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On a kinetic FitzHugh-Nagumo model of neuronal network

S. Mischler[1], C. Quiñinao[2], J. Touboul[3]

[1] Université Paris-Dauphine & IUF CEREMADE, UMR CNRS 7534
Place du Maréchal de Lattre de Tassigny, 75775, Paris Cedex 16, FRANCE
E-mail: mischler@ceremade.dauphine.fr
[2] Université Pierre et Marie Curie, Laboratoire Jacques-Louis Lions, CNRS UMR 7598 and Mathematical Neuroscience Team, CIRB - Collège de France
4 place de Jussieu, F-75005, Paris. FRANCE
E-mail: cristobal.quininao@college-de-france.fr
[3] College de France, Mathematical Neuroscience Team CIRB and INRIA Paris-Rocquencourt, Mycenae Team
11 place Marcelin-Berthelot, 75005, Paris, FRANCE
E-mail: jonathan.touboul@college-de-france.fr

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Abstract: We investigate existence and uniqueness of solutions of a McKean-Vlasov evolution PDE representing the macroscopic behaviour of interacting Fitzhugh-Nagumo neurons. This equation is hypoelliptic, nonlocal and has unbounded coefficients. We prove existence of a solution to the evolution equation and non trivial stationary solutions. Moreover, we demonstrate uniqueness of the stationary solution in the weakly nonlinear regime. Eventually, using a semi-group factorisation method, we show exponential nonlinear stability in the small connectivity regime.

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1. Introduction

This paper undertakes the analysis of the existence and uniqueness of solutions for a mean-field equation arising in the modeling of the macroscopic activity

of the brain. This equation describes the large-scale dynamics of a model of the central nervous system, taking into account the fact that it is composed of a very large number of interconnected cells that manifest highly nonlinear dynamics and are subject to noise. Non-linearities in the intrinsic dynamics of individual cells are an essential element of the neural code. Indeed, nerve cells constantly regulate their electrical potential depending on the input they receive. This regulation results from intense ionic exchanges through the cellular membranes. The modeling of these dynamics led to the development of the celebrated Hodgkin-Huxley model [21], a very precise description of ion exchanges through the membrane and their effects on the cell voltage. A simplification of this model conserving the most prominent aspects of the Hodgkin-Huxley model, the Fitzhugh-Nagumo (FhN) model [17,29], has gained the status of canonical model of excitable cells in neuroscience. This model constitutes a very good compromise between versatility and accuracy on the one hand, and relative mathematical simplicity on the other hand. It describes the evolution of the membrane potential v of the cell coupled to an auxiliary variable x, called the adaptation variable. Different neurons interact through synapses that are either chemical or electrical. In the case of electrical synapses for instance, the evolution of the pair voltage-adaptation for a set of n neurons $\{(v_t^i, x_t^i),\, 1 \le i \le n\}$ satisfy the equations:

$$\begin{cases} dv_t^i = \Big(v_t^i\,(v_t^i - \lambda)\,(1 - v_t^i) - x_t^i + \sum_{j=1}^n J_{ij}(v_t^i - v_t^j) + I_t \Big)\, dt + \sigma\, dW_t^i \\ dx_t^i = \big(-a x_t^i + b v_t^i \big)\, dt, \end{cases} \tag{1.1}$$

where the cubic nonlinearity accounts for the cell excitability, I_t is the input level, a and b are positive constants representing timescale and coupling between the two variables, and the processes $\{(W_t^i)_{t\ge 0},\, 1 \le i \le n\}$ are independent Brownian motions accounting for the intrinsic noise at the level of each cell. In the sequel, for sake of simplicity, we assume that $\sigma^2 = 2$ and $I_t = I_0 \in \mathbb{R}$ constant, but it is likely that some of our analysis can be extend to $I_t \in L^\infty(\mathbb{R}_+)$ converging rapidly when t goes to infinity. The coefficients J_{ij} represent the effect of the interconnection of cell j onto cell i. These coefficients are positive, and incorporate the information of the connectivity map. Under relatively weak assumptions on the distribution of these coefficients (see Appendix A), it is relatively classical to show that the system enjoys propagation of chaos property and finite sets of neurons converge in law towards a process whose density solves the McKean-Vlasov evolution PDE:

$$\partial_t f = Q_\varepsilon[\mathscr{J}_f]\, f := \partial_x(Af) + \partial_v\big(B_\varepsilon(\mathscr{J}_f) f\big) + \partial_{vv}^2 f, \text{ on } (0,\infty)\times\mathbb{R}^2 \tag{1.2}$$

$$A = A(x,v) = ax - bv, \quad B_\varepsilon(\mathscr{J}_f) = B(x,v\,;\varepsilon,\mathscr{J}_f), \tag{1.3}$$

$$B(x,v\,;\varepsilon,j) = v\,(v-\lambda)\,(v-1) + x - \varepsilon\,(v-j) + I_0, \tag{1.4}$$

$$\mathscr{J}_f = \mathscr{J}(f) = \int_{\mathbb{R}^2} v\, f(x,v)\, dv dx, \tag{1.5}$$

where ε denotes the averaged value of the connectivity coefficients J_{ij} and $f = f(t,x,v) \ge 0$ is the density function of finding neurons with adaptation and

voltage $(x, v) \in \mathbb{R}^2$ at time $t \geq 0$. The evolution equation (1.2) is complemented by an initial condition

$$f(0, \cdot, \cdot) = f_0(\cdot, \cdot) \geq 0 \quad \text{in } \mathbb{R}^2.$$

Since the PDE can be written in divergence form, the initial normalization of the density is conserved. In particular, consistent with the derivation of the system, we have:

$$\int_{\mathbb{R}^2} f(t, x, v)\, dxdv = \int_{\mathbb{R}^2} f_0(x, v)\, dxdv = 1,$$

when f_0 is normalized. Moreover, the nonnegativity is also a classical result of this kind of equations (for a brief discussion see Section 3), therefore we assume in the sequel that f is a probability density.

From the mathematical viewpoint, this equation presents several interests. First, the system is not Hamiltonian and the dynamics may present several equilibria, therefore, methods involving a potential and its possible convexity may not be used. Second, intrinsic noise acts as a stochastic input only into the voltage variable (since it modifies the voltage through random fluctuations of the current), leaving the adaptation equation unchanged and yielding to a hypoelliptic equation. From the phenomenological viewpoint, this system is particularly rich. It shows a number of different regimes as parameters are varied, and in particular, as a function of the connectivity level: the system goes from a non-trivial stationary regime in which several stationary solutions may exist for strong coupling, to periodic solutions, and eventually to a unique stationary solution for weak coupling. This is illustrated in section 6, in particular, we present some numerical results of (1.1) for a large number of interacting neurons.

In order to rigorously analyse equation (1.2), we restrict ourself to the latter regime, and we shall demonstrate the existence, uniqueness and stability of solutions to the McKean-Vlasov equation in the limit of weak coupling. More precisely, we shall prove existence of solution and non trivial stationary solution to the evolution equation (1.2) without restriction on the connectivity coefficient $\varepsilon > 0$, and next uniqueness of the stationary solution and its exponential NL stability in the small excitability regime.

1.1. Historical overview of macroscopic and kinetic models in neuroscience. As mentioned above, the problem we study lies within a long tradition of works in the domain of the characterization of macroscopic behaviors in large neuronal networks. First efforts to describe the macroscopic activity of large neuron ensemble can be traced back to the work of Amari, Wilson and Cowan in the 1970s [2,3,41,42], where were introduced heuristically derived equations on the averaged membrane potential of a population of neurons. These models made the assumption that populations interact through a macroscopic variable, the averaged firing rate of the population, assumed to be a sigmoidal transform of the mean voltage. This model has been extremely successful in reproducing a number of macroscopic behaviors in the cortex, one of the most striking being related to pattern formation in the cortex associated to visual hallucinations [14] (see also [7] for a recent review on the subject). The relative simplicity of the Wilson-Cowan model and the good agreement with neurological data motivated the study of the relationship between the dynamics of different individual cells

activity models and the macroscopic equations that emerge from them. This has been an important piece of work in the 1990s in the bio-physics community, using simplified (non-excitable) models and specific assumptions on the architecture of the network, including the assumption of sparse and balanced connectivity (the sum of all incoming input vanishes). The sparse connectivity assumption was used by different authors to state that the activity was uncorrelated [1,4, 9], and resulted in characterizing different neuronal states. A transition Markov two-states model governing the firing dynamics of the neurons in the network was recently introduced. In these models, the transition probability of the system, written through a master equation, is then handled using different physics techniques including van Kampen expansions or path integral methods (see for example [10,6,13,37]).

The mathematical community also undertook the analysis of the problem since the beginning of this decade. We can essentially distinguish two trends: the study of neuron models that abstract the emission of an action potential, or those considering continuous dynamics for the variables and based on the biophysics of the cells. The former approach is particularly developed for the integrate-and-fire neuron that describes the voltage of a cell through a linear equation. Spikes are fired when the voltage reaches a fixed threshold, the voltage is subsequently reset and interconnected cells evolve according to the reception of this action potential. A kinetic framework was developed for noisy integrate-and-fire neural networks and a remarkable application to the visual system was developed [12]. These models were enriched by considering conductances with different timescales [35], and fine numerical analysis of the equation were developed for the thus obtained kinetic equation [34]. In [11], the fine mathematical analysis of the time evolution of the system was developed, and it was shown in the case of excitatory networks that the firing rate of the solution of the kinetic equation blows up in finite time. These models and methods were extended to a more realistic neuron model taking into account the fact that neurons interact through changes in the conductances. This leads to a kinetic-type model [33] with complex nonlinear boundary conditions and a degenerate diffusion whose solutions are investigated therein. An alternative approach was developed with a spiking model, called the time-elapsed neuron [30,32,31], whose solution were investigated in a number of situations and it was in particular demonstrated that the limit equation in the presence of delays features periodic solutions. All these models have in common that they consider a simplification of the neuronal dynamics by separately considering input integration and spikes. This structure lead to discontinuous dynamics and interactions between cells, that results, in the limit equation, in the presence of a nonlinear implicit term in the boundary conditions. In contrast, the Fitzhugh-Nagumo model is an approximation of a biophysical model and does not abstract the spike emission, thus we have a relatively complex intrinsic dynamics for each cell. Notwithstanding, electrical synapses considered lead to a smooth interaction term. Thanks to the regularity of the interactions, we do not have blow up of the solutions. However, the non-globally Lipschitz continuous drift term introduce specific difficulties that we address here. This allows us to show in particular stability properties of stationary solutions in the nonlinear system.

1.2. Organization of the paper. The paper is organized as follows: section 2 summarizes our main results. Section 3 is interested with the existence, uniqueness and *a priori* estimates on the solutions to the evolution equation, as well as, the existence of stationary solutions. The next sections prove the stability of the unique stationary solution. Our proof uses factorization of the linearized semigroup allowing to prove linear stability, which we complete in section 5 by an analysis of the nonlinear stability of the stationary solution. Along the way, a number of open problems were identified beyond the small connectivity regime treated here that we present in section 6 together with numerical simulations: we will observe that the stationary solution splits into two stable stationary solutions as connectivity is increased, and in an intermediate regime, periodic solutions emerge. Two appendices complete the paper. Appendix A investigates the microscopic system and its convergence towards the mean-field equation (1.2) and Appendix B deals with the strict positivity of stationary solutions.

2. Summary of the main results

2.1. Functional spaces and norms. We start by introducing the functional framework in which we work throughout the paper. For any exponent $p \in [1, \infty]$ and any nonnegative weight function ω, we denote by $L^p(\omega)$ the Lebesgue space $L^p(\mathbb{R}^2; \omega\, dxdv)$ and for $k \in \mathbb{N}$ the corresponding Sobolev spaces $W^{k,p}(\mathbb{R}^2; \omega\, dxdv)$. They are associated to the norms

$$\|f\|_{L^p(\omega)} = \|f\omega^{1/p}\|_{L^p(\mathbb{R}^2)}, \quad \|f\|^p_{W^{k,p}(\omega)} = \|f\|^p_{L^p(\omega)} + \sum_{j=1}^{k} \|D^k_{x,v} f\|^p_{L^p(\omega)}.$$

For $k \geq 1$, we define the partial v-derivative space $W^k_v(\omega)$ by

$$W^{k,p}_v(\omega) \,:=\, \{\, f \,\in\, W^{k-1,p}(\omega)\,;\, D^k_v f \,\in\, L^p(\omega)\,\},$$

which are naturally associated to the norms

$$\|f\|^p_{W^{k,p}_v(\omega)} = \|f\|^p_{W^{k-1,p}(\omega)} + \|D^k_v f\|^p_{L^p(\omega)}.$$

A particularly important space in our analysis, denoted by $H^2_v(\omega)$, is

$$H^2_v(\omega) = W^{2,2}_v(\omega) = \{f \in H^1(\omega) \text{ such that } \partial^2_{vv} f \in L^2(\omega)\},$$

together with the set of functions with finite entropy

$$L^1 \log L^1 \;:=\; \big\{f \,\in\, L^1(\mathbb{R}^2) \text{ such that } f \geq 0 \text{ and } \mathscr{H}(f) < \infty\big\},$$

where we use the classical notation $\mathscr{H}(f) := \int_{\mathbb{R}^2} f \log f$. The space of probability measures on the plane will be denoted by $\mathbb{P}(\mathbb{R}^2)$. Finally for $\kappa > 0$, we introduce the exponential weight function:

$$m = e^{\kappa(M-1)} \quad \text{with} \quad M := 1 + x^2/2 + v^2/2. \tag{2.1}$$

2.2. Main results. Using classical theory of renormalisation and some nice *a priori* bounds we show later, it is not hard to see that equation (1.2) is well posed. Indeed, defining the *weak solutions* as:

Definition 2.1 *Let f_0 be a normalized nonnegative function defined on $\mathbb{R}^2$ such that $\mathscr{J}(f_0)$ is well defined. We say that $f_t(x,v) := (t,x,v) \mapsto f(t,x,v)$ is a weak solution to* (1.2) *if the following conditions are fulfilled:*

- *$f \in C([0,\infty); L^1(M^{1/2}))$;*
- *for almost any $t \geq 0$, $f \geq 0$ and*

$$\int_{\mathbb{R}^2} f(t,x,v)\,dx\,dv \;=\; \int_{\mathbb{R}^2} f_0(x,v)\,dxdv \;=\; 1;$$

- *for any $\varphi \in C^1([0,\infty); C_c^\infty(\mathbb{R}^2))$ and any $t \geq 0$ it holds*

$$\int_{\mathbb{R}^2} \varphi f_t = \int_{\mathbb{R}^2} \varphi f_0 + \int_0^t \int_{\mathbb{R}^2} \big[\partial_t \varphi + \partial^2_{vv}\varphi - A\,\partial_x\varphi - B_\varepsilon(\mathscr{J}(f_s))\partial_v\varphi\big] f_s. \quad (2.2)$$

we can state the

Theorem 2.2. *For any $f_0 \in L^1(M) \cap L^1 \log L^1 \cap \mathbb{P}(\mathbb{R}^2)$, there exists a unique global weak solution f_t to the FhN equation* (1.2)*, that moreover satisfies*

$$\|f_t\|_{L^1(M)} \leq \max(C_0, \|f_0\|_{L^1(M)}), \quad (2.3)$$

and depends continuously in $L^1(M^{1/2})$ to the initial datum. More precisely, if $f_{n,0} \to f_0$ strongly in $L^1(M^{1/2})$ and $\mathscr{H}(f_{n,0}) \leq C$ then $f_{n,t} \to f_t$ strongly in $L^1(M^{1/2})$ for any later time $t \geq 0$.

Furthermore, the regularity of the initial condition is maintained by the equation in the following sense: if $f_0 \in L^1(m)$ then for any nonnegative time t it holds

$$\|f_t\|_{L^1(m)} \leq \max(C_1, \|f_0\|_{L^1(m)}), \quad (2.4)$$

and there exist two norms $\|\cdot\|_{\mathcal{H}^1}$ and $\|\cdot\|_{\mathcal{H}^2_v}$ equivalent respectively to $\|\cdot\|_{H^1(m)}$ and $\|\cdot\|_{H^2_v(m)}$, such that if $f_0 \in H^1(m)$ then

$$\|f_t\|_{\mathcal{H}^1} \leq \max(C_2, \|f_0\|_{\mathcal{H}^1}), \quad (2.5)$$

and if $f_0 \in H^2_v(m)$ then

$$\|f_t\|_{\mathcal{H}^2_v} \leq \max(C_3, \|f_0\|_{\mathcal{H}^2_v}), \quad (2.6)$$

where C_1, C_2, C_3 are positive constants.

The other two main results of the present work are summarized in the following

Theorem 2.3. *For any $\varepsilon \geq 0$, there exists at least one stationary solution G_ε to the FhN statistical equation* (1.2)*, that is $G_\varepsilon \in H^2_v(m) \cap \mathbb{P}(\mathbb{R}^2)$ such that*

$$0 = \partial_x(AG_\varepsilon) + \partial_v(B_\varepsilon(\mathscr{J}_{G_\varepsilon})G_\varepsilon) + \partial^2_{vv}G_\varepsilon \quad \text{in} \quad \mathbb{R}^2. \quad (2.7)$$

Moreover, there exists an increasing function $\eta : \mathbb{R}_+ \to \mathbb{R}$ such that $\eta(\varepsilon) \xrightarrow[\varepsilon\to 0]{} 0$ and such that any solution to (2.7) *satisfies*

$$\|G - G_0\|_{L^2(m)} \leq \eta(\varepsilon),$$

where G_0 is the unique stationary solution corresponding to the case $\varepsilon = 0$.

Theorem 2.4. *There exists $\varepsilon^* > 0$ such that, in the small connectivity regime $\varepsilon \in (0, \varepsilon^*)$, the stationary solution is unique and exponentially stable. More precisely, there exist $\alpha^* < 0$ and $\eta^*(\varepsilon) : \mathbb{R}_+ \to \mathbb{R}$, with $\eta^*(\varepsilon) \xrightarrow[\varepsilon \to 0]{} \infty$, such that if*

$$f_0 \in H^1(m) \cap \mathbb{P}(\mathbb{R}^2) \quad \text{and} \quad \|f_0 - G\|_{H^1(m)} \leq \eta^*(\varepsilon),$$

then there exists $C^* = C^*(\varepsilon) > 0$, such that

$$\|f_t - G\|_{L^2(m)} \leq C^* \, e^{\alpha^* t}, \quad \forall \, t \geq 0,$$

where f_t is the solution to (1.2) *with initial condition f_0.*

2.3. Other notations and definitions. We prepare to the demonstration of these results by introducing a few notations that will be used throughout the paper. For two given Banach spaces $(E, \|\cdot\|_E)$ and $(\mathcal{E}, \|\cdot\|_{\mathcal{E}})$, we denote by $\mathscr{B}(E, \mathcal{E})$ the space of bounded linear operators from E to $\mathcal{E}$ and we denote by $\|\cdot\|_{\mathscr{B}(E,\mathcal{E})}$ the associated operator norm. The set of closed unbounded linear operators from E to $\mathcal{E}$ with dense domain is denoted by $\mathscr{C}(E, \mathcal{E})$. In the special case when $E = \mathcal{E}$, we simply write $\mathscr{B}(E) = \mathscr{B}(E, E)$ and $\mathscr{C}(E) = \mathscr{C}(E, E)$.

