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## ESTIMATE OF SUMS OF DIRICHLET SERIES

Shituo Lou and Qi Yao

### § 1. INTRODUCTION.

Let  $x$  be a large positive number,  $\varepsilon$  be a small positive number and  $k$  be a finite integer. Let  $y = x^\theta$ ,  $1/2 < \theta < 7/12$ ,  $T = x^{1-\theta+\varepsilon/2}$ , and let  $M_1, \dots, M_k$  and  $L$  be real numbers such that

$$M_1 \cdots M_k L = x/2. \quad (1.1)$$

$U_1, \dots, U_k$  and  $W$  be real numbers such that

$$W \ll L^{1/2-\varepsilon} \quad (1.2)$$

and

$$U_i \ll M_i^{1/2}, 1 \leq i \leq k. \quad (1.3)$$

Let  $A$  be a fixed integer, and  $|S|$  be the number of elements of set  $S$ . We discuss the set  $S(U_1, \dots, U_k, W)$  which satisfies following conditions, for  $1 \leq i \leq k$  and  $j$  be a positive integer,

$$(1.4) \quad |S(U_1, \dots, U_k, W)| \ll U_i^{-2j} (M_i^j + T) (\log x)^A \text{ and } |S(U_1, \dots, U_k, W)| \ll W^{-2j} (L^j + T) (\log x)^A;$$

$$(1.5) \quad |S(U_1, \dots, U_k, W)| \ll (U_i^{-2j} M_i^j + U_i^{-6j} M_i^j T) (\log x)^A;$$

$$(1.6) \quad |S(U_1, \dots, U_k, W)| \ll (W^{-2j} L^j + W^{-6j} L^j T) (\log x)^A;$$

$$(1.7) \quad W^4 |S(U_1, \dots, U_k, W)| \ll T(\log x)^A.$$

Let  $a_j = \log M_j / \log x$ . In § 2 we shall give some functions  $h_i(a_1, \dots, a_k)$  for  $1 \leq i \leq 3$ , such that

$$U_1 \cdots U_k W |S(U_1, \dots, U_k, W)| \ll x^{1/2-\varepsilon} + x^{h_i(a_1, \dots, a_k)}. \quad (1.8)$$

In § 3, we discuss those  $S(U_1, \dots, U_k, W)$ 's which satisfy one more inequality : for  $1 \leq j \leq k$ ,

$$|S(U_1, \dots, U_k, W)| \ll W^{-4} U_i^{-2} (M_j^2 T^{1/2} + M_j^{5/4} T^{3/4} + T) T^\varepsilon. \quad (1.9)$$

If  $T^{1/5} \leq M_j \leq T^{1/3}$ , we replace (1.9) by

$$|S(U_1, \dots, U_k, W)| \ll W^{-4} U_i^{-2} M_j^{5/4} T^{3/4+\varepsilon}, \quad (1.10)$$

and, if  $M_j \leq T^{1/5}$ , we replace (1.9) by

$$|S(U_1, \dots, U_k, W)| \ll W^{-4} U_i^{-2} T^{1+\varepsilon}. \quad (1.11)$$

In § 3 we will give  $h_i(a_1, \dots, a_k)$  for  $4 \leq i \leq 10$  with (1.8).

Heath-Brown and Iwaniec [1] discussed the gaps between consecutive primes using sieve method. The remainder term

$$R(x; M_1, \dots, M_k) = \sum_{\substack{M_i < m_i \leq 2M_i \\ 1 \leq i \leq k}} a_{m_1, 1}, \dots, a_{m_k, k} r_{m_1 \dots m_k}, \quad (1.12)$$

where

$$r_d = \left[ \frac{x}{d} \right] - \left[ \frac{x-y}{d} \right],$$

$M_i < y$  and  $|a_{m_i, i}| \leq 1$ , be considered. Applying the method which is very close to Heath-Brown and Iwaniec's, in § 5, we will show that

$$R(x; M_1, \dots, M_k) \ll x^{\theta-\varepsilon}, \quad (1.13)$$

if (1.8) holds with

$$h_i(a_1, \dots, a_k) < 1/2, \quad (1.14)$$

In fact, they proved that (see [1]) if  $\theta = 11/20 + \varepsilon$ ,  $k = 2$ ,  $M_1 \ll x^{0.46-\varepsilon}$ , and  $M_2 \ll x^{0.46-\varepsilon}$ , (1.13) holds. Consequently, they obtained that for  $y = x^\theta$ ,  $\theta \geq 11/20 + \varepsilon$ ,

$$\pi(x) - \pi(x-y) > \frac{1}{212} \frac{y}{\log x}, \quad (1.15)$$

where  $\pi(x)$  be the number of primes  $\leq x$ .

In [2], Heath-Brown discussed some kind of products of Dirichlet series

$$W(s) = X(s) \prod_{j=1}^{k_0} Y_j(s),$$

where

$$X(s) = \sum_{L_0 < n \leq 2L_0} n^{-s},$$

$$Y_j(s) = \sum_{M_j < n \leq 2M_j} b_m n^{-s}, \quad |b_m| \leq 1;$$

and

$$L_0 \prod_{j=1}^{k_0} M_j = \frac{x}{2}.$$

Using Heath-Brown's method that was used in [2], in § 5, we shall show that

$$\int_T^{2T} |W(\frac{1}{2} + it)| dt \ll x^{\frac{1}{2}} \exp(-(\log x)^{\frac{1}{3}} (\log \log x)^{\frac{2}{3}}) \quad (1.16)$$

for

$$T_1 \leq T \leq x^{1-\theta+\varepsilon},$$

where  $\theta$  is a fixed positive constant with  $1/2 < \theta < 1$ , and

$$T_1 = \exp((\log x)^{\frac{1}{3}} (\log \log x)^{-\frac{1}{3}}),$$

if (1.8) holds with (1.14).

In [3], and [4], we shall apply the results of this paper to investigate gaps between consecutive primes and prove that (1.11) holds for  $\theta = 11/20 + \varepsilon$

in [3] and  $\theta = 6/11 + \varepsilon$  in [4]. In Section 4, we also discuss the case of  $\theta = 11/20 + \varepsilon$  that will be used in [3] to prove that

$$0.99 \frac{y}{\log x} < \pi(x) - \pi(x-y) < 1.01 \frac{y}{\log x}.$$

In Section 4 we discuss the case of  $\theta = 6/11 + \varepsilon$  as well that will be used in [4] to prove that

$$\frac{0.969y}{\log x} < \pi(x) - \pi(x-y) < \frac{1.031y}{\log x},$$

for  $\theta = 6/11 + \varepsilon$ .

In this paper  $\delta$  be another positive small number with  $\delta \ll \varepsilon$  and may be difference in some paragraphs. Also, we use

$$c\delta < \delta.$$

## § 2. ESTIMATE OF $|S(U_1, \dots, U_k, W)|$ .

In this section we estimate  $|S(U_1, \dots, U_k, W)|$ . First we divide  $M_1, \dots, M_k$ , and  $L$  into  $k_0 + 1$  parts (in this paper  $2 \leq k_0 \leq 6$ ), denote the products of elements in one part by  $N_1, \dots, N_{k_0}$ , and  $L_0$ , respectively. Also denote the product of those  $U_k$ 's or  $W$  that correspond to  $N_j$  or  $L_0$  by  $U_j$  or  $W$ , again. We now replace (1.1), (1.2) and (1.3) by

$$N_1 \cdots N_{k_0} L_0 = \frac{x}{2}, \quad (2.1)$$

$$U_j \leq N_j^{\frac{1}{2}} A_j, \quad 1 \leq j \leq k_0, \quad (2.2)$$

and

$$W \leq L_0^{\frac{1}{2}} A_0 \quad (2.3)$$

where  $A_j$  (or  $A_0$ ) =  $x^{-\delta}$  if  $L$  belongs to  $N_j$  (or  $L_0$ ), otherwise  $A_j = 1$ . Thus we have that

$$W \prod_{i=1}^k U_i \ll \left( L_0 \prod_{i=1}^k M_i \right)^{\frac{1}{2} - \delta} \ll x^{\frac{1}{2} - \delta}. \quad (2.4)$$

It is clear that if  $L_0$  satisfies (1.6) with  $i = 2$  and  $L_0 \ll T^{1/2}$ , then  $L_0$  satisfies (1.7). In this paper, assume that

$$L_0 \ll T^{1/2} \text{ if } L_0 \neq L. \quad (2.5)$$

Replacing  $L$  by  $L_0$ , we shall show that the function  $|S(U_1, \dots, U_{k_0}, W)|$  with (1.4), (1.5), (1.6), (1.7) and (2.4) such that (1.8).

We now renew the notations  $a_i = \log N_i / \log x$ ,  $i = 1, \dots, k_0$ ,  $\sigma = \log L_0 / \log x = 1 - a_1 - \dots - a_{k_0}$  and  $t_0 = \log T / \log x$ . Let

$$F = |S(U_1, \dots, U_{k_0}, W)| x^{-\delta} \quad (2.6)$$

We replace  $|S(U_1, \dots, U_{k_0}, W)|$  by  $F$ , it makes that we can drop the factor  $\log^A x$  in the right hand side of (1.4), (1.5), (1.6) and (1.7), and drop  $T^\epsilon$  in (1.9).

If  $F \geq 2U_i^{-2j} N_i^j$ , then we have that

$$N_i \ll T^{1/j} \quad (2.7)$$

since  $F \leq U_i^{-2j} (N_i^j + T)$ .

Suppose  $j$  such that

$$\sigma \leq t_0/j \text{ and } j = 2 \text{ if } \sigma \geq t_0/2. \quad (2.8)$$

Thus, if  $j > 2$ , we have  $\sigma \leq t_0/j$ , i.e.  $W^j \leq T$ . Then

$$F \ll W^{-2j} T.$$

We prove following lemmas :

**Lemma 2.1.** *Let  $k_0 = 2$ , we have*

$$U_1 U_2 |S(U_1, U_2, W)| \ll x^{h_0(a_1, a_2) + \epsilon/2} + x^{1/2 - \delta}, \quad (2.9)$$

where  $h_0(a_1, a_2) = h_0(a_2, a_1)$ ,  $a'_1 = \max\{t_0, a_1\}$  and

$$h_0(a_1, a_2) = \begin{cases} \frac{1}{2} - \epsilon, & a_1 \geq a_2 \geq t_0 \\ \frac{t_0}{2} + \frac{a_1}{2} + \min\left\{\frac{a_2}{4j}, \frac{\sigma}{8} + \frac{a_2}{12j}\right\}, & \text{otherwise.} \end{cases} \quad (2.10)$$

$$(2.11)$$

**Proof.** We have that, by (1.4), (1.5) and (1.6),

$$F \leq \min\{U_1^{-2}(N_1 + T), U_2^{-2}(N_2 + T), U_1^{-2}N_1 + U_1^{-6}N_1T, U_2^{-2}N_2 + U_2^{-6}N_2T, \\ W^{-2j}L_0^j + W^{-6j}L_0^jT, W^{-2j}T\},$$

and proceed to show that

$$U_1U_2WF \ll x^{h_0(a_1, a_2)} + x^{\frac{1}{2}-\delta}. \quad (2.12)$$

We now consider four cases :

**Case 1.**  $F \leq 2U_1^{-2}N_1$  and  $F \leq 2U_2^{-2}N_2$ .

Choosing  $k'$  such that  $L_0^{k'} \geq T$ , we have  $F \ll W^{-2k'}L_0^{k'}$ , and then

$$U_1U_2WF \ll U_1U_2W(U_2^{-2}N_2)^{\frac{1}{2}-\frac{1}{2(2k'+1)}}(U_1^{-2}N_2)^{\frac{1}{2}-\frac{1}{2(2k'+1)}}(W^{-2k'}L_0^{k'})^{\frac{1}{2k'+1}} \\ \ll (U_1U_2W)^{\frac{1}{2k'+1}}(N_1N_2L_0)^{\frac{1}{2}-\frac{1}{2(2k'+1)}} \ll x^{\frac{1}{2}-\delta}$$

by (2.4). Thus (2.12) holds. (Note : when  $a_1 \geq a_2 \geq t_0$ , we have that

$$F \leq 2U_1^{-2}N_1 \text{ and } F \leq 2U_2^{-2}N_2$$

by (1.4) with  $i = 1$ . This completes the proof of (2.10).)

