



**HAL**  
open science

## Estimate of sums of Dirichlet series

Lou Shituo, Yao Qi

► **To cite this version:**

Lou Shituo, Yao Qi. Estimate of sums of Dirichlet series. Hardy-Ramanujan Journal, 1994, Volume 17 - 1994, pp.1 - 31. 10.46298/hrj.1994.128 . hal-01108730

**HAL Id: hal-01108730**

**<https://hal.science/hal-01108730v1>**

Submitted on 23 Jan 2015

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

## ESTIMATE OF SUMS OF DIRICHLET SERIES

Shituo Lou and Qi Yao

### § 1. INTRODUCTION.

Let  $x$  be a large positive number,  $\varepsilon$  be a small positive number and  $k$  be a finite integer. Let  $y = x^\theta$ ,  $1/2 < \theta < 7/12$ ,  $T = x^{1-\theta+\varepsilon/2}$ , and let  $M_1, \dots, M_k$  and  $L$  be real numbers such that

$$M_1 \cdots M_k L = x/2. \quad (1.1)$$

$U_1, \dots, U_k$  and  $W$  be real numbers such that

$$W \ll L^{1/2-\varepsilon} \quad (1.2)$$

and

$$U_i \ll M_i^{1/2}, 1 \leq i \leq k. \quad (1.3)$$

Let  $A$  be a fixed integer, and  $|S|$  be the number of elements of set  $S$ . We discuss the set  $S(U_1, \dots, U_k, W)$  which satisfies following conditions, for  $1 \leq i \leq k$  and  $j$  be a positive integer,

$$(1.4) \quad |S(U_1, \dots, U_k, W)| \ll U_i^{-2j} (M_i^j + T) (\log x)^A \text{ and } |S(U_1, \dots, U_k, W)| \ll W^{-2j} (L^j + T) (\log x)^A;$$

$$(1.5) \quad |S(U_1, \dots, U_k, W)| \ll (U_i^{-2j} M_i^j + U_i^{-6j} M_i^j T) (\log x)^A;$$

$$(1.6) \quad |S(U_1, \dots, U_k, W)| \ll (W^{-2j} L^j + W^{-6j} L^j T) (\log x)^A;$$

$$(1.7) \quad W^4 |S(U_1, \dots, U_k, W)| \ll T(\log x)^A.$$

Let  $a_j = \log M_j / \log x$ . In § 2 we shall give some functions  $h_i(a_1, \dots, a_k)$  for  $1 \leq i \leq 3$ , such that

$$U_1 \cdots U_k W |S(U_1, \dots, U_k, W)| \ll x^{1/2-\varepsilon} + x^{h_i(a_1, \dots, a_k)}. \quad (1.8)$$

In § 3, we discuss those  $S(U_1, \dots, U_k, W)$ 's which satisfy one more inequality : for  $1 \leq j \leq k$ ,

$$|S(U_1, \dots, U_k, W)| \ll W^{-4} U_i^{-2} (M_j^2 T^{1/2} + M_j^{5/4} T^{3/4} + T) T^\varepsilon. \quad (1.9)$$

If  $T^{1/5} \leq M_j \leq T^{1/3}$ , we replace (1.9) by

$$|S(U_1, \dots, U_k, W)| \ll W^{-4} U_i^{-2} M_j^{5/4} T^{3/4+\varepsilon}, \quad (1.10)$$

and, if  $M_j \leq T^{1/5}$ , we replace (1.9) by

$$|S(U_1, \dots, U_k, W)| \ll W^{-4} U_i^{-2} T^{1+\varepsilon}. \quad (1.11)$$

In § 3 we will give  $h_i(a_1, \dots, a_k)$  for  $4 \leq i \leq 10$  with (1.8).

Heath-Brown and Iwaniec [1] discussed the gaps between consecutive primes using sieve method. The remainder term

$$R(x; M_1, \dots, M_k) = \sum_{\substack{M_i < m_i \leq 2M_i \\ 1 \leq i \leq k}} a_{m_1, 1}, \dots, a_{m_k, k} r_{m_1 \dots m_k}, \quad (1.12)$$

where

$$r_d = \left[ \frac{x}{d} \right] - \left[ \frac{x-y}{d} \right],$$

$M_i < y$  and  $|a_{m_i, i}| \leq 1$ , be considered. Applying the method which is very close to Heath-Brown and Iwaniec's, in § 5, we will show that

$$R(x; M_1, \dots, M_k) \ll x^{\theta-\varepsilon}, \quad (1.13)$$

if (1.8) holds with

$$h_i(a_1, \dots, a_k) < 1/2, \quad (1.14)$$

In fact, they proved that (see [1]) if  $\theta = 11/20 + \varepsilon$ ,  $k = 2$ ,  $M_1 \ll x^{0.46-\varepsilon}$ , and  $M_2 \ll x^{0.46-\varepsilon}$ , (1.13) holds. Consequently, they obtained that for  $y = x^\theta$ ,  $\theta \geq 11/20 + \varepsilon$ ,

$$\pi(x) - \pi(x - y) > \frac{1}{212} \frac{y}{\log x}, \quad (1.15)$$

where  $\pi(x)$  be the number of primes  $\leq x$ .

In [2], Heath-Brown discussed some kind of products of Dirichlet series

$$W(s) = X(s) \prod_{j=1}^{k_0} Y_j(s),$$

where

$$X(s) = \sum_{L_0 < n \leq 2L_0} n^{-s},$$

$$Y_j(s) = \sum_{M_j < n \leq 2M_j} b_m n^{-s}, \quad |b_m| \leq 1;$$

and

$$L_0 \prod_{j=1}^{k_0} M_j = \frac{x}{2}.$$

Using Heath-Brown's method that was used in [2], in § 5, we shall show that

$$\int_T^{2T} |W(\frac{1}{2} + it)| dt \ll x^{\frac{1}{2}} \exp(-(\log x)^{\frac{1}{3}} (\log \log x)^{\frac{2}{3}}) \quad (1.16)$$

for

$$T_1 \leq T \leq x^{1-\theta+\varepsilon},$$

where  $\theta$  is a fixed positive constant with  $1/2 < \theta < 1$ , and

$$T_1 = \exp((\log x)^{\frac{1}{3}} (\log \log x)^{-\frac{1}{3}}),$$

if (1.8) holds with (1.14).

In [3], and [4], we shall apply the results of this paper to investigate gaps between consecutive primes and prove that (1.11) holds for  $\theta = 11/20 + \varepsilon$

in [3] and  $\theta = 6/11 + \varepsilon$  in [4]. In Section 4, we also discuss the case of  $\theta = 11/20 + \varepsilon$  that will be used in [3] to prove that

$$0.99 \frac{y}{\log x} < \pi(x) - \pi(x-y) < 1.01 \frac{y}{\log x}.$$

In Section 4 we discuss the case of  $\theta = 6/11 + \varepsilon$  as well that will be used in [4] to prove that

$$\frac{0.969y}{\log x} < \pi(x) - \pi(x-y) < \frac{1.031y}{\log x},$$

for  $\theta = 6/11 + \varepsilon$ .