For a given $\alpha \in \mathbb{R}$, we define the complex half plane

$$\Delta_\alpha := \{z \in \mathbb{C}, \quad \mathrm{Re}(z) > \alpha\}.$$

For a given Banach space X and $\Lambda \in \mathscr{C}(X)$ which generates a semigroup, we denote this associated semigroup by $(S_\Lambda(t),\, t \geq 0)$, by $D(\Lambda)$ its domain, by $N(\Lambda)$ its null space, by $R(\Lambda)$ its range, and by $\Sigma(\Lambda)$ its spectrum. On the resolvent set $\rho(\Lambda) = \mathbb{C} \setminus \Sigma(\Lambda)$ we may define the resolvent operator $\mathcal{R}(\Lambda)$ by

$$\forall \, z \in \mathbb{C}, \qquad \mathcal{R}_\Lambda(z) := (\Lambda - z)^{-1}.$$

Moreover, $\mathcal{R}_\Lambda(z) \in \mathscr{B}(X)$ and has range equal to $D(\Lambda)$. We recall that $\xi \in \Sigma(\Lambda)$ is called an eigenvalue of Λ if $N(\Lambda - \xi) \neq \{0\}$, and it called an isolated eigenvalue if there exists $r > 0$ such that

$$\Sigma(\Lambda) \cap \{z \in \mathbb{C},\, |z - \xi| < r\} = \{\xi\}.$$

Since the notion of convolution of semigroups will be required, we recall it here. Consider some Banach spaces X_1, X_2 and X_3 and two given functions

$$S_1 \in L^1([0, \infty); \mathscr{B}(X_1, X_2)) \quad \text{and} \quad S_2 \in L^1([0, \infty); \mathscr{B}(X_2, X_3)),$$

one define $S_2 * S_1 \in L^1([0, \infty); \mathscr{B}(X_1, X_3))$ by

$$(S_2 * S_1)(t) := \int_0^t S_2(t - s) S_1(t) \, ds, \qquad \forall \, t \geq 0.$$

In the special case $S_1 = S_2$ and $X_1 = X_2 = X_3$, $S^{(*n)}$ is defined recursively by $S^{(*1)} = S$ and $S^{(*n)} = S * S^{(*(n-1))}$ for $n > 1$. Equipped with this definition, we state the

Proposition 2.5 *Let X, Y be two Banach spaces such that $Y \subset X$ with dense inclusion. Let us consider $S(t)$ a continuous semigroup such that for all $t \geq 0$*

$$\|S(t)\|_{\mathscr{B}(\mathcal{Z})} \leq C_{\mathcal{Z}}\, e^{\alpha^* t}, \quad \mathcal{Z} \in \{X, Y\},$$

for some $\alpha^* \in \mathbb{R}$ and positive constants C_X and C_Y. If there exists $\Theta > 0$ and $C_{X,Y} > 0$ such that

$$\|S(t)f\|_Y \;\leq\; C_{X,Y}\, t^{-\Theta}\, e^{\alpha^* t}\, \|f\|_X, \quad \forall\, f \in X,\, t \in (0,1],$$

then, there exists $n \in \mathbb{N}$, and a polynomial $p_n(t)$ such that

$$\|S^{(*n)}(t)f\|_Y \;\leq\; p_n(t)\, e^{\alpha^* t}\|f\|_X, \qquad \forall\, f \in X,\, t > 0. \tag{2.8}$$

In particular, for any $\alpha > \alpha^*$, it holds

$$\|S^{(*n)}(t)f\|_Y \;\leq\; C_{\alpha,n} e^{\alpha t}\|f\|_X, \quad \forall\, f \in X,\, t > 0,$$

for some positive constant $C_{\alpha,n}$.

This general result has been already established and used in [19] and [26], but we give an alternative, and somehow simpler, proof of it.

Proof. We notice that for $\mathcal{Z} \in \{X, Y\}$, if

$$\|S^{(*n)}(t)f\|_{\mathcal{Z}} \;\leq\; p_n^{\mathcal{Z}}(t)\, e^{\alpha^* t}\|f\|_{\mathcal{Z}}, \quad \forall\, t \geq 0, \tag{2.9}$$

for $n \in \mathbb{N}$ and $p_n^{\mathcal{Z}}(t)$ a polynomial, then

$$\|S^{(*(n+1))}(t)f\|_{\mathcal{Z}} \;\leq\; \int_0^t \|S(t-s)\, S^{(*n)}(s)f\|_{\mathcal{Z}}\, ds \;\leq\; p_{n+1}^{\mathcal{Z}}(t)\, e^{\alpha^* t}\|f\|_{\mathcal{Z}},$$

for $p_{n+1}^{\mathcal{Z}} = C_{\mathcal{Z}} \int_0^t p_n^{\mathcal{Z}}(s)\, ds$. Then, by an immediate induction argument, we get (2.9) for any $n \geq 1$ and $p_n^{\mathcal{Z}}(t) := \frac{C_{\mathcal{Z}}^n t^{n-1}}{(n-1)!}$.

Now fix $t \in (0,1]$ and, without lost of generality, assume that $\Theta \notin \mathbb{N}$. In that case, if

$$\|S^{(*n)}(t)f\|_Y \leq C_n t^{-(\Theta-n+1)} e^{\alpha^* t}\|f\|_X, \quad \forall\, t \in (0,1], \tag{2.10}$$

for some $n \in \mathbb{N}$ and C_n a positive constant, then

$$\begin{aligned}
\|S^{(*(n+1))}(t)f\|_Y &\leq \int_0^{t/2} \|S(t-s)S^{(*n)}(s)f\|_Y\, ds + \int_{t/2}^t \|S(t-s)S^{(*n)}(s)f\|_Y\, ds \\
&\leq \int_0^{t/2} C_{X,Y}(t-s)^{-\Theta} e^{\alpha^*(t-s)}\|S^{(*n)}(s)f\|_X\, ds + \int_{t/2}^t C_Y e^{\alpha^*(t-s)}\|S^{(*n)}(s)f\|_Y\, ds \\
&\leq \int_0^{t/2} C_{X,Y}(t-s)^{-\Theta} e^{\alpha^* t} p_n^X(s)\|f\|_X\, ds + \int_{t/2}^t C_Y e^{\alpha^* t} C_n s^{-(\Theta-n+1)}\|f\|_X\, ds \\
&\leq \frac{C_{X,Y} C_X^n}{(n-1)!} e^{\alpha^* t}\|f\|_X \int_0^{t/2} (t-s)^{-\Theta} s^{n-1}\, ds + C_Y C_n e^{\alpha^* t}\|f\|_X \int_{t/2}^t s^{-(\Theta-n+1)}\, ds \\
&\leq C_{n+1} t^{-(\Theta-n)} e^{\alpha^* t}\|f\|_X,
\end{aligned}$$

for some C_{n+1} depending only on $C_X, C_Y, C_{X,Y}$ and C_n. Once again, by an induction argument, we get (2.10) for any n. Moreover, as soon as $\Theta - n + 1 > 0$, inequality (2.8) holds.

Finally, to conclude in the case $t > 1$, it suffices to notice that

$$\|S(t)f\|_Y \;\leq\; C_Y C_{X,Y}(t - \lfloor t \rfloor)^{-\Theta} e^{\alpha^* t} \|f\|_X,$$

where $\lfloor t \rfloor$ is the largest integer smaller than t. A similar argument that the one used for $t \in (0, 1]$, allows us to find a polynomial p_n such that (2.8) still holds when $t > 1$.

Finally, we recall the abstract notion of *hypodissipative operators*:

Definition 2.6 *Considering a Banach space $(X, \|\cdot\|_X)$, a real number $\alpha \in \mathbb{R}$ and an operator $\Lambda \in \mathscr{C}(X)$, $(\Lambda - \alpha)$ is said to be hypodissipative on X if there exists some norm $|||\cdot|||_X$ on X equivalent to the usual norm $\|\cdot\|_X$ such that*

$$\forall\, f \in D(\Lambda), \quad \exists\, \phi \in F(f) \quad \textit{such that} \quad Re\langle \phi, (\Lambda - \alpha) f\rangle \leq 0,$$

where $\langle \cdot, \cdot \rangle$ is the duality bracket in X and X^* and $F(f) \subset X^*$ is the dual set of f defined by

$$F(f) = F_{|||\cdot|||_X}(f) := \{\phi \in X^*, \quad \langle \phi, f\rangle = |||f|||_X^2 = |||\phi|||_{X^*}^2\}.$$

One classically sees (we refer to for example [19, Subsection 2.3]) that when Λ is the generator of a semigroup S_Λ, for given $\alpha \in \mathbb{R}$ and $C > 0$ constants, the following assertions are equivalent:

(a) $(\Lambda - \alpha)$ is hypodissipative;
(b) the semigroup satisfies the growth estimate $\|S_\Lambda(t)\|_{\mathscr{B}(X)} \leq Ce^{\alpha t}$, $t \geq 0$;

3. Analysis of the nonlinear evolution equation

In this section, we shall prove existence and uniqueness of solutions for the evolutionary equation, and provide some *a priori* estimates on their behavior. Before going into further details, we remark that for $\mathscr{J}$ fixed, the operator $Q_\varepsilon[\mathscr{J}]$ is linear and writes

$$Q_\varepsilon[\mathscr{J}]\, f \;=\; \partial_x(Af) + \partial_v(B_\varepsilon(\mathscr{J})\, f) + \partial^2_{vv} f.$$

In particular, for $g \in H^2_v(m)$ (or simply g regular enough and decaying fast at infinite) we have

$$\int_{\mathbb{R}^2} (Q_\varepsilon[\mathscr{J}]\, f)\, g\, dvdx \;=\; -\int_{\mathbb{R}^2} f\left(A\, \partial_x g + B_\varepsilon(\mathscr{J})\, \partial_v g - \partial^2_{vv} g\right) dvdx,$$

therefore, it is natural to define

$$Q^*_\varepsilon[\mathscr{J}]\, g \;:=\; -A\, \partial_x g - B_\varepsilon(\mathscr{J})\, \partial_v g + \partial^2_{vv} g.$$

3.1. A priori bounds. The a priori estimates that follow are uniform in ε in the bounded connectivity regime $\varepsilon \in [0, \varepsilon_0)$ where ε_0 is an arbitrary and fixed positive number, i.e., they involve constants that do not depend on ε.

Lemma 3.1 *For f_t solution to* (1.2) *with $f_0 \in L^1(M) \cap \mathbb{P}(\mathbb{R}^2)$, estimate* (2.3) *holds. Moreover, there exists $C_0' > 0$ depending on $a, b, \lambda, I_0, \varepsilon_0$ and $\|f_0\|_{L^1(M)}$ such that*

$$\sup_{t \geq 0} |\mathscr{J}(f_t)| < C_0'. \tag{3.1}$$

Proof. We first apply Cauchy-Schwarz inequality to find

$$|\mathscr{J}(f)| \leq \int_{\mathbb{R}^2} |v|\, f \leq \Big(\int_{\mathbb{R}^2} f\Big)^{1/2} \Big(\int_{\mathbb{R}^2} v^2\, f\Big)^{1/2} = \Big(\int_{\mathbb{R}^2} v^2\, f\Big)^{1/2}, \tag{3.2}$$

for any $f \in \mathbb{P}(\mathbb{R}^2) \cap L^1(M)$. Moreover, for f_t solution to (1.2), we have

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^2} f_t\, M &= \int_{\mathbb{R}^2} (Q_\varepsilon[\mathscr{J}_{f_t}]\, f_t)\, M = \int_{\mathbb{R}^2} f_t\, (Q_\varepsilon^*[\mathscr{J}_{f_t}] M) \\ &= \int_{\mathbb{R}^2} (1 - Ax - B_\varepsilon(\mathscr{J}_{f_t}) v) f_t. \end{aligned}$$

Using the definition of A and B_ε we get, thanks to (3.2), that

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^2} f_t M\, dxdv &\leq - \int_{\mathbb{R}^2} \Big(-1 + ax^2 - bxv + v^2(v-\lambda)(v-1) \\ &\qquad\qquad -\varepsilon v^2 + xv + I_0 v \Big) f_t + \varepsilon\, \mathscr{J}(f_t)^2 \\ &\leq K_1 - K_2 \int_{\mathbb{R}^2} (v^4 + x^2) f_t + \varepsilon \int_{\mathbb{R}^2} v^2\, f_t \\ &\leq K_1 - K_2 \int_{\mathbb{R}^2} f_t\, M\, dxdv, \end{aligned}$$

where K_1 and K_2 are generic constants depending only on a, b, λ, I_0 and ε_0. Using Gronwall lemma we get (2.3) for some $C_0 > 0$. Finally, coming back to (3.2), we get

$$|\mathscr{J}(f_t)|^2 \;\leq\; \int_{\mathbb{R}^2} v^2\, f_t \;\leq\; 2\, \|f_t\|_{L^1(M)} \;\leq\; 2\, \max(C_0, \|f_0\|_{L^1(M)}),$$

which is nothing but (3.1).

Lemma 3.2 *For any $\mathscr{J} \in \mathbb{R}$ fixed, there exist some constants $K_1, K_2 > 0$ depending on $a, b, \lambda, I_0, \mathscr{J}, \kappa$ and ε_0 such that*

$$\int_{\mathbb{R}^2} Q_\varepsilon[\mathscr{J}]\, f \cdot \mathrm{sign}(f)\, m \;\leq\; K_1 \|f\|_{L^1(\mathbb{R}^2)} - K_2 \|f\|_{L^1(m)}, \quad \forall\, f \in L^1(m). \tag{3.3}$$

Proof. Since $\mathscr{J} \in \mathbb{R}$ is now fixed, we drop the dependence on this parameter to simplify the notation. Using Kato inequality ($\mathrm{sign}(f)\partial^2_{vv} f \leq \partial^2_{vv}|f|$) we have

$$\begin{aligned}\int_{\mathbb{R}^2} Q_\varepsilon\, f \cdot \mathrm{sign}(f)\, m &\leq \int_{\mathbb{R}^2} |f|\, Q^*_\varepsilon\, m \\ &= -\kappa \int_{\mathbb{R}^2} |f| \big(Ax + B_\varepsilon v - (1+\kappa v^2)\big) m \\ &= -\kappa \int_{\mathbb{R}^2} |f| \big(ax^2 - bxv + v^2(v-\lambda)(v-1) \\ &\qquad\qquad -\varepsilon v(v-J) + xv + I_0 v - (1+\kappa v^2)\big) m,\end{aligned}$$

thus

$$\int_{\mathbb{R}^2} Q_\varepsilon\, f \cdot \mathrm{sign}(f)\, m \;\leq\; -\int_{\mathbb{R}^2} p(x,v)\, |f|\, m,$$

where $p(x,v)$ is a polynomial on x and v with leading term $v^4 + x^2$. Inequality (3.3) follows directly.

Remark 3.3

- *Using the same ideas stated in the proof of Lemma 3.1, one can easily extend by induction the inequality* (2.3) *to* $L^1(M^k)$ *for any* $k \in \mathbb{N}$.
- *Furthermore, we note that inequality* (3.3) *remains true for weights with a faster divergence. In particular, it is true for* $e^{\kappa(|x|^s/s + v^4/4)}$ *for any* $\kappa \in (0,1)$ *and* $s \in [5,6)$*, and also for* $s = 6$ *when the parameters* a *and* b *are well tuned and the extra restriction that* κ *is taken small enough. Therefore, the absence of diffusion in the* x *variable allows appearance of a faster decay at infinite, while not affecting its regularity.*

Corollary 3.4 *Estimate* (2.4) *holds.*

Proof. For f_t solution to (1.2), inequality (3.1) tells us that $|\mathscr{J}(f_t)| \leq C_0'$. Moreover, since the mass is unitary for almost any $t \geq 0$, it holds

$$\frac{d}{dt}\int_{\mathbb{R}^2} |f_t|\, m \;=\; \int_{\mathbb{R}^2} Q_\varepsilon[\mathscr{J}_{f_t}]\, f_t \cdot \mathrm{sign}(f_t)\, m \;\leq\; K_1 - K_2 \int_{\mathbb{R}^2} |f_t|\, m,$$

where K_1 and K_2 depend only on $a, b, \lambda, I, \varepsilon_0$ and C_0'. Finally, integrating this last inequality, we get

$$\|f_t\|_{L^1(m)} \;\leq\; \max\big(C_1, \|f_0\|_{L^1(m)}\big), \qquad \forall\, t \geq 0,$$

for some positive constant C_1 depending only on the parameters of the system, ε_0 and C_0'.

In a second series of *a priori bounds* we analyse the $H^1(m)$ and $H^2_v(m)$ norms of the solutions to (1.2), in particular, we prove the inequalities (2.5) and (2.6). Since the equation is hypodissipative, we used the ideas of "twisted spaces" and the Nash-Villani's technique (see e.g. [40]): we control the extra $L^2(\mathbb{R}^2)$ contributions by using some weaker norms.

In the sequel, we will be brought to vary the constant κ involved in the definition of m, therefore we introduce the shorthand $m_i = e^{\kappa_i(M-1)}$, $i \in \mathbb{N}$. Unless otherwise specified, these sequences are constructed under the assumption that the sequence κ_i is strictly increasing.

Lemma 3.5 *Consider two exponential weight functions m_1 and m_2 as defined in* (2.1)*. For any $\mathscr{J} \in \mathbb{R}$ fixed, there exist $K_1, K_2 > 0$ and $\delta \in (0,1)$ constants such that*

$$\langle Q_\varepsilon[\mathscr{J}]\, f, f\rangle_{\mathcal{H}^1} \le K_1 \|f\|^2_{L^2(\mathbb{R}^2)} - K_2 \|f\|^2_{\mathcal{H}^1}, \quad \forall\, f \in H^1(m_2), \tag{3.4}$$

where $\langle \cdot, \cdot\rangle_{\mathcal{H}^1}$ is the scalar product related to the Hilbert norm

$$\|f\|^2_{\mathcal{H}^1} := \|f\|^2_{L^2(m_2)} + \delta^{3/2}\|\partial_x f\|^2_{L^2(m_2)} + \delta^{4/3}\langle \partial_x f, \partial_v f\rangle_{L^2(m_1)} + \delta\, \|\partial_v f\|^2_{L^2(m_2)},$$

which is equivalent to the usual norm of $H^1(m_2)$.

Proof. The proof is presented in several steps. In a first part, we deal with inequalities in L^2 for f and its derivatives. Next, we combine those inequalities to control the $\mathcal{H}^1$ norm. Some long and tedious calculations are only outlined for the sake of clarity.

In the following we denote by k_0, k_1 and k_2 some unspecified constants and drop the dependence on $\mathscr{J}$.