**Case 2.**  $F \leq 2U_1^{-2}N_1$ ,  $F > 2U_2^{-2}N_2$ .

In this case we have that  $F \ll U_2^{-2}T$ ,  $F \ll U_2^{-6}N_2T$  and  $N_2 \ll T$  since (1.4), (1.5) (take  $M_i = N_i$ ) (2.7) and  $F > 2U_2^{-2}N_2$ . Thus

$$F \leq 2\min\{U_1^{-2}N_1, U_2^{-2}T, U_2^{-6}N_2T, W^{-2j}L_0^j, W^{-2j}T\} \\ + 2\min\{U_1^{-2}N_1, U_2^{-2}T, U_2^{-6}N_2T, W^{-2j}L_0^jT, W^{-2j}T\} \\ \leq 2(U_1^{-2}N_1)^{1/2}(U_2^{-2}T)^{(2j-3)/4j}(U_2^{-6}N_2T)^{1/4j}(W^{-2j}\min\{L_0^j, T\})^{1/2j} \\ + 2\min\{(U_1^{-2}N_1)^{1/2}(U_2^{-2}T)^{(2j-3)/4j}(U_2^{-6}N_2T)^{1/4j}(W^{-2j}T)^{1/2j}, \\ (U_1^{-2}N_1)^{1/2}(U_2^{-2}T)^{(2j-1)/4j}(U_2^{-6}N_2T)^{1/12j}(W^{-6j}L_0^jT)^{1/6j}\} \\ \ll (U_1U_2W)^{-1}N_1^{1/2}T^{1/2-1/2j}N_2^{1/4j}\min\left\{L_0^{\frac{1}{2}}, T^{\frac{1}{2j}}\right\} \\ + (U_1U_2W)^{-1}(TN_1)^{1/2}\min\left(N_2^{1/4j}, N_2^{1/12j}L_0^{1/6}\right). \quad (2.13)$$

If  $L_0 \ll T^{1/j}$ , we have that

$$L_0^{\frac{1}{2}} \ll T^{\frac{1}{2j}} \text{ and } N_2^{\frac{1}{6j}} L_0^{\frac{1}{3}} \ll T^{\frac{1}{2j}}.$$

Thus the first term on the right hand side of (2.13) is less than the second term. If  $L_0 \gg T^{1/j}$ , by (2.7), we have  $j = 2$ . We have that the first term on the right hand side of (2.13) is less than the second term again. Then

$$F \ll (U_1 U_2 W)^{-1} (T N_1)^{1/2} \min \left( N_2^{1/4j}, N_2^{1/12j} L_0^{1/6} \right).$$

Thus (2.12) and (2.11) follows.

**Case 3.**  $F \leq 2U_2^{-2} N_2, F \geq 2U_1^{-2} N_1$ .

The proof here is the same as in Case 2.

**Case 4.**  $F > 2U_1^{-2} N_1, F > 2U_2^{-2} N_1$ .

We have that

$$\begin{aligned} F &\leq 2 \min \left\{ U_1^{-2} T, U_2^{-2} T, U_1^{-6} N_1 T, U_2^{-6} N_2 T, W^{-2j} L_0^j, W^{-2j} T \right\} \\ &+ 2 \min \left\{ U_1^{-2} T, U_2^{-2} T, U_1^{-6} N_1 T, U_2^{-6} N_2 T, W^{-6j} L_0^j, W^{-2j} T \right\}. \end{aligned}$$

Then

$$\begin{aligned} F &\ll (U_1^{-2} T)^{\frac{1}{2}} (W^{-2j} L_0^j)^{\frac{1}{6j}} (W^{-2j} T)^{\frac{1}{6j}} (U_2^{-2} T)^{\frac{1}{2} - \frac{3}{4j}} (U_2^{-6} N_2 T)^{\frac{1}{4j}} \\ &+ (U_1^{-2} T)^{\frac{1}{2}} (W^{-6j} L_0^j T)^{\frac{1}{6j}} (U_2^{-2} T)^{\frac{2j-1}{4j}} (U_2^{-6} N_2 T)^{\frac{1}{4j}} \\ &\ll (U_1 U_2 W)^{-1} \left( T^{1-\frac{1}{6j}} L_0^{\frac{1}{6}} N_2^{\frac{1}{4j}} + T L_0^{\frac{1}{6}} N_2^{\frac{1}{12j}} \right) \\ &\leq (U_1 U_2 W)^{-1} T L_0^{\frac{1}{6}} N_2^{\frac{1}{12j}}, \end{aligned}$$

since  $N_2 \ll T$  by (2.7) with  $j = 1$ . Thus we get (2.12) and (2.11) again.

**Lemma 2.2.** *If  $N_2 \leq T$ , we have*

$$U_1 U_2 W |S(U_1, U_2, W)| \ll x^{\frac{1}{2}-\delta} + x^{h_1^i(a_1, a_2)}, \quad (2.14)$$

where

$$h_1(a_1, a_2) = \frac{i+1}{2(i+2)} t_0 + \frac{1}{4(i+2)} + \frac{a_1' + \sigma_i}{2} - \frac{a_1 + \sigma}{4(i+2)},$$



$$a'_1 = \max\{t_0, \log N_1 / \log x\} \text{ and } \sigma_i = \max\left\{\frac{t_0}{(i+2)}, 1 - a_1 - a_2\right\}.$$

**Proof.** We consider two cases :

**Case 1.**  $F \geq 2U_2^{-2}N_2$

In this case,  $N_2 \ll T$ , then

$$\begin{aligned} F &\ll \min\left\{U_1^{-2}x^{a'_1}, U_2^{-2}T, U_2^{-6}N_2T, W^{-2(i+2)}x^{(i+2)\sigma_i}\right\} \\ &\ll (U_1^{-2}x^{a'_1})^{1/2}(W^{-2(i+2)}x^{(i+2)\sigma_i})^{1/2(i+2)}(U_2^{-2}T)^{(2i+1)/4(i+2)}(U_2^{-6}N_2T)^{1/4(i+2)} \\ &\ll U_1^{-1}U_2^{-1}W^{-1}x^{(a'_1+\sigma_i)/2}T^{(i+1)/2(i+2)}N_2^{1/4(i+2)} \\ &\ll U_1^{-1}U_2^{-1}W^{-1}x^{1/4(i+2)}x^{a'_1/2-a_1/4(i+2)}T^{(i+1)/2(i+2)}x^{\sigma_i/2-\sigma/4(i+2)}, \end{aligned}$$

since  $N_2 = x/N_1L_0 = x^{1-a_1-\sigma}$ .

Then (2.14) follows.

**Case 2**  $F < 2U_2^{-2}N_2$

If  $F < 2U_1^{-2}N_1$ , the proof of this lemma is same as Case 1 of Lemma 2.1.

If  $F \geq 2U_1^{-2}N_1$ , then  $N_1 \leq T$ . Interchanging  $N_1$  with  $N_2$ , we may reduce this case to Case 1.

**Lemma 2.3.** Let  $k_0 = 3$  and  $\varepsilon < \sigma = 1 - a_1 - a_2 - a_3 \leq t_0/2$ , then

$$U_1U_2U_3W | S(U_1, U_2, U_3, W) | \ll x^{\frac{1}{2}-\delta} + x^{h_2(a_1, a_2, a_3)}, \quad (2.15)$$

where

$$h_2(a_1, a_2, a_3) = \frac{a'_1}{2} + \frac{a'_2 + a'_3}{2} + \frac{t_0}{6} + \frac{\sigma}{12}, \quad (2.16)$$

$$a'_1 = \frac{\log N'_1}{\log x} = \max\{a_1, t_0\}, a'_2 = \frac{\log N'_2}{\log x} = \max\left\{a_2, \frac{t_0}{3}\right\}$$

and  $a'_3 = \frac{\log N'_3}{\log x} = \max\left\{a_3, \frac{t_0}{3}\right\}$ .

**Proof.** In this case we have that  $L_0 \ll T^{1/2}$ . Then

$$\begin{aligned} F &\leq \min\{U_1^{-2}(N_1 + T), U_2^{-6}(N_2^3 + T), U_3^{-6}(N_3^3 + T), W^{-4}(L_0^2 + T), \\ &\quad W^{-4}L_0^2 + W_0^{-12}L_0^2T, U_3^{-6}N_3^3 + U_3^{-18}N_3^3T, U_1^{-2}N'_1\}. \end{aligned}$$

We consider two cases :

**Case 1.**  $F \leq 2W^{-4}L_0^2$ .

Suppose  $F \leq 2U_3^{-6}N_3^3$  and  $F \leq 2U_2^{-6}N_2^3$ . By (2.4), we have that

$$\begin{aligned} F &\ll \min \{U_1^{-2}N_1, W^{-4}L_0^2, U_2^{-6}N_2^3, U_3^{-6}N_3^3\} \\ &\ll (U_1^{-2}N_1)^{\frac{1}{2}-\frac{1}{28}}(U_2^{-6}N_2^3)^{\frac{1}{6}-\frac{1}{78}}(U_3^{-6}N_3^3)^{\frac{1}{6}-\frac{1}{78}}(W^{-4}L_0^2)^{\frac{1}{4}-\frac{1}{32}} \\ &\ll (U_1U_2U_3W)^{-1+\frac{1}{13}}(N_1N_2N_3L_0)^{\frac{1}{2}-\frac{1}{28}} \ll x^{\frac{1}{2}-\delta}. \end{aligned}$$

