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ON RIEMANN ZETA-FUNCTION AND ALLIED  
QUESTIONS-II

BY

R. BALASUBRAMANIAN AND K. RAMACHANDRA

§ 1. INTRODUCTION. It is best to begin with two definitions. We write  $s = \sigma + it$  unless otherwise stated.

**DEFINITION 1.** Let  $\{a_n\}$  ( $n = 1, 2, 3, \dots$ ) with  $a_1 = 1$  be any sequence of complex numbers ( $a_n$  may depend on two parameters  $T$  and  $H$  with  $T \geq H \geq$  a large positive constant to appear later) which vanish for all but finitely many  $n$ . Then  $F_0(s) = \sum_{n=1}^{\infty} a_n n^{-s}$  is called a Titchmarsh polynomial.

**DEFINITION 2.** Let  $\{a_n\}$  ( $n = 1, 2, 3, \dots$ ) with  $a_1 = 1$  be any sequence of complex numbers ( $a_n$  may depend on two parameters  $T$  and  $H$ , with  $T \geq H \geq$  a large positive constant, to follow). Suppose  $F(s) = \sum_{n=1}^{\infty} a_n n^{-s}$  is convergent somewhere in the complex plane (depending on  $T$  and  $H$  to follow) and can be continued analytically in  $(\sigma > 0, T - H_1 \leq t \leq T + H + H_1)$ , where  $H_1 (\leq \frac{1}{2}T)$  is a suitable function of  $T$ , and there  $|F(s)| \leq T^A \log H$  where  $A \geq 1$  is any suitable constant. Then  $F(s)$  is called an infinite Titchmarsh polynomial.

**REMARK.** In definitions 1 and 2 we may replace  $n^{-s}$  ( $n \geq 1$ ) by  $\lambda_n^{-s}$  where

$\{\lambda_n\}$  is any sequence of real numbers with  $\lambda_1 = 1$  and  $\frac{1}{A} \leq \lambda_{n+1} - \lambda_n \leq A$  ( $n = 1, 2, 3, \dots$ ),  $A \geq 1$  being a constant. We are thus led to "Generalised Titchmarsh polynomial" and "Infinite generalised Titchmarsh polynomial".

**CONJECTURE 1.** *There exist absolute numerical constants  $c_1 > 0$  and  $c_2 > 0$  such that for any Titchmarsh polynomial  $F_0(s)$ , we have*

$$\frac{1}{H} \int_T^{T+H} |F_0(it)|^2 dt \geq c_1 \sum_{n \leq c_2 H} |a_n|^2. \quad (1)$$

**CONJECTURE 2.** *There exist constants  $c_1 > 0$  and  $c_2 > 0$  (which may depend at most on  $A$ ) such that for any infinite Titchmarsh polynomial  $F(s)$ , we have*

$$\frac{1}{H} \int_T^{T+H} |F(it)|^2 dt \geq \frac{1}{2} c_1 \sum_{n \leq c_2 H} |a_n|^2 \quad (2)$$

**REMARK 1.** By LHS of (2) we mean the limit of the mean value of  $|F(s)|^2$  as  $\sigma \rightarrow 0$  from the right. We do not know how to prove Conjecture 1. We believe that it implies the stronger Conjecture 2. But we cannot prove this implication.

**REMARK 2.** For many important applications it suffices to suppose that, in Conjectures 1 and 2,  $|a_n| \leq (nH)^A$ , where  $A \geq 1$  is some constant. But we cannot prove these conjectures nor the implication referred to in Remark 1 above even under this restriction.

We first prove (in § 3) the following theorem.