Step 1. $L^2(m_2)$ norm. We start by noticing that

$$\langle \partial^2_{vv} f, f\rangle_{L^2(m_2)} = -\int_{\mathbb{R}^2} (\partial_v f)^2 m_2 + \frac{\kappa_2}{2}\int_{\mathbb{R}^2} (1+\kappa_2 v^2) f^2 m_2.$$

$$\langle \partial_x(Af), f\rangle_{L^2(m_2)} = \frac{1}{2}\int_{\mathbb{R}^2} \Big[\partial_x A - A\frac{\partial_x m_2}{m_2}\Big] f^2\, m_2 \;=\; \frac{1}{2}\int_{\mathbb{R}^2} [a - \kappa_2 x(ax - bv)] f^2\, m_2,$$

and similarly

$$\langle \partial_v(B_\varepsilon f), f\rangle_{L^2(m_2)} \;=\; \frac{1}{2}\int_{\mathbb{R}^2} \big[3v^2 - 2(1+\lambda)v + \lambda - \varepsilon - \kappa_2 v\, B_\varepsilon\big] f^2\, m_2.$$

Therefore, we get

$$\langle Q_\varepsilon f, f\rangle_{L^2(m_2)} = -\int_{\mathbb{R}^2} p(x,v) f^2\, m_2 - \|\partial_v f\|^2_{L^2(m_2)}, \tag{3.5}$$

where $p(x,v)$ is a polynomial in x and v with leading term $v^4 + x^2$. In particular, there exist some positive constants k_1 and k_2 such that

$$\langle Q_\varepsilon f, f\rangle_{L^2(m_2)} \le k_1 \|f\|^2_{L^2(\mathbb{R}^2)} - k_2 \|f\|^2_{L^2(Mm_2)} - \|\partial_v f\|^2_{L^2(m_2)}. \tag{3.6}$$

Step 2. First derivative bounds. We have

$$\begin{aligned}\langle \partial_x(\partial_x(Af)), \partial_x f\rangle_{L^2(m_2)} &= \frac{1}{2}\int_{\mathbb{R}^2} \Big[3\,\partial_x A - A\frac{\partial_x m_2}{m_2}\Big](\partial_x f)^2\, m_2 \\ &= \frac{1}{2}\int_{\mathbb{R}^2} \big[3a - \kappa_2 x(ax - bv)\big](\partial_x f)^2\, m_2,\end{aligned}$$

and

$$\langle \partial_x(\partial_v(B_\varepsilon f)), \partial_x f\rangle_{L^2(m_2)} = \int_{\mathbb{R}^2} [\partial_v B_\varepsilon \partial_x f + \partial_x B_\varepsilon \partial_v f + B_\varepsilon \partial^2_{xv} f]\partial_x f\, m_2.$$

Since $\partial_x B_\varepsilon = 1$, observing that

$$\int_{\mathbb{R}^2} [\partial_v B_\varepsilon \partial_x f + B_\varepsilon \partial^2_{xv} f] \partial_x f \, m_2 \;=\; \frac{1}{2} \int_{\mathbb{R}^2} \Big[\partial_v B_\varepsilon - B_\varepsilon \frac{\partial_v m_2}{m_2} \Big] (\partial_x f)^2 m_2,$$

we get

$$\langle \partial_x(\partial_v(B_\varepsilon f)), \partial_x f \rangle_{L^2(m_2)} \;\leq\; \int_{\mathbb{R}^2} |\partial_x f| \, |\partial_v f| m_2 + \frac{1}{2} \int_{\mathbb{R}^2} \Big[\partial_v B_\varepsilon - \kappa_2 v B_\varepsilon \Big] (\partial_x f)^2 m_2.$$

Finally, using that

$$\langle \partial_x \partial^2_{vv} f, \partial_x f \rangle_{L^2(m_2)} = - \int_{\mathbb{R}^2} |\partial^2_{xv} f|^2 m_2 + \frac{1}{2} \int_{\mathbb{R}^2} (\partial_x f)^2 \partial^2_{vv} m_2.$$

we obtain

$$\begin{aligned} \langle \partial_x (Q_\varepsilon f), \partial_x f \rangle_{L^2(m_2)} \leq k_1 \|\partial_x f\|^2_{L^2(\mathbb{R}^2)} - k_2 \|\partial_x f\|^2_{L^2(M m_2)} \\ - \|\partial^2_{xv} f\|^2_{L^2(m_2)} + \int_{\mathbb{R}^2} |\partial_x f| |\partial_v f| \, m_2. \end{aligned} \tag{3.7}$$

A similar calculation leads to

$$\begin{aligned} \langle (\partial_v Q_\varepsilon f), \partial_v f \rangle_{L^2(m_2)} \leq k_1 \|\partial_v f\|^2_{L^2(\mathbb{R}^2)} - k_2 \|\partial_v f\|^2_{L^2(M m_2)} \\ - \|\partial^2_{vv} f\|^2_{L^2(m_2)} + b \int_{\mathbb{R}^2} |\partial_x f| |\partial_v f| \, m_2 \\ + \kappa_2 (1+\lambda) \int_{\mathbb{R}^2} |v| \, f^2 m_2. \end{aligned} \tag{3.8}$$

Step 3. Cross product bound. The contribution of the cross product term is more delicate, therefore we decompose it into five quantities that we study separately:

$$\begin{aligned} &\langle \partial_x Q_\varepsilon f, \partial_v f \rangle_{L^2(m_1)} + \langle \partial_v Q_\varepsilon f, \partial_x f \rangle_{L^2(m_1)} \\ &\quad = \int_{\mathbb{R}^2} \big[(\partial_x f)(\partial^3_{vvv} f) + (\partial_v f)(\partial^3_{xvv} f) \big] \, m_1 \\ &\quad + \int_{\mathbb{R}^2} \big[\partial_x A \partial_v f + \partial_v A \partial_x f + A \partial^2_{vx} f \big] (\partial_x f) \, m_1 \\ &\quad + \int_{\mathbb{R}^2} \big[\partial^2_{vv} B_\varepsilon f + 2 \partial_v B_\varepsilon \partial_v f + B_\varepsilon \partial^2_{vv} f \big] (\partial_x f) \, m_1 \\ &\quad + \int_{\mathbb{R}^2} \big[2 \partial_x A \, \partial_x f + A \partial^2_{xx} f \big] (\partial_v f) \, m_1 \\ &\quad + \int_{\mathbb{R}^2} \big[\partial_v B_\varepsilon \partial_x f + \partial_x B_\varepsilon \partial_v f + B_\varepsilon \partial^2_{xv} f \big] (\partial_v f) \, m_1 \quad =: \quad \sum_{i=1}^{5} \mathcal{T}_i. \end{aligned}$$

We start by handling the first term on the right hand side. Using integration by parts adequately, we get

$$\mathcal{T}_1 \;=\; \int_{\mathbb{R}^2} (\partial_x f)(\partial_v f) \, \partial^2_{vv} m_1 - 2 \int_{\mathbb{R}^2} (\partial^2_{xv} f)(\partial^2_{vv} f) \, m_1.$$

Similarly, for the contributions involving A, we have

$$\mathcal{T}_2 \,=\, \frac{1}{2}\int_{\mathbb{R}^2}\Big[\partial_v A - A\frac{\partial_v m_1}{m_1}\Big](\partial_x f)^2 m_1 + a\int_{\mathbb{R}^2}(\partial_x f)(\partial_v f)\, m_1,$$

and

$$\mathcal{T}_4 \,=\, \int_{\mathbb{R}^2}\Big[\partial_x A - A\frac{\partial_x m_1}{m_1}\Big](\partial_x f)(\partial_v f) m_1 + \frac{1}{2}\int_{\mathbb{R}^2}\partial_v[Am_1](\partial_x f)^2.$$

Adding these last two expressions, it only remains

$$\int_{\mathbb{R}^2}\partial_v A(\partial_x f)^2 m_1 + \int_{\mathbb{R}^2}\Big[2a - A\frac{\partial_x m_1}{m_1}\Big](\partial_x f)(\partial_v f)\, m_1 \le -b\,\|\partial_x f\|^2_{L^2(m_1)} + k_0\int_{\mathbb{R}^2}|\partial_x f|\,|\partial_v f|\, M m_1,$$

for some constant $k_0 > 0$.

For the contributions related to B_ε, involved in $\mathcal{T}_3$ and $\mathcal{T}_5$, we have

$$\mathcal{T}_3 \,=\, -\int_{\mathbb{R}^2}\kappa_1 x(3v-1-\lambda) f^2 m_1 + 2\int_{\mathbb{R}^2}\partial_v B_\varepsilon(\partial_x f)(\partial_v f) m_1 + \int_{\mathbb{R}^2} B_\varepsilon(\partial^2_{vv} f)(\partial_x f) m_1,$$

and

$$\mathcal{T}_5 \,=\, \int_{\mathbb{R}^2}\partial_v B_\varepsilon\,(\partial_x f)(\partial_v f)\, m_1 + \frac{1}{2}\int_{\mathbb{R}^2}\Big[\partial_x B_\varepsilon - B_\varepsilon\frac{\partial_x m_1}{m_1}\Big](\partial_v f)^2 m_1,$$

Finally, for the last contribution in $\mathcal{T}_3$, it holds that

$$\int_{\mathbb{R}^2} B_\varepsilon(\partial^2_{vv} f)(\partial_x f)\, m_1^2 \,\le\, k_0\int_{\mathbb{R}^2}(\partial^2_{vv} f)(\partial_x f)\, M^{3/2}\, m_1,$$

getting that there exists $k_0 > 0$ such that

$$\begin{aligned}
&\langle \partial_x Q_\varepsilon f, \partial_v f\rangle_{L^2(m_1)} + \langle \partial_v Q_\varepsilon f, \partial_x f\rangle_{L^2(m_1)} \\
&\qquad \le k_0\int_{\mathbb{R}^2}|\partial_x f|\,|\partial_v f|\, M\, m_1 + k_0\int_{\mathbb{R}^2}|\partial^2_{xv} f|\,|\partial^2_{vv} f|\, m_1 \\
&\qquad\quad -b\,\|\partial_x f\|^2_{L^2(m_1)} + k_0\int_{\mathbb{R}^2}|\partial^2_{vv} f|\,|\partial_x f|\, M^{3/2}\, m_1 \\
&\qquad\qquad +k_0\int_{\mathbb{R}^2}|\partial_v f|^2\, M^2\, m_1 + k_0\int_{\mathbb{R}^2} f^2\, M\, m_1.
\end{aligned} \tag{3.9}$$

Step 4. Conclusion. To get (3.4), we just put together (3.6), (3.7), (3.8) and (3.9) and we use Young's inequality adequately. Indeed, the scalar product $\langle\cdot,\cdot\rangle_{\mathcal{H}^1}$ applied to any $f \in H^1(m_2)$ writes

$$\begin{aligned}
\langle Q_\varepsilon f, f\rangle_{\mathcal{H}^1} = {} & \langle Q_\varepsilon f, f\rangle_{L^2(m_2)} \\
& +\delta^{3/2}\langle \partial_x Q_\varepsilon f, \partial_x f\rangle_{L^2(m_2)} + \delta\,\langle \partial_v Q_\varepsilon f, \partial_v f\rangle_{L^2(m_2)} \\
& +\frac{\delta^{4/3}}{2}\langle \partial_x Q_\varepsilon f, \partial_v f\rangle_{L^2(m_1)} + \frac{\delta^{4/3}}{2}\langle \partial_v Q_\varepsilon f, \partial_x f\rangle_{L^2(m_1)}.
\end{aligned}$$

To give an idea of the method, we only explain how to get rid of a few terms. For example, for the positive contribution of (3.7), it holds

$$\delta^{3/2}k_1\|\partial_x f\|^2_{L^2(\mathbb{R}^2)} + \delta^{3/2}\int |\partial_x f||\partial_v f|m_2 \\ \leq \delta^{3/2}k_1\|\partial_x f\|^2_{L^2(\mathbb{R}^2)} + \delta^{7/4}\|\partial_x f\|^2_{L^2(m_2)} + \delta^{5/4}\,\|\partial_v f\|^2_{L^2(m_2)},$$

and for $\delta > 0$ small enough these terms are annihilated by the quantities

$$-\|\partial_v f\|^2_{L^2(m_2)} - \delta^{3/2}k_2\|\partial_x f\|^2_{L^2(Mm_2)} - \frac{\delta^{4/3}b}{2}\|\partial_x f\|^2_{L^2(m_1)}$$

present in the right hand side of (3.6), (3.7) and (3.9).

In (3.8), the only delicate contribution is

$$\delta\, b\int |\partial_x f||\partial_v f|m_2 \;\leq\; \frac{\delta^{5/3}\,b}{2}\|\partial_x f\|^2_{L^2(m_2)} + \frac{\delta^{1/3}b}{2}\|\partial_v f\|^2_{L^2(m_2)},$$

but the right hand sides of (3.6) and (3.7) include

$$-\|\partial_v f\|^2_{L^2(m_2)} - \delta^{3/2}k_2\|\partial_x f\|^2_{L^2(Mm_2)},$$

and once again for $\delta > 0$ small the sum is nonpositive.

The positive part of (3.9) is controlled using that $\kappa_1 < \kappa_2$. Indeed, in that situation

$$\delta^{4/3}k_0\int |\partial^2_{vv}f||\partial_x f|\,M^{3/2}\,m_1 \leq \delta^{4/3-1/4}k_0\|\partial^2_{vv}f\|^2_{L^2(m_2)} + \delta^{4/3+1/4}k_0\|\partial_x f\|^2_{L^2(m_2)},$$

replacing, if necessary, k_0 by a larger constant. If $\delta > 0$ is small we get rid of these terms thanks to the presence of

$$-\delta^{3/2}k_2\|\partial_x f\|^2_{L^2(Mm_2)} - \delta\,\|\partial^2_{vv}f\|^2_{L^2(m_2)},$$

in (3.7) and (3.8).

All remaining positive contributions can be handled in the same fashion leading to the conclusion that there exist two positive constants K_1, K_2 such that

$$\langle Q_\varepsilon f, f\rangle_{\mathcal{H}^1} \leq K_1\|f\|^2_{L^2(\mathbb{R}^2)} - K_2\|f\|^2_{\mathcal{H}^1},$$

and we get the conclusion by using the equivalence of norms, i.e., that for $0 < \delta < 1$ the norm $\mathcal{H}^1$ is such that

$$c_\delta\|f\|^2_{H^1(m_2)} \leq \|f\|^2_{L^2(m_2)} + \Big(\delta^{3/2} - \frac{\delta^{5/3}}{2}\Big)\|\partial_x f\|^2_{L^2(m_2)} + \frac{\delta}{2}\|\partial_v f\|^2_{L^2(m_2)} \leq \|f\|^2_{\mathcal{H}^1},$$

for some $c_\delta > 0$.

Corollary 3.6 *Estimate* (2.5) *holds.*

Proof. Nash's inequality implies that there exists a constant $C > 0$, such that for any $f \in L^1(\mathbb{R}^2) \cap H^1(\mathbb{R}^2)$, it holds

$$\begin{aligned}\|f\|^2_{L^2(\mathbb{R}^2)} \;&\leq\; \sqrt{C}\|f\|_{L^1(\mathbb{R}^2)}\|D_{x,v}f\|_{L^2(\mathbb{R}^2)} \\ &\leq\; \frac{C}{2\delta'}\|f\|^2_{L^1(\mathbb{R}^2)} + \frac{\delta'}{2}\|D_{x,v}f\|^2_{L^2(\mathbb{R}^2)}. \qquad (3.10)\end{aligned}$$

Coming back to the previous lemma, using the equivalence of the norms $\mathcal{H}^1$ and $H^1(m_2)$, together with the fact that a solution f_t to (1.2) is a probability measure, we get that,

$$\frac{d}{dt}\|f_t\|^2_{\mathcal{H}^1} \;=\; \langle Q_\varepsilon[\mathscr{J}_{f_t}]\, f_t, f_t\rangle_{\mathcal{H}^1} \;\leq\; k_1 - k_2\|f_t\|^2_{\mathcal{H}^1},$$

for some $k_1, k_2 > 0$ constants. Finally, integrating in time, we get

$$\|f_t\|_{\mathcal{H}^1} \leq \max(C_2, \|f_0\|_{\mathcal{H}^1}),$$

for some $C_2 > 0$ depending only on the parameters of the system and the initial condition.

Notice that it is possible to find a sharper estimate for the regularity of the solutions of (1.2). Indeed, it is not hard to see that there exists $k_0 > 0$ such that

$$\begin{aligned}&\langle \partial^2_{vv}Q_\varepsilon f, \partial^2_{vv}f\rangle_{L^2(m_2)} \\ &\quad= -\int |\partial^3_{vvv}f|\, m_2 + \frac{1}{2}\int |\partial^2_{vv}f|^2\, \partial^2_{vv}m_2 \\ &\qquad +2\int (\partial_v A)(\partial^2_{vv}f)(\partial^2_{xv}f)\, m_2 + \frac{1}{2}\int |\partial^2_{vv}f|^2\big[\partial_x A - \kappa_2 x A\big]\, m_2 \\ &\qquad + \int (\partial^3_{vvv}B) f(\partial^2_{vv}f)\, m_2 + 3\int (\partial^2_{vv}B)(\partial_v f)(\partial^2_{vv}f)\, m_2 \\ &\qquad +\frac{1}{2}\int |\partial^2_{vv}f|^2\big[5\,\partial_v B_\varepsilon - \kappa_2 v B_\varepsilon\big]\, m_2 \\ &\quad\leq -\|\partial^3_{vvv}f\|^2_{L^2(m_2)} \\ &\quad +k_0\Big[\|\partial^2_{vv}f\|^2_{L^2(\mathbb{R}^2)} + \|\partial^2_{xv}f\|^2_{L^2(m_2)} + \|f\|^2_{L^2(m_2)} + \|\partial_v f\|^2_{L^2(m_2)}\Big], \qquad (3.11)\end{aligned}$$

implying that the $H^2_v(m)$ regularity of the initial condition is maintained by the equation, i.e.

Corollary 3.7 *Estimate* (2.6) *holds.*

Proof. The proof follows the same idea already introduced in the proof of Corollary 3.6. We consider the norm

$$\|f\|^2_{\mathcal{H}^2_v} := \|f\|^2_{\mathcal{H}^1} + \delta^2\|\partial^2_{vv}f\|^2_{L^2(m_2)},$$

and notice that (3.6), (3.7), (3.8) together with (3.11) imply that

$$\begin{aligned}\frac{d}{dt}\|f_t\|^2_{\mathcal{H}^2_v} &\leq \frac{d}{dt}\|f_t\|^2_{\mathcal{H}^1} \\ &\quad +2\,\delta^2\,k_0\Big[\int|\partial^2_{vv}f_t|^2+\int|\partial^2_{xv}f_t|^2m_2+\int|f_t|^2m_2+\int|\partial_v f_t|^2m_2\Big] \\ &\leq k_1-k_2\|f_t\|^2_{\mathcal{H}^2_v},\end{aligned}$$

for some $k_1, k_2 > 0$ depending on some $\delta > 0$ small and the parameters of the system. Inequality (2.6) follows.