Suppose  $F > 2U_3^{-6}N_3^3$  and  $F > 2U_2^{-6}N_2^3$ , then

$$N_2 \ll T^{1/3}, N_3 \ll T^{1/3} \text{ and } F \ll U_2^{-18}N_2^3T,$$

since  $2U_i^{-6}N_i^3 \leq F \leq U_i^{-6}(N_i^3 + T)$ ,  $F \leq U_2^{-18}N_2^3T + U_2^{-6}N_2^3$  for  $i = 2$  or  $3$ ,  $L_0 \ll T^{1/2}$  and  $N_2 \ll T^{1/3}$ . Therefore

$$\begin{aligned} F &\ll (U_1^{-2}N_1)^{\frac{1}{2}}(W^{-4}L_0^2)^{\frac{1}{4}}(U_3^{-6}T)^{\frac{1}{6}}(U_2^{-6}T)^{\frac{1}{24}}(U_2^{-18}N_2^3T)^{\frac{1}{24}} \\ &\ll (U_1U_2U_3W)^{-1}T^{\frac{1}{4}}N_2^{\frac{1}{8}}N_1^{\frac{1}{2}}L_0^{\frac{1}{2}} \\ &\ll (U_1U_2U_3W)^{-1}x^{h_2(a_1, a_2, a_3)} \end{aligned} \quad (2.17)$$

Suppose  $F > 2U_3^{-6}N_3^3$  and  $F \leq 2U_2^{-6}N_2^3$ , then

$$\begin{aligned} F &\ll (U_1^{-2}N_1)^{\frac{1}{2}}(U_2^{-6}N_2^3)^{\frac{1}{6}}(W^{-4}L_0^2)^{\frac{1}{4}}(U_3^{-6}T)^{\frac{1}{24}}(U_3^{-18}N_3^3T)^{\frac{1}{24}} \\ &\ll (U_1U_2U_3W)^{-1}N_1^{\frac{1}{2}}N_2^{\frac{1}{2}}L_0^{\frac{1}{2}}T^{\frac{1}{12}}N_3^{\frac{1}{6}} \ll (U_1U_2U_3W)^{-1}x^{h_2(a_1, a_2, a_3)} \end{aligned} \quad (2.18)$$

since  $a'_3 = t_0/3$  and  $N_3 \ll T^{1/3}$  in this case. Suppose  $F < 2U_3^{-6}N_3^3$  and  $F \geq 2U_2^{-6}N_2^3$ .

On interchanging  $N_2$  with  $N_3$ , we may reduce this case to (2.18).

**Case 2.**  $F > 2W^{-4}L_0^2$ .

We have that

$$F \ll (U_1^{-2}N_1)^{\frac{1}{2}}(U_2^{-6}N_2^3)^{\frac{1}{6}}(U_3^{-6}N_3^3)^{\frac{1}{6}}(W^{-4}T)^{\frac{1}{6}}(W^{-12}L_0^2T)^{\frac{1}{24}},$$

then

$$F \ll (U_1U_2U_3W)^{-1}N_1^{\frac{1}{2}}N_2^{\frac{1}{2}}N_3^{\frac{1}{2}}T^{\frac{1}{6}}L_0^{\frac{1}{12}} \ll (U_1U_2U_3W)^{-1}x^{h_2(a_1, a_2, a_3)}.$$

The proof of Lemma 2.3 is now completed.

**Lemma 2.4.** *If  $k_0 = 3, a_1 \geq t_0, a_2 \geq t_0/2$ , then*

$$U_1 U_2 U_3 W | S(U_1, U_2, U_3, W) | \ll x^{\frac{1}{2}-\delta} + x^{h_3(a_1, a_2, a_3)}, \quad (2.19)$$

where

$$h_3(a_1, a_2, a_3) = \frac{a_1 + a_2}{2} + \frac{a_3'}{2} + \frac{t_0}{8} + \frac{\sigma}{6},$$

$a_3' = \log N_3'/\log x = \max\{t_0/4, a_3\}$  and  $\sigma = 1 - a_1 - a_2 - a_3$ .

**Proof.** We have that

$$F \ll \min\{U_1^{-2} N_1, U_2^{-4} N_2^2, U_3^{-8} (N_3^4 + T), W^{-6} (L_0^3 + T), W^{-18} L_0^3 T + W^{-6} L_0^3\}.$$

Suppose  $F \leq 2W^{-6} L_0^3$ , we are in Case 1 of Lemma 2.1, so that

$$U_1 U_2 U_3 W F \ll x^{\frac{1}{2}-\delta}.$$

Suppose  $F > 2W^{-6} L_0^3$ , then

$$\begin{aligned} F &\ll (U_1^{-2} N_1)^{\frac{1}{2}} (U_2^{-4} N_2^2)^{\frac{1}{4}} (U_3^{-8} x^{4a_3'})^{\frac{1}{8}} (W^{-6} T)^{\frac{1}{18}} (W^{-18} L_0^3 T)^{\frac{1}{18}} \\ &\ll (U_1 U_2 U_3 W)^{-1} N_1^{\frac{1}{2}} N_2^{\frac{1}{2}} x^{\frac{a_3'}{2}} T^{\frac{1}{9}} L_0^{\frac{1}{9}} \ll (U_1 U_2 U_3 W)^{-1} x^{h_3(a_1, a_2, a_3)}. \end{aligned}$$

Thus the lemma follows.

§ 3.  $L_0 = L$ .

In this section we discuss  $|S(U_1, \dots, U_k, W)|$  with (1.4), (1.5), (1.6), (1.7) and (1.9). We use the same notations as in § 2 and assume  $L_0 = L$ .

**Lemma 3.1.** *Let  $L = L_0$ . If  $L_0 \geq T^{1/2+\varepsilon}$ , then*

$$U_1 \cdots U_{k_0} W | S(U_1, U_{k_0}, W) | \ll x^{\frac{1}{2}-\delta}.$$

**Proof.** Take  $j$  such that

$$N_1^j \cdots N_{k_0}^j \geq T.$$

We have

$$F \ll \min\{U_1^{-2j} \cdots U_{k_0}^{-2j} N_1^{2j}, W^{-4} T\}.$$

Thus

$$\begin{aligned} F &\ll (U_1^{-2j} \dots U_{k_0}^{-2j} N_1^{2j} \dots N_{k_0}^{2j})^{3/4} (W^{-4} T)^{1/4} \\ &\ll (U_1^{-2j} \dots U_{k_0}^{-2j} N_1^{2j} \dots N_{k_0}^{2j})^{1/2j} (W^{-4} T)^{1/4} \\ &\ll (U_1 \dots U_{k_0} W)^{-1} N_1^{1/2} \dots N_{k_0}^{1/2} T^{1/4}, \end{aligned}$$

since  $U_1^{-2j} \dots U_{k_0}^{-2j} N_1^{2j} \dots N_{k_0}^{2j} \leq 1$ . We obtain

$$U_1 \dots U_{k_0} W F \ll x^{1/2-\epsilon/2}$$

by  $L_0 \geq T^{1/2+\epsilon}$ .

**Lemma 3.2.**  $k_0 = 2, N_1 \geq T, N_2 < T^{1/5}$  and

$$L_0 N_2^{1/2} \geq T^{1/2+\epsilon},$$

then

$$U_1 U_2 W |S(U_1, U_2, W)| \ll x^{\frac{1}{2}-\delta}.$$

**Proof.** We have that, choosing  $j$  such that  $N_2^j \geq T$ ,

$$F \ll \min\{W^{-4} U_2^{-2} T, U_1^{-2} N_1, U_2^{-j} N_2^j\}.$$

Thus

$$\begin{aligned} F &\ll (W^{-4} U_2^{-2} T)^{1/4} (U_1^{-2} N_1)^{1/2} (U_2^{-2j} N_2^j)^{1/4j} \\ &\ll (U_1 U_2 W)^{-1} N_1^{1/2} N_2^{1/4} T^{1/4} = (U_1 U_2 W)^{-1} (x/L_0 N_2)^{1/2} N_2^{1/4} T^{1/4} \\ &\ll x^{\frac{1}{2}-\delta}, \end{aligned}$$

since  $L_0^{1/2} N_2^{1/4} \geq T^{1/4+\epsilon/2}$ .

**Lemma 3.3.**  $k_0 = 2, N_1 \geq T, N_2 \geq T^{1/5}$  and

$$L_0 \geq T^{3/8+\epsilon} N_2^{1/8},$$

then

$$U_1 U_2 W |S(U_1, U_2, W)| \ll x^{\frac{1}{2}-\delta}.$$

**Proof.** We have that, choosing  $j$  such that  $N_2^j \geq T$ ,

$$F \ll \min\{W^{-4} U_2^{-2} N_2^{5/4} T^{3/4}, U_1^{-2} N_1, U_2^{-2j} N_2^j\}.$$

Thus, by  $U_2^{-2j} N_2^j \leq 1$ ,

$$\begin{aligned} F &\ll (W^{-4} U_2^{-2} N_2^{5/4} T^{3/4})^{1/4} (U_1^{-2} N_1)^{1/2} (U_2^{-2j} N_2^j)^{1/4j} \\ &\ll (U_1 U_2 W)^{-1} N_1^{1/2} N_2^{9/16} T^{3/16} = (U_1 U_2 W)^{-1} (x/L_0 N_2)^{1/2} N_2^{9/16} T^{3/16} \\ &\ll x^{1/2-\delta}. \end{aligned}$$

Write

$$\begin{aligned} c_1 &= \frac{1}{2j_1} + \frac{1}{2j_2} - \frac{3}{10}, c_2 = 7 + \frac{5}{j_2} + \frac{5}{j_1}, c_3 = \frac{7}{8} - \frac{3}{4j_2} - \frac{3}{4j_3} - \frac{3}{8j_4}, \\ c_4 &= \frac{1}{4} \left( \frac{1}{j_2} + \frac{1}{j_3} + \frac{1}{2j_4} - \frac{1}{2} \right), c_5 = \frac{3}{8} - \frac{1}{4j_2} - \frac{3}{4j_3} - \frac{3}{8j_4}, \end{aligned}$$

and

$$c_6 = \frac{1}{4j_2} + \frac{1}{4j_3} + \frac{1}{8j_4} - \frac{1}{8}.$$

Define

$$N_i' = \max \{ N_i, T^{1/j_i} \}, i = 3 \text{ and } 4. \quad (3.1)$$

**Lemma 3.4.** *Let  $K_0 = 4, a_3 \geq a_4, t_0/3 \geq a_4 \geq t_0/5, \sigma = 1 - a_1 - a_2 - a_3 - a_4 \geq t_0/j_1, j_1$  and  $j_2, j_3, j_4$  be integers with  $c_i \geq 0$  ( $1 \leq i \leq 6$ ), then*

$$U_1 U_2 U_3 U_4 W | S(U_1, U_2, U_3, U_4, W) | \ll x^{\frac{1}{2}-\delta} + x^{h_4(a_1, a_2, a_3, a_4)} \quad (3.2)$$

where

$$\begin{aligned} h_4(a_1, a_2, a_3, a_4) &= \frac{a_3'}{2} + \frac{a_4'}{4} + \frac{5a_4}{16} + \frac{3t_0}{16} \\ &+ \max \left\{ \frac{a_1}{2} + c_6 j_2 a_2 + (c_5 + c_6) t_0, c_4 a_1 + \frac{a_2}{2} + (c_3 + c_4) t_0 \right\}. \end{aligned}$$