**THEOREM 1.** *If  $H_1 > 0$  is a large positive constant times  $\log \log T$  and the infinite Titchmarsh polynomial  $F(s)$  converges absolutely at  $Re s = 2$  and can be continued analytically in  $(\sigma \geq -\delta, T - H_1 \leq t \leq T + H + H_1)$  and there maximum of  $|F(s)| \leq T^A \log H$ , then Conjecture 1 implies*

$$\frac{1}{H} \int_T^{T+H} |F(it)|^2 dt \geq \frac{1}{2} c_1 \sum_{n \leq c_2 H} |a_n|^2. \quad (3)$$

**REMARK.** The condition  $Re s = 2$  is unimportant. We can work with any constant in place of 2.

Next (in § 4 and § 5) we prove the following theorem. We define  $d_k(n)$  (for complex  $k$ ) by  $(\zeta(s))^k = \sum_{n=1}^{\infty} d_k(n)n^{-s}$ ,  $\text{Re } s \geq 2$ .

**THEOREM 2.** *Let  $T$  and  $H$  exceed large positive constants and let  $k$  be any complex number with  $|k| \leq \log H$  and  $\text{Re } k \geq 0$ . Then conjecture 1 implies*

$$\frac{1}{H} \int_T^{T+H} |(\zeta(\frac{1}{2} + it))^k|^2 dt \geq \frac{1}{2} c_1 \sum_{n \leq c_2 H} |d_k(n)|^2 n^{-1}. \quad (4)$$

Here we have assumed that  $\zeta(s) \neq 0$  in  $(\sigma > \frac{1}{2}, T - H_1 \leq t \leq T + H + H_1)$  where  $H_1$  is a large positive constant multiple of  $\log T \log H (\log \log T)^{-1}$ .

**REMARK 1.** We are unable, so far, to put Theorem 2 in the language mean square of infinite Titchmarsh polynomials. Also we believe that in Theorem 2 we can manage to take  $H_1$  to be a positive constant multiple of  $\log \log T$ . We have been unable to prove this.

**REMARK 2.** Theorem 2 goes through for the zeta and  $L$ -functions of algebraic number fields (in place of  $\zeta(s)$ ). In fact it has analogues for  $D(s)$  (to be introduced in § 6, and there with  $\lambda_n = n$ ) in place of  $\zeta(s)$ . For example we can select  $\alpha = \frac{1}{2}$ .

**NOTATION.** We use  $c_0, c'_0, c_1, c_2, \dots$  to denote positive constants which depend at most on  $A$  whenever it occurs. The  $O$  symbols used extensively by Hardy and Littlewood and the symbols  $\gg$  and  $\ll$  of I.M. Vinogradov have the usual meaning. Sometimes we indicate the constant parameters on which these symbols imply by writing them below these symbols.

**§ 2. APPLICATIONS OF THEOREMS 1 AND 2.** It is not hard to deduce the following two Theorems 3 and 4 as corollaries to Theorems 1 and 2 respectively. (We concentrate on results mainly with the condition that  $H$  does not exceed a large positive constant times  $\log \log T$ . We have dealt with the case  $H$  exceeding a large positive constant times  $\log \log T$  very satisfactorily elsewhere (without the help of Conjectures 1 or 2) in our previous papers (see the paper I<sup>[4]</sup> of this series of papers for references).

**THEOREM 3.** *Conjecture 1 implies*

$$\frac{1}{H} \int_T^{T+H} \left| \zeta\left(\frac{1}{2} + it\right) \right|^{2k} dt \geq \frac{1}{2} c_1 \sum_{n \leq c_2 H} (d_k(n))^2 n^{-1} \quad (5)$$

for all integers  $k \geq 1$  and so we have first that LHS of (5) is  $\gg_k (\log H)^{k^2}$  and next

$$\max_{T \leq t \leq T+H} \left| \zeta\left(\frac{1}{2} + it\right) \right| > \text{Exp} \left( \frac{3}{4} \sqrt{\frac{\log H}{\log \log H}} \right). \quad (6)$$

**THEOREM 4.** Put  $k = k_0 e^{i\theta}$  where  $\cos \theta \geq 0$  and  $k_0 \geq 1$  is any integer. Then under the conditions of Theorem 2, LHS of (4) is  $\gg_k (\log H)^{|k^2|}$  and next with  $z = e^{i\theta}$  we have

$$\max_{T \leq t \leq T+H} \left| \left( \zeta\left(\frac{1}{2} + it\right) \right)^z \right| > \text{Exp} \left( \frac{3}{4} \sqrt{\frac{\log H}{\log \log H}} \right), \quad (7)$$

LHS trivially  $\infty$  if  $\cos \theta < 0$ , of course on RH.