In this paper  $\delta$  be another positive small number with  $\delta \ll \varepsilon$  and may be difference in some paragraphs. Also, we use

$$c\delta < \delta.$$

## § 2. ESTIMATE OF $|S(U_1, \dots, U_k, W)|$ .

In this section we estimate  $|S(U_1, \dots, U_k, W)|$ . First we divide  $M_1, \dots, M_k$ , and  $L$  into  $k_0 + 1$  parts (in this paper  $2 \leq k_0 \leq 6$ ), denote the products of elements in one part by  $N_1, \dots, N_{k_0}$ , and  $L_0$ , respectively. Also denote the product of those  $U_k$ 's or  $W$  that correspond to  $N_j$  or  $L_0$  by  $U_j$  or  $W$ , again. We now replace (1.1), (1.2) and (1.3) by

$$N_1 \cdots N_{k_0} L_0 = \frac{x}{2}, \quad (2.1)$$

$$U_j \leq N_j^{\frac{1}{2}} A_j, \quad 1 \leq j \leq k_0, \quad (2.2)$$

and

$$W \leq L_0^{\frac{1}{2}} A_0 \quad (2.3)$$

where  $A_j$  (or  $A_0$ ) =  $x^{-\delta}$  if  $L$  belongs to  $N_j$  (or  $L_0$ ), otherwise  $A_j = 1$ . Thus we have that

$$W \prod_{i=1}^k U_i \ll \left( L_0 \prod_{i=1}^k M_i \right)^{\frac{1}{2} - \delta} \ll x^{\frac{1}{2} - \delta}. \quad (2.4)$$

It is clear that if  $L_0$  satisfies (1.6) with  $i = 2$  and  $L_0 \ll T^{1/2}$ , then  $L_0$  satisfies (1.7). In this paper, assume that

$$L_0 \ll T^{1/2} \text{ if } L_0 \neq L. \quad (2.5)$$

Replacing  $L$  by  $L_0$ , we shall show that the function  $|S(U_1, \dots, U_{k_0}, W)|$  with (1.4), (1.5), (1.6), (1.7) and (2.4) such that (1.8).

We now renew the notations  $a_i = \log N_i / \log x$ ,  $i = 1, \dots, k_0$ ,  $\sigma = \log L_0 / \log x = 1 - a_1 - \dots - a_{k_0}$  and  $t_0 = \log T / \log x$ . Let

$$F = |S(U_1, \dots, U_{k_0}, W)| x^{-\delta} \quad (2.6)$$

We replace  $|S(U_1, \dots, U_{k_0}, W)|$  by  $F$ , it makes that we can drop the factor  $\log^A x$  in the right hand side of (1.4), (1.5), (1.6) and (1.7), and drop  $T^\epsilon$  in (1.9).

If  $F \geq 2U_i^{-2j} N_i^j$ , then we have that

$$N_i \ll T^{1/j} \quad (2.7)$$

since  $F \leq U_i^{-2j} (N_i^j + T)$ .

Suppose  $j$  such that

$$\sigma \leq t_0/j \text{ and } j = 2 \text{ if } \sigma \geq t_0/2. \quad (2.8)$$

Thus, if  $j > 2$ , we have  $\sigma \leq t_0/j$ , i.e.  $W^j \leq T$ . Then

$$F \ll W^{-2j} T.$$

We prove following lemmas :

**Lemma 2.1.** *Let  $k_0 = 2$ , we have*

$$U_1 U_2 |S(U_1, U_2, W)| \ll x^{h_0(a_1, a_2) + \epsilon/2} + x^{1/2 - \delta}, \quad (2.9)$$

where  $h_0(a_1, a_2) = h_0(a_2, a_1)$ ,  $a'_1 = \max\{t_0, a_1\}$  and

$$h_0(a_1, a_2) = \begin{cases} \frac{1}{2} - \epsilon, & a_1 \geq a_2 \geq t_0 \\ \frac{t_0}{2} + \frac{a_1}{2} + \min\left\{\frac{a_2}{4j}, \frac{\sigma}{8} + \frac{a_2}{12j}\right\}, & \text{otherwise.} \end{cases} \quad (2.10)$$

$$(2.11)$$

**Proof.** We have that, by (1.4), (1.5) and (1.6),

$$F \leq \min\{U_1^{-2}(N_1 + T), U_2^{-2}(N_2 + T), U_1^{-2}N_1 + U_1^{-6}N_1T, U_2^{-2}N_2 + U_2^{-6}N_2T, \\ W^{-2j}L_0^j + W^{-6j}L_0^jT, W^{-2j}T\},$$

and proceed to show that

$$U_1U_2WF \ll x^{h_0(a_1, a_2)} + x^{\frac{1}{2}-\delta}. \quad (2.12)$$

We now consider four cases :

**Case 1.**  $F \leq 2U_1^{-2}N_1$  and  $F \leq 2U_2^{-2}N_2$ .

Choosing  $k'$  such that  $L_0^{k'} \geq T$ , we have  $F \ll W^{-2k'}L_0^{k'}$ , and then

$$U_1U_2WF \ll U_1U_2W(U_2^{-2}N_2)^{\frac{1}{2}-\frac{1}{2(2k'+1)}}(U_1^{-2}N_2)^{\frac{1}{2}-\frac{1}{2(2k'+1)}}(W^{-2k'}L_0^{k'})^{\frac{1}{2k'+1}} \\ \ll (U_1U_2W)^{\frac{1}{2k'+1}}(N_1N_2L_0)^{\frac{1}{2}-\frac{1}{2(2k'+1)}} \ll x^{\frac{1}{2}-\delta}$$

by (2.4). Thus (2.12) holds. (Note : when  $a_1 \geq a_2 \geq t_0$ , we have that

$$F \leq 2U_1^{-2}N_1 \text{ and } F \leq 2U_2^{-2}N_2$$

by (1.4) with  $i = 1$ . This completes the proof of (2.10).)