**Proof.** We have that, by  $t_0/3 \geq a_4 \geq t_0/5, \sigma \geq t_0/5$  and  $a_3 \geq a_4$ ,

$$\begin{aligned} F &\ll \min \{ W^{-4} U_4^{-2} N_4^{5/4} T^{3/4}, U_1^{-2} (N_1 + T), U_1^{-6} N_1 T + U_1^{-2} N_1, U_2^{-2j_2} (N_2^{j_2} + T) \\ &U_2^{-2j_2} N_2 + U_2^{-6j_2} N_2^{j_2} T, U_3^{-10} N_3^5, U_4^{-10} N_4^5, W^{-2j_1} L_0^{j_1}, U_3^{-2j_3} N_3^{j_3}, \\ &U_4^{-2j_4} (N_4^{j_4} + T) \}. \end{aligned}$$

We discuss following cases

**Case 1.**  $F \leq 2U_1^{-2}N_1$  and  $F \leq 2U_2^{-2j_2}N_2^{j_2}$ .

By (2.4), we have that

$$\begin{aligned} F &\ll (U_1^{-2}N_1)^{\frac{1}{2}-\frac{5c_1}{c_2}} (U_2^{-2j_2}N_2^{j_2})^{\frac{1}{2j_2}-\frac{5c_1}{j_2c_2}} (U_3^{-10}N_3^5)^{\frac{1}{10}-\frac{c_1}{c_2}} (U_4^{-10}N_4^5)^{\frac{1}{10}-\frac{c_1}{c_2}} \times \\ &\quad \times (W^{-2j_1}L_0^{j_1})^{\frac{1}{2j_1}-\frac{5c_1}{j_1c_2}} \\ &\ll (U_1U_2U_3U_4W)^{-1+\frac{10c_1}{c_2}} (N_1N_2N_3N_4L_0)^{\frac{1}{2}-\frac{5c_1}{c_2}} \ll (U_1U_2U_3U_4W)^{-1}x^{\frac{1}{2}-\delta}, \end{aligned}$$

since  $c_1/c_2 < 1/10$ .

**Case 2.**  $F > 2U_1^{-2}N_1$  and  $F \leq 2U_2^{-2j_2}N_2^{j_2}$ .

We use  $F \ll U_i^{-2j_i}N_i^{j_i}$  for  $i = 3$  or  $4$ . We have that, by  $2c_3 + 6c_4 = 1$ ,

$$\begin{aligned} F &\ll \left(W^{-4}U_4^{-2}N_4^{\frac{5}{4}}T^{\frac{3}{4}}\right)^{\frac{1}{4}} (U_4^{-2j_4}N_4^{j_4})^{\frac{1}{4j_4}} (U_3^{-2j_3}N_3^{j_3})^{\frac{1}{2j_3}} (U_2^{-2j_2}N_2^{j_2})^{\frac{1}{2j_2}} \times \\ &\quad \times (U_1^{-2}T)^{c_3}(U_1^{-6}N_1T)^{c_4} \\ &\ll (U_1U_2U_3U_4W)^{-1}N_1^{c_4}N_2^{\frac{1}{2}}N_3^{\frac{1}{2}}N_4^{\frac{1}{4}}N_4^{\frac{5}{16}}T^{\frac{3}{16}+c_3+c_4} \\ &\ll (U_1U_2U_3U_4W)^{-1}x^{h_4(a_1, a_2, a_3, a_4)}, \end{aligned} \tag{3.3}$$

**Case 3.**  $F \leq 2U_1^{-2}N_1$  and  $F > 2U_2^{-2j_2}N_2^{j_2}$ .

Then we have that

$$\begin{aligned} F &\ll \left(W^{-4}U_4^{-2}N_4^{\frac{5}{4}}T^{\frac{3}{4}}\right)^{\frac{1}{4}} (U_4^{-2j_4}N_4^{j_4})^{\frac{1}{4j_4}} (U_3^{-2j_3}N_3^{j_3})^{\frac{1}{2j_3}} (U_1^{-2}N_1)^{\frac{1}{2}} \times \\ &\quad \times (U_2^{-2j_2}T)^{c_5}(U_2^{-6j_2}N_2^{j_2}T)^{c_6} \\ &\ll (U_1U_2U_3U_4W)^{-1}N_1^{\frac{1}{2}}N_2^{j_2c_6}N_3^{\frac{1}{2}}N_4^{\frac{1}{4}}N_4^{\frac{5}{16}}T^{\frac{3}{16}+c_5+c_6} \\ &\ll (U_1U_2U_3U_4W)^{-1}x^{h_4(a_1, a_2, a_3, a_4)}, \end{aligned} \tag{3.4}$$

as required.

**Case 4.**  $F > 2U_1^{-2}N_1$  and  $F > 2U_2^{-2j_2}N_2^{j_2}$ .

In this case we have that  $N_1 \ll T$  and  $N_2 \ll T^{1/j_2}$ . Thus (3.3) and (3.4) are true again, i.e. we get (3.1) again.

If take  $j_2 = 2$  and  $j_3 = j_4 = 5$ , then

$$h_4(a_1, a_2, a_3, a_4) = \frac{a'_3}{2} + \frac{a'_4}{4} + \frac{5a_4}{16} + \frac{3t_0}{16} + \max \left\{ \frac{a_1}{2} + \frac{3}{20}a_2 + \frac{t_0}{10}, \frac{3}{40}a_1 + \frac{a_2}{2} + \frac{7}{20}t_0 \right\} \quad (3.5)$$

**Lemma 3.5.** Suppose  $k_0 = 4$ ,  $a_4 \leq t_0/5$ ,  $a_i \geq t_0/j_i$ , ( $i = 2, 3$  and  $4$ ) and  $\sigma \geq t_0/j$  with

$$\frac{1}{j_2} + \frac{1}{j_3} + \frac{1}{j_4} + \frac{1}{j} > \frac{1}{2},$$

then

$$U_1 U_2 U_3 U_4 W | S(U_1, U_2, U_3, U_4, W) | \ll x^{\frac{1}{2}-\delta} + x^{h_5(a_1, a_2, a_3, a_4)}$$

where

$$h_5(a_1, a_2, a_3, a_4) = \frac{a'_3}{2} + \frac{a'_4}{4} + \frac{t_0}{4} + \max \left\{ \frac{a_1}{2} + c_5 j_2 a_2 + (c_5 + c_6)t_0, a_1 + \frac{a_2}{2} + (c_3 + c_4)t_0 \right\}. \quad (3.7)$$

**Proof.** In the proof of Lemma 3.2 we replace (1.10) by (1.11). Then we replace  $5a_4/16$  by  $t_0/16$  and we get  $h_5(a_1, a_2, a_3, a_4)$  replace  $h_4(a_1, a_2, a_3, a_4)$ . The lemma follows.

**Lemma 3.6.** If  $k_0 = 3$ ,  $a_1 \geq t_0$  and  $t_0/3 \geq a_3 \geq t_0/5$ , then

$$U_1 U_2 U_3 W | S(U_1, U_2, U_3, W) | \ll x^{\frac{1}{2}-\delta} + x^{h_6(a_1, a_2, a_3)}$$

where

$$h_6(a_1, a_2, a_3) = \frac{13t_0}{48} + \frac{a_1 + a'_2}{2} + \frac{5a_3}{16}.$$

and  $a'_2 = \max\{a_2, t_0/3\}$ .

**Proof.** We have

$$\begin{aligned} F &\ll \left( W^{-4} U_3^{-2} N_3^{\frac{5}{2}} T^{\frac{3}{2}} \right)^{\frac{1}{4}} (U_1^{-2} N_1)^{\frac{1}{2}} (U_2^{-6} x^{3a'_2})^{\frac{1}{6}} (U_3^{-6} T)^{\frac{1}{12}} \\ &\ll (U_1 U_2 U_3 W)^{-1} N_1^{\frac{1}{2}} N_3^{\frac{5}{6}} x^{\frac{a'_2}{2}} T^{\frac{13}{12}} \ll (U_1 U_2 U_3 W)^{-1} x^{h_6(a_1, a_2, a_3)}. \end{aligned}$$

Thus the lemma follows.

**Lemma 3.7.** If  $k_0 = 4$  and the  $a_i$ 's satisfy : (1)  $a_1 + a_3 + a_4 \geq t_0$ ; (2)

$a_2 + a_3 + a_4 \leq t_0$ ; (3)  $a_1 + a_3 \leq t_0$ , (4)  $a_1 + a_4 \leq t_0$ ; and (5)  $a_4 \leq a_3 \leq t_0/5$ ;  
then

$$U_1 U_2 U_3 U_4 W | S(U_1, U_2, U_3, U_4, W) | \ll x^{\frac{1}{2}-\delta} + x^{h_7(a_1, a_2, a_3, a_4)},$$

where

$$h_7(a_1, a_2, a_3, a_4) = t_0 + a_2/8.$$

**Proof.** W.L.O.G. we can agree that  $U_3 \geq U_4$ , then we have

$$F \leq \min\{W^{-4}U_3^{-2}T, U_1^{-2}U_3^{-2}T, U_2^{-2}U_3^{-2}U_4^{-2}T, U_2^{-2}N_2 + U_2^{-6}N_2T, U_1^{-2}U_3^{-2}U_4^{-2}N_1N_3N_4\}.$$

If  $F \leq 2U_2^{-2}N_2$ , this is Case 1 of Lemma 2.1; if  $F \geq 2U_2^{-2}N_2$ , from  $U_3 \geq U_4$ , we have

$$\begin{aligned} F &\ll (W^{-4}U_3^{-2}T)^{\frac{1}{4}}(U_1^{-2}U_3^{-2}T)^{\frac{1}{2}}(U_2^{-2}U_3^{-2}U_4^{-2}T)^{\frac{1}{8}}(U_2^{-6}N_2T)^{\frac{1}{8}} \\ &\ll (U_1U_2U_3U_4W)^{-1}TN_2^{\frac{1}{8}}U_3^{-\frac{3}{4}}U_4^{\frac{3}{4}} \ll (U_1U_2U_3U_4W)^{-1}x^{h_7(a_1, a_2, a_3, a_4)}, \end{aligned}$$

and the lemma follows.

