Exact Controllability of Scalar Conservation Laws with an Additional Control in the Context of Entropy Solutions

Vincent Perrollaz

EXACT CONTROLLABILITY OF SCALAR CONSERVATION LAWS WITH AN ADDITIONAL CONTROL IN THE CONTEXT OF ENTROPY SOLUTIONS.*

VINCENT PERROLLAZ †

Abstract. In this paper, we study the exact controllability problem for nonlinear scalar conservation laws on a compact interval, with a regular convex flux and in the framework of entropy solutions. With the boundary data and a source term depending only on the time as controls, we provide sufficient conditions for a state to be reachable in arbitrary small time. To do so we introduce a slightly modified wave-front tracking algorithm.

Key words. Entropy solution, conservation law, controllability.

1. Introduction. This paper is concerned with the exact controllability problem of a nonlinear scalar conservation law with a source term, on a bounded interval and in the framework of entropy solutions:

$$\begin{cases} \partial_t u + \partial_x f(u) = g(t), \\ u(0,x) = u_0(x), \\ u(t,0) = u_l(t), \\ u(t,L) = u_r(t), \end{cases} \qquad (t,x) \in (0,T) \times (0,L), \tag{1.1}$$

where f is assumed to be a $\mathcal{C}^2$ strictly convex function.

Scalar conservation laws are used for instance to model traffic flow or gas networks, but their importance also consists in being a first step in the understanding of systems of conservation laws. Those systems of equations model a huge number of physical phenomena: gas dynamics, electromagnetism, magneto-hydrodynamics, shallow water theory, combustion theory... see [16, Chapter2].

In this paper, we study (1.1) from the point of view of control theory and we regard the boundary data u_l, u_r and the source term g as controls. We will provide sufficient conditions on a state u_1 in $\mathrm{BV}(0,L)$ so that for any time T and any u_0 in $\mathrm{BV}(0,L)$ there exist u_l and u_r in $\mathrm{BV}(0,T)$ and g in $\mathcal{C}^1([0,T])$ such that $u(T,.) = u_1$.

For equations such as (1.1), the Cauchy problem on the whole line is well posed in small time in the framework of classical solutions and with a classical initial value. However those solutions generally blow up in finite time: shock waves appear. Hence to get global in time results, a weaker notion of solution is called for. In [33] Oleinik proved that given a flux $f \in \mathcal{C}^2$ such that $f'' > 0$ and any $u_0 \in L^\infty(\mathbb{R})$ there exists one and only one weak solution to:

$$u_t + (f(u))_x = 0, \qquad x \in \mathbb{R} \text{ and } t > 0, \tag{1.2}$$

$$u(0,.) = u_0, \tag{1.3}$$

satisfying the additional condition:

$$\frac{u(t,x+a) - u(t,x)}{a} < \frac{E}{t} \qquad \text{for } x \in \mathbb{R}, \quad t > 0, \text{ and } a > 0, \tag{1.4}$$

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† Université Pierre et Marie Curie–Paris6, UMR 7598 Laboratoire Jacques-Louis Lions, Paris, F-75005, France. E-mail: perrollaz@ann.jussieu.fr

and where E depends only on the quantities $\inf(f'')$ and $\sup(f')$ taken on the interval $[-||u_0||_{L^\infty}, ||u_0||_{L^\infty}]$ but not on u_0. Later in [28], Kruzkov extended this global result to the multidimensional problem, with a C^1 flux $f : \mathbb{R} \to \mathbb{R}^n$ not necessarily convex and with a different entropy condition:

$$u_t + \operatorname{div}(f(u)) = 0, \qquad \text{for } t > 0 \text{ and } x \in \mathbb{R}^n. \tag{1.5}$$

This time the weak entropy solution is defined as satisfying the following integral inequality:

$$\text{for all real numbers } k \text{ and all positive functions } \phi \text{ in } \mathcal{C}^1_c(\mathbb{R}^2), \tag{1.6}$$

$$\iint_{\mathbb{R}^2} |u-k|\phi_t + \operatorname{sgn}(u-k)(f(u)-f(k))\nabla\phi dtdx + \int_{\mathbb{R}} u_0(x)\phi(0,x)dx \geq 0. \tag{1.7}$$

The initial boundary value problem for equation (1.1) is also well posed as shown by Leroux in [30] for the one dimensional case with BV data, by Bardos Leroux and Nedelec in [4] for the multidimensional case with C^2 data and later by Otto in [34] (see also [32]) for L^∞ data. However the meaning of the boundary condition is quite intricate and the Dirichlet condition may not be fulfilled pointwise a.e. in time. In the following, we will use the fact that the restriction of a weak entropy solution of (1.1) on the whole line is the weak entropy solution to the IBVP on an interval with boundary data given by its trace at the boundary points (which exists the solution being in BV).

In the framework of entropy solutions, only a few controllability results exist for equation (1.1). In [2], Ancona and Marson characterized exactly the reachable states of

$$\begin{cases} \partial_t u + \partial_x(f(u)) = 0, & t > 0, \quad x > 0, \\ u(0,x) = 0, \quad x > 0, & u(t,0) = c(t), \quad t > 0. \end{cases} \tag{1.8}$$

where f is strictly convex and with a boundary control c. A state $w \in L^\infty(0,+\infty)$ is reachable in time T if and only if the following conditions hold:

$$\begin{gathered} w(x) \neq 0 \Rightarrow f'(w(x)) \geq \frac{x}{T}, \\ \big(w(x^-) \neq 0 \text{ and for every } y \text{ greater than } x,\ w(y) = 0\big) \Rightarrow f'(w(x^-)) > \frac{x}{T}, \\ \limsup_{h\to 0} \frac{w(x+h)-w(x)}{h} \leq \frac{f'(w(x))}{xf''(w(x))}, \end{gathered} \tag{1.9}$$

for every $x > 0$. The first two conditions are related to the propagation speed of (1.8) and the third is analogous to (1.4) but in the presence of a boundary.
In [24] Horsin provided sufficient conditions (related to (1.9)) on a state to be reachable for the Burgers equation posed on a compact interval and with a general initial data and where the controls are the two boundary values.
There are also some results on the controllability and noncontrollability of systems of conservation laws in the context of entropy solutions by Bressan and Coclite [6], Ancona and Coclite [1], Ancona and Marson [3] and by Glass [21]. In all those cases, some very reasonable looking states cannot be reached in any time using only boundary control. For example in the case of Burgers' equation on a compact interval, the constant state 0 cannot be reached from most initial states in any given time.

However with an additional control $g(t)$ as in (1.1) and with $f(z) = \frac{z^2}{2}$ (Burgers equation), Chapouly showed in [9] that in the framework of classical solutions, any state is reachable from any initial data and in any time (note that in this context, the controls also had to prevent the blow up of the solution, which will not be a concern for entropy solutions). This is the kind of improvement we want to obtain in the framework of entropy weak solutions, and for more general convex fluxes. Compared to the work of Chapouly, one can hope that working directly in the framework of entropy solution may provide more robust controls with respect to perturbation of the initial data. Indeed with regular solution one has to have an error term small in the $\mathcal{C}^1$ norm so has to prevent the blow up, while in the framework of entropy solution the smallness is required only in the L^∞ norm.

Let us now address the question of the relevance of the problem considered here. While the equation considered here is too simple to describe any "real" problem with enough precision, the appearance in a hyperbolic conservation law (or system) of an additional control (besides the boundary conditions) as a source term depending only on time is a rather general situation. And our toy model already exhibits many difficulties found in those more complex situations. Let us now give a few examples of those models.

We begin with the Camassa-Holm introduced in [11] and which describes the motion of hydrodynamical waves. It reads:

$$\partial_t v - \partial^3_{txx} v + 2\kappa \partial_x v + 3v\partial_x v = 2\partial_x v \partial^2_{xx} v + v\partial^3_{xxx} v. \tag{1.10}$$

However this can also be written:

$$(1 - \partial^2_{xx})\Big(v_t + \Big(\frac{v^2}{2}\Big)_x\Big) = -\partial_x \Big(v^2 + \frac{(v_x)^2}{2} + 2\kappa v\Big). \tag{1.11}$$

And if the equation is posed on an interval $(0, L)$ the linear operator $1 - \partial^2_{xx}$ has a kernel of dimension two so we may rather solve:

$$v_t + \Big(\frac{v^2}{2}\Big)_x = \alpha(t)\cosh(x) + \beta(t)\sinh(x) - \int_0^L \frac{e^{-|x-y|}}{2} \partial_x \Big(v^2 + \frac{(v_x)^2}{2} + 2\kappa v\Big)(t, y)dy, \tag{1.12}$$

for $t > 0$ and $x \in (0, L)$ and where α and β are two functions to be specified. Compared to (1.1) we have two controls α, β and an additional non-local source term, . It has been shown in [10] that singularities may occur for solutions to the Camassa-Holm equation. Thus different notions of weak solutions were developed in [38] and [7] and were used to obtain global in time existence results. Note that the non local term prevents the formation of true shock waves and those weak solutions are continuous functions. The Camassa-Holm equation has already been investigated from the viewpoint of control theory in [22] and [35] but only in the framework of regular solutions, the control law making sure that the solutions remain regular. It would be interesting however to work with the weak framework to get more robust controllability results.

