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**A LARGE VALUE THEOREM FOR  $\zeta(s)$** 

BY

**K. RAMACHANDRA****(TO PROFESSOR MATTI JUTILA ON HIS FIFTY-SECOND BIRTHDAY)**

§ 1. **INTRODUCTION.** The first object of this paper is to prove the following theorem.

**THEOREM 1.** *Let  $\varepsilon > 0$  be a sufficiently small constant and let  $T$  exceed a large positive constant depending on  $\varepsilon$ . We write  $a = a(\varepsilon) = \varepsilon^{-5}$ . Let  $T \leq t_1 < t_2 < \dots < t_R \leq 2T, t_{j+1} - t_j \geq 1$  ( $j = 1, 2, \dots, R-1$ ) and  $|\log \zeta(1 + it_j)| \geq \log \log \log T + \log a$ , ( $j = 1, 2, \dots, R$ ). Then  $R \ll_{\varepsilon} T^{\varepsilon}$ .*

**COROLLARY.** *Let  $-T \leq q_1 < q_2 < \dots < q_{R'} \leq T, q_{j+1} - q_j \geq 1$  ( $j = 1, 2, \dots, R'-1$ ) and*

$$|\zeta(iq_j)| \geq \varepsilon^{-6} (q_j^*)^{\frac{1}{2}} \log \log q_j^* \quad (j = 1, 2, \dots, R')$$

where  $q_j^* = |q_j| + 100$ . Then  $R' \ll_{\varepsilon} T^{\varepsilon}$ .

**PROOF OF THE COROLLARY.** The proof follows from the functional equation.

We next use this corollary to deduce the following large value theorem from Montgomery's work (see the fundamental Lemma 8.1 in [1]).

**THEOREM 2.** *Let  $S$  be a set of complex numbers with the following*

properties. (i)  $\operatorname{Re} s \geq \sigma_0$  for all  $s \in S$ . (ii)  $1 \leq |\operatorname{Im}(s - s')| \leq T_0$  for all pairs  $(s, s')$  with  $s \neq s', s \in S, s' \in S$ , where  $T_0 \geq 100$ . Then for any integer  $N \geq 1$  and any set  $\{a_n\}$  ( $n = N, N + 1, \dots, 2N$ ) of complex numbers we have

$$\sum_{s \in S} \left| \sum a_n n^{-s} \right|^2 \ll_{\epsilon} GN + (T_0^{\epsilon} + |S| \log \log T_0) GT_0^{\frac{1}{2}}$$

where  $G = \sum |a_n|^2 n^{-2\sigma_0}$ ,  $\epsilon$  is a sufficiently small positive constant and  $T_0$  exceeds a constant depending only on  $\epsilon$ .

**REMARKS.** It suffices to prove the theorem in the case  $N > 1$ . For, when  $N = 1$  we have  $|\sum a_n n^{-s}|^2 \leq 2 \sum |a_n|^2 n^{-2\sigma_0}$ . Next it suffices to prove this theorem when  $\sigma_0 \leq \operatorname{Re} s \leq \sigma_0 + \frac{1}{10}$  for all  $s \in S$ . The general case follows by dividing the range for  $\operatorname{Re} s$  into intervals of length  $\frac{1}{10}$  and then adding up the various RHS with  $|S|$  in each case replaced by the sum of the various  $|S|$ .

Given an inequality of the type of Theorem 2 with  $\log \log T_0$  replaced by  $(\log(NT_0))^2$ , M.N. Huxley made an important deduction which is well-known. We deduce from Theorem 2 the following important theorem by his method.