3.2. Entropy estimates and uniqueness of the solution. Now, we prove that solutions to the equation (1.2) remain in the space of functions with finite entropy. To that aim, for any positive function f, we define

$$I_v(f) \;:=\; \int_{\mathbb{R}^2}\frac{|\partial_v f(x,v)|^2}{f(x,v)}\,dxdv,$$

which is understood as a *partial Fisher information.* When the previous quantity is not well defined we use the convention $I_v(f) = +\infty$. Equipped with this definition we can state:

Lemma 3.8 *For any $f_0 \in L^1(M)\cap L^1\log L^1\cap\mathbb{P}(\mathbb{R}^2)$ we denote by f_t the associated solution to the FhN statistical equation* (1.2) *with initial condition f_0. It holds*

$$\sup_{t\in[0,T]}\mathscr{H}(f_t)+\int_0^t I_v(f_s)\,ds\leq C(T), \tag{3.12}$$

where $C(T)$ depend on f_0 and the coefficients of the problem.

Proof. It is well known that for functions with finite moments, the entropy can be bounded from below. Indeed, since

$$\begin{aligned}f\,\log f &\geq -f\,|\log f|\,\mathbf{1}_{0\leq f\leq e^{-M}}-f\,|\log f|\,\mathbf{1}_{e^{-M}\leq f\leq 1}\\ &\geq -M^{1/2}-f\,M,\end{aligned}$$

it holds

$$\mathscr{H}(f_t)\geq-\int_{\mathbb{R}^2}e^{-M}-\int_{\mathbb{R}^2}f_t\,M\geq-4\pi e^{-1/2}-\max(C_0,\|f_0\|_{L^1(M)}).$$

On the other hand, for any solution of (1.2) with initial datum f_0 there exists a positive constant C, depending on the parameters of the system, ε_0 and C_0', such that

$$\begin{aligned}\frac{d}{dt}\mathscr{H}(f_t) &= \int(1+\log(f_t))\,Q_\varepsilon[\mathscr{J}_{f_t}]\,f_t\\ &= -I_v(f_t)+\int\big(\partial_x A+\partial_v B_\varepsilon(\mathscr{J}_{f_t})\big)\,f_t\\ &\leq -I_v(f_t)+C\|f_t\|_{L^1(M)}.\end{aligned}$$

Fix $T > 0$ and take any $t < T$, thanks to estimate (2.3), we get that

$$\mathscr{H}(f_t) \leq -\int_0^t I_v(f_s)\, ds + \mathscr{H}(f_0) + C\,T\, \max(C_0, \|f_0\|_{L^1(M)}),$$

and also that $I_v(f_t) \in L^1([0,T])$, by simply using that $\mathscr{H}$ is bounded by below. Finally, taking the supremum on the last relationship, we get

$$\sup_{t\in[0,T]} \mathscr{H}(f_t) \;\leq\; \mathscr{H}(f_0) + C\,T \max(C_0, \|f_0\|_{L^1(M)}).$$

Corollary 3.9 *For any two initial data $f_0, g_0 \in L^1(M) \cap L^1 \log L^1 \cap \mathbb{P}(\mathbb{R}^2)$ the associated solutions f_t and g_t to the FhN statistical equation* (1.2)*, satisfy*

$$\sup_{[0,T]} \|f_t - g_t\|_{L^1(M^{1/2})} \leq C(T)\, \|f_0 - g_0\|_{L^1(M^{1/2})},$$

for some positive $C(T)$. In particular, equation (1.2) *with initial datum in $L^1(M) \cap L^1 \log L^1 \cap \mathbb{P}(\mathbb{R}^2)$ has at most one solution.*

Proof. We write

$$\partial_t(f_t - g_t) = Q_\varepsilon[\mathscr{J}(f_t)]\,(f_t - g_t) + \varepsilon\, \mathscr{J}(f_t - g_t)\, \partial_v g_t$$

from which we deduce

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^2} |f_t - g_t| M^{1/2} &\leq \int_{\mathbb{R}^2} |f_t - g_t|\, Q_\varepsilon^*[\mathscr{J}(f_t)] M^{1/2} + \varepsilon\, |\mathscr{J}(f_t - g_t)| \int_{\mathbb{R}^2} |\partial_v g_t| M^{1/2} \\ &\leq C \int_{\mathbb{R}^2} |f_t - g_t| + \varepsilon\, |I(g_t)|^{1/2} \|g_t\|_{L^1(M)}^{1/2} \int_{\mathbb{R}^2} |f_t - g_t|\, M^{1/2},\end{aligned}$$

where the generic constant C is an upper bound of

$$\begin{aligned}Q_\varepsilon^*[\mathscr{J}(f_t)] M^{1/2} &= \partial_{vv}^2 M^{1/2} - A\partial_x M^{1/2} - B_\varepsilon \partial_v M^{1/2} \\ &= \frac{1}{2M^{1/2}} - \frac{v^2}{4M^{3/2}} - \frac{Ax + B_\varepsilon v}{2M^{1/2}}.\end{aligned}$$

The rest of the proof is a direct application of the time integrability of $I_v(g_t)$, inequality (2.3) and Gronwall lemma.

We finish this section by giving some insights of the proofs of the existence of solutions and stationary solutions to equation (1.2) which are, however, classical.

Proof (Theorem 2.2).
Consider an exponential weight m and $\mathscr{J} \in L^\infty(\mathbb{R}_+)$ such that

$$\sup_{t\geq 0} |\mathscr{J}| \leq C_0',$$

where C_0' is given by (3.1). We first notice that the equation (1.2) is degenerate and has non bounded drift terms A and B_ε, therefore we cannot use directly the classical results on existence. However, for any $R > 0$, we define a regular truncation function

$$\chi_R(x,v) = \chi(x/R, v/R), \quad \chi \in \mathcal{D}(\mathbb{R}^2), \quad \mathbf{1}_{B(0,1)} \leq \chi \leq \mathbf{1}_{B(0,2)}. \tag{3.13}$$

and the following family of bilinear forms

$$\begin{aligned} a_\sigma(t;f,g) := \langle \partial_v f, \partial_v g\rangle_{L^2(m)} + \langle \partial_v f, g\,\chi_R\,\kappa v\rangle_{L^2(m)} \\ +\sigma\,\langle \partial_x f, \partial_x g\rangle_{L^2(m)} + \sigma\,\langle \partial_x f, g\,\chi_R \kappa x\rangle_{L^2(m)} \\ + \textstyle\int_{\mathbb{R}^2} f\,\chi_R \partial_x(gAm) + \int_{\mathbb{R}^2} f\,\chi_R \partial_v(gB_\varepsilon(\mathscr{J}_t)m) \end{aligned}$$

indexed by $\sigma > 0$, which for any $f, g \in H^1(m)$ are well defined a.e. $t \geq 0$.

Moreover, for any $\sigma > 0$, a_σ is continuous

$$\exists\, C_R > 0 \quad \text{s.t.} \quad |\,a_\sigma(t;f,g)| \;\leq\; C_R \|f\|_{H^1(m)} \|g\|_{H^1(m)},$$

and coercive

$$a_\sigma(t;f,f) \;\geq\; \frac{1}{2}\|\partial_v f\|^2_{L^2(m)} + \frac{\sigma}{2}\|\partial_x f\|^2_{L^2(m)} - k_1 \|f\|^2_{L^2(m)},$$

with $k_1 > 0$ not depending on t, nor on R and nor on σ. The J. L. Lions theorem [8, Theorem X.9] implies that for any $f_0 \in L^2(m)$ there exists a unique

$$f \in L^2((0,\infty); H^1(m)) \cap C([0,\infty); L^2(m)); \qquad \frac{d}{dt} f \in L^2((0,\infty); H^1(m)')$$

such that $f(0) = f_0$ and

$$\langle \frac{d}{dt} f, g\rangle_{L^2(m)} + a_\sigma(f(t), g) \;=\; 0, \qquad \text{a.e. } t \geq 0,\, \forall\, g \in H^1(m).$$

We recall that $f_- := \min(f, 0)$ belongs to $H^1(m)$, therefore we can use it as a test function to find that

$$f_0 \geq 0 \qquad \Rightarrow \qquad f(t) \geq 0 \quad \text{a.e.} \quad t \geq 0,$$

and for any $T > 0$ fixed, using f as a test function, we recover the inequality

$$\|f_t\|^2_{L^2(m)} + \int_0^T \|\partial_v f_s\|^2_{L^2(m)}\, ds \;\leq\; e^{k_1 T} \|f_0\|^2_{L^2(m)},$$

therefore, we are allowed to take the limits $\sigma \to 0$ and $R \to \infty$, to find that for any $\varphi \in C^1([0,T]; C^2_c(\mathbb{R}^2))$

$$\int_{\mathbb{R}^2} \varphi_t f_t = \int_{\mathbb{R}^2} \varphi_0 f_0 + \int_0^t \int_{\mathbb{R}^2} \big[\partial_t \varphi_s + \partial^2_{vv}\varphi_s - A\,\partial_x \varphi_s - B_\varepsilon(\mathscr{J}_s)\partial_v \varphi_s\big] f_s\, ds,$$

holds, i.e., f_t is a solution for the equation (1.2) when the nonlinearity $\mathscr{J}$ is replaced by a given term. Moreover, taking a well chosen sequence $\varphi^n \to M$, we deduce that

$$\sup_{t\in[0,T]} \|f_t\|_{L^1(M)} \;\leq\; \max\big(C_0, \|f_0\|_{L^1(M)}\big),$$

for C_0 the constant introduced in Lemma 3.1. Finally, by using renormalisation concepts, we also recover the entropy inequality

$$\sup_{t\in[0,T]} \mathscr{H}(f_t) + \int_0^t I_v(f_s)\, ds \;\leq\; \mathscr{H}(f_0) + K_0 T \max(C_0, \|f_0\|_{L^1(M)}).$$

Take $f_0 \in L^1(M) \cap L^1 \log L^1 \cap \mathbb{P}(\mathbb{R}^2)$, and a sequence $\{f_{n,0}\} \subset L^2(m)$ such that $f_{n,0} \to f_0$ in $L^1(M)$. Assume also that there is a positive constant $C > 0$ such that $\mathscr{H}(f_{n,0}) \leq C$, for any $n \in \mathbb{N}$. From the previous analysis we get a family $\{f_n\} \in C((0,T); L^1(M))$ of functions related to the initial conditions $\{f_{n,0}\}$. Using the Dunford-Pettis criterium we can pass to the limit in $L^1(M)$ finding a solution to the linear problem

$$\partial_t f \;=\; \partial_x(Af) + \partial_v(B_\varepsilon(\mathscr{J}_t)f) + \partial^2_{vv} f. \tag{3.14}$$

that depends continuously to the initial datum (in the sense defined in Theorem 2.2). Moreover, from Corollary 3.9 we get that this solution is necessarily unique.

Finally, we use again the ideas of Corollary 3.9 to find a solution to the NL equation (1.2). Indeed, it suffices to notice that the mapping

$$\begin{cases} L^\infty([0,T]) \longrightarrow C([0,T]; L^1(M^{1/2})) \\ \qquad\qquad \mathscr{J} \longmapsto f, \end{cases}$$

with f solution of (3.14) for $\mathscr{J}$ given, is Lipschitz and contracting when $T > 0$ is small enough.

Existence of stationary solutions will be shown as a result of an abstract version of the Brouwer fixed point theorem (a variant of [15, Theorem 1.2] and [18]):

Theorem 3.10. *Consider $\mathcal{Z}$ a convex and compact subset of a Banach space X and $S(t)$ a continuous semigroup on $\mathcal{Z}$. Let us assume that $\mathcal{Z}$ is invariant under the action of $S(t)$ (that is $S(t)z \in \mathcal{Z}$ for any $z \in \mathcal{Z}$ and $t \geq 0$). Then, there exists $z_0 \in \mathcal{Z}$ which is stationary under the action of $S(t)$, i.e, $S(t)z_0 = z_0$ for any $t \geq 0$.*

We present the argument briefly in this section. Our aim is to find a fixed point for the nonlinear semigroup $S_{Q_\varepsilon}(t)$ related to equation (1.2). At this point we do not have any hint on the number of functions solving

$$Q_\varepsilon[\mathscr{J}_F]\, F = 0,$$

and the nonlinearity could lead to the presence of more than one. However, in the disconnected regime $\varepsilon = 0$ the nonlinearity disappears, and the multiplicity problem is no longer present.

Proof (Existence of stationary solutions).

Let us fix m an exponential weight and define for any $t \geq 0$

$$S(t) : X \to X \quad \text{with} \quad X \;=\; H^2_v(m) \cap L^1 \log L^1 \cap \mathbb{P}(\mathbb{R}^2),$$

such that $S(t)f_0$ is the solution to (1.2) given by Theorem 2.2 associated to the initial condition f_0. Estimates (2.6) and (3.12) imply that $S(t)$ is well defined. Moreover, the continuity of S in the Banach space $L^1(\mathbb{R}^2)$ is direct from the definition of weak solutions, in particular,

$$S(t)f_0 \in C([0,\infty); L^1(\mathbb{R}^2)),$$

with the topology of compact subsets in time.

Finally, defining

$$\mathcal{Z} := \mathcal{Z}(\varepsilon) = \{f \in X \text{ such that (2.3) and (2.6) hold}\} \subset L^1(\mathbb{R}^2),$$

which is invariant under S_t for any $t \geq 0$ and convex. Moreover, the compactness of the inclusion $\mathcal{Z} \subset H^1(m) \hookrightarrow L^1(\mathbb{R}^2)$ allows us to apply Theorem 3.10 and find the existence of a fixed point for $S(t)$ and by consequence a stationary solution to (1.2).

It is worth emphasising that the above proof show yet that the map $\varepsilon \mapsto G_\varepsilon$ is locally bounded in $[0, \infty)$, i.e., if $\varepsilon_0 > 0$ is fixed, then

$$G_\varepsilon \in \mathcal{Z}(\varepsilon_0) \quad \text{for any} \quad \varepsilon \in (0, \varepsilon_0).$$

4. The linearized equation

To illustrate the main ideas of the *splitting method* we use in the following, let us assume that an operator Λ on a Banach space X can be written as

$$\Lambda = \mathcal{A} + \mathcal{B},$$

where $\mathcal{A}$ is *much more regular than* $\mathcal{B}$, and $\mathcal{B}$ has some *dissipative property.* If $\mathcal{B}$ has a good localisation of its spectrum, under some reasonable hypotheses on $\mathcal{A}$, we expect $\Sigma(\Lambda)$ to be close to $\Sigma(\mathcal{B})$.

This is nothing but the Weyl's abstract theorem (and/or the generalisation of the Krein-Rutman theorem) from Mischler and Scher [28], that we recall here:

Theorem 4.1. *We consider a semigroup generator Λ on a "Banach lattice of functions" X, and we assume that*

1. *there exists some $\alpha^* \in \mathbb{R}$ and two operators $\mathcal{A}, \mathcal{B} \in \mathscr{C}(X)$, such that $\Lambda = \mathcal{A} + \mathcal{B}$ and*
   (a) for any $\alpha > \alpha^*, \ell \geq 0$, there exists a constant $C_{\alpha,\ell} > 0$ such that
   $$\forall\, t \geq 0, \qquad \|S_{\mathcal{B}} * (\mathcal{A} S_{\mathcal{B}})^{(*\ell)}(t)\|_{\mathscr{B}(X)} \leq C_{\alpha,\ell}\, e^{\alpha t}.$$
   (b) $\mathcal{A}$ is bounded, and there exists an integer $n \geq 1$ such that for any $\alpha > \alpha^*$, there exists a constant $C_{\alpha,n} > 0$ such that
   $$\forall\, t \geq 0, \qquad \|(\mathcal{A} S_{\mathcal{B}})^{(*n)}(t)\|_{\mathscr{B}(X,Y)} \leq C_{\alpha,n} e^{\alpha t},$$
   with $Y \subset D(\Lambda)$ and $Y \subset X$ with compact embedding;
2. *for Λ^* the dual operator of Λ defined in X', there exists $\beta > \alpha^*$ and $\psi \in D(\Lambda^*) \cap X'_+ \setminus \{0\}$ such that*
   $$\Lambda^* \psi \geq \beta \psi;$$
3. *S_Λ satisfies Kato's inequalities, i.e,*
   $$\forall\, f \in D(\Lambda), \quad \Lambda\theta(f) \geq \theta'(f)\, \Lambda f,$$
   holds for $\theta(s) = |s|$ or $\theta(s) = s_+$.

4. $-\Lambda$ satisfies a strong maximum principle: for any given f and $\gamma \in \mathbb{R}$, there holds,

$$|f| \in D(\Lambda) \setminus \{0\} \text{ and } (-\Lambda+\gamma)|f| \geq 0 \quad \text{imply} \quad f>0 \text{ or } f<0.$$

Defining

$$\lambda := s(\Lambda) = \sup\big\{Re(\xi) \ : \ \xi \in \Sigma(\Lambda)\big\},$$

there exists $0 < f_\infty \in D(\Lambda)$ and $0 < \phi \in D(\Lambda^*)$ such that

$$\Lambda f_\infty = \lambda\, f_\infty, \qquad \Lambda^* \phi = \lambda\, \phi.$$

Moreover, there is some $\bar\alpha \in (\alpha^*, \lambda)$ and $C>0$ such that for any $f_0 \in X$

$$\|S_\Lambda(t) f_0 - e^{\lambda t}\langle f_0, \phi\rangle f_\infty\|_X \leq C e^{\bar\alpha t}\|f_0 - \langle f_0,\phi\rangle f_\infty\|_X.$$

From Theorem 2.3 we know that for any value of ε there exists at least one G_ε non zero stationary solution of the FhN kinetic equation (1.2). The linearized equation, on the variation $h := f - G_\varepsilon$, induces the linearized operator

$$\mathscr{L}_\varepsilon h = Q_\varepsilon(\mathscr{J}(G_\varepsilon))h + \varepsilon\, \mathscr{J}(h)\partial_v G_\varepsilon.$$

Moreover, let us recall that in Section 3 we proved that

$$\langle Q_\varepsilon[\mathscr{J}(G_\varepsilon)]\, f, f\rangle_{L^2(m)} \ \leq \ K_1\|f\|_{L^2(\mathbb{R}^2)} - K_2\|f\|_{L^2(m)},$$

if we could make $K_1 = 0$, then the operator Q_ε together with $\mathscr{L}_\varepsilon$ would be dissipative. Since it is not the case, we fix a constant $N>0$ and define

$$\mathcal{B}_\varepsilon := \mathscr{L}_\varepsilon - \mathcal{A}, \quad \text{where} \quad \mathcal{A} = N\,\chi_R(x,v); \tag{4.1}$$

with χ_R given by (3.13). We remark that $\mathcal{A} \in \mathscr{B}(H^2_v(m))$, and that $\mathcal{A}f$ vanishes outside a ball of radius $2R$ for any $f \in H^2_v(m)$.