**REMARK 1.** It is well-known that for complex  $k$  and all  $x \geq 2$ , we have  $\sum_{n \leq x} |d_k(n)|^2 n^{-1} \gg_k (\log x)^{|k^2|}$ . The inequalities (6) and (7) with some positive constant  $c_0$  in place of  $\frac{3}{4}$  follow from (5) and (4). All that we have to prove is that for all  $x$  exceeding a large positive constant, we have

$$\max_{|k| \leq \log x} \left( \sum_{n \leq x} |d_k(n)|^2 n^{-1} \right)^{|2k|^{-1}} > \text{Exp} \left( c_0 \sqrt{\frac{\log x}{\log \log x}} \right). \quad (8)$$

This is proved by us in [2]. But in [1] R. Balasubramanian has shown by an ingenious method that the logarithms of both sides of (8) are asymptotic, as  $x \rightarrow \infty$ , with  $c_0 = 0.75 \dots$

**REMARK 2.** In Theorems 3 and 4 we have stated results on the critical line. But we can also state these for any  $\sigma$  with  $\frac{1}{2} < \sigma < 1$ . But the result corresponding to (8) is

$$\max_{|k| \leq \log x} \left( \sum_{n \leq x} |d_k(n)|^2 n^{-2\sigma} \right)^{|2k|^{-1}} > \text{Exp} \left( c'_0 \frac{(\log x)^{1-\sigma}}{\log \log x} \right) \quad (9)$$

(where  $c'_0 > 0$  is constant) and so the results although valid for short intervals are not so good as those of H.L. Montgomery [3] who treats only long intervals like  $0 \leq t \leq T$ . His method fails for short intervals. Actually in [2] we have proved that the logarithm of LHS of (9) is  $O((\log x)^{1-\sigma}(\log \log x)^{-1})$ .

**REMARK 3.** It may be noted that the maximums in (8) and (9) even though we do not impose  $|k| \leq \log x$  are really attained in the range  $|k| \leq \log x$  and precisely when  $|k|$  exceeds a positive constant power of  $\log x$  (see [2]). Hence for Theorems 3 and 4 Conjectures 1 and 2 with  $c_1(\log H)^{-100}$  in place of  $c_1$  are enough.

**REMARK 4.** On the line  $\sigma = 1$  there are excellent results (see [5]).

§ 3. **PROOF OF THEOREM 1.** Let  $s = it$  and let  $t$  exceed a large positive constant. For  $X \geq 1$  we define  $A_X(s)$  by

$$A_X(s) = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} F(s+w) X^2 \text{Exp} \left( \left( \sin \frac{w}{100} \right)^2 \right) \frac{dw}{w} \quad (10)$$

$$= \sum_{n=1}^{\infty} \frac{a_n}{n^{it}} \Delta \left( \frac{X}{n} \right) \quad (11)$$

where for real  $\chi > 0$ ,  $\Delta(\chi)$  is defined by

$$\Delta(\chi) = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \chi^w \text{Exp} \left( \left( \sin \frac{w}{100} \right)^2 \right) \frac{dw}{w}. \quad (12)$$

Here  $w = u + iv$  is a complex variable and  $|\chi^w| = \chi^u$ . By moving the line of integration to  $u = 4$  and  $u = -4$  respectively we have

$$\Delta(\chi) = O(\chi^4) = 1 + O(\chi^{-4}) \quad (13)$$

and so the contribution to  $A_X(s)$  from very large  $n$  is very small. Notice that our condition on  $a_n$  in definition 2 implies that  $|a_n|$  does not exceed a fixed power (depending on  $T$  and  $H$ ) of  $n$ . Hence the expression (11) is a Titchmarsh polynomial plus a term whose absolute value is very small. We truncate the integral for  $A_X(s)$  suitably and we have (with  $Y \leq T^{\frac{1}{2}}$ )

$$A_X(s) = \frac{1}{2\pi i} \int_{u=2, |v| \leq Y} \dots + O \left( T^{2A} \log H X^2 \left( \text{ExpExp} \left( \frac{Y}{400} \right) \right)^{-1} \right). \quad (14)$$