The same kind of phenomenon also happens with the Hunter-Saxton equation introduced in [26] and which describes planar oscillation in nematic liquid crystals:

$$\partial_x \Big(u_t + \Big(\frac{u^2}{2}\Big)_x\Big) = \frac{(u_x)^2}{2}, \tag{1.13}$$

where u is the perturbation of the director angle from a constant equilibrium position. Once again when we consider the equation posed on an interval $(0, L)$ we have a constant to choose before inverting the differential operator ∂_x. More precisely we may choose a function γ depending only on the time t such that:

$$u_t + \Big(\frac{u^2}{2}\Big)_x = \int_0^x \frac{(u_x(t,z))^2}{2} dz + \gamma(t). \tag{1.14}$$

It has been shown that regular solutions may degenerate in finite time. And different framework of weak solutions have been introduced to obtain global in time results [26], [8], [27]. Furthermore it was shown in [18] that generalized characteristics are a very useful tool to study this equation. Once again it would be interesting to consider the exact controllability problem in this weak framework.

We may also consider the transverse motion of a tank which can be described as follows:

$$\begin{cases} H_t + (Hv)_x = 0, \\ (Hv)_t + (gH^2 + \frac{Hv^2}{2})_x = -u(t), \\ \ddot{D}(t) = gu(t). \end{cases} \tag{1.15}$$

where H is the height of the fluid, v is the horizontal velocity of the fluid, D is the displacement of the tank and u is the force applied to the tank. From the viewpoint of control theory our control is u and the goal is to move the tank from a given point and with the liquid at rest to another given point with the liquid once again at rest. It was already investigated by Dubois & Petit & Rouchon in [20] and also by Coron in [14] in the framework of regular solutions. However the framework of entropy solutions would hopefully provide more efficient motion planning and more robustness with regard to perturbations and errors.

Finally, we consider the Euler-Poisson system:

$$\begin{cases} \rho_t + j_x = 0, \\ j_t + (\frac{j^2}{\rho} + P(\rho))_x = -\sigma j + \frac{q}{\mu}\rho\phi_x, \\ \phi_{xx} = \frac{q}{\epsilon}(\rho - n), \end{cases} \tag{1.16}$$

where ρ is the electrons density, j the current density, ϕ the electric potential, q the charge of an electron, μ the weight of an electron, ϵ the permittivity of the medium, σ a damping coefficient and n the doping profile. This system models the behavior of electrons in semi conductors and has been studied in the following articles (among others):

- Degond & Markowich [19] proved the existence of many regular stationary states,
- Poupaud [36] showed how to derive the model from the Boltzman equation,
- Bo Zhang [39] proved the existence of a global entropy solution to the system using the method of compensated compactness and a Godunov scheme,
- Poupaud & Rascle & Vila [37] managed to prove the global existence of an entropy solution using the Glimm scheme.

Once we introduce the boundary conditions $\phi_r(t) = \phi(t,1)$ and $\phi_l(t) = \phi(t,0)$ and use the Green function associated to the stationary equation of the electrostatic potential we get the following system:

$$\begin{cases} \rho_t + j_x = 0, \\ j_t + (\frac{j^2}{\rho} + P(\rho))_x = -\sigma j + \frac{q}{\mu}\rho(\phi_r(t) - \phi_l(t)) + \frac{q^2}{\mu\epsilon}\int_0^1\int_y^x \rho(t,z) - n(z) dz dy. \end{cases} \tag{1.17}$$

Once again we need to control an hyperbolic equation using the boundary conditions and an additional source term depending only on the time variable.

2. Statement of the results.. In what follows f is a $\mathcal{C}^2$ strictly convex function and L and T are positive numbers. We consider the equation:

$$\partial_t u + \partial_x(f(u)) = g(t), \qquad \text{for } (t,x) \in (0,T) \times (0,L), \tag{2.1}$$

where g is a $\mathcal{C}^1$ function that we can specify, that is, a control.

We begin by recalling the definition of an entropy solution for a scalar conservation law.

DEFINITION 1. *A couple of $\mathcal{C}^1(\mathbb{R},\mathbb{R})$ functions (η, q) is a convex entropy-flux pair for f if:*

$$\eta \textit{ is convex and } \forall z \in \mathbb{R},\ \eta'(z)f'(z) = q'(z).$$

Now we say that $u \in L^\infty((0,T)\times(0,L))$ is an entropy solution of (2.1) if for all non-negative functions ϕ in $\mathcal{C}^1_c((0,T)\times(0,L))$ and all convex entropy-flux pairs (η,q) we have:

$$\int_0^T\int_0^L \eta(u(t,x))\partial_t\phi(t,x) + q(u(t,x))\partial_x\phi(t,x) + \eta'(u(t,x))\phi(t,x)g(t)dxdt \geq 0. \tag{2.2}$$

It will be useful to consider only the class representatives of BV functions that are right-continuous, which is possible since the discontinuity points of such a function are countable, we will do so in all the paper. We now provide our first controllability result concerning (2.1).

THEOREM 1. *Let $u_1 \in \mathrm{BV}(0,L)$ satisfy:*

$$\sup_{\substack{0<h<L \\ 0<x<L-h}} \frac{u_1(x+h)-u_1(x)}{h} < +\infty, \tag{2.3}$$

and suppose that f satisfy one of the following conditions:

$$\frac{f'(M)}{\sup\limits_{z\in[0,M]} f''(z)} \to +\infty \textit{ as } M \to +\infty \quad \textit{or} \quad \frac{f'(M)}{\sup\limits_{z\in[M,0]} f''(z)} \to -\infty \textit{ as } M \to -\infty. \tag{2.4}$$

Then for any positive time T and any u_0 in $\mathrm{BV}(0,L)$ there exist two functions g and u respectively in $\mathcal{C}^1([0,T])$ and $L^\infty((0,T);\mathrm{BV}(0,L)) \cap \mathrm{Lip}([0,T];L^1(0,L))$, such that u is an entropy solution of (2.1) on $(0,T)\times(0,L)$ and

$$u(0,.) = u_0 \quad \textit{and} \quad u(T,.) = u_1 \quad \textit{in } (0,L)$$

.

REMARK 1.

- *Estimates (2.3) and (1.4) are of similar nature but (2.3) is much less restrictive since this supremum can be arbitrarily large.*
- *The first two conditions of (1.9) are replaced here by (2.4) which concerns only the flux. Therefore many more states are reachable with the additional control g. Furthermore they are reachable in arbitrarily small time.*

We now provide some results in the case where the semi-Lipschitz condition (2.3) degenerates near one boundary point. Indeed we can see that in the third condition of (1.9), the right-hand side can blow up as $x \to 0^+$, which is not the case for (2.3). Since the transformation:

$$X = L - x, \quad F(z) = f(-z), \quad v(t, X) = -u(t, x), \tag{2.5}$$

transforms an entropy solution u of (2.1) in an entropy solution v of $\partial_t v + \partial_X F(v) = -g$ with $F(z) = f(-z)$ also a convex function, and exchanges the boundary points, we will only consider the case where the degeneracy takes place at 0.

Now to quantify this degeneracy, we introduce the following function K:

$$\forall x \in (0, L), \qquad K(x) = \left(\sup_{\substack{x \le y < L \\ 0 < h < L-y}} \frac{u_1(y+h) - u_1(y)}{h} \right)_+ . \tag{2.6}$$

From now on, we will always suppose that K is finite at each point of $(0, L)$. It is obviously non-increasing and non-negative therefore it may only blow up at 0. In the case of such a blow-up we have the following sufficient condition for controllability.