**Case 2.**  $F \leq 2U_1^{-2}N_1$ ,  $F > 2U_2^{-2}N_2$ .

In this case we have that  $F \ll U_2^{-2}T$ ,  $F \ll U_2^{-6}N_2T$  and  $N_2 \ll T$  since (1.4), (1.5) (take  $M_i = N_i$ ) (2.7) and  $F > 2U_2^{-2}N_2$ . Thus

$$F \leq 2\min\{U_1^{-2}N_1, U_2^{-2}T, U_2^{-6}N_2T, W^{-2j}L_0^j, W^{-2j}T\} \\ + 2\min\{U_1^{-2}N_1, U_2^{-2}T, U_2^{-6}N_2T, W^{-2j}L_0^jT, W^{-2j}T\} \\ \leq 2(U_1^{-2}N_1)^{1/2}(U_2^{-2}T)^{(2j-3)/4j}(U_2^{-6}N_2T)^{1/4j}(W^{-2j}\min\{L_0^j, T\})^{1/2j} \\ + 2\min\{(U_1^{-2}N_1)^{1/2}(U_2^{-2}T)^{(2j-3)/4j}(U_2^{-6}N_2T)^{1/4j}(W^{-2j}T)^{1/2j}, \\ (U_1^{-2}N_1)^{1/2}(U_2^{-2}T)^{(2j-1)/4j}(U_2^{-6}N_2T)^{1/12j}(W^{-6j}L_0^jT)^{1/6j}\} \\ \ll (U_1U_2W)^{-1}N_1^{1/2}T^{1/2-1/2j}N_2^{1/4j}\min\left\{L_0^{\frac{1}{2}}, T^{\frac{1}{2j}}\right\} \\ + (U_1U_2W)^{-1}(TN_1)^{1/2}\min\left(N_2^{1/4j}, N_2^{1/12j}L_0^{1/6}\right). \quad (2.13)$$

If  $L_0 \ll T^{1/j}$ , we have that

$$L_0^{\frac{1}{2}} \ll T^{\frac{1}{2j}} \text{ and } N_2^{\frac{1}{6j}} L_0^{\frac{1}{3}} \ll T^{\frac{1}{2j}}.$$

Thus the first term on the right hand side of (2.13) is less than the second term. If  $L_0 \gg T^{1/j}$ , by (2.7), we have  $j = 2$ . We have that the first term on the right hand side of (2.13) is less than the second term again. Then

$$F \ll (U_1 U_2 W)^{-1} (T N_1)^{1/2} \min \left( N_2^{1/4j}, N_2^{1/12j} L_0^{1/6} \right).$$

Thus (2.12) and (2.11) follows.

**Case 3.**  $F \leq 2U_2^{-2} N_2, F \geq 2U_1^{-2} N_1$ .

The proof here is the same as in Case 2.

**Case 4.**  $F > 2U_1^{-2} N_1, F > 2U_2^{-2} N_1$ .

We have that

$$\begin{aligned} F &\leq 2 \min \left\{ U_1^{-2} T, U_2^{-2} T, U_1^{-6} N_1 T, U_2^{-6} N_2 T, W^{-2j} L_0^j, W^{-2j} T \right\} \\ &+ 2 \min \left\{ U_1^{-2} T, U_2^{-2} T, U_1^{-6} N_1 T, U_2^{-6} N_2 T, W^{-6j} L_0^j, W^{-2j} T \right\}. \end{aligned}$$

Then

$$\begin{aligned} F &\ll (U_1^{-2} T)^{\frac{1}{2}} (W^{-2j} L_0^j)^{\frac{1}{6j}} (W^{-2j} T)^{\frac{1}{6j}} (U_2^{-2} T)^{\frac{1}{2} - \frac{3}{4j}} (U_2^{-6} N_2 T)^{\frac{1}{4j}} \\ &+ (U_1^{-2} T)^{\frac{1}{2}} (W^{-6j} L_0^j T)^{\frac{1}{6j}} (U_2^{-2} T)^{\frac{2j-1}{4j}} (U_2^{-6} N_2 T)^{\frac{1}{4j}} \\ &\ll (U_1 U_2 W)^{-1} \left( T^{1-\frac{1}{6j}} L_0^{\frac{1}{6}} N_2^{\frac{1}{4j}} + T L_0^{\frac{1}{6}} N_2^{\frac{1}{12j}} \right) \\ &\leq (U_1 U_2 W)^{-1} T L_0^{\frac{1}{6}} N_2^{\frac{1}{12j}}, \end{aligned}$$

since  $N_2 \ll T$  by (2.7) with  $j = 1$ . Thus we get (2.12) and (2.11) again.

**Lemma 2.2.** *If  $N_2 \leq T$ , we have*

$$U_1 U_2 W |S(U_1, U_2, W)| \ll x^{\frac{1}{2}-\delta} + x^{h_1^i(a_1, a_2)}, \quad (2.14)$$

where

$$h_1(a_1, a_2) = \frac{i+1}{2(i+2)} t_0 + \frac{1}{4(i+2)} + \frac{a_1' + \sigma_i}{2} - \frac{a_1 + \sigma}{4(i+2)},$$



$$a'_1 = \max\{t_0, \log N_1 / \log x\} \text{ and } \sigma_i = \max\left\{\frac{t_0}{(i+2)}, 1 - a_1 - a_2\right\}.$$

**Proof.** We consider two cases :

**Case 1.**  $F \geq 2U_2^{-2}N_2$

In this case,  $N_2 \ll T$ , then

$$\begin{aligned} F &\ll \min\left\{U_1^{-2}x^{a'_1}, U_2^{-2}T, U_2^{-6}N_2T, W^{-2(i+2)}x^{(i+2)\sigma_i}\right\} \\ &\ll (U_1^{-2}x^{a'_1})^{1/2}(W^{-2(i+2)}x^{(i+2)\sigma_i})^{1/2(i+2)}(U_2^{-2}T)^{(2i+1)/4(i+2)}(U_2^{-6}N_2T)^{1/4(i+2)} \\ &\ll U_1^{-1}U_2^{-1}W^{-1}x^{(a'_1+\sigma_i)/2}T^{(i+1)/2(i+2)}N_2^{1/4(i+2)} \\ &\ll U_1^{-1}U_2^{-1}W^{-1}x^{1/4(i+2)}x^{a'_1/2-a_1/4(i+2)}T^{(i+1)/2(i+2)}x^{\sigma_i/2-\sigma/4(i+2)}, \end{aligned}$$

since  $N_2 = x/N_1L_0 = x^{1-a_1-\sigma}$ .

Then (2.14) follows.

**Case 2**  $F < 2U_2^{-2}N_2$

If  $F < 2U_1^{-2}N_1$ , the proof of this lemma is same as Case 1 of Lemma 2.1.

If  $F \geq 2U_1^{-2}N_1$ , then  $N_1 \leq T$ . Interchanging  $N_1$  with  $N_2$ , we may reduce this case to Case 1.

**Lemma 2.3.** Let  $k_0 = 3$  and  $\varepsilon < \sigma = 1 - a_1 - a_2 - a_3 \leq t_0/2$ , then

$$U_1U_2U_3W | S(U_1, U_2, U_3, W) | \ll x^{\frac{1}{2}-\delta} + x^{h_2(a_1, a_2, a_3)}, \quad (2.15)$$

where

$$h_2(a_1, a_2, a_3) = \frac{a'_1}{2} + \frac{a'_2 + a'_3}{2} + \frac{t_0}{6} + \frac{\sigma}{12}, \quad (2.16)$$

$$a'_1 = \frac{\log N'_1}{\log x} = \max\{a_1, t_0\}, a'_2 = \frac{\log N'_2}{\log x} = \max\left\{a_2, \frac{t_0}{3}\right\}$$

and  $a'_3 = \frac{\log N'_3}{\log x} = \max\left\{a_3, \frac{t_0}{3}\right\}$ .