Here moving the line of integration to  $u = -\delta$ , we have,

$$A_X(s) = F(s) + O(T^{2A \log H} X^{2(\dots)^{-1}}) + O\left(\int_{u=-\delta} T^{2A \log H} X^{-\delta} \frac{|dw|}{|w^2|}\right). \quad (15)$$

We note that  $H$  is not more than a positive constant times  $\log \log T$  and so by first choosing  $X$  to be a very large positive constant power of  $T^{A \log H}$  and then  $Y$  to be a large positive constant times  $\log \log T$ , we see that

$$A_X(s) = F(s) + \text{a very small term} \quad (16)$$

Thus on using Conjecture 1 we have

$$\frac{1}{H} \int_T^{T+H} |F(it)|^2 dt \geq \frac{3}{4} H^{-1} \int_T^{T+H} |A_X(s)|^2 dt \geq \frac{1}{2} c_1 \sum_{n \leq c_2 H} |a_n|^2. \quad (17)$$

This proves Theorem 1 completely.

§ 4. **FIRST PART OF THE PROOF OF THEOREM 2.** Let now  $s = \frac{1}{2} + \delta + it$  ( $t > 0$  large,  $\delta > 0$  now not a constant) and  $X \geq 1$ . With  $B_X(s)$  now defined by

$$B_X(s) = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} (\zeta(s+w))^k X^w \text{Exp}\left(\left(\sin \frac{w}{100}\right)^2\right) \frac{dw}{w} \quad (18)$$

$$= \sum_{n=1}^{\infty} \frac{d_k(n)}{n^s} \Delta\left(\frac{X}{n}\right), \quad (19)$$

We have as before

$$\Delta(\chi) = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \chi^w \text{Exp}\left(\left(\sin \frac{w}{100}\right)^2\right) \frac{dw}{w}, (\chi > 0) \quad (20)$$

and so

$$\Delta(\chi) = O(\chi^4) = 1 + O(\chi^{-4}). \quad (21)$$

Under the conditions of Theorem 2 we have for

$$z = \sigma + i\tau, (\sigma > \frac{1}{2}, |t - \tau| \leq c_3 \log \log \log T),$$

$$T - H_1 + c_3 \log \log \log T \leq \tau \leq T + H + H_1 - c_3 \log \log \log T,$$

$$-c_4 \frac{\log T}{\log \log T} \max \left( 1, \log \left( \frac{c_5}{\left(\sigma - \frac{1}{2}\right) \log \log T} \right) \right) \leq \log |\zeta(z)| \leq c_6 \frac{\log T}{\log \log T} \quad (22)$$

and

$$|\arg \zeta(z)| \leq c_7 \frac{\log T}{\log \log T}. \quad (23)$$

For these results see [8] and also [6]. These results are due to K. Ramachandra and A. Sankaranarayanan. For convenient reference we state in § 6 a very general result of theirs for use by others for further work if necessary. Note that if  $k = k_1 + ik_2$  ( $k_1, k_2$  real and  $k_1 \geq 0$ ) we have

$$|\zeta(z)^k| = |\text{Exp}((k_1 + ik_2)(\log |\zeta(z)| + i \arg \zeta(z)))|$$

$$= |\zeta(z)|^{k_1} \text{Exp}(-k_2 \arg \zeta(z)) \quad (24)$$

$$\leq \text{Exp} \left( c_8 \frac{\log T \log H}{\log \log T} \right) \quad (25)$$