Theorem 2. *Let p be a real number in $(0,1)$, and let $u_1 \in \mathrm{BV}(0, L)$ satisfy both*

$$K(x) = O\left(\frac{1}{x^p}\right) \quad \text{and} \quad \left(u_1(0) - \inf_{0<y\le x} u_1(y)\right) = O(x^{2p}) \quad \text{as } x \to 0^+. \tag{2.7}$$

Let us define:

$$\forall M > 0, \ I_M = [\inf_{x\in(0,L)} u_1(x), \sup_{x\in(0,L)} u_1(x) + M], \tag{2.8}$$

and suppose that for a certain $q > 0$, such that $p(2q+1) \le 1$, the flux f satisfies both:

$$\frac{M^q}{\sup_{z\in I_M} f''(z)} \xrightarrow[M\to+\infty]{} +\infty \qquad \text{and} \qquad \frac{|h|^q}{|f'(u_1(0)+h)|} = O(1) \text{ at } 0 \text{ and at } +\infty. \tag{2.9}$$

Then for any T positive and any u_0 in $\mathrm{BV}(0, L)$ there exist two functions g and u respectively in $\mathcal{C}^1([0,T])$ and $L^\infty((0,T); \mathrm{BV}(0, L)) \cap \mathrm{Lip}([0,T]; L^1(0,L))$ such that the following holds:

- *u is an entropy solution of (2.1) on $(0,T) \times (0, L)$ with $u(0,.) = u_0$,*
- *at the final time T we have both $u(T,.) = u_1$ and $g(T) = 0$.*

Remark 2.

- *This contains the fluxes of shape $f(z) = |z|^{q+1}$ with q less than $\frac{1}{2p} - \frac{1}{2}$.*
- *The fact that at time T, g is C^1 and equal to zero is restrictive. The compatibility condition $p(2q+1) \le 1$ could be improved by removing either hypotheses.*
- *Note that in comparison to conditions (1.9), there is is a new phenomenon: the compatibility condition $p(2q+1) \le 1$.*
- *Using the theory of generalized characteristics of Dafermos [15], it is easy to show that an entropy solution u of (2.1) satisfies the following necessary condition. For $0 < x < y < L$ let us take:*

$$V = \max(|f'(||u(t,.)||_{L^\infty(0,L)} + ||g||_{L^1(0,t)})|, |f'(-||u(t,.)||_{L^\infty(0,L)} - ||g||_{L^1(0,t)})|),$$

then we have: $\quad u(t,y) - u(t,x) \le \dfrac{V}{\inf(f''(z))} \dfrac{1}{\min(L-y, x, Vt)} (y-x).$ (2.10)

So we see that the semi-Lipschitz condition (2.3) *may a priori blow up at both endpoints.*

We conclude this part with the most general result on controllability properties for equation (2.1).

THEOREM 3. *Suppose that $f'(z)$ tends to infinity as z does. Let u_1 be in* $\mathrm{BV}(0,L)$ *and let $\bar{T}$ be a positive number. We introduce the following notations:*

$$\forall x \in (0,L),\ \tau(x) = \min\left(\bar{T}, \frac{1}{2K(x)} \frac{1}{\sup\limits_{z \in I_M} f''(z)}\right). \tag{2.11}$$

Suppose that there exists a non-positive function $\bar{g} \in \mathcal{C}^1([0,\bar{T}])$ such that:

$$\liminf_{\beta \to 0^+} \sup_{\frac{3\beta}{2} \le \alpha < L} \left(\alpha - \int_{\bar{T}-\tau(\alpha-\beta)}^{\bar{T}} f'\left(\inf_{\alpha-\beta \le x \le \alpha} u_1(x) - \int_s^{\bar{T}} \bar{g}(r)dr\right) ds\right) \le 0. \tag{2.12}$$

Then for any time T larger that $\bar{T}$ and any function u_0 in $\mathrm{BV}(0,L)$ *there exist two functions g in $\mathcal{C}^1([0,T])$ and u in $L^\infty((0,T);\mathrm{BV}(0,L)) \cap \mathrm{Lip}([0,T];L^1(0,L))$ such that:*

u is an entropy solution of (2.1) *on* $(0,T) \times (0,L)$,

$$u(0,.) = u_0 \quad \textit{and} \quad u(T,.) = u_1 \quad \textit{in } (0,L).$$

REMARK 3. *The condition* (2.12) *has a geometrical meaning. The number β is a discretization parameter for the wave-front tracking algorithm. The function $\tau(x)$ is a duration for which we know that the backward discontinuity fronts coming from points in (x,L) will not collide. The number:*

$$\alpha - \int_{\bar{T}-\tau(\alpha-\beta)}^{\bar{T}} f'\left(\inf_{\alpha-\beta \le x \le \alpha} u_1(x) - \int_s^{\bar{T}} \bar{g}(r)dr\right) ds,$$

is an upper bound on the position of a discontinuity front coming from somewhere between $[\alpha-\beta,\alpha]$ at the time $\bar{T}-\tau(\alpha-\beta)$. Thus the condition eq. (22) means that as the discretization size of the wave front tracking algorithm goes to zero every discontinuity fronts leaves the domain $(0,L)$ before colliding with any discontinuity front coming from its right. Thus when β tends to 0 *we get a trajectory of the system with a constant initial datum and u_1 as the final state at time $\bar{T}$.*

Before proving the results above let us make a few general comments on the problem and on the method which we will use. The linearization of equation (2.1) is problematic because of the lack of regularity and also because the linearized equation is no longer in conservative form. Therefore we will rather construct approximate solutions using a wave-front tracking algorithm and then use a classical compactness argument to get a trajectory solving the exact controllability problem. It should be noted that another approach would be to control the viscous equation and then let the viscosity tend to zero while keeping uniformly bounded controls, as in [23] or [29].

A first obvious remark is that when both the initial and final states u_0 and u_1 are constant functions on $(0,L)$, the exact controllability problem for (2.1) is reduced to finding g in $\mathcal{C}^1([0,T])$ such that $\int_0^T g(s)ds = u_1 - u_0$ which is trivial for any choice of T, $g(0)$ and $g(T)$. Now we follow the strategy of the return method J.-M. Coron introduced in [12] (see also [13]): rather than keeping the control small and use a linearization argument, we use large controls and the nonlinearity to perturb the system. More precisely, we proceed in two steps, in the first we begin with a general initial value and end with a chosen constant one, in the second with begin with a constant initial value and end with a more general one

The rest of the paper is organized as follows. In the next section, we will prove that for any initial condition u_0 in $\mathrm{BV}(0,L)$ and any positive T, we can find g in $C^1([0,T])$ and an entropy solution of (2.1) such that both $u(0,.) = u_0$ and $u(T,.)$ is constant on $(0,L)$. In Section 4 we will prove the remaining part of Theorem 3: given $\bar{T}$ positive, u_1 in $\mathrm{BV}(0,L)$ and a flux f satisfying the hypotheses of the theorem we can construct g and an entropy solution u of (2.1) such that $u(\bar{T},.) = u_1$ and $u(0,.)$ is constant on $(0,L)$. In Section 5 we show how we deduce Theorem 1 from Theorem 3. And in Section 6 we prove Theorem 2 using Theorem 3. Finally we collect useful results on our wave-front tracking algorithm in an appendix.

3. Control toward a constant.. The aim of this section is to prove the following result dealing with the exact controllability problem from a general initial data toward a final constant state in arbitrarily small time.

PROPOSITION 3.1. *Let u_0 be in* $\mathrm{BV}(0,L)$*, and T be a positive number. There exist g and u respectively in* $\mathcal{C}^1([0,T])$ *and* $L^\infty((0,T);\mathrm{BV}(0,L)) \cap \mathrm{Lip}([0,T];L^1(0,L))$ *such that:*

- *u is an entropy solution of* (2.1)*,*
- $u(0,.) = u_0$ *on* $(0,L)$*,*
- $u(T,.)$ *is constant on* $(0,L)$*.*

Proof. Take g non-negative in $\mathcal{C}^1([0,T])$ such that the following condition is satisfied:

$$\int_0^T f'\left(\int_0^t g(s)ds - ||u_0||_{L^\infty(0,1)}\right) dt \geq L, \tag{3.1}$$

and define:

$$c(t) = \int_0^t f'\left(\int_0^r g(s)ds - ||u_0||_{L^\infty(0,1)}\right) dr. \tag{3.2}$$

We first recall a classical lemma.