**Proof.** In this case we have that  $L_0 \ll T^{1/2}$ . Then

$$\begin{aligned} F &\leq \min\{U_1^{-2}(N_1 + T), U_2^{-6}(N_2^3 + T), U_3^{-6}(N_3^3 + T), W^{-4}(L_0^2 + T), \\ &\quad W^{-4}L_0^2 + W_0^{-12}L_0^2T, U_3^{-6}N_3^3 + U_3^{-18}N_3^3T, U_1^{-2}N'_1\}. \end{aligned}$$

We consider two cases :

**Case 1.**  $F \leq 2W^{-4}L_0^2$ .

Suppose  $F \leq 2U_3^{-6}N_3^3$  and  $F \leq 2U_2^{-6}N_2^3$ . By (2.4), we have that

$$\begin{aligned} F &\ll \min \{U_1^{-2}N_1, W^{-4}L_0^2, U_2^{-6}N_2^3, U_3^{-6}N_3^3\} \\ &\ll (U_1^{-2}N_1)^{\frac{1}{2}-\frac{1}{28}}(U_2^{-6}N_2^3)^{\frac{1}{6}-\frac{1}{78}}(U_3^{-6}N_3^3)^{\frac{1}{6}-\frac{1}{78}}(W^{-4}L_0^2)^{\frac{1}{4}-\frac{1}{32}} \\ &\ll (U_1U_2U_3W)^{-1+\frac{1}{13}}(N_1N_2N_3L_0)^{\frac{1}{2}-\frac{1}{28}} \ll x^{\frac{1}{2}-\delta}. \end{aligned}$$

Suppose  $F > 2U_3^{-6}N_3^3$  and  $F > 2U_2^{-6}N_2^3$ , then

$$N_2 \ll T^{1/3}, N_3 \ll T^{1/3} \text{ and } F \ll U_2^{-18}N_2^3T,$$

since  $2U_i^{-6}N_i^3 \leq F \leq U_i^{-6}(N_i^3 + T)$ ,  $F \leq U_2^{-18}N_2^3T + U_2^{-6}N_2^3$  for  $i = 2$  or  $3$ ,  $L_0 \ll T^{1/2}$  and  $N_2 \ll T^{1/3}$ . Therefore

$$\begin{aligned} F &\ll (U_1^{-2}N_1)^{\frac{1}{2}}(W^{-4}L_0^2)^{\frac{1}{4}}(U_3^{-6}T)^{\frac{1}{6}}(U_2^{-6}T)^{\frac{1}{24}}(U_2^{-18}N_2^3T)^{\frac{1}{24}} \\ &\ll (U_1U_2U_3W)^{-1}T^{\frac{1}{4}}N_2^{\frac{1}{8}}N_1^{\frac{1}{2}}L_0^{\frac{1}{2}} \\ &\ll (U_1U_2U_3W)^{-1}x^{h_2(a_1, a_2, a_3)} \end{aligned} \quad (2.17)$$

Suppose  $F > 2U_3^{-6}N_3^3$  and  $F \leq 2U_2^{-6}N_2^3$ , then

$$\begin{aligned} F &\ll (U_1^{-2}N_1)^{\frac{1}{2}}(U_2^{-6}N_2^3)^{\frac{1}{6}}(W^{-4}L_0^2)^{\frac{1}{4}}(U_3^{-6}T)^{\frac{1}{24}}(U_3^{-18}N_3^3T)^{\frac{1}{24}} \\ &\ll (U_1U_2U_3W)^{-1}N_1^{\frac{1}{2}}N_2^{\frac{1}{2}}L_0^{\frac{1}{2}}T^{\frac{1}{12}}N_3^{\frac{1}{6}} \ll (U_1U_2U_3W)^{-1}x^{h_2(a_1, a_2, a_3)} \end{aligned} \quad (2.18)$$

since  $a'_3 = t_0/3$  and  $N_3 \ll T^{1/3}$  in this case. Suppose  $F < 2U_3^{-6}N_3^3$  and  $F \geq 2U_2^{-6}N_2^3$ .

On interchanging  $N_2$  with  $N_3$ , we may reduce this case to (2.18).

**Case 2.**  $F > 2W^{-4}L_0^2$ .

We have that

$$F \ll (U_1^{-2}N_1)^{\frac{1}{2}}(U_2^{-6}N_2^3)^{\frac{1}{6}}(U_3^{-6}N_3^3)^{\frac{1}{6}}(W^{-4}T)^{\frac{1}{6}}(W^{-12}L_0^2T)^{\frac{1}{24}},$$

then

$$F \ll (U_1U_2U_3W)^{-1}N_1^{\frac{1}{2}}N_2^{\frac{1}{2}}N_3^{\frac{1}{2}}T^{\frac{1}{6}}L_0^{\frac{1}{12}} \ll (U_1U_2U_3W)^{-1}x^{h_2(a_1, a_2, a_3)}.$$

The proof of Lemma 2.3 is now completed.

**Lemma 2.4.** *If  $k_0 = 3, a_1 \geq t_0, a_2 \geq t_0/2$ , then*

$$U_1 U_2 U_3 W | S(U_1, U_2, U_3, W) | \ll x^{\frac{1}{2}-\delta} + x^{h_3(a_1, a_2, a_3)}, \quad (2.19)$$

where

$$h_3(a_1, a_2, a_3) = \frac{a_1 + a_2}{2} + \frac{a_3'}{2} + \frac{t_0}{8} + \frac{\sigma}{6},$$

$a_3' = \log N_3'/\log x = \max\{t_0/4, a_3\}$  and  $\sigma = 1 - a_1 - a_2 - a_3$ .

**Proof.** We have that

$$F \ll \min\{U_1^{-2} N_1, U_2^{-4} N_2^2, U_3^{-8} (N_3^4 + T), W^{-6} (L_0^3 + T), W^{-18} L_0^3 T + W^{-6} L_0^3\}.$$

Suppose  $F \leq 2W^{-6} L_0^3$ , we are in Case 1 of Lemma 2.1, so that

$$U_1 U_2 U_3 W F \ll x^{\frac{1}{2}-\delta}.$$

Suppose  $F > 2W^{-6} L_0^3$ , then

$$\begin{aligned} F &\ll (U_1^{-2} N_1)^{\frac{1}{2}} (U_2^{-4} N_2^2)^{\frac{1}{4}} (U_3^{-8} x^{4a_3'})^{\frac{1}{8}} (W^{-6} T)^{\frac{1}{18}} (W^{-18} L_0^3 T)^{\frac{1}{18}} \\ &\ll (U_1 U_2 U_3 W)^{-1} N_1^{\frac{1}{2}} N_2^{\frac{1}{2}} x^{\frac{a_3'}{2}} T^{\frac{1}{9}} L_0^{\frac{1}{9}} \ll (U_1 U_2 U_3 W)^{-1} x^{h_3(a_1, a_2, a_3)}. \end{aligned}$$

Thus the lemma follows.