**Lemma 3.8.** Suppose that  $k_0 = 6, t_0/3 \geq a_1 \geq \dots \geq a_6 \geq t_0/5$  and  $\sigma + a_1 + a_2 \geq t_0$ , then

$$U_1 U_2 \dots U_6 W | S(U_1, \dots, U_6) | \ll x^{\frac{1}{2}-\delta} + x^{h_8(a_1, \dots, a_6)}$$

where

$$h_8(a_1, \dots, a_6) = \frac{19}{48}a_6 + \frac{a_5}{12} + \frac{a_4}{12} + \frac{15}{16}t_0.$$

**Proof.** If  $F \leq U_6^{-2}U_5^{-2}U_4^{-2}N_6N_5N_4$ , then

$$\begin{aligned} F &\ll (U_6^{-2}U_5^{-2}U_4^{-2}N_6N_5N_4)^{\frac{1}{2}-\frac{1}{11}}(W^{-2}U_1^{-2}U_2^{-2}L_0N_1N_2)^{\frac{1}{2}-\frac{1}{11}}(U_3^{-10}N_3^5)^{\frac{1}{16}-\frac{1}{11}} \\ &\ll (WU_6 \dots U_1)^{-1+\frac{1}{11}}(L_0N_6 \dots N_1)^{\frac{1}{2}-\frac{1}{11}} \ll x^{\frac{1}{2}-\delta}. \end{aligned}$$

If  $F > U_6^{-2}U_5^{-2}U_4^{-2}N_6N_5N_4$ , then

$$F \leq U_6^{-6}U_5^{-6}U_4^{-6}N_6N_5N_4T,$$



and

$$\begin{aligned} F &\ll \left( W^{-4} U_6^{-2} N_6^{\frac{5}{4}} T^{\frac{3}{4}} \right)^{\frac{1}{4}} (U_1^{-2} U_2^{-2} U_3^{-2} T)^{\frac{1}{2}} (U_4^{-6} U_5^{-6} U_6^{-6} N_4 N_5 N_6 T)^{\frac{1}{12}} \times \\ &\times (U_4^{-4} U_5^{-2} T)^{\frac{1}{8}} (U_5^{-6} T)^{\frac{1}{24}} \\ &\ll (W U_1 \cdots U_6)^{-1} T^{\frac{15}{16}} N_4^{\frac{1}{2}} N_5^{\frac{1}{2}} N_6^{\frac{19}{16}} \ll (W U_1 \cdots U_6)^{-1} x^{h_8(a_1, \dots, a_6)}. \end{aligned}$$

**Lemma 3.9.** *Suppose that  $k_0 = 4, a_1 \geq a_2 \geq a_3 \geq a_4, a_1 + a_3 \leq t_0 < a_1 + a_3 + a_4, \sigma \geq t_0/3$  and  $2a_4 \leq t_0/3$ , then*

$$U_1 \cdots U_4 W \mid S(U_1, \dots, U_4) \ll x^{\frac{1}{2}-\delta} + x^{h_9(a_1, \dots, a_4)}$$

where

$$h_9(a_1, \dots, a_4) = \frac{15}{16} t_0 + \frac{a_2}{8} + \frac{5a_4}{8}.$$

**Proof.** If  $F \leq 2U_2^{-2} N_2$ , then

$$F \ll (U_1^{-2} U_3^{-2} U_4^2 N_1 N_3 N_4)^{\frac{1}{2}-\frac{1}{30}} (U_2^{-2} N_2)^{\frac{1}{2}-\frac{1}{30}} (W^{-6} L_0^3)^{\frac{1}{8}-\frac{1}{10}} \ll x^{\frac{1}{2}-\delta}$$

by (2.4).

If  $F > 2U_2^{-2} N_2$ , then  $F \ll \min\{U_2^{-2} T, U_2^{-6} N_2 T\}$ . Thus

$$\begin{aligned} F &\ll \left( W^{-4} U_4^{-4} N_4^{\frac{5}{2}} T^{\frac{3}{4}} \right)^{\frac{1}{4}} (U_1^{-2} U_3^{-2} T)^{\frac{1}{2}} (U_2^{-2} T)^{\frac{1}{8}} (U_2^{-6} N_2 T)^{\frac{1}{8}} \\ &\ll T^{\frac{15}{16}} N_2^{\frac{1}{8}} N_4^{\frac{5}{8}} \ll x^{h_9(a_1, \dots, a_4)}, \end{aligned}$$

as required.

**Lemma 3.10.** *Suppose that  $k_0 = 3, a_2 + a_3 \leq t_0, a_1 + \sigma \geq t_0$  and  $a_3 \leq t_0/5$ , then*

$$U_1 \cdots U_4 W \mid S(U_1, \dots, U_4) \ll x^{\frac{1}{2}-\delta} + x^{h_{10}(a_1, \dots, a_4)}$$

where

$$h_{10}(a_1, \dots, a_4) = t_0 + \frac{a_2}{8} + \frac{a_3}{24}.$$

**Proof.** We have that

$$F \ll \min\{W^{-4} U_3^{-2} T, U_1^{-2} T, U_2^{-2} U_3^{-2} (T + N_2 N_3), U_2^{-6} U_3^{-6} N_2 N_3 T + U_2^{-2} U_3^{-2} N_2 N_3,$$

$$W^{-2}U_1^{-2}L_0N_1\}.$$

If  $F \leq 2U_2^{-2}U_3^{-2}N_2N_3$ , then we go back to the Case 1 on Lemma 3.1, since  $F \ll W^{-2}U_1^{-2}L_0N_1$ , and  $N_4 \geq x^\varepsilon$ ; if  $F > 2U_2^{-2}U_3^{-2}N_2N_3$ , then

$$\begin{aligned} F &\ll (W^{-4}U_3^{-2}T)^{\frac{1}{4}} (U_1^{-2}T)^{\frac{1}{2}} (U_2^{-2}U_3^{-2}T)^{\frac{1}{4}} (U_2^{-6}U_3^{-6}N_2N_3T)^{\frac{1}{24}} (U_2^{-6}N_2T)^{\frac{1}{12}} \\ &\ll (U_1U_2U_3W)^{-1}x^{h_{10}(a_1, a_2, a_3)}, \end{aligned}$$

as required.

#### § 4. SET $E(\theta)$ .

Let  $\theta_j (1 \leq j \leq r)$  be positive numbers with

$$\sum_{1 \leq i \leq r} \theta_i < 1.$$

Denote  $\{\theta_j\} = \{\theta_0, \theta_1, \dots, \theta_r\}$  and

$$\theta_0 = 1 - \sum_{1 \leq i \leq r} \theta_i.$$

Divide the set of numbers

$$\{\theta_0, \theta_1, \dots, \theta_r\}$$

into three subsets and call the sums of terms in each subset  $a_1, a_2$  and  $\sigma$ , where  $\sigma$  is distinguished by being  $\theta_0$  if  $\sigma > t_0/2$ , and otherwise  $\sigma \leq t_0/2$ . Since  $a_1 + a_2 + \sigma = 1$ , any two of  $a_1, a_2, \sigma$  determine the third. We attach an exactly similar meaning to  $\{a_1, a_2, a_3, \sigma\}$  and  $\{a_1, a_2, a_3, a_4, \sigma\}$ . We refer to  $\{a_1, a_2, \sigma\}$ , or  $\{a_1, a_2, a_3, \sigma\}$ , or  $\{a_1, a_2, a_3, a_4, \sigma\}$  as set of complementary partial sums. For  $\{a_1, \dots, a_k, \sigma\}$  if there exists some  $h_i(a_1, \dots, a_k)$  which satisfies (1.8) with

$$h_i(a_1, \dots, a_k) < 1/2,$$

then we call it  $\{a_1, \dots, a_k, \sigma\} \in E(\theta)$ . For short, we write that  $\{\theta_j\} \in E(\theta)$  instead of  $\{a_1, \dots, a_k, \sigma\} \in E(\theta)$ .

**Lemma 4.1.** *If there exists at least one set of complementary partial sums*

$\{a_1, a_2, \sigma\}$  (or  $\{a_1, a_2, a_3, \sigma\}$ ) of  $\{\theta_j\}$  such that at least one of conditions (4.1.1) - (4.1.4) (or (4.1.5)) holds, then  $\{\theta_j\} \in E(\theta)$ .

(4.1.1)  $a_1 \geq a_2 \geq t_0$  and  $\sigma > \varepsilon$ ;

This situation is covered by (2.9) and (2.10) of Lemma 2.1

(4.1.2)  $a_2 \leq a_1 \leq t_0$  and  $a_2 < 4 - 8t_0$ ;

Using (2.11) of Lemma 2.1 with  $a'_1 = t_0$  and  $j = 2$ , we have that

$$h_0(a_1, a_2) = t_0 + \min\{a_2/8, \sigma/6 + a_2/24\} \leq t_0 + a_2/8 < 1/2.$$

(4.1.3)  $a_1 \geq t_0, a_2 > \frac{2i+2}{2i+3}t_0$  and  $\sigma \geq \frac{t_0}{i+2}$ ;

In this case, we use Lemma 2.2,  $\sigma_i = \max\{t_0/(i+2), 1 - a_1 - a_2\} = \max\{t_0/(i+2), \sigma\} = \sigma$ , and  $a'_1 = a_1$ , then

$$\begin{aligned} h_1^i(a_1, a_2) &= \frac{i+1}{2(i+2)}t_0 + \frac{1}{4(i+2)} + \frac{2i+3}{4(i+2)}(a_1 + \sigma) \\ &= \frac{i+1}{2(i+2)}t_0 + \frac{1}{4(i+2)} + \frac{2i+3}{4(i+2)}(1 - a_2) < \frac{1}{2}. \end{aligned}$$

(4.1.4)  $a_3 \leq a_2 \leq t_0/3 \leq a_1 \leq t_0, \sigma = 1 - a_1 - a_2 - a_3$  and  $a_1 + \sigma/6 < 1 - t_0$ .

By Lemma 2.3 with  $a'_2 = \max\{t_0/3, a_2\} = t_0/3, a'_3 = \max\{t_0/3, a_3\} = t_0/3$ ; then

$$\begin{aligned} h_2(a_1, a_2, a_3) &= a_1/2 + (a'_2 + a'_3)/2 + t_0/6 + \sigma/12 \\ &= a_1/2 + t_0/2 + \sigma/12 < 1/2. \end{aligned}$$

**Lemma 4.2.** *If there exists at least one set of complementary partial sums  $\{a_1, a_2, a_3, \sigma\}$  of  $\{\theta_j\}$  such that at least one of conditions (4.2.1) - (4.2.3) holds, then  $\{\theta_j\} \in E(\theta)$ .*

(4.2.1).  $a_1 \geq t_0, a_2 \geq t_0/2, a_3 \geq t_0/4$  and  $\sigma > 2t_0/7$ .

In this case, Lemma 2.4 is applied, we have  $a'_3 = \max\{a_3, t_0/4\} = a_3$ , then

$$\begin{aligned} h_3(a_1, a_2, a_3) &= (a_1 + a_2 + a_3)/2 + t_0/8 + \sigma/16 \\ &= 1/2 + t_0/8 - 7\sigma/16 < 1/2, \end{aligned}$$

since  $\sigma > 2t_0/7$ .

(4.2.2).  $a_1 \geq t_0, a_2 \geq a_3 \geq t_0/3$  and  $\sigma > 2t_0/5$ .