4.1. Properties of $\mathcal{A}$ and $\mathcal{B}_\varepsilon$. We now present two lemmas dealing with the hypodissipativity and regularisation properties of the sppliting $\mathcal{A}$ and $\mathcal{B}_\varepsilon$. To that aim, we use some ideas developed in [27,19] and [26].

Lemma 4.2 *For any exponential weight m, there exist some constants $N, R > 0$ such that $(\mathcal{B}_\varepsilon + 1)$ is hypodissipative in $H^2_v(m)$.*

Proof. From the characterisation of hypodissipativity given in Section 2, it suffices to show that there exists a constant $C>0$ such that

$$\|S_{\mathcal{B}_\varepsilon}(t)\|_{\mathscr{B}(H^2_v(m))} \leq C\, e^{-t}, \quad t \geq 0,$$

or simply, to show that for any $h \in H^2_v(m)$, it holds

$$\langle \mathcal{B}_\varepsilon h, h\rangle_{\bar H^2_v(m)} \leq -\|h\|^2_{\bar H^2_v(m)}, \tag{4.2}$$

for some norm $\|\cdot\|_{\bar H^2_v(m)}$ equivalent to the usual norm $\|\cdot\|_{H^2_v(m)}$. Recall that the operator $\mathcal{B}_\varepsilon$ writes

$$\mathcal{B}_\varepsilon \ = \ \mathscr{L}_\varepsilon - \mathcal{A} \ = \ (Q_\varepsilon[\mathscr{J}_{G_\varepsilon}] - N\chi_R)h + \varepsilon\, \mathscr{J}(h)\, \partial_v G_\varepsilon,$$

and since $\mathscr{J}_{G_\varepsilon} \in \mathbb{R}$ is a real constant, we can use all a priori estimates on Q_ε directly. As usual, when no confusion is possible, we drop the dependence on $\mathscr{J}_\varepsilon$. Three steps complete the proof:

Step 1. Dissipativity in $L^2(m)$. Recall that for any $h \in L^2(m)$ we have

$$|\mathscr{J}(h)| \leq C\|h\|_{L^2(m)},$$

for some constant $C > 0$. It follows that

$$\mathscr{J}(h)\int_{\mathbb{R}^2}(\partial_v G_\varepsilon)\, h\, m \leq |\mathscr{J}(h)|\|\partial_v G_\varepsilon\|_{L^2(m)}\|h\|_{L^2(m)} \leq C\,\|\partial_v G_\varepsilon\|_{L^2(m)}\int_{\mathbb{R}^2} h^2 m.$$

Thus, coming back to (3.5), we find that for N and R large enough one can assume $k_1 = -1$, getting

$$\langle \mathcal{B}_\varepsilon h, h\rangle_{L^2(m)} \leq -\|h\|^2_{L^2(m)} - k_2\|h\|^2_{L^2(Mm)} - \|\partial_v h\|^2_{L^2(m)}, \tag{4.3}$$

as a consequence, $(\mathcal{B}_\varepsilon + 1)$ is dissipative in $L^2(m)$.

Step 2. Bounds on the derivatives of $\mathcal{B}_\varepsilon$. For the x-derivative we see that there exists some constant C' depending on R and on N, such that

$$-N\langle \partial_x(\chi_R h), \partial_x h\rangle_{L^2(m)} \leq C'\|h\|^2_{L^2(m)} - N\|(\partial_x h)\sqrt{\chi_R}\|^2_{L^2(m)}.$$

On the other hand, thanks to Young's inequality, we get

$$\begin{aligned}\mathscr{J}(h)\int_{\mathbb{R}^2}(\partial^2_{xv}G_\varepsilon)(\partial_x h)\, m &= -\mathscr{J}(h)\int_{\mathbb{R}^2}\partial_x G_\varepsilon\left[\partial^2_{vx}h + \kappa v\,\partial_x h\right] m \\ &\leq \mathscr{J}(h)^2\,\|\partial_x G_\varepsilon\|^2_{L^2(m)} + \frac{1}{2}\|\partial^2_{xv}h\|^2_{L^2(m)} + \|\kappa|v|\,\partial_x h\|^2_{L^2(m)}.\end{aligned}$$

These two inequalities, together with (3.7), imply that for N and R large enough

$$\langle \partial_x(\mathcal{B}_\varepsilon h), \partial_x h\rangle_{L^2(m)} \leq -\|\partial_x h\|^2_{L^2(m)} - \frac{1}{2}\|\partial^2_{xv}h\|^2_{L^2(m)} + C'\|h\|^2_{L^2(m)} + \int_{\mathbb{R}^2}|\partial_x h|\,|\partial_v h| m.$$

Proceeding similarly with the v-derivative we get

$$\begin{aligned}\mathscr{J}(h)\int_{\mathbb{R}^2}(\partial^2_{vv}G_\varepsilon)(\partial_v h)\, m^2 &= |\mathscr{J}(h)|\|\partial^2_{vv}G_\varepsilon\|_{L^2(m)}\|\partial_v h\|_{L^2(m)} \\ &\leq \frac{1}{2}\,\|\partial^2_{vv}G_\varepsilon\|_{L^2(m)}(C^2\|h\|^2_{L^2(m)} + \|\partial_v h\|^2_{L^2(m)}),\end{aligned}$$

then, coming back to (3.8), we find $N, R > 0$ such that

$$\langle \partial_v(\mathcal{B}_\varepsilon h), \partial_v h\rangle_{L^2(m)} \leq -\|\partial_v h\|^2_{L^2(m)} - \|\partial^2_{vv}h\|^2_{L^2(m)} + C'\,\|h\|^2_{L^2(m)} + \int_{\mathbb{R}^2}|\partial_x h|\,|\partial_v h|\, m.$$

Finally, for the second v-derivative we find C' such that

$$-N\langle \partial^2_{vv}(\chi_R h), \partial^2_{vv}h\rangle_{L^2(m)} \leq -N\int_{\mathbb{R}^2}\chi_R(\partial^2_{vv}h)^2 m + C'\int_{\mathbb{R}^2}(\partial_v h)^2 m + C'\int_{\mathbb{R}^2} h\,|\partial^2_{vv}h|\, m,$$

and for any $\epsilon > 0$

$$\mathscr{J}(h)\int_{\mathbb{R}^2}(\partial^3_{vvv}G_\varepsilon)(\partial^2_{vv}h)\,m \le \frac{\mathscr{J}(h)^2}{2\epsilon} + \epsilon\left(\|\partial^2_{vv}G_\varepsilon\|^2_{L^2(m)}\|\partial^3_{vvv}h\|^2_{L^2(m)} + \right.$$
$$\left. + \|\partial^2_{vv}G_\varepsilon\|^2_{L^2(m)}\|\kappa|v|\,(\partial^2_{vv}h)\|^2_{L^2(m)}\right).$$

If $\epsilon > 0$ is small and N, R large enough, we obtain as an application of (3.11), that there is a constant $C' > 0$ such that

$$\langle \partial^2_{vv}(\mathcal{B}_\varepsilon h), \partial^2_{vv}h\rangle_{L^2(m)} \le -\|\partial^2_{vv}h\|^2_{L^2(m)}$$
$$+ C'\left[\|h\|^2_{L^2(m)} + \|\partial_v h\|^2_{L^2(m)} + \|\partial^2_{xv}h\|^2_{L^2(m)} + \|\partial^2_{vv}h\|^2_{L^2(m)}\right].$$

Step 3. Equivalent norm and conclusion. Let $\delta > 0$ and $h_1, h_2 \in H^2_v(m)$, we can define the bilinear product

$$\langle h_1, h_2\rangle_{\bar{H}^2_v(m)} := \langle h_1, h_2\rangle_{L^2(m)}$$
$$+ \delta\langle \partial_x h_1, \partial_x h_2\rangle_{L^2(m)} + \delta\langle \partial_v h_1, \partial_v h_2\rangle_{L^2(m)} + \delta^2\langle \partial^2_{vv}h_1, \partial^2_{vv}h_2\rangle_{L^2(m)}.$$

and the relative norm

$$\|h\|^2_{\bar{H}^2_v(m)} := \|h\|^2_{L^2(m)} + \delta\,\|D_{x,v}h\|^2_{L^2(m)} + \delta^2\,\|\partial^2_{vv}h\|^2_{L^2(m)}.$$

Choosing $\delta > 0$ small enough we conclude that for any $\alpha \in (0, 1]$ one find δ_α such that

$$\langle \mathcal{B}_\varepsilon h, h\rangle_{\bar{H}^2_v(m)} \le -\alpha\,\|h\|^2_{\bar{H}^2_v(m)},$$

and since the norm related to $\bar{H}^2_v(m)$ is equivalent to the usual norm in $H^2_v(m)$, we conclude that $(\mathcal{B}_\varepsilon + 1)$ is hypodissipative in $H^2_v(m)$.

Lemma 4.3 *There are positive constants N, R large enough and some $C_{\mathcal{B}_\varepsilon} > 0$, such that the semigroup $S_{\mathcal{B}_\varepsilon}$ satisfies*

$$\|S_{\mathcal{B}_\varepsilon}(t)h\|_{H^2_v(m_1)} \le C_{\mathcal{B}_\varepsilon}t^{-9/2}\|h\|_{L^2(m_2)}, \quad \forall\, t \in (0, 1].$$

As a consequence, for any $\alpha > -1$, and any exponential weight m, there exists $n \ge 1$ and $C_{n,\varepsilon}$ such that of any $t > 0$ it holds

$$\|(\mathcal{A}S_{\mathcal{B}_\varepsilon})^{(*n)}(t)h\|_{H^2_v(m)} \le C_{n,\varepsilon}\,e^{\alpha t}\|h\|_{L^2(m)}. \tag{4.4}$$

Proof. We split the proof in three steps, in the first one we refine the previous estimates on the norm of the semigroup associated to the operator $\mathcal{B}_\varepsilon$. In the following steps we use Hormander-Hérau technique (see e.g. [20]) to get the first inequality, and finally we prove (4.4).

Step 1. Sharper estimates on $\mathcal{B}_\varepsilon$. In the whole proof we denote for $C > 0$ a generic constant. From the proof of the previous Lemma, we know that there

are N, R large enough such that for any $h \in D(\mathcal{B}_\varepsilon)$ and for any $\delta, t \in (0,1)$, it holds

$$\begin{aligned}
\langle \mathcal{B}_\varepsilon h, h\rangle_{L^2(m_2)} &\leq -C\|h\|^2_{L^2(m_2)} - \|\partial_v h\|^2_{L^2(m_2)}\\
\langle \partial_x \mathcal{B}_\varepsilon h, \partial_x h\rangle_{L^2(m_1)} &\leq -\frac{1}{2}\|\partial_x h\|^2_{L^2(m_1)} - \frac{1}{2}\|\partial^2_{xv} h\|^2_{L^2(m_1)} + C\|h\|^2_{L^2(m_1)} + \frac{1}{2\delta t}\|\partial_v h\|^2_{L^2(m_1)}\\
\langle \partial_v \mathcal{B}_\varepsilon h, \partial_v h\rangle_{L^2(m_1)} &\leq -\|\partial^2_{vv} h\|^2_{L^2(m_1)} + C\|h\|^2_{L^2(m_1)} + \frac{1}{2\delta t}\|\partial_v h\|^2_{L^2(m_1)} + \frac{\delta t}{2}\|\partial_x h\|^2_{L^2(m_1)}\\
\langle \partial^2_{vv} \mathcal{B}_\varepsilon h, \partial^2_{vv} h\rangle_{L^2(m_1)} &\leq C\|h\|^2_{L^2(m_1)} + C\|\partial_v h\|^2_{L^2(m_1)} + \frac{1}{2t\delta}\|\partial^2_{xv} h\|^2_{L^2(m_1)}.
\end{aligned}$$

We also notice that

$$\begin{aligned}
\langle \partial_x(Q_\varepsilon[\mathscr{J}_{G_\varepsilon}] - \mathcal{A})h, \partial_v h\rangle_{L^2(m_1)} + \langle \partial_v(Q_\varepsilon[\mathscr{J}_{G_\varepsilon}] - \mathcal{A})h, \partial_x h\rangle_{L^2(m_1)} \leq\\
-\frac{b}{2}\|\partial_x h\|^2_{L^2(m_1)} + C\|h\|^2_{L^2(m_2)} + \frac{C}{t\delta}\|\partial_v h\|^2_{L^2(m_2)}\\
+\frac{1}{t\delta^{1/3}}\|\partial^2_{vv} h\|^2_{L^2(m_1)} + Ct\delta^{1/3}\|\partial^2_{xv} h\|^2_{L^2(m_1)},
\end{aligned}$$

and

$$\begin{aligned}
\langle \mathscr{J}(h)\partial^2_{xv} G_\varepsilon, \partial_v h\rangle_{L^2(m_1)} + \langle \mathscr{J}(h)\partial^2_{vv} G_\varepsilon, \partial_x h\rangle_{L^2(m_1)} \leq \frac{\|\partial_x G_\varepsilon\|_{L^2(m_1)}}{2}\big[2\mathscr{J}(h)^2\\
+\|\partial^2_{vv} h\|^2_{L^2(m_1)} + \|\partial_v h\|^2_{L^2(m_2)}\big] + \frac{\|\partial^2_{vv} G_\varepsilon\|_{L^2(m_1)}}{2}\Big[\frac{\mathscr{J}(h)^2}{t\delta} + t\delta\|\partial_x h\|^2_{L^2(m_1)}\Big],
\end{aligned}$$

yielding to

$$\begin{aligned}
\langle \partial_x \mathcal{B}_\varepsilon h, \partial_v h\rangle_{L^2(m)} + \langle \partial_v \mathcal{B}_\varepsilon h, \partial_x h\rangle_{L^2(m)} \leq -\frac{b}{4}\|\partial_x h\|^2_{L^2(m_1)} + \frac{C}{t\delta}\|h\|^2_{L^2(m_2)} + \frac{C}{t\delta}\|\partial_v h\|^2_{L^2(m_2)}\\
+\frac{1}{t\delta^{1/3}}\|\partial^2_{vv} h\|^2_{L^2(m_1)} + Ct\delta^{1/3}\|\partial^2_{xv} h\|^2_{L^2(m_1)}.
\end{aligned}$$

Step 2. Hormander-Hérau technique. For a given $h \in H^2_v(m_1) \cap L^2(m_2)$ we denote $h_t := S_{\mathcal{B}_\varepsilon}(t)h$, and define $\mathcal{F}$ by

$$\begin{aligned}
\mathcal{F}(h,t) := \|h\|^2_{L^2(m_2)} + c_1 t^3\|\partial_x h\|^2_{L^2(m_1)}\\
+ c_2 t\|\partial_v h\|^2_{L^2(m_1)} + c_3 t^2\langle \partial_x h, \partial_v h\rangle_{L^2(m_1)} + c_4 t^4\|\partial^2_{vv} h\|^2_{L^2(m_1)},
\end{aligned}$$

which is decreasing when the parameters are well chosen. Indeed, thanks to the first step, we have

$$\frac{d}{dt}\mathcal{F}(t, h_t) \leq \sum_{i=1}^{5} \mathcal{T}_i,$$

with

$$\mathcal{T}_1 = K\int_{\mathbb{R}^2}\big[-2m_2 + 2(c_1t^3 + c_2t + c_4t^4)m_1 + \frac{c_3t}{\delta}m_2\big]h_t^2,$$
$$\mathcal{T}_2 = \int_{R^2}\big[(3c_1 + c_2\delta - \frac{b}{4}c_3 + c_3\delta)t^2 - c_1t^3\big](\partial_x h_t)^2 m_1,$$
$$\mathcal{T}_3 = \int_{\mathbb{R}^2}\big[-2m_2 + c_2m_1 + \frac{c_2}{\delta}m_1 + \frac{c_3}{\delta}m_1 + \frac{c_1t^2}{\delta}m_1 + c_4t^4Cm_1 + \frac{c_3tC}{\delta}m_2\big](\partial_v h_t)^2,$$
$$\mathcal{T}_4 = \int_{\mathbb{R}^2} t^3\big[-c_1 + \frac{c_4}{2\delta} + c_3C\delta^{1/3}\big](\partial^2_{xv}h_t)^2 m_1,$$
$$\mathcal{T}_5 = \int_{\mathbb{R}^2}\big[-2c_2t + \frac{c_3t}{\delta^{1/3}} + 4c_4t^3\big](\partial^2_{vv}h_t)^2 m_1.$$

Choosing

$$c_1 = \delta^{3/2}, \quad c_2 = \delta \quad c_3 = \delta^{4/3} \quad \text{and} \quad c_4 = \delta^4,$$

we get that for $\delta \in (0,1]$ small enough and for any $t \in (0,1]$, it holds

$$\frac{d}{dt}\mathcal{F}(t,h_t) \le 0.$$

Since $0 < c_4 \le c_1 \le c_3 \le c_2$ and the choice of c_1, c_2 and c_3, we finally get that

$$c_4\, t^{9/2}\Big(\|\partial_{x,v}h_t\|^2_{L^2(m_1)} + \|\partial^2_{vv}h_t\|^2_{L^2(m_1)}\Big) \le \mathcal{F}(t,h_t) \le F(0,h_0) = \|h_0\|^2_{L^2(m_2)}.$$

Step 3. Proof of inequality (4.4). From the definition of $\mathcal{A}$ we notice that

$$\|\mathcal{A}\, S_{\mathcal{B}_\varepsilon}(t)h\|_{H^2_v(m)} \le C' t^{-9/2}e^{-t}\|h\|_{L^2(m)}, \quad \forall\, t \in (0,1],$$

for some constant C'. It is important to remark that since $\mathcal{A}$ lies in a compact, we do not need anymore two different weights m_1 and m_2. Therefore, we apply Proposition 2.5 with $X = L^2(m)$, $Y = H^2_v(m)$, $\Theta = 9/2$ and $\alpha^* = -1$ to get (4.4).

4.2. Spectral analysis on the linear operator in the disconnected case. The FhN kinetic equation in the case $\varepsilon = 0$ is linear and writes

$$\begin{aligned}\partial_t g &= \partial_x(Ag) + \partial_v(B_0\, g) + \partial^2_{vv}g\\ B_0 &= v\,(v-\lambda)\,(v-1) + x,\end{aligned}$$

Theorem 2.3 states that there exists at least one function $G_0 \in \mathbb{P}(\mathbb{R}^2) \cap H^2_v(m)$ which is a solution to the associated (linear) stationary problem

$$\mathscr{L}_0 G_0 = \partial_x(AG_0) + \partial_v(B_0G_0) + \partial^2_{vv}G_0 = 0.$$

Since the operator now enjoys a *positive structure* (it generates a positive semi-group $S_{\mathscr{L}_0}$), we can perform a more accurate analysis. Indeed, we can apply the the abstract Krein-Rutman theorem 4.1 previously stated.