§ 3.  $L_0 = L$ .

In this section we discuss  $|S(U_1, \dots, U_k, W)|$  with (1.4), (1.5), (1.6), (1.7) and (1.9). We use the same notations as in § 2 and assume  $L_0 = L$ .

**Lemma 3.1.** *Let  $L = L_0$ . If  $L_0 \geq T^{1/2+\varepsilon}$ , then*

$$U_1 \cdots U_{k_0} W | S(U_1, U_{k_0}, W) | \ll x^{\frac{1}{2}-\delta}.$$

**Proof.** Take  $j$  such that

$$N_1^j \cdots N_{k_0}^j \geq T.$$

We have

$$F \ll \min\{U_1^{-2j} \cdots U_{k_0}^{-2j} N_1^{2j}, W^{-4} T\}.$$

Thus

$$\begin{aligned} F &\ll (U_1^{-2j} \dots U_{k_0}^{-2j} N_1^{2j} \dots N_{k_0}^{2j})^{3/4} (W^{-4} T)^{1/4} \\ &\ll (U_1^{-2j} \dots U_{k_0}^{-2j} N_1^{2j} \dots N_{k_0}^{2j})^{1/2j} (W^{-4} T)^{1/4} \\ &\ll (U_1 \dots U_{k_0} W)^{-1} N_1^{1/2} \dots N_{k_0}^{1/2} T^{1/4}, \end{aligned}$$

since  $U_1^{-2j} \dots U_{k_0}^{-2j} N_1^{2j} \dots N_{k_0}^{2j} \leq 1$ . We obtain

$$U_1 \dots U_{k_0} W F \ll x^{1/2-\epsilon/2}$$

by  $L_0 \geq T^{1/2+\epsilon}$ .

**Lemma 3.2.**  $k_0 = 2, N_1 \geq T, N_2 < T^{1/5}$  and

$$L_0 N_2^{1/2} \geq T^{1/2+\epsilon},$$

then

$$U_1 U_2 W |S(U_1, U_2, W)| \ll x^{\frac{1}{2}-\delta}.$$

**Proof.** We have that, choosing  $j$  such that  $N_2^j \geq T$ ,

$$F \ll \min\{W^{-4} U_2^{-2} T, U_1^{-2} N_1, U_2^{-j} N_2^j\}.$$

Thus

$$\begin{aligned} F &\ll (W^{-4} U_2^{-2} T)^{1/4} (U_1^{-2} N_1)^{1/2} (U_2^{-2j} N_2^j)^{1/4j} \\ &\ll (U_1 U_2 W)^{-1} N_1^{1/2} N_2^{1/4} T^{1/4} = (U_1 U_2 W)^{-1} (x/L_0 N_2)^{1/2} N_2^{1/4} T^{1/4} \\ &\ll x^{\frac{1}{2}-\delta}, \end{aligned}$$

since  $L_0^{1/2} N_2^{1/4} \geq T^{1/4+\epsilon/2}$ .

**Lemma 3.3.**  $k_0 = 2, N_1 \geq T, N_2 \geq T^{1/5}$  and

$$L_0 \geq T^{3/8+\epsilon} N_2^{1/8},$$

then

$$U_1 U_2 W |S(U_1, U_2, W)| \ll x^{\frac{1}{2}-\delta}.$$

**Proof.** We have that, choosing  $j$  such that  $N_2^j \geq T$ ,

$$F \ll \min\{W^{-4} U_2^{-2} N_2^{5/4} T^{3/4}, U_1^{-2} N_1, U_2^{-2j} N_2^j\}.$$

Thus, by  $U_2^{-2j} N_2^j \leq 1$ ,

$$\begin{aligned} F &\ll (W^{-4} U_2^{-2} N_2^{5/4} T^{3/4})^{1/4} (U_1^{-2} N_1)^{1/2} (U_2^{-2j} N_2^j)^{1/4j} \\ &\ll (U_1 U_2 W)^{-1} N_1^{1/2} N_2^{9/16} T^{3/16} = (U_1 U_2 W)^{-1} (x/L_0 N_2)^{1/2} N_2^{9/16} T^{3/16} \\ &\ll x^{1/2-\delta}. \end{aligned}$$

Write

$$\begin{aligned} c_1 &= \frac{1}{2j_1} + \frac{1}{2j_2} - \frac{3}{10}, c_2 = 7 + \frac{5}{j_2} + \frac{5}{j_1}, c_3 = \frac{7}{8} - \frac{3}{4j_2} - \frac{3}{4j_3} - \frac{3}{8j_4}, \\ c_4 &= \frac{1}{4} \left( \frac{1}{j_2} + \frac{1}{j_3} + \frac{1}{2j_4} - \frac{1}{2} \right), c_5 = \frac{3}{8} - \frac{1}{4j_2} - \frac{3}{4j_3} - \frac{3}{8j_4}, \end{aligned}$$

and

$$c_6 = \frac{1}{4j_2} + \frac{1}{4j_3} + \frac{1}{8j_4} - \frac{1}{8}.$$

Define

$$N'_i = \max \{ N_i, T^{1/i} \}, i = 3 \text{ and } 4. \quad (3.1)$$

**Lemma 3.4.** *Let  $K_0 = 4, a_3 \geq a_4, t_0/3 \geq a_4 \geq t_0/5, \sigma = 1 - a_1 - a_2 - a_3 - a_4 \geq t_0/j_1, j_1$  and  $j_2, j_3, j_4$  be integers with  $c_i \geq 0$  ( $1 \leq i \leq 6$ ), then*

$$U_1 U_2 U_3 U_4 W | S(U_1, U_2, U_3, U_4, W) | \ll x^{\frac{1}{2}-\delta} + x^{h_4(a_1, a_2, a_3, a_4)} \quad (3.2)$$

where

$$\begin{aligned} h_4(a_1, a_2, a_3, a_4) &= \frac{a'_3}{2} + \frac{a'_4}{4} + \frac{5a_4}{16} + \frac{3t_0}{16} \\ &+ \max \left\{ \frac{a_1}{2} + c_6 j_2 a_2 + (c_5 + c_6) t_0, c_4 a_1 + \frac{a_2}{2} + (c_3 + c_4) t_0 \right\}. \end{aligned}$$