LEMMA 1 (Helly's theorem). *If (u_n) is a family of functions defined on $[t_1,t_2] \times (a,b)$ and C a constant independent of n such that:*

$$\forall t \in [t_1,t_2], \qquad ||u_n(t,.)||_{\mathrm{BV}((a,b))} \leq C, \tag{3.3}$$

$$\forall t,s \in [t_1,t_2], \qquad \int_a^b |u_n(t,z) - u_n(s,z)|dz \leq C|t-s|. \tag{3.4}$$

Then we can extract $(u_{\psi(n)})_{n\geq 0}$ and get u satisfying (3.3)*,* (3.4) *and such that:*

$$\left.\begin{array}{r} ||u_{\psi(n)} - u||_{L^1_{loc}((0,T)\times(a,b))} \to 0 \\ \forall t \in [0,T], \quad ||u_{\psi(n)}(t,.) - u(t,.)||_{L^1_{loc}(a,b)} \to 0 \end{array}\right. \quad \text{when } n \to +\infty. \tag{3.5}$$

Proof. See [5] Chapter 2 Section 4. ☐ Now we will to construct u by approximation.

LEMMA 2. *Suppose that we have a sequence $(u_n)_{n\geq 1}$ satisfying the following properties:*

1. *the family* $(u_n)_{n\geq 1}$ *is bounded in* $L^\infty((0,T);\mathrm{BV}(\mathbb{R}))\cap \mathrm{Lip}([0,T],L^1_{loc}(\mathbb{R}))$,
2. *we have* $||u_n(0,.)-u_0||_{L^1(0,L)}\to 0$ *when* $n\to+\infty$,
3. *for every entropy-flux pair* (η,q), *we have:*

$$\limsup_{n\to+\infty}\int_0^T\int_0^L \eta(u_n(t,x))\partial_t\phi(t,x)+q(u_n(t,x))\partial_x\phi(t,x) \\ +\eta'(u_n(t,x))\phi(t,x)g(t)dtdx\leq 0, \quad (3.6)$$

4. *all the functions of the form* $u_n(t,x)-\int_0^t g(s)ds$ *are constant on the set* $\{(t,x)\in[0,T]\times\mathbb{R}\mid x\leq c(t)\}$.

Then there exists u *as in Proposition 3.1.*

Proof. Using the first property above and the standard compactness result of lemma 1, we can extract a subsequence $(u_{\phi(n)})_{n\geq 1}$ that converges in $L^1_{loc}(\mathbb{R}^2)$ toward a function u which belongs to the space $L^\infty((0,T);\mathrm{BV}(\mathbb{R}))\cap\mathrm{Lip}([0,T],L^1_{loc}(\mathbb{R}))$. Furthermore we can also suppose that for every t in $[0,T]$ we have

$$||u_{\phi(n)}(t,)-u(t,.)||_{L^1(0,L)}\to 0.$$

The third property satisfied by $(u_n)_{n\geq 1}$ implies that u is an entropy solution of (2.1), the second that $u_0=u(0,.)$ on $(0,L)$ and the last property together with (3.1) and (3.2) implies that $u(T,.)$ is constant on $(0,L)$. □ It only remains to construct such a family. We will do so using a wave-front tracking algorithm. Compared to the classical wave-front tracking algorithm (see [17] or [5] chapter 6 or [16] chapter 14) the discontinuities travel along piecewise $\mathcal{C}^1$ curves and not polygonal lines. Note that while we use this modification to deal with a source term $g(t)$, the same ideas might be used with $g(t,u)$.

To be more precise take

$$G_1(t)=\int_0^t g(s)ds \quad (3.7)$$

and introduce the following notion of wave-front tracking approximation.

Definition 2. *If* ϵ *is a positive number and* u_ϵ *a function defined on* $[0,T]\times(0,L)$ *we say that* u_ϵ *is an* ϵ*-approximate front tracking solution of* (2.1) *if:*

- *as a function of two variables* $u_\epsilon(t,x)-G_1(t)$ *is locally constant except on a finite number of curves* $x=x_\alpha(t)$ *which are* $\mathcal{C}^1$ *where the discontinuities are located and which we will call discontinuity fronts,*
- *for each curve* x_α *we have for a.e.* t*:*

$$\dot{x}_\alpha(t)=\frac{f(u_\epsilon(t,x_\alpha(t)^+))-f(u_\epsilon(t,x_\alpha(t)^-))}{u(t,x_\alpha(t)^+)-u(t,x_\alpha(t)^-)}, \quad (3.8)$$

- *for each curve* x_α *and a.e. time* t *we have*

$$u_\epsilon(t,x_\alpha(t)^+)\leq u_\epsilon(t,x_\alpha(t)^-)+\epsilon. \quad (3.9)$$

We have the following key property of the ϵ-approximate front tracking solution.

Lemma 3. *If* u_ϵ *is an* ϵ*-approximate front tracking solution of* (2.1) *and* (η,q) *is an entropy-flux pair then we have for every positive function* ϕ *in* $\mathcal{C}^1_c((0,T)\times(0,L))$*:*

$$\int_0^T\int_0^L (\eta(u_\epsilon(t,x))\partial_t\phi(t,x)+q(u_\epsilon(t,x))\partial_x\phi(t,x) \\ +\eta'(u_\epsilon(t,x))\phi(t,x)g(t)dtdx\geq -C||\phi||_{\mathcal{C}^0((0,T)\times(0,L))}\epsilon. \quad (3.10)$$

Where the constant C depends only on f, η, $||g||_{L^1(0,T)}$ and $||u_\epsilon||_{L^\infty((0,T),\mathrm{BV}(0,L))}$.

Proof. See the Appendix. □ Now to construct such a wave-front tracking approximation we proceed as follows.

Let n be a positive integer, we define:

$$u_n^k = u_0(\frac{2k-1}{2n}L), \quad \text{for } 1 \leq k \leq n, \qquad u_n^0 = u_0(0), \quad u_n^{n+1} = u_0(L). \tag{3.11}$$

We take $u_n(0,x)$ on $\mathbb{R}$ equal to:

- u_n^k for $\frac{k-1}{n}L < x < \frac{k}{n}L$ and $1 \leq k \leq n$,
- u_n^0 for $x < 0$,
- u_n^{n+1} for $x > L$.

Now at each discontinuity point of $u^n(0,.)$, we approximately solve the Riemann problem as follows. We suppose that the discontinuity is at $x = 0$ and that the left and right state are respectively v^- and v^+. Then

- if $v^- > v^+$ the discontinuity is a shock and defining

$$\gamma(t) = \int_0^t \frac{f(v^- + \int_0^s g(r)dr) - f(v^+ + \int_0^s g(r)dr)}{v^- - v^+} ds, \tag{3.12}$$

we take:

$$v(t,x) = \begin{cases} v^- + \int_0^t g(r)dr & \text{if } x < \gamma(t) \\ v^+ + \int_0^t g(r)dr & \text{if } x > \gamma(t) \end{cases}. \tag{3.13}$$

- if $v^- < v^+$ take $p = \lceil n(v^+ - v^-) \rceil + 1$ and define

$$\text{for } 0 \leq l \leq p, \quad v^l = \frac{p-l}{p} v^- + \frac{l}{p} v^+, \tag{3.14}$$

$$\text{and for} 1 \leq l \leq p, \quad \gamma_l(t) = \int_0^t \frac{f(v^l + \int_0^s g(r)dr) - f(v^{l-1} + \int_0^s g(r)dr)}{v^l - v^{l-1}} ds. \tag{3.15}$$

Finally we define:

$$v(t,x) = \begin{cases} v^0 + \int_0^t g(r)dr & \text{if } x < \gamma_1(t) \\ v^l + \int_0^t g(r)dr & \text{if } \gamma_{l-1}(t) < x < \gamma_l(t) \text{ and } 1 \leq l \leq p-1 \\ v^p + \int_0^t g(r)dr & \text{if } \gamma_p(t) < x. \end{cases} \tag{3.16}$$

Now there is a small time during which all the discontinuity fronts created at time $t = 0$ do not intersect. And when two or more fronts interact at a time $t > 0$ we use the same procedure. It should be noted that only one front leaves the interaction point. In order to see that we begin with the following simple lemma.