Proof (of the stability around $\varepsilon = 0$ in Theorem 2.3). Let us assume for a first moment that hypotheses of the abstract Theorem 4.1 hold for $\mathscr{L}_0$ with $\alpha^* = -1$. We easily remark that

$$\lambda = 0, \quad f_\infty = G_0 \quad \phi = 1,$$

therefore, there exists $\bar{\alpha} \in (-1, 0)$ such that

$$\Sigma(\mathscr{L}_0) \cap \Delta_{\bar{\alpha}} = \{0\},$$

and

$$\forall\, f_0 \in L^2(m),\ \forall\, t \geq 0 \qquad \|S_{\mathscr{L}_0}(t) f_0 - \langle f_0 \rangle G_0\|_{L^2(m)} \leq C\, e^{\bar{\alpha} t} \|f_0 - \langle f_0 \rangle G_0\|_{L^2(m)}.$$

For $\varepsilon > 0$, we recall that any G_ε such that

$$Q_\varepsilon[\mathscr{J}_{G_\varepsilon}]\, G_\varepsilon = 0,$$

then

$$\frac{\partial}{\partial t}(G_\varepsilon - G_0) + \mathscr{L}_0(G_\varepsilon - G_0) = h, \qquad h = \varepsilon\, \partial_v((v - \mathscr{J}(G_\varepsilon))G_\varepsilon),$$

and thanks to Duhamel's formula, we get that

$$\|G_\varepsilon - G_0\|_{L^2(m)} \leq \|S_{\mathscr{L}_0}(t)(G_\varepsilon - G_0)\|_{L^2(m)} + \int_0^t \|S_{\mathscr{L}_0}(t-s)h\|_{L^2(m)}\, ds.$$

Since $G_\varepsilon - G_0$ and h have zero mean, it follows that

$$\|G_\varepsilon - G_0\|_{L^2(m)} \leq C\|G_\varepsilon - G_0\|_{L^2(m)} e^{\bar{\alpha} t} + \varepsilon\, \frac{C}{|\bar{\alpha}|} \|G_\varepsilon\|_{H^1_v(Mm)} (1 - e^{\bar{\alpha} t}),$$

and letting $t \to \infty$ we conclude that there exists $C_{\bar{\alpha}} > 0$ such that

$$\|G_\varepsilon - G_0\|_{L^2(m)} \leq \varepsilon\, C_{\bar{\alpha}} \|G_\varepsilon\|_{H^1_v(Mm)}.$$

Finally, thanks to Corollary 3.6, we have

$$0 \;=\; \langle Q_\varepsilon[\mathscr{J}_{G_\varepsilon}] G_\varepsilon, G_\varepsilon \rangle_{\mathcal{H}^1} \;\leq\; K_1 - K_2 \|G_\varepsilon\|^2_{\mathcal{H}^1} \;\leq\; K_1 - c_\delta\, K_2 \|G_\varepsilon\|^2_{H^1(m_2)},$$

for any exponential weight m_2, then for $\kappa_2 > \kappa$, we have that

$$\|G_\varepsilon\|^2_{H^1_v(Mm)} \;\leq\; C_{\kappa,\kappa_2} \|G_\varepsilon\|^2_{H^1(m_2)} \;\leq\; C_{\kappa,\kappa_2} K_1 / c_\delta K_2,$$

and in the small connectivity regime $\varepsilon \in (0, \varepsilon_0)$, constants K_1 and K_2 do not depend on ε. Defining $\eta(\varepsilon) = \varepsilon\, C_{\bar{\alpha}} C_{\kappa,\kappa_2} K_1 / c_\delta K_2$ we get the stability part of Theorem 2.3.

It only remains to verify that the requirement of Theorem 4.1 are fulfilled for $\mathscr{L}_0$ in the Banach lattice $X = L^2(m)$.

1.(a) the splitting (4.1) has the nice structure. Indeed, the Lemma 4.2 implies that $\mathcal{B}_0 + 1$ is hypodissipative in $L^2(m)$, therefore

$$\|S_{\mathcal{B}_0}(t)\|_{\mathscr{B}(L^2(m))} \leq C e^{-t}, \quad \forall\, t \geq 0,$$

i.e., it suffices to take $\alpha^* = -1$.

(b) if $Y = H^2_v(m)$ and $X = L^2(m)$, the desired inequality is consequence of Lemma 4.3.

2. The requirement is obtained for $\beta = 0$ and $\psi = 1$. Indeed, in that case

$$\mathscr{L}_0^* \psi \;=\; Q_0^* 1 \;=\; 0 \;\geq\; \beta\psi.$$

3. A side consequence of (3.6) is the positivity of the semigroup:

$$f_0 \geq 0 \qquad \Rightarrow \qquad S_{\mathscr{L}_0} f_0(t) \geq 0, \quad \forall\, t \geq 0.$$

Moreover, using that $L^2(m)$ is also a Hilbert space, we deduce the Kato's inequalities.

4. The strict positivity (or strong maximum principle) is a straightforward consequence of Theorem B.1 in Appendix B.

We finish this section by summarizing the properties of the spectrum of $\mathscr{L}_0$ in the space $L^2(m)$ and with a useful result on the regularisation properties of $\mathcal{R}_{\mathscr{L}_0}(z)$.

Proposition 4.4

(i) There exists $\bar\alpha < 0$ such that the spectrum $\Sigma(\mathscr{L}_0)$ of $\mathscr{L}_0$ in $L^2(m)$ writes

$$\Sigma(\mathscr{L}_0) \cap \Delta_{\bar\alpha} \;=\; \{0\},$$

with 0 a simple eigenvalue.

(ii) For any $\alpha > \bar\alpha$, there exists a constant $C_{H^1_v} > 0$ depending on $(\alpha - \bar\alpha)$, such that

$$\|\mathcal{R}_{\mathscr{L}_0}(z)\|_{\mathscr{B}(L^2(m), H^1_v(m))} \leq C_{H^1_v}(1 + |z|^{-1}), \qquad \forall\, z \in \mathbb{C} \setminus \{0\}, Re(z) > \alpha.$$

Proof. It only remains to prove (ii). Let us consider $z \in \Delta_\alpha \setminus \{0\}$, and take $f, g \in L^2(m)$ such that

$$(\mathscr{L}_0 - z) f \;=\; g.$$

Thanks to Lemma 4.2 and the definition of $\mathcal{A}$, we get

$$(\mathrm{Re}(z) - \bar\alpha)\|f\|^2_{L^2(m)} + \|\partial_v f\|^2_{L^2(m)} \;\leq\; \|g\|_{L^2(m)}\, \|f\|_{L^2(m)} + N\, \|f\|^2_{L^2(m)}.$$

Moreover, (i) tells us that 0 is an isolated simple eigenvalue for $\mathscr{L}_0$ in $L^2(m)$, then $\mathcal{R}_{\mathscr{L}_0}(z)$ writes as the Laurent series (see for example [23, Section 3.5])

$$\mathcal{R}_{\mathscr{L}_0}(z) = \sum_{k=-1}^{\infty} z^k \mathcal{C}_k, \qquad \mathcal{C}_k \in \mathscr{B}(L^2(m)),$$

which on a small disc around 0 converges. Thus, there is some $C^0 > 0$ such that $\|\mathcal{R}_{\mathscr{L}_0}(z)\|_{\mathscr{B}(L^2(m))} \leq C^0\, |z|^{-1}$ for any $z \in \Delta_\alpha$, $z \neq 0$. Finally, we notice that

$$\min(1, \alpha - \bar\alpha)\|f\|_{H^1_v(m)} \;\leq\; 2(1 + (1+N)C^0|z|^{-1})\, \|g\|_{L^2(m)},$$

therefore it suffices to take $C_{H^1_v} = 2(1 + (1+N)C^0)/\min(1, \alpha - \bar\alpha)$.

5. Stability of the stationary solution in the small connectivity regime

In this final analytic part of the paper, we establish the exponential convergence of the nonlinear equation. To that aim, we first show that in the small connectivity regime, the linear operator $\mathscr{L}_\varepsilon$ inherits (in a sense that we precise later on) the stability properties of $\mathscr{L}_0$.

5.1. Uniqueness of the stationary solution in the weak connectivity regime. As a first step in the proof of Theorem 2.4, we need a uniqueness condition that, for instance, can be settled as a consequence of the following estimate:

Lemma 5.1 *Let $f \in L^2(m)$ be the solution to the linear equation $\mathscr{L}_0 f = g$, then there exists a constant $C_{\mathcal{V}}$ such that for any $g \in L^2(m)$, $\langle g \rangle = 0$ there holds*

$$\|f\|_{\mathcal{V}} := \|f\|_{L^2(M^2 m)} + \|\nabla_v f\|_{L^2(Mm)} \leq C_{\mathcal{V}} \, \|g\|_{L^2(m)}. \tag{5.1}$$

Proof. We easily compute

$$\int_{\mathbb{R}^2} (\mathscr{L}_0 f) f \, Mm \; = \; - \int_{\mathbb{R}^2} p(x,v) f^2 \, m - \int_{\mathbb{R}^2} (\partial_v f)^2 \, Mm,$$

for some $p(x,v)$ polynomial in x and v with leading term $v^6 + x^4$. Therefore, there exists some constants $K_1 > 0$ and $0 < K_2 < 1$, such that

$$\int_{\mathbb{R}^2} (\mathscr{L}_0 f) f M m \; \leq \; K_1 \int_{\mathbb{R}^2} f^2 m - K_2 \int_{\mathbb{R}^2} f^2 M^2 m - K_2 \int_{\mathbb{R}^2} (\partial_v f)^2 M m.$$

The invertibility of $\mathscr{L}_0$ in $L^2(m)$ for zero mean functions, writes

$$\mathscr{L}_0 f = g \in L^2(m), \; \langle g \rangle = 0 \quad \Rightarrow \quad \|f\|_{L^2(m)} \leq C_{\bar{\alpha}} \, \|g\|_{L^2(m)},$$

with $C_{\bar{\alpha}}$ given in the proof of the stability part of Theorem 2.3. As a consequence, for any f and g as in the statement of the lemma, we have

$$\begin{aligned} \int_{\mathbb{R}^2} f^2 M^2 m + \int_{\mathbb{R}^2} (\partial_v f)^2 M m \; &\leq \; -\frac{1}{K_2} \int_{\mathbb{R}^2} g \, f M m + \frac{K_1}{K_2} \int_{\mathbb{R}^2} f^2 m \\ &\leq \; \frac{1}{2} \int_{\mathbb{R}^2} f^2 \, M^2 m + \frac{1}{2K_2^2} \int_{\mathbb{R}^2} g^2 \, m + \frac{K_1 C}{K_2} \int_{\mathbb{R}^2} g^2 \, m, \end{aligned}$$

from which (5.1) immediately follows.

Corollary 5.2 *There exists $\varepsilon_1 \in (0, \varepsilon_0)$ such that in the small connectivity regime $\varepsilon \in (0, \varepsilon_1)$ the stationary solution is unique.*

Proof. We write

$$\begin{aligned} G_\varepsilon - F_\varepsilon &= \varepsilon \, \mathscr{L}_0^{-1} \Big[\partial_v \Big((v - \mathscr{J}(F_\varepsilon)) F_\varepsilon - (v - \mathscr{J}(G_\varepsilon)) G_\varepsilon \Big) \Big] \\ &= \varepsilon \, \mathscr{L}_0^{-1} \Big[\partial_v \Big((v - \mathscr{J}(F_\varepsilon))(F_\varepsilon - G_\varepsilon) + (\mathscr{J}(F_\varepsilon) - \mathscr{J}(G_\varepsilon)) G_\varepsilon \Big) \Big] \end{aligned} \tag{5.2}$$

As a consequence, using the invertibility property of $\mathscr{L}_0$ for zero mean functions, and the uniform bound (2.5) on G_ε, F_ε, we get

$$\begin{aligned}\|F_\varepsilon - G_\varepsilon\|_{\mathcal{V}} &\le \varepsilon\, C_{\bar\alpha} \left\|\partial_v\big((v-\mathscr{J}(F_\varepsilon))(F_\varepsilon - G_\varepsilon) + (\mathscr{J}(F_\varepsilon)-\mathscr{J}(G_\varepsilon))G_\varepsilon\big)\right\|_{L^2(m)}\\ &\le \varepsilon\, C\, \|F_\varepsilon - G_\varepsilon\|_{\mathcal{V}},\end{aligned}$$

for some C depending on the parameters of the system and ε_0. The previous relationship implies, in particular, that $\|F_\varepsilon - G_\varepsilon\|_{\mathcal{V}} = 0$ for $\varepsilon < \varepsilon_1 = 1/C$.

5.2. Study of the Spectrum and Semigroup for the Linear Problem. We now turn into a generalisation of Proposition 4.4 in the case $\varepsilon > 0$ small. Since the positivity of the operator is lost, Krein-Rutman theory does not apply anymore, however we can prove the following result based on a perturbation argument

Theorem 5.3. *Let us fix $\alpha \in (\bar\alpha, 0)$. Then there exists $\varepsilon_2 \in (0, \varepsilon_1)$ such that for any $\varepsilon \in [0, \varepsilon_2]$, there hold*

(i) The spectrum $\Sigma(\mathscr{L}_\varepsilon)$ of $\mathscr{L}_\varepsilon$ in $L^2(m)$ writes

$$\Sigma(\mathscr{L}_\varepsilon) \cup \Delta_\alpha = \{\mu_\varepsilon\},$$

where μ_ε is a eigenvalue simple. Moreover, since $\mathscr{L}_\varepsilon$ remains in divergence form, we still have

$$\mathscr{L}_\varepsilon^* 1 = 0$$

and then $\mu_\varepsilon = 0$.

(ii) The linear semigroup $S_{\mathscr{L}_\varepsilon}(t)$ associated to $\mathscr{L}_\varepsilon$ in $L^2(m)$ writes

$$S_{\mathscr{L}_\varepsilon}(t) = e^{\mu_\varepsilon t}\Pi_\varepsilon + R_\varepsilon(t),$$

where Π_ε is the projection on the eigenspace associated to μ_ε and where $R_\varepsilon(t)$ is a semigroup which satisfies

$$\|R_\varepsilon(t)\|_{\mathscr{B}(L^2(m))} \le C_{\mathscr{L}_{\varepsilon_1}}\, e^{\alpha t},$$

for some positive constant $C_{\mathscr{L}_{\varepsilon_1}}$ independent of ε.

To enlighten the key points of the proof we present it in three steps: *accurate preliminaries, geometry of the spectrum* of the linear operator in the small connectivity regime and *sharp study of the spectrum close to 0:*

Step 1. Accurate preliminaries: Let us introduce the operator

$$P_\varepsilon = \mathscr{L}_\varepsilon - \mathscr{L}_0 = -\,\varepsilon\,\partial_v((v-\mathscr{J}(G_\varepsilon))\,\cdot) + \varepsilon\,\mathscr{J}(\cdot)\,\partial_v G_\varepsilon.$$

Our aim is to estimate the convergence to 0 of this operator in a suitable norm. We notice that, for two exponential weights m_1, m_2 as in (2.1) with $\kappa_1 < \kappa_2$, it holds

$$\begin{aligned}\|P_\varepsilon h\|^2_{L^2(m_1)} &\le C\,\varepsilon^2 \int_{\mathbb{R}^2} \big(h^2 + v^2|\partial_v h|^2\big)\, m_1 + C\,\varepsilon^2\,\mathscr{J}(h)^2\\ &\le C\,\varepsilon^2\big(\|h\|^2_{L^2(m_1)} + \|\partial_v h\|^2_{L^2(m_2)}\big),\end{aligned}$$

where C depends only on the parameters of the system and, in the small connectivity regime, on ε_1. Therefore, there exists $C_{P_{\varepsilon_1}} > 0$ such that

$$\|P_\varepsilon h\|_{L^2(m_1)} \leq C_{P_{\varepsilon_1}}\, \varepsilon \|h\|_{H^1_v(m_2)}.$$

Step 2. Geometry of the spectrum of $\mathscr{L}_\varepsilon$.

Lemma 5.4 *For any $z \in \Delta_\alpha$, $z \neq 0$ let us define $K_\varepsilon(z)$ by*

$$K_\varepsilon(z) = -\, P_\varepsilon\, \mathcal{R}_{\mathscr{L}_0}(z)\, \mathcal{A}\mathcal{R}_{\mathcal{B}_\varepsilon}(z).$$

Then, there exists $\eta_2(\varepsilon) \xrightarrow[\varepsilon\to 0]{} 0$, such that

$$\forall\, z \in \Omega_\varepsilon := \Delta_\alpha \setminus \bar{B}(0, \eta_2(\varepsilon)), \quad \|K_\varepsilon(z)\|_{\mathscr{B}(L^2(m))} \leq \eta_2(\varepsilon)(1+\eta_2(\varepsilon)).$$

Moreover, there exists $\varepsilon_2 \in (0, \varepsilon_1]$ such that for any $\varepsilon \in [0, \varepsilon_2]$ we have

1. $I + K_\varepsilon(z)$ is invertible for any $z \in \Omega_\varepsilon$
2. $\mathscr{L}_\varepsilon - z$ is also invertible for any $z \in \Omega_\varepsilon$ and

$$\forall\, z \in \Omega_\varepsilon, \quad \mathcal{R}_{\mathscr{L}_\varepsilon}(z) = \mathcal{U}_\varepsilon(z)\big(I + K_\varepsilon(z)\big)^{-1}$$

where

$$\mathcal{U}_\varepsilon(z) = \mathcal{R}_{\mathcal{B}_\varepsilon}(z) - \mathcal{R}_{\mathscr{L}_0}(z)\, \mathcal{A}\, \mathcal{R}_{\mathcal{B}_\varepsilon}(z).$$

We thus deduce that

$$\Sigma(\mathscr{L}_\varepsilon) \cap \Delta_\alpha \subset B(0, \eta_2(\varepsilon)).$$

Proof. We define m_1 and m_2 two exponential weights with $m_1 = m$. From Lemma 4.2, Proposition 4.4 and the Step 1 we get that for any $z \in \Omega_\varepsilon$, any $h \in L^2(m)$

$$\begin{aligned}
\|K_\varepsilon(z)h\|_{L^2(m)} &\leq \varepsilon\, C_{P_{\varepsilon_1}} \|\mathcal{R}_{\mathscr{L}_0}(z)\mathcal{A}\, \mathcal{R}_{\mathcal{B}_\varepsilon}(z)h\|_{H^1_v(m_2)} \\
&\leq \varepsilon\, C_{P_{\varepsilon_1}}\, C_{H^1_v}(1+|z|^{-1})\|\mathcal{A}\, \mathcal{R}_{\mathcal{B}_\varepsilon}(z)h\|_{L^2(m_2)} \\
&\leq \varepsilon\, C_{P_{\varepsilon_1}}\, C_{H^1_v}(1+|z|^{-1})\, C_{\varepsilon_1}\|h\|_{L^2(m)},
\end{aligned}$$

where C_{ε_1} is an upper bound of $\|\mathcal{A}\mathcal{R}_{\mathcal{B}_\varepsilon}\|_{\mathscr{B}(L^2(m), L^2(m_2))}$ that does not depend on ε. Defining

$$\eta_2(\varepsilon) := (\varepsilon\, C_{P_{\varepsilon_1}}\, C_{H^1_v}\, C_{\varepsilon_1})^{1/2},$$

it holds

$$\|K_\varepsilon(z)\|_{\mathscr{B}(L^2(m))} \leq \eta_2(\varepsilon)^2(1+\eta_2(\varepsilon)^{-1}) = \eta_2(\varepsilon)(1+\eta_2(\varepsilon)), \quad \forall\, z \in \Omega_\varepsilon,$$

therefore, fixing $\varepsilon_2 > 0$ such that $\eta_2(\varepsilon) < 1/2, \forall\, \varepsilon \in (0, \varepsilon_2]$, we obtain the invertibility of $I + K_\varepsilon(z)$.