**Proof.** We have that, by  $t_0/3 \geq a_4 \geq t_0/5, \sigma \geq t_0/5$  and  $a_3 \geq a_4$ ,

$$\begin{aligned} F &\ll \min \{ W^{-4} U_4^{-2} N_4^{5/4} T^{3/4}, U_1^{-2} (N_1 + T), U_1^{-6} N_1 T + U_1^{-2} N_1, U_2^{-2j_2} (N_2^{j_2} + T) \\ &U_2^{-2j_2} N_2 + U_2^{-6j_2} N_2^{j_2} T, U_3^{-10} N_3^5, U_4^{-10} N_4^5, W^{-2j_1} L_0^{j_1}, U_3^{-2j_3} N_3^{j_3}, \\ &U_4^{-2j_4} (N_4^{j_4} + T) \}. \end{aligned}$$

We discuss following cases

**Case 1.**  $F \leq 2U_1^{-2}N_1$  and  $F \leq 2U_2^{-2j_2}N_2^{j_2}$ .

By (2.4), we have that

$$\begin{aligned} F &\ll (U_1^{-2}N_1)^{\frac{1}{2}-\frac{5c_1}{c_2}} (U_2^{-2j_2}N_2^{j_2})^{\frac{1}{2j_2}-\frac{5c_1}{j_2c_2}} (U_3^{-10}N_3^5)^{\frac{1}{10}-\frac{c_1}{c_2}} (U_4^{-10}N_4^5)^{\frac{1}{10}-\frac{c_1}{c_2}} \times \\ &\quad \times (W^{-2j_1}L_0^{j_1})^{\frac{1}{2j_1}-\frac{5c_1}{j_1c_2}} \\ &\ll (U_1U_2U_3U_4W)^{-1+\frac{10c_1}{c_2}} (N_1N_2N_3N_4L_0)^{\frac{1}{2}-\frac{5c_1}{c_2}} \ll (U_1U_2U_3U_4W)^{-1}x^{\frac{1}{2}-\delta}, \end{aligned}$$

since  $c_1/c_2 < 1/10$ .

**Case 2.**  $F > 2U_1^{-2}N_1$  and  $F \leq 2U_2^{-2j_2}N_2^{j_2}$ .

We use  $F \ll U_i^{-2j_i}N_i^{j_i}$  for  $i = 3$  or  $4$ . We have that, by  $2c_3 + 6c_4 = 1$ ,

$$\begin{aligned} F &\ll \left(W^{-4}U_4^{-2}N_4^{\frac{5}{4}}T^{\frac{3}{4}}\right)^{\frac{1}{4}} (U_4^{-2j_4}N_4^{j_4})^{\frac{1}{4j_4}} (U_3^{-2j_3}N_3^{j_3})^{\frac{1}{2j_3}} (U_2^{-2j_2}N_2^{j_2})^{\frac{1}{2j_2}} \times \\ &\quad \times (U_1^{-2}T)^{c_3}(U_1^{-6}N_1T)^{c_4} \\ &\ll (U_1U_2U_3U_4W)^{-1}N_1^{c_4}N_2^{\frac{1}{2}}N_3^{\frac{1}{2}}N_4^{\frac{1}{4}}N_4^{\frac{5}{16}}T^{\frac{3}{16}+c_3+c_4} \\ &\ll (U_1U_2U_3U_4W)^{-1}x^{h_4(a_1, a_2, a_3, a_4)}, \end{aligned} \tag{3.3}$$

**Case 3.**  $F \leq 2U_1^{-2}N_1$  and  $F > 2U_2^{-2j_2}N_2^{j_2}$ .

Then we have that

$$\begin{aligned} F &\ll \left(W^{-4}U_4^{-2}N_4^{\frac{5}{4}}T^{\frac{3}{4}}\right)^{\frac{1}{4}} (U_4^{-2j_4}N_4^{j_4})^{\frac{1}{4j_4}} (U_3^{-2j_3}N_3^{j_3})^{\frac{1}{2j_3}} (U_1^{-2}N_1)^{\frac{1}{2}} \times \\ &\quad \times (U_2^{-2j_2}T)^{c_5}(U_2^{-6j_2}N_2^{j_2}T)^{c_6} \\ &\ll (U_1U_2U_3U_4W)^{-1}N_1^{\frac{1}{2}}N_2^{j_2c_6}N_3^{\frac{1}{2}}N_4^{\frac{1}{4}}N_4^{\frac{5}{16}}T^{\frac{3}{16}+c_5+c_6} \\ &\ll (U_1U_2U_3U_4W)^{-1}x^{h_4(a_1, a_2, a_3, a_4)}, \end{aligned} \tag{3.4}$$

as required.

**Case 4.**  $F > 2U_1^{-2}N_1$  and  $F > 2U_2^{-2j_2}N_2^{j_2}$ .

In this case we have that  $N_1 \ll T$  and  $N_2 \ll T^{1/j_2}$ . Thus (3.3) and (3.4) are true again, i.e. we get (3.1) again.

If take  $j_2 = 2$  and  $j_3 = j_4 = 5$ , then

$$h_4(a_1, a_2, a_3, a_4) = \frac{a'_3}{2} + \frac{a'_4}{4} + \frac{5a_4}{16} + \frac{3t_0}{16} + \max \left\{ \frac{a_1}{2} + \frac{3}{20}a_2 + \frac{t_0}{10}, \frac{3}{40}a_1 + \frac{a_2}{2} + \frac{7}{20}t_0 \right\} \quad (3.5)$$

**Lemma 3.5.** Suppose  $k_0 = 4$ ,  $a_4 \leq t_0/5$ ,  $a_i \geq t_0/j_i$ , ( $i = 2, 3$  and  $4$ ) and  $\sigma \geq t_0/j$  with

$$\frac{1}{j_2} + \frac{1}{j_3} + \frac{1}{j_4} + \frac{1}{j} > \frac{1}{2},$$

then

$$U_1 U_2 U_3 U_4 W | S(U_1, U_2, U_3, U_4, W) | \ll x^{\frac{1}{2}-\delta} + x^{h_5(a_1, a_2, a_3, a_4)}$$

where

$$h_5(a_1, a_2, a_3, a_4) = \frac{a'_3}{2} + \frac{a'_4}{4} + \frac{t_0}{4} + \max \left\{ \frac{a_1}{2} + c_5 j_2 a_2 + (c_5 + c_6)t_0, a_1 + \frac{a_2}{2} + (c_3 + c_4)t_0 \right\}. \quad (3.7)$$

**Proof.** In the proof of Lemma 3.2 we replace (1.10) by (1.11). Then we replace  $5a_4/16$  by  $t_0/16$  and we get  $h_5(a_1, a_2, a_3, a_4)$  replace  $h_4(a_1, a_2, a_3, a_4)$ . The lemma follows.