**THEOREM 3.** Let  $S^*$  be a set of complex numbers with the following properties. (i)  $\operatorname{Re} s \geq \sigma_0$  for all  $s \in S^*$ . (ii)  $|\operatorname{Im} s| \leq T$  for all  $s \in S^*$ . (iii)  $|\operatorname{Im}(s - s')| \geq 1$  for all pairs  $(s, s')$  with  $s \neq s', s \in S^*, s' \in S^*$ . Let  $N (\geq 1)$  be an integer and let  $\{a_n\}$  ( $n = N, N + 1, \dots, 2N$ ) be any set of complex numbers. Then for any real  $V > 0$ , the number of elements of  $S^*$  with  $|\sum a_n n^{-s}| \geq V$  is

$$\ll_{\epsilon} GNV^{-2} + TG^3NV^{-6}(\log \log T)^2 + T^{\epsilon}(1 + TG^2V^{-4}),$$

where  $G = \sum |a_n|^2 n^{-2\sigma_0}$  and  $T$  exceeds a large positive constant depending only on  $\epsilon, \epsilon (> 0)$  being any constant.

Since Huxley's method is very simple we give the deduction here itself. We divide the range  $(-T, T)$  for  $\operatorname{Im} s (s \in S^*)$  into at most  $2(1 + TT_0^{-1})$  intervals of length  $T_0$ . Suppose we fix any  $\epsilon$  and make the following assumptions. (i)  $T_0^{\epsilon} \leq |S|$  and (ii)  $GT_0^{\frac{1}{2}} \log \log T = \eta V^2$ , where  $\eta > 0$  is a small constant.

Then from Theorem 2 it follows that  $|S| \ll_{\varepsilon, \eta} GNV^{-2}$ . If (i) is not satisfied then we have  $|S| \leq T_0^\varepsilon$  and so in any case  $|S| \ll_{\varepsilon, \eta} GNV^{-2} + T_0^\varepsilon$ , provided only that we fix  $T_0$  by (ii). Thus (since  $T_0 \leq T$ )

$$\begin{aligned} |S^*| &\ll_{\varepsilon, \eta} (1 + TT_0^{-1})(GNV^{-2} + T^\varepsilon) \\ &\ll_{\varepsilon, \eta} (GNV^{-2} + T^\varepsilon)(1 + TG^2V^{-4}(\log \log T)^2). \end{aligned}$$

(We have assumed in this reasoning that  $T_0 \gg_\varepsilon 1$ . But if  $T_0 \ll_\varepsilon 1$  the term  $TT_0^{-1}T^\varepsilon$  is a trivial upper bound for  $|S^*|$ ). Thus Theorem 3 is proved since  $\eta$  depends only on  $\varepsilon$  and we can replace  $\varepsilon$  by  $\frac{1}{2}\varepsilon$ .

**REMARK.** From Theorem 3 we can (proceeding as in the appendix, § A.3 of [3]) prove that (if  $\frac{3}{4} \leq \sigma \leq \frac{3}{4} + \frac{D \log \log T}{\log T}$ ) then

$$N(\sigma, T) \ll_D T^{A_2(\sigma)(1-\sigma)} (\log T)^{8-\frac{1}{5}} (\log \log T)^{2-\frac{2}{5}}, \quad (D > 0 \text{ is any constant}),$$

with  $A_2(\sigma) = (5\sigma - 3)(\sigma^2 + \sigma - 1)^{-1}$ , which is a minor improvement over the result

$$N(\sigma, T) \ll T^{A_2(\sigma)(1-\sigma)} (\log T)^9$$

due to M.N. Huxley. A slightly cruder form of this improvement (with  $8 - \frac{1}{5} + \varepsilon$  in place of  $8 - \frac{1}{5}$  and  $\ll_{\varepsilon, D}$  in place of  $\ll_D$ ) was announced in the post-script to [3]. These refinements have also application to prime number theory similar to what has been done in [3].

**NOTATION.** The notation is mostly standard and whenever there is a departure it will be explained in the respective sections.