As before we move the line of integration to  $u = -\delta$  and we have

$$B_X(s) = (\zeta(s))^k + O \left( T^{\frac{c_9 \log H}{\log \log T}} X^2(\dots)^{-1} \right) + O \left( \int_{u=-\delta, |v| \leq Y} T^{\frac{c_9 \log H}{\log \log T}} X^{-\delta} \frac{|dw|}{|w|} \right) \quad (26)$$

We note that  $H$  is not more than a constant times  $\log \log T$  and so by first choosing  $X$  by

$$X^{\frac{\delta}{2}} = T^{\frac{c_9 \log H}{\log \log T}} \quad (27)$$

and then choosing  $Y$  such that

$$T^{\frac{c_9 \log H}{\log \log T}} \cdot T^{\frac{\delta}{2}} \frac{c_9 \log H}{\log \log T} \left( \text{Exp Exp} \frac{Y}{100} \right)^{-1} \quad (28)$$

is very small we find that

$$B_X(s) = (\zeta(s))^k + \text{a small quantity} + O \left( X^{-\frac{\delta}{2}} \left( \log \frac{1}{\delta} + \log Y \right) \right). \quad (29)$$

The last  $O$ -term in (29) is small if  $T^{-\frac{c_9 \log H}{\log \log T}} (\log \frac{1}{\delta} + \log T)$  is small since  $1 \leq Y \leq T^{\frac{1}{2}}$ . In the next section we show that we can choose  $\delta$  to be a large



positive constant times  $T^{-\frac{\log H}{\log \log T}}$  in order to prove Theorem 2. Thus we have to choose  $Y$  to be greater than a large positive constant multiple of

$$\log \log \left( T^{\frac{8}{\delta} c_9 \frac{\log H}{\log \log T}} \right) = \log \left( \frac{8}{\delta} c_9 \frac{\log H \log T}{\log \log T} \right).$$

Thus (with  $s = \frac{1}{2} + \delta + it$ ) we have

$$\begin{aligned} \frac{1}{H} \int_T^{T+H} |(\zeta(s))^k|^2 dt &\gg \frac{99}{100} c_1 \sum_{n \leq c_2 H} |d_k(n)|^2 n^{-1-2\delta} \\ &\geq \frac{3}{4} c_1 \sum_{n \leq c_2 H} |d_k(n)|^2 n^{-1}. \end{aligned} \tag{30}$$

§ 5. DIFFERENCE OF TWO INTEGRALS. The rest of the work consists in showing that  $D$  defined by

$$D = \frac{1}{H} \int_T^{T+H} \left( |(\zeta(\frac{1}{2} + it))^k|^2 - |(\zeta(\frac{1}{2} + \delta + it))^k|^2 \right) dt, \tag{31}$$

has a small absolute value provided  $\delta$  is small, but not too small since we want  $\log \frac{1}{\delta}$  to be small. This would prove that

$$\frac{1}{H} \int_T^{T+H} |(\zeta(\frac{1}{2} + it))^k|^2 dt \geq \frac{1}{2} c_1 \sum_{n \leq c_2 H} |d_k(n)|^2 n^{-1}, \tag{32}$$

which is precisely Theorem 2.

By (24) we have

$$\begin{aligned} |D| &\leq \frac{1}{H} \int_T^{T+H} \left| \int_{\frac{1}{2}}^{\frac{1}{2}+\delta} \frac{\partial}{\partial \sigma} (\zeta(\sigma + it))^{2k} d\sigma \right| dt \\ &\leq \frac{|2k|}{H} \text{Exp} \left( 2c_8 \frac{\log T \log H}{\log \log T} \right) \int_T^{T+H} \int_{\frac{1}{2}}^{\frac{1}{2}+\delta} \left| \frac{\zeta'(\sigma + it)}{\zeta(\sigma + it)} \right| d\sigma dt. \end{aligned} \tag{33}$$

To estimate the double integral in (33) we need a few lemmas. We begin with

**LEMMA 1.** *Let  $f(z)$  be analytic in  $|z - z_0| \leq R$  and on the boundary of this disc let  $\text{Re } f(z) \leq U$ . Then*

$$f'(z_0) = O((U + |f(z_0)|)R^{-1}) \tag{34}$$

where the implied constant is absolute.