**Lemma 3.6.** If  $k_0 = 3$ ,  $a_1 \geq t_0$  and  $t_0/3 \geq a_3 \geq t_0/5$ , then

$$U_1 U_2 U_3 W | S(U_1, U_2, U_3, W) | \ll x^{\frac{1}{2}-\delta} + x^{h_6(a_1, a_2, a_3)}$$

where

$$h_6(a_1, a_2, a_3) = \frac{13t_0}{48} + \frac{a_1 + a'_2}{2} + \frac{5a_3}{16}.$$

and  $a'_2 = \max\{a_2, t_0/3\}$ .

**Proof.** We have

$$\begin{aligned} F &\ll \left( W^{-4} U_3^{-2} N_3^{\frac{5}{2}} T^{\frac{3}{2}} \right)^{\frac{1}{4}} (U_1^{-2} N_1)^{\frac{1}{2}} (U_2^{-6} x^{3a'_2})^{\frac{1}{6}} (U_3^{-6} T)^{\frac{1}{12}} \\ &\ll (U_1 U_2 U_3 W)^{-1} N_1^{\frac{1}{2}} N_3^{\frac{5}{6}} x^{\frac{a'_2}{2}} T^{\frac{13}{12}} \ll (U_1 U_2 U_3 W)^{-1} x^{h_6(a_1, a_2, a_3)}. \end{aligned}$$

Thus the lemma follows.

**Lemma 3.7.** If  $k_0 = 4$  and the  $a_i$ 's satisfy : (1)  $a_1 + a_3 + a_4 \geq t_0$ ; (2)

$a_2 + a_3 + a_4 \leq t_0$ ; (3)  $a_1 + a_3 \leq t_0$ , (4)  $a_1 + a_4 \leq t_0$ ; and (5)  $a_4 \leq a_3 \leq t_0/5$ ;  
then

$$U_1 U_2 U_3 U_4 W | S(U_1, U_2, U_3, U_4, W) | \ll x^{\frac{1}{2}-\delta} + x^{h_7(a_1, a_2, a_3, a_4)},$$

where

$$h_7(a_1, a_2, a_3, a_4) = t_0 + a_2/8.$$

**Proof.** W.L.O.G. we can agree that  $U_3 \geq U_4$ , then we have

$$F \leq \min\{W^{-4}U_3^{-2}T, U_1^{-2}U_3^{-2}T, U_2^{-2}U_3^{-2}U_4^{-2}T, U_2^{-2}N_2 + U_2^{-6}N_2T, U_1^{-2}U_3^{-2}U_4^{-2}N_1N_3N_4\}.$$

If  $F \leq 2U_2^{-2}N_2$ , this is Case 1 of Lemma 2.1; if  $F \geq 2U_2^{-2}N_2$ , from  $U_3 \geq U_4$ , we have

$$\begin{aligned} F &\ll (W^{-4}U_3^{-2}T)^{\frac{1}{4}}(U_1^{-2}U_3^{-2}T)^{\frac{1}{2}}(U_2^{-2}U_3^{-2}U_4^{-2}T)^{\frac{1}{8}}(U_2^{-6}N_2T)^{\frac{1}{8}} \\ &\ll (U_1U_2U_3U_4W)^{-1}TN_2^{\frac{1}{8}}U_3^{-\frac{3}{8}}U_4^{\frac{3}{8}} \ll (U_1U_2U_3U_4W)^{-1}x^{h_7(a_1, a_2, a_3, a_4)}, \end{aligned}$$

and the lemma follows.

**Lemma 3.8.** Suppose that  $k_0 = 6, t_0/3 \geq a_1 \geq \dots \geq a_6 \geq t_0/5$  and  $\sigma + a_1 + a_2 \geq t_0$ , then

$$U_1 U_2 \dots U_6 W | S(U_1, \dots, U_6) | \ll x^{\frac{1}{2}-\delta} + x^{h_8(a_1, \dots, a_6)}$$

where

$$h_8(a_1, \dots, a_6) = \frac{19}{48}a_6 + \frac{a_5}{12} + \frac{a_4}{12} + \frac{15}{16}t_0.$$

**Proof.** If  $F \leq U_6^{-2}U_5^{-2}U_4^{-2}N_6N_5N_4$ , then

$$\begin{aligned} F &\ll (U_6^{-2}U_5^{-2}U_4^{-2}N_6N_5N_4)^{\frac{1}{2}-\frac{1}{22}}(W^{-2}U_1^{-2}U_2^{-2}L_0N_1N_2)^{\frac{1}{2}-\frac{1}{22}}(U_3^{-10}N_3^5)^{\frac{1}{16}-\frac{1}{116}} \\ &\ll (WU_6 \dots U_1)^{-1+\frac{1}{11}}(L_0N_6 \dots N_1)^{\frac{1}{2}-\frac{1}{22}} \ll x^{\frac{1}{2}-\delta}. \end{aligned}$$

If  $F > U_6^{-2}U_5^{-2}U_4^{-2}N_6N_5N_4$ , then

$$F \leq U_6^{-6}U_5^{-6}U_4^{-6}N_6N_5N_4T,$$



and

$$\begin{aligned} F &\ll \left( W^{-4} U_6^{-2} N_6^{\frac{5}{4}} T^{\frac{3}{4}} \right)^{\frac{1}{4}} (U_1^{-2} U_2^{-2} U_3^{-2} T)^{\frac{1}{2}} (U_4^{-6} U_5^{-6} U_6^{-6} N_4 N_5 N_6 T)^{\frac{1}{12}} \times \\ &\times (U_4^{-4} U_5^{-2} T)^{\frac{1}{8}} (U_5^{-6} T)^{\frac{1}{24}} \\ &\ll (W U_1 \cdots U_6)^{-1} T^{\frac{15}{16}} N_4^{\frac{1}{2}} N_5^{\frac{1}{2}} N_6^{\frac{19}{16}} \ll (W U_1 \cdots U_6)^{-1} x^{h_8(a_1, \dots, a_6)}. \end{aligned}$$

**Lemma 3.9.** Suppose that  $k_0 = 4, a_1 \geq a_2 \geq a_3 \geq a_4, a_1 + a_3 \leq t_0 < a_1 + a_3 + a_4, \sigma \geq t_0/3$  and  $2a_4 \leq t_0/3$ , then

$$U_1 \cdots U_4 W \mid S(U_1, \dots, U_4) \ll x^{\frac{1}{2}-\delta} + x^{h_9(a_1, \dots, a_4)}$$

where

$$h_9(a_1, \dots, a_4) = \frac{15}{16} t_0 + \frac{a_2}{8} + \frac{5a_4}{8}.$$

**Proof.** If  $F \leq 2U_2^{-2} N_2$ , then

$$F \ll (U_1^{-2} U_3^{-2} U_4^2 N_1 N_3 N_4)^{\frac{1}{2}-\frac{1}{30}} (U_2^{-2} N_2)^{\frac{1}{2}-\frac{1}{30}} (W^{-6} L_0^3)^{\frac{1}{8}-\frac{1}{10}} \ll x^{\frac{1}{2}-\delta}$$

by (2.4).