§ 2. **PROOF OF THEOREM 1.** Throughout this section  $T \geq 100$  (since we are interested in the bound  $\ll_\varepsilon T^\varepsilon$ , we can assume that  $T$  exceeds a positive constant depending only on  $\varepsilon$ ),  $T \leq t \leq 2T$ ,  $e(u) = \text{Exp}(-u)$ ,  $X = \text{Exp}((\log T)^{200})$ ,  $r = (\log X)^{-2}$ ,  $z = x + iy$  and finally with real  $A, B$  (with  $B > A > 0$ ) and  $\text{Re } z \geq 1 - r$  we write

$$f(A, B, z) = \sum_{A \leq p < B} p^{-z} e\left(\frac{p}{X}\right).$$

**LEMMA 1.** *We have*

$$\log \zeta(1 + it) = \sum p^{-1-it} e\left(\frac{p}{X}\right) + O(1),$$

where  $p$  runs over all primes and the  $O$ -constant is absolute.

**PROOF.** Follows (since higher powers of  $p$  contribute  $O(1)$ ) from

$$\sum p^{-1-it} e\left(\frac{p}{X}\right) + O(1) = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} (\log \zeta(1 + it + w)) X^{wT} \Gamma(w) dw$$

(where  $w = u + iv$  is a complex variable) on moving the line of integration suitably, using the zero-free region  $\sigma \geq 1 - (\log T)^{-10}$  and the estimates for  $\log \zeta(1 + it + w)$  in the zero-free region.

**LEMMA 2.** *The inequality  $|\log \zeta(1 + it)| \geq \log \log \log T + \log a$  implies at least one of the following inequalities.*

(i)  $|f(T, \infty, 1 + it)| \geq 1$

(ii)  $|f(Y, T, 1 + it)| \geq \frac{1}{3} \log a$ , where  $\log \log Y = \log \log \log T + \frac{1}{4} \log a$ .

**PROOF.** Trivial since  $\sum_{p \leq Y} p^{-1} = \log \log Y + \gamma_0 + O((\log Y)^{-1})$ , where  $\gamma_0$  is an absolute constant.

**LEMMA 3.** *If  $f(z)$  is analytic in  $|z - z_0| \leq r$ , we have*

$$|f(z_0)| \leq \frac{1}{\pi r^2} \iint |f(z)| dx dy$$

where the integration is over the disc  $|z - z_0| \leq r$ .

**PROOF.** Follows from

$$|f(z_0)| \leq \frac{1}{2\pi} \int_0^{2\pi} |f(z_0 + \lambda e^{i\theta})| d\theta$$

valid for all  $\lambda$  with  $0 < \lambda \leq r$ . (We have only to multiply by  $\lambda d\lambda$  and integrate from  $\lambda = 0$  to  $\lambda = r$ ).

**LEMMA 4.** *The number  $R_1$  of points  $t_j$  with  $|f(T, \infty, 1 + it_j)| \geq 1$  satisfies*

$$R_1 \leq \frac{1}{\pi r^2} \iint |f(T, \infty, z)|^2 dx dy$$

where the integration is over the rectangle  $|x-1| \leq r, T-1 \leq y \leq 2T+1$ .

**PROOF.** Follows from Lemma 3 (since the discs with the centres  $1+it$ , and radii  $r$  are all disjoint).

**LEMMA 5.** *Uniformly in  $|x-1| \leq r$ , we have*

$$\int_{T-1}^{2T+1} |f(T, \infty, z)|^2 dy \ll \log \log X.$$

**PROOF.** By the well-known Montgomery Vaughan theorem we have

$$\int_T^{2T} \left| \sum_n d_n n^{-it} \right|^2 dt \ll \sum_n (T+n) |d_n|^2$$

where  $\{d_n\} (n=1, 2, \dots)$  is any sequence of complex numbers such that  $\sum n |d_n|^2$  is convergent. From this the LHS of the lemma is

$$\ll \sum p^{-1+2r} e\left(\frac{2p}{X}\right) \ll \sum_{p \leq X^2} p^{-1} \ll \log \log X.$$

This proves the lemma.