**PROOF.** The lemma is well-known (see page 8, Theorem 2.4.1 of [7]).

**LEMMA 2.** We have, with  $s = \sigma + it$  as in (33),

$$\frac{\zeta'(\sigma + it)}{\zeta(\sigma + it)} = O \left( \log T + |\log \zeta(s)| + \sum_{\rho \in D_0} |\rho - s|^{-1} \right) \quad (35)$$

where  $\rho$  runs over the zeros in the disc  $D_0$  given by  $|z - s| \leq R = \frac{1}{100}$ .

**PROOF.** Put  $z_0 = s$  and

$$f_0(z) = \frac{\zeta(z)}{\prod_{\rho \in D_0} \left(1 - \frac{z-s}{\rho-s}\right)}$$

we have

$$\max_{z \in D_0} |f_0(z)| \leq \max_{z \in D_1} |f_0(z)|,$$

where  $D_1$  is the disc  $|z - s| \leq \frac{1}{4}$ . Hence by applying Lemma 1 to  $f(z) = \log f_0(z)$  we obtain the lemma.

**LEMMA 3.** We have

$$\frac{1}{H} \int_T^{T+H} \int_{\frac{1}{2}}^{\frac{1}{2}+\delta} (\log T + |\arg \zeta(s)|) d\sigma dt = O(\delta \log T). \quad (36)$$

**PROOF.** Follows from the result of Ramachandra and Sankaranarayanan mentioned in (23).

**LEMMA 4.** We have

$$\frac{1}{H} \int_T^{T+H} \int_{\frac{1}{2}}^{\frac{1}{2}+\delta} \max \left( 1, \log \frac{c_5}{(\sigma - \frac{1}{2}) \log \log T} \right) d\sigma dt = O(\delta \log \log T + \delta^{\frac{1}{2}}). \quad (37)$$

**PROOF.** Follows from

$$\int_{\frac{1}{2}}^{\frac{1}{2}+\delta} \log \left( \frac{1}{\sigma - \frac{1}{2}} \right) d\sigma \leq \sqrt{\delta} \left( \int_{\frac{1}{2}}^{\frac{1}{2}+\delta} \left( \log \frac{1}{\sigma - \frac{1}{2}} \right)^2 d\sigma \right)^{\frac{1}{2}}$$

**LEMMA 5.** *We have*

$$\frac{1}{H} \int_T^{T+H} \int_{\frac{1}{2}}^{\frac{1}{2}+\delta} |\log |\zeta(s)|| \, d\sigma dt = O\left(\left(\delta \log \log T + \delta^{\frac{1}{2}}\right) \frac{\log T}{\log \log T}\right). \quad (38)$$

**PROOF.** Follows from Lemma 4 and the result of Ramachandra and Sankaranarayanan mentioned in (23).

**LEMMA 6.** *Let  $\rho$  be a zero of the Riemann zeta-function with  $\operatorname{Re} \rho \leq \frac{1}{2}$ .*

*Then*

$$\frac{1}{H} \int_T^{T+H} \int_{\frac{1}{2}}^{\frac{1}{2}+\delta} \frac{d\sigma dt}{|\rho - s|} = O(\delta^{\frac{1}{2}} \log T). \quad (39)$$

**PROOF.** Let  $\rho = \beta + i\gamma$  with  $\beta \leq \frac{1}{2}$ . Then  $|\rho - s|^{-1} \leq |\frac{1}{2} - \sigma + i\gamma|^{-1}$  and hence

$$\begin{aligned} \int_T^{T+H} \left| \frac{1}{2} - \sigma - it + i\gamma \right|^{-1} dt &= \int_{|t-\gamma| \leq \sigma - \frac{1}{2}} \dots + \int_{T \geq |t-\gamma| \geq \sigma - \frac{1}{2}} \dots \\ &= O(1) + O(\log T) + O\left(\log \frac{1}{c - \frac{1}{2}}\right) \end{aligned}$$

and hence the lemma follows.