Lemma 4. *let u^1, u^2, u^3 be three real numbers such that:*

$$\frac{f(u^1) - f(u^2)}{u^1 - u^2} > \frac{f(u^2) - f(u^3)}{u^2 - u^3}, \tag{3.17}$$

then $u^3 < u^1$

Proof. Straightforward from the convexity of f. □ Now if m fronts separating $m+1$ states $u^1, ..., u^{m+1}$ are interacting at time τ we have, thanks to the order of their respective speed:

$$\frac{f(u^{i-1}(\tau)) - f(u^i(\tau))}{u^{i-1}(\tau) - u^i(\tau)} > \frac{f(u^i(\tau)) - f(u^{i+1}(\tau))}{u^i(\tau) - u^{i+1}(\tau)}, \qquad \forall 1 \leq i \leq m. \tag{3.18}$$

Now using the lemma we get if m is even $u^1 > u^3 > u^5 > \cdots > u^{m+1}$ and the resulting front is a shock, if m is odd we have $u^1 > u^3 > u^5 > \cdots > u^m$ and since $u^{m+1} \leq u^m + \frac{1}{n}$ we can conclude that $u^{m+1} \leq u^1 + \frac{1}{n}$ and we have either a shock or a single rarefaction front.

Since the number of discontinuity fronts decreases at each interaction, this scheme allows us to define u_n on $\mathbb{R}+\times\mathbb{R}$ and produce a $\frac{1}{n}$-approximate wave-front tracking solution. Furthermore since all the states separated by the discontinuity fronts are translated by the same value $\int_0^t g(s)ds$, the quantity $\mathrm{TotVar}(u^n(t,.))$ does not increase with time and the quantity $||u^n(t,.)||_{L^\infty(\mathbb{R})}$ is bounded by $||u_0||_{L^\infty(0,L)} + ||g||_{L^1(0,T)}$. Since u_n is a $\frac{1}{n}$-approximate wave-front tracking solution we can apply Lemma 3. And we see that the third property of Lemma 2 is satisfied. Obviously the second property is satisfied. And we also have the following estimate:

$$||u_n||_{L^\infty(0,T;\mathrm{BV}(0,L))} \leq ||u_0||_{\mathrm{BV}(0,L)} + ||g||_{L^1(0,T)}. \tag{3.19}$$

Furthermore since the speed of any discontinuity front is bounded by the quantity $f'(||u_n||_{L^\infty((0,T)\times(0,L)})$ we have:

$$||u_n(t+h,.)-u_n(t,.)||_{L^1_{loc}(\mathbb{R})} \leq |h| \left(||g||_{L^\infty(0,T)} + 2||u_n||_{L^\infty((0,T)\times\mathbb{R})} f'(||u_n||_{L^\infty((0,T)\times\mathbb{R})})\right), \tag{3.20}$$

and we see that the first property of Lemma 2 is satisfied.

Finally for any n larger than 0 the leftmost discontinuity front $\gamma(t)$ satisfies the following:

$$\dot{\gamma}(t) = \frac{f(u_n^0 + \int_0^t g(r)dr) - f(u_n^k + \int_0^t g(r)dr)}{u_n^0 - u_n^k} \geq f'\left(-||u_0||_{L^\infty(0,L)} + \int_0^t g(r)dr\right) = \dot{c}(t), \tag{3.21}$$

where k may depend on t. Since $c(0) \leq \gamma(0)$ we end up with $\gamma(t) \geq c(t)$ for all positive time t. And using (3.1) and (3.2) we see that the fourth property of Lemma 2 is satisfied by $(u_n)_{n\geq 1}$. □

4. Proof of Theorem 3.. We now prove the following result which deals with the exact controllability from a constant state toward a state u_1 belonging to $\mathrm{BV}(0,L)$ and satisfying the hypotheses of Theorem 3. This concludes the proof of Theorem 3. We recall that the function τ is defined in (2.11), and for a given number M the interval I_M in (2.8).

PROPOSITION 4.1. *Consider $T > 0$, $M > 0$ and u_1 in $\mathrm{BV}(0,L)$. Suppose that there exists g in $\mathcal{C}^1([0,T])$ a non-positive function satisfying:*

$$||g||_{L^1(0,T)} \leq M, \qquad g(T) = 0, \tag{4.1}$$

$$\liminf_{\beta\to 0^+} \sup_{\alpha\in[\frac{3\beta}{2},L]} \left(\alpha - \int_{T-\tau(\alpha-\beta)}^T f'\left(\inf_{x\in[\alpha-\beta,\alpha]}(u_1(x)) - \int_s^{\bar{T}} g(r)dr\right) ds\right) \leq 0. \tag{4.2}$$

Then there exists u in $L^\infty((0,T);\mathrm{BV}(0,L)) \cap \mathrm{Lip}([0,T];L^1(0,L))$ an entropy solution of (2.1), such that:

$$u(T,.) = u_1 \quad \textit{and} \quad u(0,.) \textit{ constant on } (0,L).$$

From now on we let G_2 be the function defined by:

$$G_2(s) = -\int_s^T g(r)dr. \tag{4.3}$$

We begin with a lemma dealing with two discontinuity fronts of a wave-front tracking approximation:

LEMMA 5. *For $\alpha \in (0,L)$ and $0 < \beta < min(\alpha, L-\alpha)$ consider γ_+ and γ_- as follows:*

$$\gamma_\pm(T) = \alpha \pm \frac{\beta}{2}, \qquad \dot{\gamma}_\pm(t) = \frac{f(u_1(\alpha\pm\beta)+G_2(t)) - f(u_1(\alpha)+G_2(t))}{u_1(\alpha\pm\beta)-u_1(\alpha)}, \tag{4.4}$$

we have the two following properties:

$$\forall t \in [T-\tau(\alpha-\beta), T], \qquad \gamma_-(t) \le \gamma_+(t), \tag{4.5}$$

$$\gamma_-(T-\tau(\alpha-\beta)) \le \alpha - \frac{\beta}{2} - \int_{T-\tau(\alpha-\beta)}^{T} f'(\inf_{x\in[\alpha-\beta,\alpha]}(u_1(x)) + G_2(s))ds. \tag{4.6}$$

Proof. Both properties are consequences of the convexity of f. The first one follows from:

$$\begin{aligned}
\gamma_+(t) - \gamma_-(t) = \beta - \int_t^T & \frac{f(u_1(\alpha+\beta)+G_2(s)) - f(u_1(\alpha)+G_2(s))}{u_1(\alpha+\beta)-u_1(\alpha)} \\
& - \frac{f(u_1(\alpha-\beta)+G_2(s)) - f(u_1(\alpha)+G_2(s))}{u_1(\alpha-\beta)-u_1(\alpha)} ds \\
\ge \beta - \int_t^T & f'(\max(u_1(\alpha), u_1(\alpha+\beta)) + G_2(s)) \\
& - f'(\min(u_1(\alpha-\beta), u_1(\alpha)) + G_2(s))ds \\
\ge \beta - \int_t^T & f'(u_1(\alpha) + K(\alpha)\beta + G_2(s)) \\
& - f'(u_1(\alpha) - K(\alpha-\beta)\beta + G_2(s))ds \\
\ge \beta - \int_t^T & \sup_{z\in I_M}(f''(z))(K(\alpha)+K(\alpha-\beta))\beta \\
\ge \beta(1 - (T-t) & \sup_{z\in I_M}(f''(z))(K(\alpha)+K(\alpha-\beta))) \\
\ge 0 \qquad & \text{when } T-t \le \tau(\alpha-\beta).
\end{aligned}$$

And the second one comes from:

$$\begin{aligned}
\gamma_-(t) &= \alpha - \frac{\beta}{2} - \int_t^T \frac{f(u_1(\alpha-\beta)+G_2(s)) - f(u_1(\alpha)+G_2(s))}{u_1(\alpha-\beta)-u_1(\alpha)} ds \\
&\le \alpha - \frac{\beta}{2} - \int_t^T f'[\min(u_1(\alpha), u_1(\alpha-\beta)) + G_2(s)]ds \\
&\le \alpha - \frac{\beta}{2} - \int_t^T f'[\inf_{x\in[\alpha-\beta,\alpha]}(u_1(x)) + G_2(s)]ds.
\end{aligned}$$

□

We prove Proposition 4.1 by constructing appropriate wave-front tracking approximations.