Using Lemma 2.3, we have  $a'_2 = \max\{t_0/3, a_2\} = a_2$  and  $a'_3 = \max\{t_0/3, a_3\} = a_3$ , then

$$\begin{aligned} h_3(a_1, a_2, a_3) &= (a_1 + a_2 + a_3)/2 + t_0/6 + \sigma/12 \\ &= (1 - \sigma)/2 + t_0/6 + \sigma/12 \\ &= 1/2 + t_0/6 - 5\sigma/12 < 1/2 \end{aligned}$$

since  $\sigma > 2t_0/5$ .

(4.2.3).  $a_1 \geq t_0, a_2 \geq t_0/3 \geq a_3$  and  $t_0/2 \geq \sigma > \max\{2t_0/5, 4t_0/5 - 6a_3/5\}$ .

We use Lemma 2.3 with  $a'_2 = \max\{t_0/3, a_2\} = a_2$ , and  $a'_3 = \max\{t_0/3, a_3\} = t_0/3$ , then

$$\begin{aligned} h_3(a_1, a_2, a_3) &= (a_1 + a_2)/2 + t_0/6 + t_0/6 + \sigma/12 \\ &= (1 - \sigma - a_3)/2 + t_0/3 + \sigma/12 \\ &= 1/2 + t_0/3 - 5\sigma/12 - a_3/3 < 1/2 \end{aligned}$$

since  $\sigma > 4t_0/5 - 6a_3/5$ .

**Lemma 4.3.** Suppose  $L = L_0$  (i.e. (1.9) holds). If  $\sigma > t_0/2$ ; then  $\{\theta_j\} \in E(\theta)$ .

It is covered by Lemma 3.1.

For a fixed  $\sigma$ , denote

$$M_\sigma = \sup_{\substack{\{a_1, a_2, \sigma\} \in \{\theta\} \\ a_2 \leq t_0 \leq a_1}} \{a_1\} \text{ and } m_\sigma = \inf_{\substack{\{a_1, a_2, \sigma\} \in \{\theta\} \\ a_2 \leq t_0 \leq a_1}} \{a_2\}.$$

**Lemma 4.4.** Suppose that  $\{\theta_j\}$  satisfies the condition (4.4.1) : (4.4.1) all of complementary partial sums  $\{a_1, a_2, \sigma\}$  with  $a_2 \leq a_1 \leq t_0$  satisfies  $\{\theta_j\} \in E(\theta)$ , ( $a_2 \leq a_1$  is not necessary). Then for a fixed  $\sigma$ ,  $\{\theta_j\} \in E(\theta)$  if

$$m_\sigma < a_2 < M_\sigma. \quad (4.4.2)$$

Moreover, we have that

$$m_\sigma + a_2 + M_\sigma = 1, \quad (4.4.3)$$

$$m_\sigma < t_0 < M_\sigma; \quad (4.4.4)$$

for  $t_0/j \geq \sigma \geq 2t_0/(2j+1)$ ,

$$M_\sigma \geq 1 - \frac{6j}{6j-1}t_0 - \frac{2j-1}{6j-1}\sigma, \quad (4.4.5)$$

$$m_\sigma \leq \frac{6j}{6j-1}t_0 - \frac{4j}{6j-1}\sigma; \quad (4.4.6)$$

and for  $2t_0/(2j+1) \geq \sigma \geq t_0/(j+1)$ ,

$$M_\sigma \geq 1 - \frac{2j}{2j+1}t_0 - \sigma, \quad (4.4.7)$$

$$m_\sigma \leq \frac{2j}{2j+1}t_0. \quad (4.4.8)$$

**Proof.** We use (4.13) with  $j = i + 1$  to prove (4.4.8). Then we have (4.4.7) by (4.4.3).

For  $\theta > 11/20$ , we can get much simple conditions for  $\{a_1, a_2, \sigma\} \in E(\theta)$ . We will use it to discuss the gap between consecutive primes in [3].

**Lemma 4.5.** *Suppose that  $t_0 < 9/20$ . If there exists at least one set of complementary partial sums  $\{a_1, a_2, \sigma\}$  with  $a_1 \leq t_0$ , and  $a_2 \leq t_0$ , then  $\{\theta_j\} \in E(\theta)$ . Moreover, we have that*

$$M_\sigma - m_\sigma > \sigma, \text{ if } \sigma < 0.1; \quad (4.5.1)$$

and

$$M_\sigma - m_\sigma > 1 - 2t_0, \text{ if } \sigma \geq 0.1. \quad (4.5.2)$$

**Proof.** If  $a_2 \leq a_1$ , and  $a_2 < 4 - 8t_0$ , it is same as (5.1.8). If  $a_1 \geq a_2 \geq 4 - 8t_0$ , using (2.11) of Lemma 2.1 with  $j = 2$ , we have

$$\begin{aligned} h_0(a_1, a_2) &= t_0 + \min\{a_2/8, \sigma/6 + a_2/24\} \\ &\leq t_0 + (4 - 8t_0)/24 + (1 - 2(4 - 8t_0))/6 < 1/2. \end{aligned}$$

By (4.4.4), (4.4.5) and (4.4.6), we have that

$$M_\sigma - m_\sigma \geq 1 - \frac{28}{29}t_0 - \frac{9}{29}\sigma - \left(\frac{28}{29}t_0 - \frac{20}{29}t_0\right) \geq \sigma, \quad (4.5.3)$$

if  $\sigma < t_0/5$ . When  $t_0/5 \leq \sigma < 0.1$ , we have that

$$M_\sigma - m_\sigma \geq 1 - \frac{8}{9}t_0 - \sigma - \frac{8}{9}t_0 > 1 - 2t_0.$$

Then (4.5.1) holds. When  $t_0/(j+1) \leq \sigma \leq 2t_0/(2j+1)$ , by (4.4.4) and (4.4.6),

$$M_\sigma - m_\sigma \geq 1 - \frac{2j}{2j+1}t_0 - \sigma - \frac{2j}{2j+1}t_0 \geq 1 - 2t_0. \quad (4.5.4)$$

When  $2t_0/(2j+1) \leq \sigma \leq t_0/(j+1)$ , by (4.4.5) and (4.4.7)

$$M_\sigma - m_\sigma \geq 1 - \frac{6j}{6j-1}t_0 - \frac{2j-1}{6j-1}\sigma - \left( \frac{6j}{6j-1}t_0' - \frac{4j}{6j-1}\sigma \right) \geq 1 - 2t_0 \quad (4.5.5)$$

Thus (4.5.2) holds.

For  $\theta > 6/11$ , we can get some simple conditions for  $\{a_1, a_2, \sigma\} \in E(\theta)$ . We will use it to discuss the gap between consecutive primes in [3]. Moreover, we have that

**Corollary 4.5.1.** *Suppose that  $\theta > 6/11$ . (4.5.1) and (4.5.2) hold. If  $a_2 \leq a_1 \leq 1/2$  and  $\sigma = 1 - a_1 - a_2 < 1/2 - 8t_0/9$ , then  $\{\theta_j\} \in E(\theta)$ .*

**Proof.** If  $a_2 \geq t_0$ , it is covered by (4.1.1); if  $a_1 < t_0$ , we have that  $\{\theta_j\} \in E(\theta)$  by (4.5.1); if  $a_2 \leq t_0 < a_1 < 1/2$  and  $\sigma \geq t_0/5$ , then

$$a_2 = 1 - a_1 - \sigma > 1 - 1/2 - (1/2 - 8t_0/9) = 8t_0/9,$$

thus  $\{\theta_j\} \in E(\theta)$  by (4.1.3). If  $a_2 \leq t_0 < a_1 < 1/2$  and  $\sigma < t_0/5$ , using (2.11) in Lemma 2.1 with  $j = 5$ , then

$$\begin{aligned} h_0(a_1, a_2) &= t_0/2 + a_1/2 + \min\{a_2/8, \sigma/6 + a_2/60\} \\ &\leq t_0/2 + a_1/2 + \sigma/6 + a_2/60 \\ &= t_0/2 + a_1/2 + \sigma/6 + (1 - a_1 - \sigma)/60 \\ &= t_0/2 + 1/60 + 29a_1/60 + 3\sigma/20 \\ &\leq t_0/2 + 1/60 + 29/120 + 3t_0/100 < 1/2, \end{aligned}$$

since  $a_1 \leq 1/2$  and  $\sigma \leq t_0/5$ .

**Lemma 4.6.** *Suppose  $t_0 < 5/11$ . If there exists at least one set of complementary partial sums  $\{a_1, a_2, \sigma\}$  (or  $\{a_1, a_2, a_3, \sigma\}$ ) of  $\{\theta_j\}$  such that at least one of conditions (4.6.1) - (4.6.5) holds, then  $\{\theta_j\} \in E(\theta)$ .*

(4.6.1)  $a_2 \leq a_1 \leq t_0$  and  $\sigma < 1 - 20t_0/11$ .

**Proof.** When  $\sigma \geq t_0/3$ , we apply Lemma 2.2 with  $i = 1$ ,  $\sigma_1 = \max\{t_0/3, 1 - a_1 - a_2\} = \max\{t_0/3, \sigma\} = \sigma$  and  $a'_1 = t_0$ , then

$$\begin{aligned} h_1^i(a_1, a_2) &= t_0/3 + 1/12 + t_0/2 + \sigma/2 - (a_1 + \sigma)/12 \\ &\leq 5t_0/6 + 1/12 + 5\sigma/12 - a_1/12 \\ &\leq 5t_0/6 + 1/24 + 11\sigma/24 \\ &< 5t_0/6 + 1/24 + 11(1 - 20t_0/11)/24 = 1/2, \end{aligned}$$

since  $a_1 \geq (1 - \sigma)/2$ . ( $a_1 \geq a_2$  and  $a_1 + a_2 + \sigma = 1$ ).

When  $\sigma < t_0/3$ , we turn to (2.10) of Lemma 2.1 with  $j = 2$

$$\begin{aligned} h_0(a_1, a_2) &\leq t_0 + \sigma/6 + a_2/24 < t_0 + \sigma + (1 - \sigma)/48 \\ &\leq t_0 + \frac{1}{48} + 7\sigma/48 \leq 1/2. \end{aligned}$$

(4.6.2).  $a_1 \geq a_2 > (2i + 2)t_0/(2i + 3)$  and  $1 - 20t_0/11 > \sigma \geq t_0/i$ .

**Proof.** If  $a_2 \geq t_0$ , it is already covered by (4.1.1). If  $a_1 \geq t_0 > a_2 > (2i + 2)t_0/(2i + 3)$ , using (4.1.3). If  $t_0 > a_1 \geq a_2 > 8t_0/9$ , using (4.2.1).

We can write (4.2.2) to be

(4.6.3).  $t_0 \geq a_1 > (2i + 2)t_0/(2i + 3)$  and  $1 - 20t_0/11 > \sigma \geq t_0/i$ . (It is not necessary  $a_2 \leq a_1$ ).