Finally, for any $z \in \Omega_\varepsilon$:

$$(\mathscr{L}_\varepsilon - z)\mathcal{U}_\varepsilon(z) = I + K_\varepsilon(z),$$

then there exists a right inverse of $\mathscr{L}_\varepsilon - z$. The rest of the proof is similar to the proof of [39, Lemma 2.16].

Step 3. Sharp study of spectrum close to 0.

Let us fix $r \in (0, -\alpha]$ and choose any $\varepsilon_r \in [0, \varepsilon_2]$ such that $\eta_2(\varepsilon_r) < r$ in such a way that $\Sigma(\mathscr{L}_\varepsilon) \cap \Delta_\alpha \subset B(0, r)$ for any $\varepsilon \in [0, \varepsilon_r]$. We may define the spectral projection operator

$$\Pi_\varepsilon := -\frac{1}{2\pi i} \int_{|z'|=r} \mathcal{R}_{\mathscr{L}_\varepsilon}(z')\, dz'.$$

We have then the

Lemma 5.5 *The operator Π_ε is well defined and bounded in $L^2(m)$. Moreover, for any $\varepsilon \in [0, \varepsilon_r]$, it holds*

$$\|\Pi_\varepsilon - \Pi_0\|_{\mathscr{B}(L^2(m))} \le \eta_3(\varepsilon),$$

for some $\eta_3(\varepsilon) \underset{\varepsilon \to 0}{\longrightarrow} 0$.

Proof. Let us notice that

$$\Pi_0 \;=\; -\frac{1}{2\pi i} \int_{|z'|=r} (\mathcal{R}_{\mathcal{B}_0}(z') - \mathcal{R}_{\mathscr{L}_0}\, \mathcal{A}\, \mathcal{R}_{\mathcal{B}_0}(z'))\, dz' \;=\; \frac{1}{2\pi i} \int_{|z'|=r} \mathcal{R}_{\mathscr{L}_0}\, \mathcal{A}\, \mathcal{R}_{\mathcal{B}_0}(z')\, dz'$$

and

$$\begin{aligned}
\Pi_\varepsilon &= -\frac{1}{2\pi i} \int_{|z'|=r} (\mathcal{R}_{\mathcal{B}_\varepsilon}(z') - \mathcal{R}_{\mathscr{L}_0}\, \mathcal{A}\, \mathcal{R}_{\mathcal{B}_\varepsilon}(z'))(I + K_\varepsilon(z'))^{-1}\, dz' \\
&= -\frac{1}{2\pi i} \int_{|z'|=r} \mathcal{R}_{\mathcal{B}_\varepsilon}(z')\, K_\varepsilon(z')(I + K_\varepsilon(z'))^{-1}\, dz' \\
&\quad + \frac{1}{2\pi i} \int_{|z'|=r} \mathcal{R}_{\mathscr{L}_0}\, \mathcal{A}\, \mathcal{R}_{\mathcal{B}_\varepsilon}(z')(I + K_\varepsilon(z'))^{-1}\, dz'.
\end{aligned}$$

Then, we deduce that

$$\begin{aligned}
\Pi_\varepsilon - \Pi_0 = &-\frac{1}{2\pi i} \int_{|z'|=r} \mathcal{R}_{\mathcal{B}_\varepsilon}(z')\, K_\varepsilon(z')(I + K_\varepsilon(z'))^{-1}\, dz' \\
&+ \frac{1}{2\pi i} \int_{|z'|=r} \mathcal{R}_{\mathscr{L}_0}\, \mathcal{A}\, (\mathcal{R}_{\mathcal{B}_\varepsilon}(z') - \mathcal{R}_{\mathcal{B}_0}(z'))\, dz' \\
&- \frac{1}{2\pi i} \int_{|z'|=r} \mathcal{R}_{\mathscr{L}_0}\, \mathcal{A}\, \mathcal{R}_{\mathcal{B}_\varepsilon}(z')(I - (I + K_\varepsilon(z'))^{-1})\, dz',
\end{aligned}$$

here, the first and third terms are going to 0 because of the upper bounds of $K_\varepsilon(z)$. For the second term, it suffices to notice that

$$\mathcal{R}_{\mathcal{B}_\varepsilon}(z') - \mathcal{R}_{\mathcal{B}_0}(z') \;=\; \mathcal{R}_{\mathcal{B}_0}(z')\, (\mathcal{B}_\varepsilon - \mathcal{B}_0)\, \mathcal{R}_{\mathcal{B}_\varepsilon}(z'),$$

and use that $(\mathcal{B}_\varepsilon - \mathcal{B}_0) = P_\varepsilon$.

To conclude the proof we recall the following lemma from [23, paragraph I.4.6]

Lemma 5.6 *Let X be a Banach space and P, Q two projectors in $\mathscr{B}(X)$ such that $\|P - Q\|_{\mathscr{B}(X)} < 1$. Then the ranges of P and Q are isomorphic. In particular, $dim(R(P)) = dim(R(Q))$.*

Provided with this lemma and fixing ε' such that $\eta_3(\varepsilon') < 1$, we get the

Corollary 5.7 *There exists $\varepsilon' > 0$ such that for any $\varepsilon \in [0, \varepsilon']$ there holds*

$$\Sigma(\mathscr{L}_\varepsilon) \cap \Delta_\alpha = \{\mu_\varepsilon\} \quad \textit{and the eigenspace associated to } \mu_\varepsilon \textit{ is 1-dimensional.}$$

5.3. Exponential stability of the NL equation. In the small connectivity regime $\varepsilon \in (0, \varepsilon')$, let us consider the variation $h := f_\varepsilon - G_\varepsilon$, with f_ε solution to (1.2) and G_ε the unique solution to (2.7) given by Theorem 2.3. By definition, h satisfies the evolution PDE:

$$\partial_t h \;=\; \mathscr{L}_0 h - \varepsilon \partial_v(vh) + \varepsilon\, \mathscr{J}(f_\varepsilon)\partial_v f_\varepsilon - \varepsilon\, \mathscr{J}(G_\varepsilon)\partial_v G_\varepsilon \;=\; \mathscr{L}_\varepsilon h + \varepsilon\, \mathscr{J}(h)\partial_v h,$$

moreover, the nonlinear part is such that

$$\|\varepsilon\, \mathscr{J}(h)\partial_v h\|_{L^2(m)} \le C\, \varepsilon\, \|h\|_{L^2(m)} \|\partial_v h\|_{L^2(m)}$$

for some positive constant C.

Proof (of Theorem 2.4). Let us first notice that, thanks to inequality (2.5) and the definition of $\mathscr{J}(\cdot)$, we have that

$$\|\varepsilon\, \mathscr{J}(h)\partial_v h\|_{L^2(m)} \le C_{NL}\, \varepsilon\, \|h\|_{L^2(m)}, \qquad \forall\, h_0 \in H^1(m),$$

where

$$C_{NL} \;=\; c_\delta^{-1} \max(C_2, \|h_0\|_{H^1(m)}).$$

On the other hand, Duhamel's formula reads

$$h = S_{\mathscr{L}_\varepsilon}(t)h_0 + \int_0^t S_{\mathscr{L}_\varepsilon}(t-s)\big(\varepsilon\, \mathscr{J}(h)\partial_v h\big)\, ds,$$

then, we have that

$$\begin{aligned} u(t) \;:=\; \|h\|_{L^2(m)} &\le \|S_{\mathscr{L}_\varepsilon}(t)h_0\|_{L^2(m)} + \int_0^t \|S_{\mathscr{L}_\varepsilon}(t-s)\big(\varepsilon\, \mathscr{J}(h)\partial_v h\big)\|_{L^2(m)}\, ds \\ &\le C_{\mathscr{L}_{\varepsilon_1}}\, e^{\alpha t}\|h_0\|_{L^2(m)} + C_{\mathscr{L}_{\varepsilon_1}} C_{NL}\, \varepsilon \int_0^t e^{\alpha(t-s)}\|h\|_{L^2(m)}\, ds \\ &= C_{\mathscr{L}_{\varepsilon_1}}\, e^{\alpha t} u(0) + C_{\mathscr{L}_{\varepsilon_1}} C_{NL}\, \varepsilon \int_0^t e^{\alpha(t-s)} u(s)\, ds. \end{aligned}$$

In particular,

$$u(t) \;\le\; C_{\mathscr{L}_{\varepsilon_1}}\, u(0)\, e^{(\alpha + C_{\mathscr{L}_{\varepsilon_1}} C_{NL}\varepsilon)t},$$

Summarising, it suffices to define $\eta^*(\varepsilon) := C_2/\sqrt{\varepsilon}$ to get that for any f_0 such that

$$\|f_0 - G_\varepsilon\|_{H^1(m)} \;\le\; \eta^*(\varepsilon),$$

it holds

$$\|f_\varepsilon(t) - G_\varepsilon\|_{L^2(m)} \;\le\; C_{\mathscr{L}_{\varepsilon_1}} \|f_0 - G_\varepsilon\|_{L^2(m)} e^{\alpha^* t},$$

with

$$\alpha^* \;=\; \alpha + C_{\mathscr{L}_{\varepsilon_1}} c_\delta^{-1} C_2 \sqrt{\varepsilon^*} < 0,$$

if ε^* is small enough.

6. Open problems beyond the weak coupling regime

In the weak coupling regime, we have demonstrated that existence and uniqueness of solutions persist. In that regime, noise overcomes nonlinear effects and the system is mixing: one finds a unique distribution with an everywhere strictly positive density. As coupling increases, highly non-trivial phenomena may emerge as nonlinear effects of the McKean-Vlasov equation. For instance, it is likely that in another asymptotic regime in which coupling is non-trivial and noise goes to zero, Dirac-delta distributed solutions shall emerge (in which all neurons are synchronized and their voltage and adaptation variable are equal to one of the stable fixed point of the deterministic Fitzhugh-Nagumo ODE).

Here, we numerically explore the dynamics of the Fitzhugh-Nagumo McKean-Vlasov equation using a Monte-Carlo algorithm. We observe that complex phenomena occur as the coupling is varied. That numerical evidence tends to show that several additional equilibria may emerge, the stability of stationary solutions may change as a function of connectivity levels, and attractive periodic solution in time may emerge. These regimes are particularly interesting from the application viewpoint: indeed, among important collective effects in biology, from large networks often emerge bistable high-state of down-states (characterized by high or low firing rates), and even oscillations. These two phenomena are particularly important in developing and storing memories, and this occurs by slowly reinforcing connections [22]. Interestingly, these two types of behaviors emerge naturally in the FhN McKean Vlasov equation beyond weak coupling. For instance, for fixed $\sigma = 0.5$, we present the solutions of the particle system varying the connectivity weight beyond small values, both in the bistable case (in which the FhN model presents two stable attractors) and the excitable regime, the most relevant for biological applications, characterized by a single stable equilibrium and a manifold separating those trajectories doing large excursions (spikes) from those returning to the resting state directly. In both cases, we observe (i) that the unique stationary solution is not centered close from a fixed point of the dynamical system: neurons intermittently fire in an asynchronous manner for small coupling. As coupling increases, a periodic attractive solution emerges, before the appearance of distinct stationary solutions (two in the bistable case, one in the excitable case). These phenomena are depicted in Fig. 1. Proving, for larger coupling, the existence and stability of a periodic solution or distinct and multiple stationary solutions constitute exciting perspectives of this work.

These phenomena are actually conjectured to be generic in coupled excitable systems subject to noise.

A. Mean-Field limit for Fitzhugh-Nagumo neurons

Let us start by a well known result with is a simple application of global existence and path wise uniqueness for system of SDE, see [16, Chapter 5, Theorems 3.7 and 3.11] for example. Consider the particle system for $1 \leq i \leq N$:

$$\begin{cases} dv_t^i = \left(v_t^i\left(v_t^i - \lambda\right)\left(1 - v_t^i\right) - x_t^i + I_0\right) dt + \dfrac{J}{N}\displaystyle\sum_{j=1}^{N}\left(v_t^i - v_t^j\right) dt + dW_t^i \\ dx_t^i = (-ax_t^i + bv_t^i)dt, \end{cases} \tag{A.1}$$

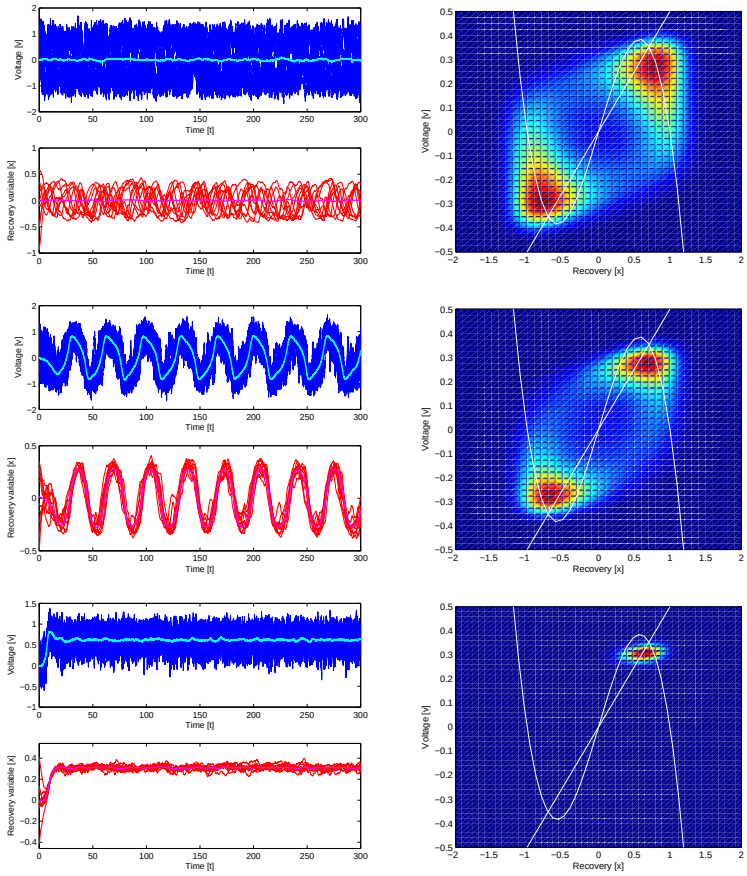

Fig. 1. Permanent (non-transient) regimes of the FhN particle system for $N = 2\,000$. Top row $\varepsilon = 0.1$, middle row $\varepsilon = 1$, and bottom row $\varepsilon = 3$. The unique stationary solution in the small coupling limit analyzed in the manuscript visits both attractors transiently (middle), while in the high coupling regime (bottom), the system remains around one of the attractors (the system has at least two such solutions). In an intermediate regime, the system shows periodic oscillations (see voltage and recovery evolution in the middle row).

with initial data (X_0^i, V_0^i) for $1 \le i \le N$ distributed according to $f_0 \in \mathbb{P}_2(\mathbb{R}^2)$, i.e., a probability measure in $\mathbb{R}^2$ with finite second moment. Here the $(W_t^i)_{t\ge 0}$ are n independent standard Brownian motions in $\mathbb{R}$. This result was stated in [5], but the proof used a localization argument which is not completely justified. We prove here a simpler version of the result based on *a-priori* estimates.

Lemma A.1 *Let $f_0 \in \mathbb{P}(\mathbb{R}^2)$ be a probability with finite second moment, and a set of random variables (X_0^i, V_0^i) with law f_0. Then* (A.1) *admits a path wise unique global solution with initial datum (X_0^i, V_0^i) for $1 \le i \le N$.*

Proof. The system (A.1) can be written in $\mathbb{R}^{2N}$ as the SDE

$$d\mathbf{Z}_t^N = \sigma^N \, d\mathbf{B}_t^N + \mathbf{b}(\mathbf{Z}_t^N)\, dt,$$

where $\mathbf{Z}_t^N = (x_t^1, v_t^1, \dots, x_t^N, v_t^N)$, σ^N is a constant $2N \times 2N$ sparse matrix, $(\mathbf{B}_t^N)_{t\geq 0}$ is a standard Brownian motion on $\mathbb{R}^{2N}$, and $\mathbf{b} : \mathbb{R}^{2N} \to \mathbb{R}^{2N}$ is a function defined in the obvious way. It is easy to see that $\mathbf{b}$ is a locally Lipschitz function, moreover, letting $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ the scalar product and the Euclidean norm on $\mathbb{R}^{2N}$ respectively, then for all $\mathbf{Z}^N = (x^1, v^1, \dots, x^N, v^N)$,

$$\begin{aligned}
\langle \mathbf{Z}^N, \mathbf{b}(\mathbf{Z}^N) \rangle &= \sum_{i=1}^N x^i(-ax^i + bv^i) + \sum_{i=1}^N v^i\big(v^i\,(v^i - \lambda)\,(1 - v_t^i) - x^i + I_0\big) + \frac{J}{N}\sum_{i.j=1}^N v^i\big(v^i - v^j\big) \\
&\leq \sum_{i=1}^N (b-1)x^i v^i + \sum_{i=1}^N \big(J|v^i|^2 - a|x^i|^2\big) - \frac{J}{N}\sum_{i.j=1}^N v^i v^j + CN \\
&\leq C(1 + \|\mathbf{Z}^N\|^2).
\end{aligned}$$

This is a sufficient condition for global existence and pathwise uniqueness (see e.g. [25]).

