 - 4x_5^2 - x_6^2 - 4x_7^2 - x_8^2 + 1.25; & g_5 &= -2x_1^2 - 3x_2^2 - 2x_3^2 - 3x_4^2 - 2x_5^2 - 3x_6^2 - 2x_7^2 - 3x_8^2 + 1.1; \\
& & & x \geq 0.
\end{aligned}$$

Example C8_4 (8 variables, degree 4):

$$\begin{aligned}
f &= x_1^4 + x_2^4 + x_3^4 + x_4^4 + x_5^4 + x_6^4 + x_7^4 + x_8^4 + 3x_1^2x_2^2 - x_1 - x_2; & g_1 &= -x_1^4 - 2x_2^4 - x_3^4 - 2x_4^4 - x_5^4 - 2x_6^4 - x_7^4 - 2x_8^4 + 1; \\
g_2 &= -2x_1^4 - x_2^4 - 2x_3^4 - x_4^4 - 2x_5^4 - x_6^4 - 2x_7^4 - x_8^4 + 1; & g_3 &= -x_1^4 - 4x_2^4 - x_3^4 - 4x_4^4 - x_5^4 - 4x_6^4 - x_7^4 - 4x_8^4 + 1.25; \\
g_4 &= -4x_1^4 - x_2^4 - 4x_3^4 - x_4^4 - 4x_5^4 - x_6^4 - 4x_7^4 - x_8^4 + 1.25; & g_5 &= -2x_1^4 - 3x_2^4 - 2x_3^4 - 3x_4^4 - 2x_5^4 - 3x_6^4 - 2x_7^4 - 3x_8^4 + 1.1; \\
& & & x \geq 0.
\end{aligned}$$

Example C10_2 (10 variables, degree 2):

$$\begin{aligned}
f &= x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2 + x_7^2 + x_8^2 + x_9^2 + x_{10}^2 + 2x_1x_2 - x_1 - x_2; \\
g_1 &= -x_1^2 - 2x_2^2 - x_3^2 - 2x_4^2 - x_5^2 - 2x_6^2 - x_7^2 - 2x_8^2 - x_9^2 - 2x_{10}^2 + 1; \\
g_2 &= -2x_1^2 - x_2^2 - 2x_3^2 - x_4^2 - 2x_5^2 - x_6^2 - 2x_7^2 - x_8^2 - 2x_9^2 - x_{10}^2 + 1; \\
g_3 &= -x_1^2 - 4x_2^2 - x_3^2 - 4x_4^2 - x_5^2 - 4x_6^2 - x_7^2 - 4x_8^2 - x_9^2 - 4x_{10}^2 + 1.25; \\
g_4 &= -4x_1^2 - x_2^2 - 4x_3^2 - x_4^2 - 4x_5^2 - x_6^2 - 4x_7^2 - x_8^2 - 4x_9^2 - x_{10}^2 + 1.25; \\
g_5 &= -2x_1^2 - 3x_2^2 - 2x_3^2 - 3x_4^2 - 2x_5^2 - 3x_6^2 - 2x_7^2 - 3x_8^2 - 2x_9^2 - 3x_{10}^2 + 1.1; \\
& & & x \geq 0.
\end{aligned}$$

Example C10_4 (10 variables, degree 4):