**Proof.** If  $a_2 \geq t_0$ , we are back to (4.1.3). If  $a_1 < t_0$ , it is covered by (4.2.2).

(4.6.4).  $t_0 \leq a_1 \leq 1/2$ , and  $\varepsilon < \sigma < 1/2 - 8t_0/9$ ;

**Proof.** If  $\sigma \geq t_0/5$ , then  $a_2 = 1 - \sigma - a_1 > 8t_0/9$ ,  $\{\theta_j\} \in E(\theta)$  by (4.1.3)

with  $i = 3$ . If  $\sigma < t_0/5$ , we apply (2.10) with  $j = 5$  in Lemma 2.1. We have

$$\begin{aligned} h_0(a_1, a_2) &= t_0/2 + a_1/2 + \sigma/6 + a_2/60 \\ &= t_0/2 + a_1/2 + \sigma/6 + (1 - \sigma - a_1)/60 \\ &< t_0/2 + 29/120 + (9/60)(1/2 - 8t_0/9) + 1/60 < 1/2. \end{aligned}$$

(4.6.5).  $a_2 \leq a_1 \leq 1/2$  and  $\sigma < 1/2 - 8t_0/9$ .

**Proof.** If  $a_2 \geq t_0$ , it is covered by (4.1.1). If  $a_1 < t_0$ ,  $\{\theta_j\} \in E(\theta)$  by (4.1.5) since  $\sigma < 1/2 - 8t_0/9 < 1 - 20t_0/11$ . If  $a_2 \leq t_0 < a_1 < 1/2$ , the proof is same as (4.5.6).

From Lemma 4.3 and (4.6.1), we have, for  $\sigma < 1 - 20t_0/11$ , (4.4.3) - (4.4.8) hold. Moreover, (4.5.3), (4.5.4) and (4.5.5) with  $t_0 < 5/11$  can imply that

$$M_\sigma - m_\sigma > \sigma, \text{ if } \sigma < t_0/5; \quad (4.6.6)$$

and

$$M_\sigma - m_\sigma > t_0/5, \text{ if } \sigma \geq t_0/5 \quad (4.6.7)$$

since  $1 - 2t_0 > t_0/5$ .

(4.6.6)  $t_0 \leq a_1 \leq 1/2$  and  $a_2 < 2 - 4t_0$ ;

In this case, we use Lemma 2.1. By (2.11) with  $j = 2$  and  $a'_1 = a_1$ ,

$$h_0(a_1, a_2) \leq a_1/2 + t_0/2 + a_2/8 \leq 1/4 + t_0/2 + t_0/20 < 1/2.$$

**Lemma 4.7.** Suppose that  $t_0 < 5/11$  and  $L = L_0$ . If there exists at least one set of complementary partial sums  $\{a_1, a_2, \sigma\}$  (or  $\{a_1, a_2, a_3, \sigma\}$ ) of  $\{\theta_j\}$  such that at least one of conditions (4.7.1) - (4.7.6) holds, then  $\{\theta_j\} \in E(\theta)$ .

(4.7.1).  $k_0 = 3, 1/2 \geq a_1 \geq t_0, 2t_0/5 \geq a_2 \geq t_0/3$ , and  $t_0/4 \geq a_3 \geq t_0/5$ .

**Proof.** We use Lemma 3.6 with  $a'_2 = \max\{a_2, t_0/3\} = a_2$ ; then

$$\begin{aligned} h_8(a_1, a_2, a_3) &= 13t_0/48 + (a_1 + a_2)/2 + 5a_3/16 \\ &\leq 13t_0/48 + 1/4 + t_0/5 + 5t_0/64 < 1/2. \end{aligned}$$



(4.7.2).  $k_0 = 4, a_1 \leq 8t_0/9, a_2 \leq 4t_0/9$ , and  $a_4 \leq a_3 \leq t_0/4$ .

**Proof.** Using (3.5), we have that,  $a'_3 = a'_4 = t_0/4$  and

$$h_4(a_1, a_2, a_3, a_4) = \frac{29}{64}t_0 + \max \left\{ \frac{3}{40}a_1 + \frac{a_2}{2} + \frac{7t_0}{20}, \frac{a_1}{2} + \frac{3a_2}{20} + \frac{t_0}{10} \right\} < \frac{1}{2}.$$

(4.7.3).  $k_0 = 4, a_1 + a_3 + a_4 \geq t_0, a_2 + a_3 + a_4 \leq t_0, a_1 + a_3 \leq t_0, a_1 + a_4 \leq t_0, a_4 \leq a_3 \leq t_0/5$  and  $a_2 \leq 4 - 8t_0$ .

**Proof.** Using Lemma 3.7, since  $a_2 \leq 4 - 8t_0$ , we have

$$h_7(a_1, a_2, a_3, a_4) = t_0 + a_2/8 < 1/2.$$

(4.7.4).  $k_0 = 6, 2t_0/7 \geq a_1 \geq \dots \geq a_6 \geq t_0/5$  and  $\sigma + a_1 + a_2 \geq t_0$ .

**Proof.** Using Lemma 3.8, we have that

$$h_8(a_1, \dots, a_6) = \frac{15}{16}t_0 + \frac{a_4}{12} + \frac{a_5}{12} + \frac{19}{48}a_6 \leq \frac{15}{16}t_0 + \frac{t_0}{21} + \frac{19}{168}t_0 < \frac{1}{2}.$$

(4.7.5).  $2t_0/5 \geq \theta_1 \geq \dots \geq \theta_4 \geq 1 - 20t_0/11, 2t_0/5 \geq 1 - \theta_1 - \dots - \theta_6 \geq \theta_1, \geq 1 - 20t_0/11$ , and  $\theta_5 \geq \theta_6, 2\theta_6 \leq t_0/3, \theta_1 + \theta_2 + \theta_5 < t_0 < \theta_1 + \theta_2 + \theta_5 + \theta_6$ .

**Proof.** In Lemma 3.9, take  $a_1 = \theta_1 + \theta_2, a_2 = \theta_3 + \theta_4, a_3 = \theta_5$ , and  $a_4 = \theta_6$ , then

$$\begin{aligned} a_4 &= \theta_6 \leq \frac{1}{2}(\theta_5 + \theta_6) \leq \frac{1}{2} \left( 1 - \theta_1 - \dots - \theta_4 - \left( 1 - \sum_{1 \leq i \leq 6} \theta_i \right) \right) \\ &\leq (1 - 5a_2)/2 \end{aligned}$$

and

$$\begin{aligned} h_9(a_1, \dots, a_4) &\leq 15t_0/16 + a_2/8 + 5(1 - 5a_2)/16 \\ &< 15t_0/16 + 5/16 - 23a_2/16 < 1/2. \end{aligned}$$

(4.7.6).  $k_0 = 4, a_1 + a_4 \leq t_0, a_2 = a_3 \leq t_0, \sigma + a_1 \geq t_0, a_3 < t_0/5, a_2 < 1/3$  and  $a_3 < t_0/5$ .

**Proof.** In Lemma 3.10,

$$h_{10}(a_1, \dots, a_4) = t_0 + a_2/8 + a_3/24 < t_0 + 1/24 + t_0/120 < 1/2.$$

(4.7.7).  $k_0 = 5, a_3 \leq a_2 \leq a_1 \leq t_0/2$ , and  $a_5 \leq a_4 \leq t_0/4$ .

**Proof.** If  $a_2 + a_1 > 8t_0/9$ , we replace  $a_1$  by  $a_2 + a_1$  and  $s$  by  $a_5$  with  $k_0 = 2$ , then  $\{\theta_j\} \in E$  by (4.6.3). If  $a_2 + a_1 \leq 8t_0/9$ , then  $a_3 > 4t_0/9$  it is covered by (4.7.2).

(4.7.8)  $k_0 = 2, a_1 \geq t_0/2, a_2 \leq t_0/5$ , and  $\sigma + a_2/2 > t_0/2$ .

See Lemma 3.2.

(4.7.9).  $k_0 = 2, a_1 \geq t_0, t_0/3 \geq a_2 \geq t_0/5$ , and  $\sigma > a_2/8 + 3t_0/8$ .

See Lemma 3.3.

### § 5. ANALYTIC FORM OF $R(x; M_1 \cdots M_k)$ .

We shall examine the remainder term  $R(x; M_1, \dots, M_k)$  (see (1.12)) which was used in [1]. Let  $x$  be a large number,  $y = x^\theta$  with  $1/2 < \theta < 7/12$ , and  $A = \{n : x - y < n < x\}$ . For convenience we define  $a_{m_i, i} = 0$ , unless  $M_i < m \leq 2M_i$ . We rewrite (1.12) :

$$R(x; M_1, \dots, M_k) = \sum_{m_1, \dots, m_k} a_{m_1, 1} \cdots a_{m_k, k} \left( \left[ \frac{x}{m_1 \cdots m_k} \right] - \left[ \frac{x - y}{m_1 \cdots m_k} \right] \right)$$

and we write

$$\begin{aligned} L &= \frac{x}{2M_1 \cdots M_k}, \\ L(s) &= \sum_{L/2^k < I \leq 3L} I^{-s}, \\ M_i(s) &= \sum_{M_i < m_i \leq 2M_i} a_{m_i, i} m_i^{-s}, \text{ for } 1 \leq i \leq k, \end{aligned}$$

and

$$g(s) = L(s) \prod_{1 \leq i \leq k} M_i(s),$$

where  $k$  is a positive integer.