**LEMMA 6.** *Divide the interval  $[Y, T]$  into  $\leq 3 \log T$  intervals  $I$  by interposing the points  $U$  which are powers of 2. The extreme two intervals are of the type  $[Y, Y+V_1)$  and  $[U, T)$  with  $0 \leq V_1 \leq Y$ . Then for at least one interval  $I$  we have*

$$\left| \sum_{p \in I} p^{-1-it} e\left(\frac{p}{X}\right) \right| \geq (\log T)^{-1}$$

**PROOF.** Trivial.

**LEMMA 7.** *For any interval  $I$  define an integer  $k = k(I) \geq 1$  to be the least integer such that  $(\min_{p \in I} p)^k \geq T$ . Define  $a_k(n)$  by*

$$\left( \sum_{p \in I} p^{-z} e\left(\frac{p}{X}\right) \right)^k = \sum_{T \leq n \leq 2^{2k} T} a_k(n) n^{-z}.$$

Then  $1 \leq k \leq (\log T)(\log Y)^{-1}$  and

$$0 \leq a_k(m) \leq k^k.$$

**PROOF.** The lemma is obvious since whenever  $a_k(n) \neq 0$ ,  $n$  is the power product of primes with  $k$  as the sum of the exponents.

**LEMMA 8.** Let  $R(I)$  denote the number of  $t_j$  satisfying the final inequality of Lemma 6. Then

$$R(I) \leq (\log T)^{2k} \frac{1}{\pi r^2} \int \int \left| \sum_{p \in I} p^{-z} e\left(\frac{p}{X}\right) \right|^{2k} dx dy$$

where the integration is over the rectangle  $|x-1| \leq r, T-1 \leq t \leq 2T+1$ .

**PROOF.** Follows from Lemma 3 just as Lemma 4 followed from Lemma 3.

**LEMMA 9.** We have

$$\begin{aligned} R(I) &\ll (\log T)^{2k+500} \sum_{T \leq n \leq 2^{2k}T} |a_k(n)|^2 n \\ &\ll (\log T)^{2k+500} k^{2k+2}. \end{aligned}$$

**PROOF.** Follows from the well-known theorem of Montgomery and Vaughan (in view of Lemma 7).

**LEMMA 10.** We have

$$R(I) \ll T^\epsilon.$$

**PROOF.** Follows from

$$k \leq (\log T)(\log Y)^{-1} \leq (\log T)(\log \log T)^{-1} a^{-\frac{1}{4}}$$

and hence

$$(\log T)^k \cdot k^k \ll \text{Exp}(3k \log \log T) \ll \text{Exp}(3a^{-\frac{1}{4}} \log T)$$

where the implied constants are absolute.

**LEMMA 11.** *The number  $R_2$  of numbers  $t = t_j$  which satisfy the second alternative of Lemma 2 satisfies*

$$R_2 \leq \sum_I R(I) \leq T^{2\epsilon}$$

**LEMMA 12.** *We have*

$$R \ll_{\epsilon} T^{\epsilon}.$$

**PROOF.** Since  $R \leq R_1 + R_2$ , the lemma follows by Lemmas 4,5 and 11.

§ 3. **PROOF THEOREM 2.** We begin by stating the following theorem of H.L. Montgomery (which is a special case of his fundamental Lemma 8.1 of [1]).

**THEOREM 4 (H.L. MONTGOMERY).** *In the notation of Theorem 2 above, we have*

$$\sum_{s \in S} \left| \sum a_n n^{-s} \right|^2 \ll \left( \sum |a_n|^2 b_n^{-1} \right) \max_{s \in S} \sum_{s' \in S} |B(\bar{s} + s')|,$$

where  $b_n = \left( \text{Exp}\left(-\frac{n}{2N}\right) - \text{Exp}\left(-\frac{n}{N}\right) \right) n^{2\sigma_0}$  and for any complex  $z$  we define  $B(z)$  by