Proof. Thanks to (2.12) we can take (β_n) and (δ_n) two decreasing sequences such that $\beta_n \to 0$, $\delta_n \to 0$ and

$$\sup_{\alpha\in[\frac{3\beta_n}{2},L]} \left(\alpha - \int_{T-\tau(\alpha-\beta_n)}^{T} f'\left(\inf_{x\in[\alpha-\beta_n,\alpha]}(u_1(x)) + G_2(s)\right) ds \right) \le \delta_n. \tag{4.7}$$

We can also suppose that:

$$K(\delta_n)\beta_n \to 0. \tag{4.8}$$

Now we construct $u_n \in L^\infty((0,T);\mathrm{BV}(\delta_n, L))$ as follows. Let $p = \lceil \frac{L-\delta_n}{\beta_n} \rceil$. For $k \in \{1,..,p\}$, we take:

$$x_k = \delta_n + k\beta_n, \tag{4.9}$$

$$v_k(t) = u_1(\delta_n + (k+\frac{1}{2})\beta_n) + G_2(t). \tag{4.10}$$

For $k \in \{1,..,p-1\}$ we define the curve γ_k by:

$$\gamma_k(T) = x_k, \tag{4.11}$$

$$\dot{\gamma}_k(t) = \frac{f(v_k(t)) - f(v_{k-1}(t))}{v_k(t) - v_{k-1}(t)}. \tag{4.12}$$

Thanks to (4.5), (4.6) and (4.7) we see that the curves γ_k do not cross each other inside $[0,T]\times[\delta_n, L]$ therefore we can define u_n as follows:

$$u_n(t,x) = \begin{cases} v_1(t) \text{ if } x \le \gamma_1(t), \\ v_k(t) \text{ if } \gamma_{k-1}(t) \le x \le \gamma_k(t) \text{ and } 2 \le k \le p-1, \\ v_p(t) \text{ if } \gamma_{p-1}(t) \le x. \end{cases} \tag{4.13}$$

Furthermore thanks to (2.11) and (4.6) we see that:

$$\forall x \in [\delta_n, L], \qquad u_n(0,x) = u_1(\delta_n + (p-\frac{1}{2})\beta_n) + G_2(0). \tag{4.14}$$

We also have the estimates:

$$||u_n||_{L^\infty((0,T);\mathrm{BV}(\delta_n,L))} \le ||u_1||_{\mathrm{BV}(0,L)} + M, \tag{4.15}$$

$$||u_n||_{\mathrm{Lip}([0,T];L^1(\delta_n,L))} \le \max(L(||u_1||_{L^\infty(0,L)} + M), ||g||_{\mathcal{C}^0([0,T])} \\ + ||u_1||_{\mathrm{BV}(0,L)} f'(||u_1||_{L^\infty(0,L)} + M)). \tag{4.16}$$

Finally thanks to (2.6), (4.8) and (3.10), for every convex entropy-flux couple (η, q) we get a constant C independent of n such that $\forall \phi \in \mathcal{C}_c^1((0,T)\times(\delta_n, L))$ non-negative we have:

$$\int_0^T \int_{\delta_n}^L (\eta(u_n(t,x))\partial_t\phi(t,x) + q(u_n(t,x))\partial_x\phi(t,x) + \eta'(u_n(t,x))g(t)\phi(t,x))dtdx \\ \ge -C.||\phi||_{\mathcal{C}^0((0,T)\times(0,L))}\beta_n K(\delta_n). \tag{4.17}$$

Using lemma 1, we extract a subsequence from (u_n) and get a solution to equation (2.1). Since $\delta_n \to 0$ the limit u is defined on $(0,T)\times(0,L)$ and is an entropy solution of (2.1). And taking the limit $n \to +\infty$ in (4.14) we see that $u(0,.)$ is constant on $(0,L)$. □

5. Proof of Theorem 1.. We now show that under the hypotheses of Theorem 1, condition (2.12) is satisfied. We know that:

$$\forall \alpha \in (0,L), \quad K(\alpha) \leq K < +\infty, \tag{5.1}$$

$$\frac{f'(M)}{\sup\limits_{z\in[0,M]} f''(z)} \underset{M\to+\infty}{\to} +\infty. \tag{5.2}$$

Recalling the definitions of I_M in (2.8) and τ in (2.11), it is clear that

$$\forall \alpha \in (0,L) \quad \tau(\alpha) \geq \tau_M := \frac{1}{2K}\frac{1}{\sup\limits_{z\in I_M} f''(z)}.$$

Therefore with g non-positive and such that G_2 satisfies:

$$G_2(T) = G_2'(T) = 0, \tag{5.3}$$

$$\forall t \in [0, T - \frac{\tau_M}{2}] \qquad G_2(t) = M, \tag{5.4}$$

we have when M is large enough so that $f'(\inf\limits_{x\in(0,L)} u_1(x) + M) \geq 0$:

$$\sup_{\alpha\in[\frac{3\beta}{2},L]}\left(\alpha - \int_{T-\tau(\alpha-\beta)}^{T} f'(\inf_{x\in[\alpha-\beta,\alpha]}(u_1(x)) + G_2(s))ds\right)$$
$$\leq L - \frac{\tau_M}{2}\left(f'(\inf_{x\in(0,L)}(u_1(x))) + f'(\inf_{x\in(0,L)}(u_1(x)) + M)\right). \tag{5.5}$$

But now we can get:

$$\begin{aligned}\liminf_{\beta\to 0^+}\sup_{\alpha\in[\frac{3\beta}{2},L]}&\left(\alpha - \int_{T-\tau(\alpha-\beta)}^{T} f'(\min_{x\in[\alpha-\beta,\alpha]}(u_1(x)) + G_2(s))ds\right)\\ &\leq L - \frac{\tau_M}{2}\left(f'(\inf_{x\in(0,L)}(u_1(x))) + f'(\inf_{x\in(0,L)}(u_1(x)) + M)\right)\\ &\leq \frac{L}{4K}\left(1 - \frac{f'(\inf\limits_{x\in(0,L)}(u_1(x))) + f'(\inf\limits_{x\in(0,L)}(u_1(x)) + M)}{\sup\limits_{z\in I_M} f''(z)}\right).\end{aligned}$$

And thanks to (5.2) this expression is non-positive for M large enough.

6. Proof of Theorem 2.. In this part we prove that under the hypotheses of Theorem 2 condition (2.12) is satisfied and therefore that we have exact controllability in arbitrarily small time.
We recall that $u_1 \in \mathrm{BV}(0,L)$, $M > 0$ and $0 < \epsilon < T$, where ϵ is the amount of time needed to get controllability and therefore is as small as we want. Indeed if we can control in time T_1 we can obviously control in any time $T_2 \geq T_1$ since in our strategy we can spend an arbitrary amount of time between the two intermediate states constant spaces.
We use once again the functions K, τ and the interval I_M defined in (2.6) (2.11) and (2.8), and also the following:

$$U_i(\alpha) = \inf_{z\in(0,\alpha]}(u_1(z)) \quad \text{and } \alpha_c(\epsilon) = \sup\{\alpha \in (0,L) \mid \tau(\alpha) < \epsilon\}. \tag{6.1}$$

It is clear that K and U_i are non-increasing, that τ is non-increasing in α but non-decreasing in M and that α_c is non-decreasing in ϵ but non-increasing in M. We can suppose that $\tau(\alpha) \underset{\alpha\to 0^+}{\to} 0$. It implies that $\alpha_c(\epsilon) \underset{\epsilon\to 0^+}{\to} 0$.

The assumptions made in Theorem 2 can be reformulated as follows, for any $H>0$ there exist $C>0$ and $\bar{U}_i>0$ such that the following holds:

$$\frac{M^q}{\sup\limits_{z\in[0,M]} f''(z)} \underset{M\to+\infty}{\to} +\infty, \qquad \forall h>0,\ f'(U_i(0^+)+h) \geq \frac{h^q}{C}$$
$$\text{and} \quad \forall h\in(0,H]\ f'(U_i(0^+)-h) \geq -Ch^q, \tag{6.2}$$

$$\forall\alpha\in(0,L), \qquad K(\alpha) \leq \frac{\bar{K}}{\alpha^p} \text{ and } \quad U_i(0^+)-U_i(\alpha) \leq \bar{U}_i\alpha^{2p}, \tag{6.3}$$

and finally we have the compatibility condition:

$$p(2q+1) \leq 1. \tag{6.4}$$

Now we define:

$$\tilde{\tau}(\alpha) = \frac{\alpha^p}{2\bar{K}} \frac{1}{\sup\limits_{z\in I_M} f''(z)}, \tag{6.5}$$

$$\mathcal{F}(\alpha,\beta) = \alpha - \int_{T-\tau(\alpha-\beta)}^{T} \left(U_i(0^+) - \bar{U}_i\alpha^{2p} + G_2(s)\right) ds. \tag{6.6}$$

Finally we take g such that:

$$G_2(t) = -\int_t^T g(s)ds \text{ is decreasing,} \tag{6.7}$$

$$\forall s\in[T-\epsilon,T] \quad G_2(s) = \frac{M}{2}\left(\frac{T-s}{\epsilon}\right)^2, \tag{6.8}$$

$$\forall s\in[0,T] \quad G_2(s) \leq M. \tag{6.9}$$

We want to prove that (2.12) holds, but since f' is monotone and using the bound on U_i in (6.3) it is sufficient to prove:

$$\liminf_{\beta\to 0^+} \sup_{\frac{3\beta}{2}\leq\alpha\leq L} \mathcal{F}(\alpha,\beta) \leq 0, \tag{6.10}$$

for M large enough and ϵ given by:

$$\epsilon = \frac{1}{\sqrt{f'\left(U_i(0^+) - \bar{U}_iL^{2p} + \frac{M}{8}\right) . \sup\limits_{z\in I_M} f''(z)}}. \tag{6.11}$$

Note that with this choice $\epsilon\to 0$ when $M\to+\infty$.