**LEMMA 7.** *The number of zeros  $\rho$  counted with  $t \leq \operatorname{Im} \rho \leq T + 1$  and  $0 \leq \operatorname{Re} \rho \leq 1$ , is  $O(\log T)$ .*

**PROOF.** This is well-known. See for example Theorem 3.1.1 on page 11 of [7].

**LEMMA 8.** *We have*

$$\frac{1}{H} \int_T^{T+H} \int_{\frac{1}{2}}^{\frac{1}{2}+\delta} \left( \sum_{\rho \in D_0} |\rho - s|^{-1} \right) d\sigma dt = O(\delta^{\frac{1}{2}} (\log T)^3). \quad (40)$$

**PROOF.** Of course the sum over  $\rho$  depends on  $s$ . But we extend the sum over all the zeros of  $\zeta(s)$  in  $T-1 \leq t \leq T+H+1$ . Now we can interchange the sum over  $\rho$  and the double integral. The number of zeros is now  $O(H \log T)$ . The lemma now follows from Lemma 6.

**LEMMA 8.** *We have*

$$D = O\left(\delta^{\frac{1}{2}}(\log T)^4 \text{Exp}\left(2c_8 \frac{\log T \log H}{\log \log T}\right)\right) \quad (41)$$

**PROOF.** Follows from (33) and Lemmas 2 to 7.

**LEMMA 9.** *|D| is very small if  $\delta$  is taken to be a large positive constant power of  $\text{Exp}\left(-\frac{\log T \log H}{\log \log T}\right)$ .*

**PROOF.** Follows from Lemma 8.

Theorem 2 now follows from the results of § 4 and § 5.

§ 6. **A RESULT OF K. RAMACHANDRA AND A. SANKARA-NARAYANAN.** While stating Lemma 1 of § 2 on page 392 of [6] the condition (4) of that paper is not necessary. It was meant for other purposes. Accordingly we state this once again.

**THEOREM 5. (K. RAMACHANDRA AND A. SANKARA-NARAYANAN).** *Let*

$$D(s) = \sum_{n=1}^{\infty} a_n \lambda_n^{-s} \quad (42)$$

where  $a_1 = 1 = \lambda_1$ ,  $\frac{1}{A} \leq \lambda_{n+1} - \lambda_n \leq A$  ( $A \geq 1$  is a constant)  $\{\lambda_n\}$  is any sequence of real numbers and  $\{a_n\}$  is any sequence of complex numbers with  $|a_n| \leq n^A$ . Let  $\alpha > \delta$  ( $\delta > 0$  a constant) and let  $R(H, \alpha)$  denote the rectangle ( $\sigma \geq \alpha, T_1 - H \leq t \leq T_1 + H$ ). Let  $D(s)$  be continuable analytically in  $R(H, \alpha - \delta)$  and there  $\max |D(s)| \leq T^A$  (where  $A_5 \log \log \log T \leq H \leq \frac{1}{2}T$ ) and  $T_1$  is any number lying between  $T - H$  and  $2T + H$ . Let  $D(s) \neq 0$  in  $R(H, \alpha)$ . Then for  $t = T_1, s = \sigma + it$  in  $R(H, \alpha)$  we have uniformly for  $\sigma \geq \alpha, t = T_1$ ,

$$-A_1 \frac{\log T}{\log \log T} \max \left[ 1, \log \left( \frac{A_2}{(\sigma - \alpha) \log \log T} \right) \right] \leq \log |D(s)| \leq A_3 \frac{\log T}{\log \log T} \quad (43)$$

and

$$|\arg D(s)| \leq A_4 \frac{\log T}{\log \log T}. \quad (44)$$

Here  $A_1, A_2, \dots, A_5$  are positive constants depending only on  $\delta$  and  $A$ .

**Note.** It is enough to assume  $D(s) \neq 0$  in  $(\sigma > \alpha, T_1 - H \leq t \leq T_1 + H)$ .

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