$$B(z) = \sum_{n=1}^{\infty} b_n n^{-z}.$$

**PROOF.** This is a special case of Lemma 8.1 of [1].

We continue with the proof of Theorem 2. Note that we can assume  $N > 1$  and also  $\sigma_0 \leq \text{Re } s \leq \sigma_0 + \frac{1}{10}$  for all  $s \in S$ . Also note that  $\sum |a_n|^2 b_n^{-1} \ll G$ . We write  $s = \sigma + it$ ,  $s' = \sigma' + it'$  and we have

$$B(\bar{s} + s') = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \zeta(w + \bar{s} + s' - 2\sigma_0) N^w (2^w - 1) \Gamma(w) dw$$

where  $w = u + iv$  is a complex variable. Moving the line of integration to  $u$  given by  $u + \sigma + \sigma' - 2\sigma_0 = 0$  we obtain

$$|B(\bar{s} + s')| \ll N \text{Exp}(-|t - t'|) + \int_{-\infty}^{\infty} |\zeta(iv - it + it')| \text{Exp}(-|v|) dv.$$



For any fixed  $t$ , we have uniformly

$$\sum_{s'} |B(\bar{s} + s')| \ll N + \sum_{t'} \int_{-\infty}^{\infty} |\zeta(it - t' + it')| \text{Exp}(-|v|) dv.$$

When  $|t - t'|$  is bounded above the last integral is bounded. Also for the remaining  $t'$  the contribution from  $|v| \geq (\log |t - t'|)^2$  is negligible. Thus

$$\sum_{s'} |B(\bar{s} + s')| \ll N + \sum_{t', |t-t'| \geq 1000} \int_{|v| \leq (\log |t-t'|)^2} |\zeta(i(v-t+t'))| \text{Exp}(-|v|) dv + |S|$$

We look at the maximum  $M(t - t')$  of  $|\zeta(i(v - t + t'))|$  in the range of integration. Suppose this is attained at  $v = \tau(t')$  say. For fixed  $t$ , the points  $\tau(t') - t + t' = q_j$  with  $M(t - t') \geq \varepsilon^{-7} (q_j^*)^{\frac{1}{2}} \log \log q_j^*$  (for varying  $t'$  form a union of  $\leq (\log T_0)^3$  sets of points (at mutual distances  $\geq 1$ ) and by corollary to Theorem 1 the total number of such points  $t'$  (note that  $T = 2T_0$ ) is  $O((\log T_0)^3 T_0^\varepsilon) = O(T_0^{2\varepsilon})$ . Hence with the exception of  $O(T_0^{2\varepsilon})$  points  $t'$  we have  $M(t - t') \leq \varepsilon^{-7} (|t - t'| + 100)^{\frac{1}{2}} \log \log (|t - t'| + 100)$ . Thus the contribution to  $\sum_{s'} |B(\bar{s} + s')|$  from the exceptional points  $t'$  is  $O(T_0^{\frac{1}{2} + 3\varepsilon})$ , (we employ the trivial bound  $M(t - t') = O(T_0^{\frac{1}{2}} \log T_0)$  for exceptional points). The other points contribute  $O(|S| T_0^{\frac{1}{2}} \log \log T_0)$  and this proves Theorem 2 completely (on noting that we can replace  $\varepsilon$  by  $\frac{1}{4}\varepsilon$  if necessary).

**FINAL REMARK.** The analogue of Theorem 1 to "short intervals" is somewhat delicate and was investigated in [2]. The upper bound therein is in fact  $O_\varepsilon(1)$ , provided  $|\log \zeta(1 + it)| \geq \varepsilon \log \log T$ . The present work although self contained may in some sense be considered as a continuation of [2].

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P.S.: After writing up the paper I noticed that Theorem 1 can be deduced easily from any zero density estimate which ensures that

$$\limlim_{\sigma, T} \{(\log T)^{-1} \log(N(\sigma, T) + 1)\} = 0$$

as  $T \rightarrow \infty$  and then  $\sigma \rightarrow 1 - 0$ .