We will get upper bounds of $\mathcal{F}$ in two different ways.

Lemma 6. *There exists M_0 such that for all (α,β) such that $\alpha\geq\alpha_c(\epsilon)+\beta$ and for any $M\geq M_0$ we have $\mathcal{F}(\alpha,\beta)\leq 0$.*

Proof. If $\alpha - \beta \geq \alpha_c(\epsilon)$ we have $\tau(\alpha-\beta) = \epsilon$ and therefore:

$$\mathcal{F}(\alpha,\beta) = \alpha - \int_{T-\epsilon}^{T} \big(U_i(0^+) - \bar{U}_i\alpha^{2p} + G_2(s)\big)\, ds \tag{6.12}$$

$$\leq L - \frac{\epsilon}{2}\left(f'\left(U_i(0^+) - \bar{U}_i L^{2p}\right) + f'\left(U_i(0^+) - \bar{U}_i L^{2p} + \frac{M}{8}\right)\right) \tag{6.13}$$

$$\leq L - \frac{\epsilon}{2} f'\left(U_i(0^+) - \bar{U}_i L^{2p}\right) - \left(\frac{f'\left(U_i(0^+) - \bar{U}_i L^{2p} + \frac{M}{8}\right)}{\sup\limits_{z\in I_M} f''(z)}\right)^{\frac{1}{2}}. \tag{6.14}$$

And using the definition of ϵ and (6.2) we can conclude. □ It remains to obtain a bound on $\mathcal{F}$ on the part of the domain where $\frac{\beta}{2} \leq \alpha - \beta < \alpha_c(\epsilon)$, we begin by a few observations.

LEMMA 7. *The following properties hold:*

- *when $M \to +\infty$ we have $\alpha_c(\epsilon) \to 0$,*
- *for $\alpha \leq \alpha_c(\epsilon) + \beta$ we have $\tilde{\tau}(\alpha-\beta) \leq \tau(\alpha-\beta)$*
- *for $\alpha \leq \alpha_c(\epsilon) + \beta$ we have:*

$$\alpha - \int_{T-\tilde{\tau}(\alpha-\beta)}^{T} f'\left(U_i(0^+) - \bar{U}_i\alpha^{2p} + \frac{M}{2}\left(\frac{T-s}{\epsilon}\right)^2\right) ds \leq 0 \Rightarrow \mathcal{F}(\alpha,\beta) \leq 0. \tag{6.15}$$

Proof. The first part of the lemma comes from the definitions of α_c and ϵ through the following calculations:

$$\begin{aligned}
\alpha_c(\epsilon) &= \sup\{\alpha \in (0,L) \mid \tau(\alpha) < \epsilon\} \\
&= \sup\Big\{\alpha \in (0,L) \;\Big|\; \frac{1}{2K(\alpha)} \frac{1}{\sup\limits_{z\in I_M} f''(z)} < \epsilon\Big\} \\
&= \sup\Big\{\alpha \in (0,L) \;\Big|\; K(\alpha) > \frac{1}{2\epsilon \sup\limits_{z\in I_M} f''(z)}\Big\} \\
&= \sup\Big\{\alpha \in (0,L) \;\Big|\; K(\alpha) > \frac{1}{2} \frac{f'\left(U_i(0^+) - \bar{U}_i L^{2p} + \frac{M}{8}\right)}{\sup\limits_{z\in I_M} f''(z)}\Big\}.
\end{aligned}$$

The second part is an immediate consequence of the definitions of $\tilde{\tau}$, τ, α_c and of the first part of (6.3).

And for the third part we have, thanks to the monotonicity of f':

$$\begin{aligned}
&\alpha - \int_{T-\tilde{\tau}(\alpha-\beta)}^{T} f'\left(U_i(0^+) - \bar{U}_i\alpha^{2p} + \frac{M}{2}\left(\frac{T-s}{\epsilon}\right)^2\right) ds \leq 0 \\
&\Rightarrow f'\left(U_i(0^+) - \bar{U}_i\alpha^{2p} + \frac{M}{2}\left(\frac{\tilde{\tau}(\alpha-\beta)}{\epsilon}\right)^2\right) \geq 0 \\
&\Rightarrow \int_{T-\tilde{(}\alpha-\beta)}^{T-\tilde{\tau}(\alpha-\beta)} f'\left(U_i(0^+) - \bar{U}_i\alpha^{2p} + \frac{M}{2}\left(\frac{T-s}{\epsilon}\right)^2\right) ds \geq 0.
\end{aligned} \tag{6.16}$$

Hence:

$$\mathcal{F}(\alpha,\beta) \leq \alpha - \int_{T-\tilde{\tau}(\alpha-\beta)}^{T} f'\left(U_i(0^+) - \bar{U}_i\alpha^{2p} + \frac{M}{2}\left(\frac{T-s}{\epsilon}\right)^2\right) ds \leq 0. \tag{6.17}$$

□

LEMMA 8. *There exists M_1 such that for any $M \geq M_1$ and if $\dfrac{3\beta}{2} \leq \alpha \leq \alpha_c(\epsilon)+\beta$ we have:*

$$\alpha - \int_{T-\tilde{\tau}(\alpha-\beta)}^{T} f'\left(U_i(0^+) - \bar{U}_i\alpha^{2p} + \frac{M}{2}\left(\frac{T-s}{\epsilon}\right)^2\right) ds \leq 0. \tag{6.18}$$

Proof. Let

$$\begin{aligned} Q &:= \alpha - \int_{T-\tilde{\tau}(\alpha-\beta)}^{T} f'\left(U_i(0^+) - \bar{U}_i\alpha^{2p} + \frac{M}{2}\left(\frac{T-s}{\epsilon}\right)^2\right) ds \\ &\leq \alpha - \frac{\tilde{\tau}(\alpha-\beta)}{2}\left(f'\left(U_i(0^+) - \bar{U}_i\alpha^{2p}\right) + f'\left(U_i(0^+) - \bar{U}_i\alpha^{2p} + \frac{M}{2}\left(\frac{\tilde{\tau}(\alpha-\beta)}{2\epsilon}\right)^2\right)\right). \end{aligned}$$

Using the definition of $\tilde{\tau}$ and (6.2) we get:

$$\begin{aligned} Q \leq \alpha - \frac{(\alpha-\beta)^p}{4\bar{K}\sup_{z\in I_M} f''(z)}\Bigg(&-C\bar{U}_i\alpha^{2pq} \\ &+\frac{1}{C}\left(\frac{M(\alpha-\beta)^{2p}}{8\bar{K}^2}\frac{f'\left(U_i(0^+) - \bar{U}_iL^{2p} + \frac{M}{8}\right)}{\sup_{z\in I_M} f''(z)} - \bar{U}_i\alpha^{2p}\right)^q\Bigg). \end{aligned}$$

But then we can deduce:

$$\begin{aligned} Q \leq \alpha - \alpha^{p(2q+1)}\frac{\left(1-\frac{\beta}{\alpha}\right)^p}{4\bar{K}\sup_{z\in I_M} f''(z)}\Bigg(&-C\bar{U}_i \\ &+\frac{1}{C}\left(\frac{M\left(1-\frac{\beta}{\alpha}\right)^{2p}}{8\bar{K}^2}\frac{f'\left(U_i(0^+) - \bar{U}_iL^{2p} + \frac{M}{8}\right)}{\sup_{z\in I_M} f''(z)} - \bar{U}_i\right)^q\Bigg). \end{aligned}$$