$$\begin{aligned}
f &= x_1^4 + x_2^4 + x_3^4 + x_4^4 + x_5^4 + x_6^4 + x_7^4 + x_8^4 + x_9^4 + x_{10}^4 + 3x_1^2x_2^2 - x_1 - x_2; \\
g_1 &= -x_1^4 - 2x_2^4 - x_3^4 - 2x_4^4 - x_5^4 - 2x_6^4 - x_7^4 - 2x_8^4 - x_9^4 - 2x_{10}^4 + 1; \\
g_2 &= -2x_1^4 - x_2^4 - 2x_3^4 - x_4^4 - 2x_5^4 - x_6^4 - 2x_7^4 - x_8^4 - 2x_9^4 - x_{10}^4 + 1; \\
g_3 &= -x_1^4 - 4x_2^4 - x_3^4 - 4x_4^4 - x_5^4 - 4x_6^4 - x_7^4 - 4x_8^4 - x_9^4 - 4x_{10}^4 + 1.25; \\
g_4 &= -4x_1^4 - x_2^4 - 4x_3^4 - x_4^4 - 4x_5^4 - x_6^4 - 4x_7^4 - x_8^4 - 4x_9^4 - x_{10}^4 + 1.25; \\
g_5 &= -2x_1^4 - 3x_2^4 - 2x_3^4 - 3x_4^4 - 2x_5^4 - 3x_6^4 - 2x_7^4 - 3x_8^4 - 2x_9^4 - 3x_{10}^4 + 1.1; \\
& & & x \geq 0.
\end{aligned}$$

Example C20_2 (20 variables, degree 2):

$$\begin{aligned}
f &= x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2 + x_7^2 + x_8^2 + x_9^2 + x_{10}^2 + 2x_1x_2 - x_1 - x_2 \\
&\quad + x_{11}^2 + x_{12}^2 + x_{13}^2 + x_{14}^2 + x_{15}^2 + x_{16}^2 + x_{17}^2 + x_{18}^2 + x_{19}^2 + x_{20}^2; \\
g_1 &= -x_1^2 - 2x_2^2 - x_3^2 - 2x_4^2 - x_5^2 - 2x_6^2 - x_7^2 - 2x_8^2 - x_9^2 - 2x_{10}^2 \\
&\quad - x_{11}^2 - 2x_{12}^2 - x_{13}^2 - 2x_{14}^2 - x_{15}^2 - 2x_{16}^2 - x_{17}^2 - 2x_{18}^2 - x_{19}^2 - 2x_{20}^2 + 1; \\
g_2 &= -2x_1^2 - x_2^2 - 2x_3^2 - x_4^2 - 2x_5^2 - x_6^2 - 2x_7^2 - x_8^2 - 2x_9^2 - x_{10}^2 \\
&\quad - 2x_{11}^2 - x_{12}^2 - 2x_{13}^2 - x_{14}^2 - 2x_{15}^2 - x_{16}^2 - 2x_{17}^2 - x_{18}^2 - 2x_{19}^2 - x_{20}^2 + 1; \\
g_3 &= -x_1^2 - 4x_2^2 - x_3^2 - 4x_4^2 - x_5^2 - 4x_6^2 - x_7^2 - 4x_8^2 - x_9^2 - 4x_{10}^2 \\
&\quad - x_{11}^2 - 4x_{12}^2 - x_{13}^2 - 4x_{14}^2 - x_{15}^2 - 4x_{16}^2 - x_{17}^2 - 4x_{18}^2 - x_{19}^2 - 4x_{20}^2 + 1.25; \\
g_4 &= -4x_1^2 - x_2^2 - 4x_3^2 - x_4^2 - 4x_5^2 - x_6^2 - 4x_7^2 - x_8^2 - 4x_9^2 - x_{10}^2 \\
&\quad - 4x_{11}^2 - x_{12}^2 - 4x_{13}^2 - x_{14}^2 - 4x_{15}^2 - x_{16}^2 - 4x_{17}^2 - x_{18}^2 - 4x_{19}^2 - x_{20}^2 + 1.25; \\
g_5 &= -2x_1^2 - 3x_2^2 - 2x_3^2 - 3x_4^2 - 2x_5^2 - 3x_6^2 - 2x_7^2 - 3x_8^2 - 2x_9^2 - 3x_{10}^2 \\
&\quad - 2x_{11}^2 - 3x_{12}^2 - 2x_{13}^2 - 3x_{14}^2 - 2x_{15}^2 - 3x_{16}^2 - 2x_{17}^2 - 3x_{18}^2 - 2x_{19}^2 - 3x_{20}^2 + 1.1; \\
& & & x \geq 0.
\end{aligned}$$