Mean-Field limit. Now we turn to the propagation of chaos property. We already know the existence and uniqueness of the particle system (A.1), moreover the nonlinear SDE:

$$\begin{cases}
d\bar{v}_t = \big(\bar{v}_t(\bar{v}_t - \lambda)(1 - \bar{v}_t) - \bar{x}_t + I\big)\,dt + J\displaystyle\int_{\mathbb{R}^2} (\bar{v}_t - v)\,df_t(x,v)\,dt + dW_t, \\
d\bar{x}_t = (-a\bar{x} + b\bar{v}_t)dt \\
f_t = \mathrm{law}(\bar{x}_t, \bar{v}_t), \quad \mathrm{law}(\bar{x}_0, \bar{v}_0) = f_0.
\end{cases} \tag{A.2}$$

is also well-posed for $f_0 \in L^1(M) \cap L^1 \log L^1 \cap \mathbb{P}(\mathbb{R}^2)$, as a consequence of Theorem 2.2. Then, for instance, we can sate the

Theorem A.2. *Let f_0 be a Borel probability measure and (X_0^i, V_0^i) for $1 \leq i \leq N$ be N independent variables with law f_0. Let us assume that the solutions to* (A.1) *and* (A.2) *with initial data (X_0^i, V_0^i) and f_0 are well defined on $[0,T]$ and such that*

$$\sup_{[0,T]} \Big\{ \int_{\mathbb{R}^2} (|x|^2 + |v|^2)\,df_t(x,v) \Big\} < +\infty, \tag{A.3}$$

with $f_t = law(\bar{x}_t^i, \bar{v}_t^i)$ (which actually does not depend on i by exchangeability). Then there exists a constant $C > 0$ such that

$$\mathbb{E}\big[|x_t^i - \bar{x}_t^i|^2 + |v_t^i - \bar{v}_t^i|^2\big] \leq \frac{C}{N} e^{Ct}. \tag{A.4}$$

Proof. We start by writing $X_t^i = x_t^i - \bar{x}_t^i$ and $V_t^i = v_t^i - \bar{v}_t^i$. For notational convenience we drop the time dependence subindex, take $J = 1$, and use the letter C to denote a finite constant that may change in the different inequalities. Because x_t^i and $\bar{x}_t^i$ are driven by the same Brownian motion, we have that

$$\begin{cases}
dV^i = \big(v^i(v^i - \lambda)(1 - v^i) - \bar{v}^i(\bar{v}^i - \lambda)(1 - \bar{v}^i) - X^i\big)\,dt \\
\qquad + \dfrac{1}{N}\displaystyle\sum_{j=1}^N \big(v_t^i - v_t^j\big)\,dt - \int_{\mathbb{R}^2} (\bar{v}^i - v)\,df_t(x,v)\,dt \\
dX^i = (-aX^i + bV^i)dt,
\end{cases}$$

We define $\alpha(t) = \mathbb{E}\big[|X^i|^2 + |V^i|^2\big]$ which is independent of the label i by symmetry and exchangeability of the system. It is not hard to see that

$$\frac{1}{2}\frac{d}{dt}\mathbb{E}\left[|X^i|^2\right] = \mathbb{E}\big[b|X^i|\,|V^i| - a|X^i|^2\big] \leq \frac{b}{2}\alpha(t),$$

and

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\mathbb{E}\left[|V^i|^2\right] &= \mathbb{E}\left[V^i\big(v^i(v^i-\lambda)(1-v^i) - \bar{v}^i(\bar{v}^i-\lambda)(1-\bar{v}^i) - X^i\big)\right] \\ &\quad + \mathbb{E}\left[\frac{V^i}{N}\sum_{j=1}^N \big(v_t^i - v_t^j\big)\,dt - V^i\int_{\mathbb{R}^2}(\bar{v}^i - v)\,df_t(x,v)\right] =: \mathcal{S}_1 + \mathcal{S}_2.\end{aligned}$$

Estimate for $\mathcal{S}_1$: Let us first notice that

$$\begin{aligned}v^i(v^i-\lambda)(1-v^i) - \bar{v}^i(\bar{v}^i-\lambda)(1-\bar{v}^i) &= -(|v^i|^3 - |\bar{v}^i|^3) + (1+\lambda)(|v^i|^2 - |\bar{v}^i|^2) - \lambda V^i \\ &= -V^i(|v^i|^2 + v^i\,\bar{v}^i + |\bar{v}^i|^2) + (1+\lambda)V^i(|v^i| + |\bar{v}^i|) - \lambda V^i,\end{aligned}$$

therefore

$$\mathcal{S}_1 \;=\; \mathbb{E}[|V^i|^2(-|v^i|^2 - v^i\,\bar{v}^i - |\bar{v}^i|^2 + (1+\lambda)(|v^i| + |\bar{v}^i|) - \lambda)] - \mathbb{E}[V^iX^i],$$

and by consequence there is some constant $C > 0$ such that

$$\mathcal{S}_1 \;\leq\; C\alpha(t). \tag{A.5}$$

Estimate for $\mathcal{S}_2$: By definition, it holds

$$\begin{aligned}\mathcal{S}_2 &= \mathbb{E}\left[V^i(v_t^i - \bar{v}^i) - \frac{V^i}{N}\sum_{j=1}^N\big(v_t^j - \int_{\mathbb{R}^2} v\,df_t(x,v)\big)\right] \\ &= \mathbb{E}\left[|V^i|^2\right] - \frac{1}{N}\sum_{j=1}^N \mathbb{E}\left[V^iV^j\right] - \frac{1}{N}\,\mathbb{E}\left[V^i\sum_{j=1}^N\left(\bar{v}_t^j - \int_{\mathbb{R}^2} v\,df_t(x,v)\right)\right].\end{aligned}$$

Using Cauchy-Schwarz inequality and the independence of the processes $(\bar{v}^j)_{j=1\cdots N}$, we find:

$$\begin{aligned}\mathcal{S}_2 &\leq 2\,\mathbb{E}\left[|V^1|^2\right] + \tfrac{1}{N}\Big(\mathbb{E}\left[|V^1|^2\right]\Big)^{1/2}\Big(\mathbb{E}\Big[\big|\textstyle\sum_{j=1}^N\left(\bar{v}_t^j - \int_{\mathbb{R}^2} v\,df_t(x,v)\right)\big|^2\Big]\Big)^{1/2} \\ &\leq 2\,\mathbb{E}\left[|V^1|^2\right] + \tfrac{1}{\sqrt{N}}\Big(\mathbb{E}\left[|V^1|^2\right]\Big)^{1/2}\left(\int_{\mathbb{R}^2}\left(w - \int_{\mathbb{R}^2} v\,df_t(x,v)\right)^2 df_t(y,w)\right)^{1/2}\end{aligned}$$

Since the second moment of f_t is uniformly bounded on $[0,T]$, we conclude that

$$\mathcal{S}_2 \;\leq\; C\Big(\alpha(t) + \frac{\alpha(t)^{1/2}}{\sqrt{N}}\Big). \tag{A.6}$$

Combining (A.5) and (A.6), we thus find:

$$\frac{d}{dt}\alpha(t) \leq C\Big(\alpha(t) + \frac{\alpha(t)^{1/2}}{\sqrt{N}}\Big) \leq C\Big(\alpha(t) + \frac{1}{N}\Big),$$

and using Grönwal's Lemma,

$$\alpha(t) \leq \left(\alpha(0) + \frac{C}{N}\right)e^{Ct} = \frac{C}{N}e^{Ct}$$

which finishes the proof.

B. Strong maximum principle for the linearized operator

In this final appendix we shall extend the result provided in [40, Corollary A.20] to our framework. These local positivity estimates are classical in hypoelliptic equations and they are a necessary condition for Theorem 2.3. Here, our result is time dependant and by consequence more general than it is needed in the applications.

In the sequel, we shall use the notation

$$B_r(x_0, v_0) := \{(x, v) \in \mathbb{R}^2\,; \quad |v - v_0| \le r,\, |x - x_0| \le r^3\},$$

and come back to the classical notation $\nabla_{x,v} = D_{x,v}$ and $\partial^2_{vv} = \Delta_v$. Also, we simplify the problem by choosing $a = b = 1$, but the proof can be easily extended to the general case.

Theorem B.1. *Let $f(t, x, v)$ be a classical nonnegative solution of*

$$\frac{\partial}{\partial t} f - \Delta_v f = A(t, x, v)\, \nabla_v f + B(x, v)\, \nabla_x f + C(t, x, v)\, f \tag{B.1}$$

in $[0, T) \times \Omega$, where Ω is an open subset of $\mathbb{R}^2$, and $A, C : [0, T) \times \mathbb{R}^2$ and bounded continuous functions and $B(x, v) = x - v$. Let $(x_0, v_0) \in \Omega$ and $\bar{A}$ and $\bar{C}$ upper bounds of respectively $\|A\|_{L^\infty}$ and $\|C\|_{L^\infty}$.

Then, for any $r, \tau > 0$ there are constants λ, $K > 0$, only depending on $\bar{A}$, $\bar{C}$ and r^2/τ such that the following holds: If $B_{\lambda r}(x_0, v_0) \subset \Omega$, $\tau < \min(1/2, -\log(r^3/2|x_0 - v_0|))$ and $f \ge \delta > 0$ in $[\tau/2, \tau) \times B_r(x_0, v_0)$, then $f \ge K\delta$ in $[\tau/2, \tau) \times B_{2r}(x_0, v_0)$.

Theorem B.1 implies, via covering arguments in variables t, x, v the

Corollary B.2 *If $f \ge 0$ solves* (B.1) *in $[0, T) \times \Omega$ and $f \ge \delta > 0$ in $[0, T) \times B_r(x_0, v_0)$, then for any compact set $K \subset \Omega$ containing (x_0, v_0) and for any $t_0 \in (0, T)$, we have $f \ge \delta' > 0$ in $[t_0, T) \times K$ where δ' only depends on $\bar{A}, \bar{C}, K, \Omega, x_0, v_0, r, t_0, \delta$.*

Proof (Proof of Theorem B.1).

We only explain how to adapt the proof of Theorem A.19 given in [40]. Let $g = e^{\bar{C}t} f(t, x, v)$; then $g \ge f$ and $\mathcal{L}\, g \ge 0$ in $(0, T) \times \Omega$, where

$$\mathcal{L} = \partial_t + (v - x)\, \nabla_x - \Delta_v - A(t, x, v)\, \nabla_v.$$

Next, we construct a particular subsolution for $\mathcal{L}$. In the sequel, B_r stands for $B_r(x_0, v_0)$ and we define $X_t(x_0, v_0) = v_0 + (x_0 - v_0)e^{-t}$.

Step 1. Construction of the subsolution.

For $t \in (0, \tau]$ and $(x, v) \in \Omega \setminus B_r$ let

$$P(t, x, v) = \alpha \frac{(v - v_0)^2}{2t} - \frac{\beta}{t^2}(v - v_0)(x - X_t) + \gamma \frac{(x - X_t)^2}{2t^3},$$

with $\alpha, \beta, \gamma > 0$ to be chosen later on. Let further define

$$\varphi(t, x, v) = \delta\, e^{-\mu\, P(t,x,v)} - \varepsilon,$$

where $\mu, \varepsilon > 0$ will also be chosen later on. If we assume that $\beta^2 < \alpha\,\gamma$, then P is a positive quadratic form in the variables $v - v_0$ and $x - X_t$. Clearly

$$\mathcal{L}\,\varphi = -\mu\,\delta\,e^{-\mu\,P}\mathcal{E}(P),$$

where

$$\mathcal{E}(P) = \partial_t P + (v - x)\,\nabla_x P - \triangle_v P + \mu\,|\nabla_v P|^2 - A(t,x,v)\,\nabla_v P.$$

By straightforward computation we find that $\mathcal{E} = \mathcal{E}_1 + \mathcal{E}_2$, with

$$\mathcal{E}_1(P) = \Big(\mu\,\alpha^2 - \frac{\alpha}{2} - \beta\Big)\frac{(v-v_0)^2}{t^2} + 2\Big(\beta + \frac{\gamma}{2} - \mu\,\alpha\,\beta\Big)\frac{(v-v_0)(x-X_t)}{t^3} + \Big(\mu\,\beta^2 - \frac{3\,\gamma}{2}\Big)\frac{(x-X_t)^2}{t^4}$$

and

$$\mathcal{E}_2(P) = \beta\,\frac{(v-v_0)(x-X_t)}{t^2} - \alpha\,\frac{1}{t} - \gamma\,\frac{(x-X_t)^2}{t^3} - \alpha\,\frac{A(t,x,v)(v-v_0)}{t} + \beta\,\frac{A(t,x,v)(x-X_t)}{t^2}.$$

Now we notice that $\mathcal{E}_1$ is defined by the quadratic form

$$M_q = \begin{bmatrix} \mu\,\alpha^2 - \frac{\alpha}{2} - \beta & \beta + \frac{\gamma}{2} - \mu\,\alpha\,\beta \\ \beta + \frac{\gamma}{2} - \mu\,\alpha\,\beta & \mu\,\beta^2 - \frac{3\,\gamma}{2} \end{bmatrix}$$

which is nothing but a quadratic polynomial on $(v - v_0)/t$ and $(x - X_t)/t^2$. As $\mu \to \infty$

$$\begin{cases} \operatorname{tr} M_q \;=\; \mu(\alpha^2 + \beta^2) + O(1) \\ \det M_q \;=\; \mu\Big[\,\dfrac{3\,\alpha\,\beta^2}{2} + \alpha\,\beta\,\gamma - \beta^3 - \dfrac{3\alpha^2\gamma}{2}\Big] + O(1), \end{cases}$$

both positive quantities if $\beta > \alpha$ and $\alpha\,\gamma > \beta^2$. In particular, for $\beta = 2\,\alpha$ and $\gamma = 8\,\alpha$,

$$\begin{cases} \operatorname{tr} M_q \;= 5\,\alpha^2\,\mu + O(1) \\ \det M_q \;= 2\,\alpha^3\,\mu + O(1), \end{cases}$$

and letting $\mu \to \infty$ the eigenvalues of M_q are of order $\mu\,\beta^2$ and β. So, for any fixed $C > 0$ we may choose α, β, γ and μ such that

$$\mathcal{E}_1(P) \;\geq\; C\beta\Big(\frac{(v-v_0)^2}{t^2} + \frac{(x-X_t)^2}{t^4}\Big).$$

Second, if $t \in (0,1)$ then

$$\mathcal{E}_2(P) \;\geq\; -4\beta\frac{(x-X_t)^2}{t^4} - \frac{3\beta(v-v_0)^2}{2} - \frac{3\beta(x-X_t)^2}{2t^4} - 2\beta\bar{A}^2 - \frac{\beta}{2t},$$

and making $\tau \leq 1$, we get,

$$\mathcal{E}(P) \geq \text{const}\,\frac{\beta}{t}\Big[C\Big(\frac{(v-v_0)^2}{t} + \frac{(x-X_t)^2}{t^3}\Big) - 1\Big],$$

with C arbitrarily large.

Let us briefly describe the rest of the proof. Recall that $(x,v) \notin B_r$ so

1. either $|v-v_0|\geq r$, then $\mathcal{E}(P)\geq \text{const.}(\beta/t)[Cr^2/\tau-1]$, which is positive for $C>\tau/r^2$;
2. or $|x-x_0|\geq r^3$, and then, if $\tau\leq \frac{1}{2}\min(1,-\log(\frac{r^3}{|x_0-v_0|}))$ then for any $t\in[0,\tau)$

$$|X_t-x_0|\leq r^3/2 \quad \text{and} \quad \frac{|x-X_t|^2}{t^2}\geq \frac{|x-x_0|^2}{2t^2}-\frac{|X_t-x_0|^2}{t^2}\geq \frac{r^6}{4\tau^2},$$

so $\mathcal{E}(P)\geq \text{const.}(\beta/t)[Cr^6/4\tau^3-1]$, which is positive as soon as $C>4\tau^3/r^6$.

Summarizing: under the assumptions, we can always choose constants $\gamma>\beta>\alpha>1$ and $\alpha\,\gamma>\beta^2$, depending only on $\bar{A}$ and r^2/τ, so that

$$\mathcal{L}\,\varphi\geq 0, \qquad \text{in } [0,\tau)\times(B_{\lambda r}\setminus B_r),$$

as soon as $\tau<\min(1/2,-\log(r^3/2|x_0-v_0|))$.

Step 2. Boundary conditions. We now wish to prove that $\varphi\leq g$ for $t=0$ and for any $(x,v)\in\partial(B_{\lambda r}\setminus B_r)$; then classical maximum principle will do the rest.

Let us first notice that the boundary condition at $t=0$ is obvious (φ can be extended by continuity by 0 at the initial time). The condition at ∂B_r is also true since $\forall\,(x,v)\in\partial B_r$: $\varphi\leq\delta\leq g$.

It remains to fix the remaining parameters in order to conclude that $\varphi\leq g$ in $\partial B_{\lambda r}$. From the choice of α,β and γ, it is easy to see that for any $(x,v)\in\partial B_{\lambda r}$:

$$P(t,x,v)\geq\frac{\alpha}{4}\Big(\frac{(v-v_0)^2}{t}+\frac{(x-X_t)^2}{t^3}\Big)\geq\frac{\alpha}{4}\min\Big(\frac{\lambda^2r^2}{\tau},\frac{\lambda^6r^6}{4\tau^3}\Big)\geq\frac{\alpha\,\lambda^2}{16}\min\Big(\frac{r^2}{\tau},\frac{r^6}{\tau^3}\Big),$$

notice that we are imposing $\lambda>1$. Choosing

$$\varepsilon\,=\,\delta\exp\Big(-\frac{\mu\,\alpha\,\lambda^2}{16}\min\Big(\frac{r^2}{\tau},\frac{r^6}{\tau^3}\Big)\Big),$$

we get $\varphi=\delta\,e^{-\mu P(t,x,v)}-\varepsilon\leq 0$ on $\partial B_{\lambda r}$. By consequence $\varphi\leq g$ on the whole set $B_{\lambda r}$.

Let us finally notice that at this point we have uniform bounds for g on $B_{2r}\setminus B_r$ for any $t\in[\tau/2,\tau)$. Indeed,

$$P(t,x,v)\leq 2\,\gamma\left(\frac{(v-v_0)^2}{t}+\frac{(x-X_t)^2}{t^3}\right)\leq 2\,\gamma\Big(\frac{8\,r^2}{\tau}+\frac{1026\,r^6}{\tau^3}\Big)\leq 2068\,\gamma\,\max\Big(\frac{r^2}{\tau},\frac{r^6}{\tau^3}\Big)$$

Then, for λ big enough we find $K_0>0$ such that

$$\varphi(t,x,v)\geq\delta\Big[\exp\Big(-2068\,\mu\,\gamma\,\max\Big(\frac{r^2}{\tau},\frac{r^6}{\tau^3}\Big)\Big)-\exp\Big(-\frac{\mu\,\alpha\,\lambda^2}{16}\min\Big(\frac{r^2}{\tau},\frac{r^6}{\tau^3}\Big)\Big)\Big]\geq K_0\,\delta,$$

because $\gamma=8\,\alpha$, to find such λ it suffices that

$$2068\times 16\times 8\,\max\Big(\frac{r^2}{\tau},\frac{r^6}{\tau^3}\Big)\leq\lambda^2\,\min\Big(\frac{r^2}{\tau},\frac{r^6}{\tau^3}\Big),$$

by consequence λ depends only on r^2/τ.

Finally, we find $K,\lambda>0$ depending on $\bar{A}$, $\bar{C}$ and r^2/τ such that

$$f\geq K_0\,\delta\,e^{-\tau\,\bar{C}} \quad \text{on} \quad [\tau/2,\tau)\times(B_{2r}\setminus B_r).$$

Remark B.3 *Let us notice that we can extend Theorem B.1 to some cases when A or C are not necessarily bounded and $\Omega = \mathbb{R}^2$. It suffices to take any $r, \tau > 0$ and fix λ (which as we saw only depends on a numerical constant and the ratio r^2/τ). We can then fix $R > 0$ big enough, in order to have that $\lambda r < R$ and study the equation into B_R, where by continuity A and C attain their maximum in the compact set $[0, \tau] \times \bar{B}_R$.*

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