Now using (6.4) and the fact that $1 - \frac{\beta}{\alpha} \geq \frac{1}{3}$ we see that, for any $A > 0$ and for M large enough (independently of α and β) we have:

$$\frac{\left(1-\frac{\beta}{\alpha}\right)^p}{4\bar{K}\sup_{z\in I_M} f''(z)}\left(-C\bar{U}_i + \frac{1}{C}\left(\frac{M\left(1-\frac{\beta}{\alpha}\right)^{2p}}{8\bar{K}^2}\frac{f'\left(U_i(0^+) - \bar{U}_iL^{2p} + \frac{M}{8}\right)}{\sup_{z\in I_M} f''(z)} - \bar{U}_i\right)^q\right) \geq A.$$

Finally thanks to (6.4) we see that for any $A > 1$, $\alpha - A\alpha^{p(2q+1)} \leq 0$ and since α_c tends to 0 when M goes to infinity and β is arbitrarily small we have the lemma. □

Appendix A. Wave-front tracking approximations.. In this appendix we prove Lemma 3. It consists of a rather straightforward generalization of the classical result corresponding to the case where there is no source term and which can be found in [5] or [25].
Let $T > 0$, $a, b \in \mathbb{R}$ such that $a < b$. We consider g a continuous function on $[0, T]$ and

recall that $G_1(t) = \int_0^t g(s)ds$. And finally we suppose that f is a $\mathcal{C}^2$ convex function defined on $\mathbb{R}$.
We will be interested in the entropic solutions of the equation:

$$\partial_t u(t,x) + \partial_x(f(u(t,x))) = g(t) \text{ on } [0,T] \times (a,b). \tag{A.1}$$

Note that while here we deal only with $g(t)$, the same idea could be used to deal with a source term $g(t,u)$ though of course one would need some additional informations on g to get the existence in large time. We recall that approximate wave-front tracking approximations were defined in Definition 2, and we will now prove Lemma 3.

Proof. [Proof of Lemma 3] We evaluate the left hand side of (3.10) using the fact that $v = u - G_1$ is piecewise constant.
More precisely we apply Green's theorem to the vector field $X = (\phi\eta(v+G_1), q(v+G_1)\phi)$ on the parts where it is regular. We know that η, q and G_1 are regular. Furthermore we know that v is piecewise constant therefore regular except on the curves x_α.
Now consider D a connected component of the open subset of $(0,T)\times(a,b)$ constituted of the points (t,x) on a neighborhood of which, v is constant. Thanks to the definition of approximate front tracking solutions we know that the boundary of D is constituted of

$D = \{(t,x) \in (t_1,t_2) \times (a,b) | x_{\alpha_1}(t) < x < x_{\alpha_2}(t)\}$ such that no other curve x_α lies in D, and that we have the following alternatives:

1. either $t_1 = 0$ or $x_{\alpha_1}(t_1) = x_{\alpha_2}(t_1)$ or $x_1(t_1) = a$ or $x_2(t_1) = b$,
2. either $t_2 = T$ or $x_{\alpha_1}(t_2) = x_{\alpha_2}(t_2)$ or $x_1(t_2) = a$ or $x_2(t_2) = b$,

When we apply Green's theorem to X on D we get the following:

$$\begin{aligned}
\iint_D \operatorname{div}(X)dxdt &= \int_{\partial D} X.n \; ds \\
&= \int_{t_1}^{t_2} \left(X(t,x_{\alpha_2}(t)).(-\dot{x}_{\alpha_2}(t),1) + X(t,x_{\alpha_1}(t)).(\dot{x}_{\alpha_1}(t),-1)\right) dt \\
&+ \int_{x_{\alpha_1}(t_2)}^{x_{\alpha_2}(t_2)} q(u(t_2,x))\phi(t_2,x)dx - \int_{x_{\alpha_1}(t_1)}^{x_{\alpha_2}(t_1)} q(u(t_1,x))\phi(t_1,x)dx.
\end{aligned}$$

Now since either $x_{\alpha_1}(t_2) = x_{\alpha_2}(t_2)$ or $\phi(T,.) = 0$ we get:

$$\int_{x_{\alpha_1}(t_2)}^{x_{\alpha_2}(t_2)} q(u(t_2,x))\phi(t_2,x)dx = 0.$$

And with the same kind of reasoning we also have

$$\int_{x_{\alpha_1}(t_1)}^{x_{\alpha_2}(t_1)} q(u(t_1,x))\phi(t_1,x)dx = 0.$$

On the other hand we have:

$$\begin{aligned}
\iint_D \operatorname{div}(X)dxdt = \iint_D & \eta(u(t,x))\partial_t\phi(t,x) + q(u(t,x))\partial_x\phi(t,x) \\
&+ \eta'(u(t,x))\phi(t,x)g(t)dtdx.
\end{aligned} \tag{A.2}$$

In the end we obtain:

$$\iint_D \eta(u(t,x))\partial_t\phi(t,x) + q(u(t,x))\partial_x\phi(t,x) + \eta'(u(t,x))\phi(t,x)g(t)dtdx =$$
$$\int_{t_1}^{t_2} \left(q(u(t,x_{\alpha_2}(t)^-)) - \dot{x}_{\alpha_2}(t)\eta(u(t,x_{\alpha_2}(t)^-))\right)\phi(t,x_{\alpha_2}(t))$$
$$- \left(q(u(t,x_{\alpha_1}(t)^+)) - \dot{x}_{\alpha_1}(t)\eta(u(t,x_{\alpha_1}(t)^+))\right)\phi(t,x_{\alpha_1}(t))dt. \quad \text{(A.3)}$$

Furthermore by adding arbitrary fronts to the family $\{x_\alpha\}$ on the parts of the domain where v is constant (since there is no jump those artificial fronts automatically satisfy (3.8) and (3.9)), we can obtain a partition of $(a,b)\times(0,T)$ by sets such as D. As a consequence we get:

$$\int_a^b\int_0^T \eta(u(t,x))\partial_t\phi(t,x)+q(u(t,x))\partial_x\phi(t,x)+\eta'(u(t,x))\phi(t,x)g(t)dtdx =$$
$$\sum_\alpha\int_0^T\Big(\dot{x}_\alpha(t)\big(\eta(u(t,x_\alpha(t)^+))-\eta(u(t,x_\alpha(t)^-))\big)$$
$$+ q(u(t,x_\alpha(t)^-))-q(u(t,x_\alpha(t)^+))\Big)\phi(t,x_\alpha(t))dt. \quad \text{(A.4)}$$

Now we replace $\dot{x}_\alpha$ by its value in terms of u and we use the following trivial lemma:

Lemma 9. *Let z^- and z^+ be real numbers such that $z^+ \leq z^- + \epsilon$ then we have:*

$$\frac{f(z^+)-f(z^-)}{z^+-z^-}\left(\eta(z^+)-\eta(z^-)\right) - \left(q(z^+)-q(z^-)\right) \geq -C\epsilon|z^+-z^-|, \quad \text{(A.5)}$$

with $C = (||f''||_{\mathcal{C}^0([z^-,z^+])}||\eta'||_{\mathcal{C}^0([z^-,z^+])} + ||f'||_{\mathcal{C}^0([z^-,z^+])}||\eta''||_{\mathcal{C}^0([z^-,z^+])})$. Thus we get:

$$\int_a^b\int_0^T \eta(u(t,x))\partial_t\phi(t,x) + q(u(t,x))\partial_x\phi(t,x) + \eta'(u(t,x))\phi(t,x)g(t)dtdx$$
$$\geq \int_0^T \sum_\alpha -C\epsilon|u(t,x_\alpha(t)^+) - u(t,x_\alpha(t)^-)|||\phi||_{\mathcal{C}^0((0,T)\times(a,b))}$$
$$\geq \int_0^T -C\epsilon||\phi||_{\mathcal{C}^0((0,T)\times(a,b))}\mathrm{TotVar}(u(t,.))dt$$
$$\geq -\tilde{C}\epsilon||\phi||_{\mathcal{C}^0((0,T)\times(a,b))}||u||_{L^\infty((0,T);\mathrm{BV}(a,b))}. \quad \text{(A.6)}$$

Where the constant $\tilde{C}$ is given by:

$$\tilde{C} = T\left(||f''||_{\mathcal{C}^0([-||u||_{L^\infty},||u||_{L^\infty}])}||\eta'||_{\mathcal{C}^0([-||u||_{L^\infty},||u||_{L^\infty}])}\right.$$
$$\left.+ ||f'||_{\mathcal{C}^0([-||u||_{L^\infty},||u||_{L^\infty}])}||\eta''||_{\mathcal{C}^0([-||u||_{L^\infty},||u||_{L^\infty}])}\right). \quad \text{(A.7)}$$

□

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