**Theorem 2.** Suppose  $\{\theta_j\} \in E(\theta)$ , then

$$R(x; M_1, \dots, M_k) \ll x^{\theta - \epsilon}.$$

**Proof.** Lemma 3.12 of Titchmarsh [7] is applied to the function  $g(s)$ , to yield

$$\sum_{\substack{m_1, \dots, m_k \\ Im_1 \dots m_k \in A}} a_{m_1,1} \dots a_{m_k,k} = \frac{1}{2\pi i} \int_{c-iT}^{c+iT} g(s) \frac{x^s - (x-y)^s}{s} ds + O\left(\frac{x^{1-\eta}}{T}\right), \quad (5.1)$$

where  $c = 1 + (\log x)^{-1}$  and  $T > 0$ . The conditions  $Im_1 \dots m_k \in A$  and  $M_i < m_i \leq 2M_i$  imply that  $L/2^k < I \leq 3L$ , and so that the sum on the left of (5.1) becomes

$$\begin{aligned} & \sum_{m_1, \dots, m_k} a_{m_1,1} \dots a_{m_k,k} \left( \left[ \frac{x}{m_1 \dots m_k} \right] - \left[ \frac{x-y}{m_1 \dots m_k} \right] \right) \\ &= R(x; M_1, \dots, M_k) - y \sum_{m_1, \dots, m_k} \frac{a_{m_1,1} \dots a_{m_k,k}}{m_1 \dots m_k} \end{aligned}$$

For  $T_0 \leq L$ , in the range  $\{s = c + it : |t| \leq T_0 \leq L\}$ , we have

$$L(s) = \frac{(3L)^{1-s} - (L/2^k)^{1-s}}{s-1} + O(L^{-c})$$

by Theorem 4.11 of Titchmarsh [7]. Moreover if  $T_0 y \leq x$ ,

$$\frac{x^s - (x-y)^s}{s} = yx^{s-1} - O(|s| y^2 x^{c-2}) \ll yx^{c-1}.$$

Hence

$$\begin{aligned} & \frac{1}{2\pi i} \int_{c-iT_0}^{c+iT_0} g(s) \frac{x^s - (x-y)^s}{s} ds \\ &= \frac{1}{2\pi i} \int_{c-iT_0}^{c+iT_0} \frac{x^s - (x-y)^s}{s} M_1(s) \dots M_k(s) x^{s-1} ds + O(E_1) + O(E_2), \end{aligned}$$

where

$$E_1 = \int_{-T_0}^{T_0} L^{-c} \left| \prod_{1 \leq j \leq k} M_j(c-it) \right| yx^{c-1} dt \ll T_0 M_1 \dots M_k yx^{-1},$$

and

$$\begin{aligned} E_2 &= \int_{-T_0}^{T_0} L^{-c} \left| \prod_{1 \leq j \leq k} M_j(c-it) \right| y^2 x^{c-2} dt \ll T_0 L^{1-c} (M_1 \dots M_k)^{1-c} y^2 x^{c-2} \\ &\leq T_0 y^2 x^{-1}. \end{aligned}$$

Moreover, on integrating termwise as in the proof of Lemma 3.1 in Titchmarsh [7], we have

$$\begin{aligned} & \frac{1}{2\pi i} \int_{c-iT_0}^{c+iT_0} \frac{x^{-s} - (x-y)^s}{s} M_1(s) \cdots M_k(s) x^{s-1} ds \\ &= \sum_{\substack{m_1, \dots, m_k \\ 1 \leq m_1 \cdots m_k \in A}} \frac{a_{m_1,1} \cdots a_{m_k,k}}{m_1 \cdots m_k} + O\left(\frac{1}{T_0}\right). \end{aligned}$$

It follows from the above estimate, choosing  $T_0 = L^{1/2}$ , that

$$R(x; M_1, \dots, M_k) = \frac{1}{2\pi i} \left( \int_{c-iT}^{c+iT_0} + \int_{c+iT_0}^{c+iT} \right) g(s) \frac{x^s - (x-y)^s}{s} ds + O(x^{\theta-\epsilon}). \quad (5.2)$$

We divide the latter range into at most  $2 \log x$  subintervals of type  $[T_1, T_2]$ , where  $T_2 \leq 2T_1$  and bound the interval over such a range by a sum over well spaced points  $T_1 \leq t_1 < \cdots < t_k \leq T_2 (t_{r+1} - t_r \geq 1)$ . Thus, for some  $T_1$  with  $T_0 \leq T_1 \leq T$ , and some set of  $t_r$ , we have

$$R(x; M_1 \cdots M_k) \ll x^{\theta-\epsilon} + (\log x) y x^{c-1} \sum_{1 \leq r \leq k} g(c + it_r). \quad (5.3)$$

In the following sections we shall estimate (5.3).

We denote the right hand-side of (5.3) by  $R(x; N_1, \dots, N_{k_0})$ . We have

$$g(s) = N_1(s) \cdots N_{k_0}(s) L(s), \quad (5.4)$$

where the definition of  $N_i(s)$  or  $L_0(s)$  is the same as (5.1) and one of  $N_i(s)$  or  $L_0(s)$  is equal  $L(s)M_{i_1}(s) \cdots M_{i_n}(s)$ .

We trivially have

$$|L_0(c + it_r)| \leq 4, \text{ and } |N_j(c + it_r)| \leq 4 \quad (j \leq k).$$

Hence those  $t_r$  for which

$$|L_0(c + it_r)| \leq x^{-2}, \text{ or } |N_j(c + it_r)| \leq x^{-2} \quad (j \leq k_0)$$

contribute a total

$$\ll (\log x) y x^{c-1} T_2 x^{-2} \ll x^{\theta-\epsilon}$$

to  $R(x; N_1, \dots, N_{k_0})$ . Such points may therefore be neglected. We now divide the remaining  $t_r$  into at most  $(4 \log x)^{k_0+1}$  sets  $S(U_1, \dots, U_{k_0}, W)$  for which

$$\begin{aligned} U_i &< N_i^{c-1/2} |N_i(c+it_r)| < 2U_i, \quad i = 1, \dots, k_0, \\ W &\leq L_0^{c-1/2} |L_0(c+it_r)| < 2W, \end{aligned}$$

where  $x^{-2} \leq L^{1/2-c}W \leq 2^{-u} \leq 1$ , for some integer  $u$  and similarly for  $U_1, \dots, U_{k_0}$ . It follows that

$$R(x; N_1, \dots, N_{k_0}) \ll x^{\theta-\varepsilon} + (\log x)^{k_0+2} y x^{-1/2} U_1 \dots U_{k_0} W |S(U_1, \dots, U_{k_0}, W)| \quad (5.5)$$

for some  $T$ , some  $U_1, \dots, U_{k_0}, W$  and some well-spaced set  $S(U_1, \dots, U_{k_0}, W)$  having the property (1.4) - (1.7): the mean-value technique of Montgomery [8]

$$|S(U_1, \dots, U_{k_0}, W)| \ll U_i^{-2j} (N_i^j + T) \log^A x, \quad 1 \leq i \leq k_0, \quad (5.6)$$

where  $A$  is constant; Halasz's method in the form due to Huxley

$$|S(U_1, \dots, U_{k_0}, W)| \ll (U_i^{-2j} N_i + U_i^{-6j} N_i T) \log^A x; \quad (5.7)$$

while from the inequality

$$|L_0(c+it_r)|^2 \geq U^2 L_0^{1-2c},$$

we have

$$|S(U_1, \dots, U_{k_0}, W)| \ll (W^{-4} L_0^2 + W^{-12} L_0^2 T) (\log x)^A. \quad (5.8)$$

Our last estimate for  $|S(U_1, \dots, U_{k_0}, W)|$  depend on the formula

$$L(s) = \frac{1}{2\pi i} \int_{\frac{1}{2}-c-iT}^{\frac{1}{2}-c+iT} \zeta(s+z) \left( (3L)^z - \left( \frac{L}{2^k} \right)^z \right) \frac{dz}{z} + O\left(L^{\frac{1}{2}-c} \log x\right)$$

for  $s = c + it, T_1 \leq t \leq 2T_1$ . This yield, if  $L_0 = L$ ,

$$W^4 |S(U_1, \dots, U_{k_0}, W)| \leq L_0^{4c-2} S |L_0(c+it_r)|^4 \ll T (\log x)^A. \quad (5.9)$$

Deshouillers and Iwaniec [6] proved: if  $N \geq 1, T \geq 1$ , the coefficients  $d_n$  are complex numbers and  $\varepsilon > 0$ , then

$$\int_{-2T}^{2T} \left| \zeta\left(\frac{1}{2} + it\right) \right|^4 \sum_{n < n \leq 2N} |d_n n^{it}|^2 dt \ll T^\varepsilon (N_2 T^{1/2} + N^{5/4} T^{3/4} + T) \sum_{n \leq n \leq 2N} |d_n|^2 \quad (5.10)$$

From (4.1), we have

$$\begin{aligned} W^4 U_j^2 |S(U_1, \dots, U_{k_0}, W)| &\leq L^{4c-2} N^{2c-1} \sum_{\tau} |L(c + it_{\tau})|^4 |N_j(c + it_{\tau})|^2 \\ &\ll L^{4c-2} N_j^{2c-1} \sum_{\tau} \int_{-2T}^{2T} \left| \zeta\left(\frac{1}{2} + it\right) \right|^4 L^{2-4c} N_j^{1-2c} |N_j\left(\frac{1}{2} + it\right)|^2 \frac{dt}{1 + |\tau - t_{\tau}|} \\ &\quad + |S(U_1, \dots, U_{k_0}, W)| |(\log x)|^4 \ll T^\varepsilon (N_k^2 T^{1/2} + N_k^{5/4} T^{3/4} + T) \quad (5.11) \end{aligned}$$

whence, for  $1 \leq k \leq j$ ,

$$|S(U_1, \dots, U_{k_0}, W)| \ll W^{-4} U_k^{-2} (N_k^2 T^{1/2} + N_k^{5/4} T^{3/4} + T) T^\varepsilon. \quad (5.12)$$

We have that

$$U_1 \cdots U_k W |S(U_1, \dots, U_{k_0}, W)| \ll x^{1/2-\varepsilon} + x^{h(a_1, \dots, a_{k_0})} \quad (5.13)$$

with  $h(a_1, \dots, a_{k_0}) < 1/2$  for a certain constant  $\varepsilon > 0$  since  $\{\theta_j\} \in E(\theta)$ . This will complete the proof of

$$R(x; M_1, \dots, M_k) \ll x^{\theta-\varepsilon}.$$

**Theorem 3.** Suppose  $\{\theta_j\} \in E(\theta)$ . then

$$\int_T^{2T} |W\left(\frac{1}{2} + it\right)| dt \ll x^{\frac{1}{2}-\varepsilon}, \quad (5.14)$$

for

$$T_1 \leq T \leq x^{1-\theta+\varepsilon}$$

where  $\theta$  is a fixed positive constant, and

$$T_1 = \exp\left((\log x)^{\frac{1}{3}} (\log \log x)^{-\frac{1}{3}}\right).$$

**Proof.** Defined  $S(U_1, \dots, U_k, W)$  be a set of  $t_r$  into for which

$$U_i < M_i^{c-1/2} |M_i(c + it_r)| < 2U_i, \quad i = 1, \dots, k_0,$$

$$W \leq L_0^{c-1/2} |L_0(c + it_r)| < 2W,$$

where  $x^{-2} \leq L^{1/2-c}W \leq 2^{-u} \leq 1$ , for some integer  $u$  and similarly for  $U_1, \dots, U_{k_0}$ . In Lemma 16 of [8], Heath-Brown proved

$$\int_T^{2T} |W(\frac{1}{2} + it)| dt \ll x^{\frac{1}{2}-\epsilon} + x^{-\delta} \sum |S(U_1, \dots, U_k, W)|$$

where  $\sum$  runs over  $(U_1, \dots, U_k, W)$  such that (1.1), (1.2) and (1.3), and  $|S(U_1, \dots, U_k, W)|$  satisfies (1.4), (1.5), (1.6), (1.7) and (1.9). Since  $\{\theta_j\} \in E(\theta)$ , we have that

$$|S(U_1, \dots, U_k, W)| < x^{\frac{1}{2}-\epsilon},$$

then

$$\sum |S(U_1, \dots, U_k, W)| < x^{\frac{1}{2}-\epsilon},$$

since at most  $(4 \log x)^{k+1}$  terms in above " $\sum$ ". Thus (5.14) follows.

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