If  $F > 2U_2^{-2} N_2$ , then  $F \ll \min\{U_2^{-2} T, U_2^{-6} N_2 T\}$ . Thus

$$\begin{aligned} F &\ll \left( W^{-4} U_4^{-4} N_4^{\frac{5}{2}} T^{\frac{3}{4}} \right)^{\frac{1}{4}} (U_1^{-2} U_3^{-2} T)^{\frac{1}{2}} (U_2^{-2} T)^{\frac{1}{8}} (U_2^{-6} N_2 T)^{\frac{1}{8}} \\ &\ll T^{\frac{15}{16}} N_2^{\frac{1}{8}} N_4^{\frac{5}{8}} \ll x^{h_9(a_1, \dots, a_4)}, \end{aligned}$$

as required.

**Lemma 3.10.** Suppose that  $k_0 = 3, a_2 + a_3 \leq t_0, a_1 + \sigma \geq t_0$  and  $a_3 \leq t_0/5$ , then

$$U_1 \cdots U_4 W \mid S(U_1, \dots, U_4) \ll x^{\frac{1}{2}-\delta} + x^{h_{10}(a_1, \dots, a_4)}$$

where

$$h_{10}(a_1, \dots, a_4) = t_0 + \frac{a_2}{8} + \frac{a_3}{24}.$$

**Proof.** We have that

$$F \ll \min\{W^{-4} U_3^{-2} T, U_1^{-2} T, U_2^{-2} U_3^{-2} (T + N_2 N_3), U_2^{-6} U_3^{-6} N_2 N_3 T + U_2^{-2} U_3^{-2} N_2 N_3,$$

$$W^{-2}U_1^{-2}L_0N_1\}.$$

If  $F \leq 2U_2^{-2}U_3^{-2}N_2N_3$ , then we go back to the Case 1 on Lemma 3.1, since  $F \ll W^{-2}U_1^{-2}L_0N_1$ , and  $N_4 \geq x^\varepsilon$ ; if  $F > 2U_2^{-2}U_3^{-2}N_2N_3$ , then

$$\begin{aligned} F &\ll (W^{-4}U_3^{-2}T)^{\frac{1}{4}} (U_1^{-2}T)^{\frac{1}{2}} (U_2^{-2}U_3^{-2}T)^{\frac{1}{4}} (U_2^{-6}U_3^{-6}N_2N_3T)^{\frac{1}{24}} (U_2^{-6}N_2T)^{\frac{1}{12}} \\ &\ll (U_1U_2U_3W)^{-1}x^{h_{10}(a_1, a_2, a_3)}, \end{aligned}$$

as required.

#### § 4. SET $E(\theta)$ .

Let  $\theta_j (1 \leq j \leq r)$  be positive numbers with

$$\sum_{1 \leq i \leq r} \theta_i < 1.$$

Denote  $\{\theta_j\} = \{\theta_0, \theta_1, \dots, \theta_r\}$  and

$$\theta_0 = 1 - \sum_{1 \leq i \leq r} \theta_i.$$

Divide the set of numbers

$$\{\theta_0, \theta_1, \dots, \theta_r\}$$

into three subsets and call the sums of terms in each subset  $a_1, a_2$  and  $\sigma$ , where  $\sigma$  is distinguished by being  $\theta_0$  if  $\sigma > t_0/2$ , and otherwise  $\sigma \leq t_0/2$ . Since  $a_1 + a_2 + \sigma = 1$ , any two of  $a_1, a_2, \sigma$  determine the third. We attach an exactly similar meaning to  $\{a_1, a_2, a_3, \sigma\}$  and  $\{a_1, a_2, a_3, a_4, \sigma\}$ . We refer to  $\{a_1, a_2, \sigma\}$ , or  $\{a_1, a_2, a_3, \sigma\}$ , or  $\{a_1, a_2, a_3, a_4, \sigma\}$  as set of complementary partial sums. For  $\{a_1, \dots, a_k, \sigma\}$  if there exists some  $h_i(a_1, \dots, a_k)$  which satisfies (1.8) with

$$h_i(a_1, \dots, a_k) < 1/2,$$

then we call it  $\{a_1, \dots, a_k, \sigma\} \in E(\theta)$ . For short, we write that  $\{\theta_j\} \in E(\theta)$  instead of  $\{a_1, \dots, a_k, \sigma\} \in E(\theta)$ .

**Lemma 4.1.** *If there exists at least one set of complementary partial sums*

$\{a_1, a_2, \sigma\}$  (or  $\{a_1, a_2, a_3, \sigma\}$ ) of  $\{\theta_j\}$  such that at least one of conditions (4.1.1) - (4.1.4) (or (4.1.5)) holds, then  $\{\theta_j\} \in E(\theta)$ .

(4.1.1)  $a_1 \geq a_2 \geq t_0$  and  $\sigma > \varepsilon$ ;

This situation is covered by (2.9) and (2.10) of Lemma 2.1

(4.1.2)  $a_2 \leq a_1 \leq t_0$  and  $a_2 < 4 - 8t_0$ ;

Using (2.11) of Lemma 2.1 with  $a'_1 = t_0$  and  $j = 2$ , we have that

$$h_0(a_1, a_2) = t_0 + \min\{a_2/8, \sigma/6 + a_2/24\} \leq t_0 + a_2/8 < 1/2.$$

(4.1.3)  $a_1 \geq t_0, a_2 > \frac{2i+2}{2i+3}t_0$  and  $\sigma \geq \frac{t_0}{i+2}$ ;

In this case, we use Lemma 2.2,  $\sigma_i = \max\{t_0/(i+2), 1 - a_1 - a_2\} = \max\{t_0/(i+2), \sigma\} = \sigma$ , and  $a'_1 = a_1$ , then

$$\begin{aligned} h_1^i(a_1, a_2) &= \frac{i+1}{2(i+2)}t_0 + \frac{1}{4(i+2)} + \frac{2i+3}{4(i+2)}(a_1 + \sigma) \\ &= \frac{i+1}{2(i+2)}t_0 + \frac{1}{4(i+2)} + \frac{2i+3}{4(i+2)}(1 - a_2) < \frac{1}{2}. \end{aligned}$$

(4.1.4)  $a_3 \leq a_2 \leq t_0/3 \leq a_1 \leq t_0, \sigma = 1 - a_1 - a_2 - a_3$  and  $a_1 + \sigma/6 < 1 - t_0$ .

By Lemma 2.3 with  $a'_2 = \max\{t_0/3, a_2\} = t_0/3, a'_3 = \max\{t_0/3, a_3\} = t_0/3$ ; then

$$\begin{aligned} h_2(a_1, a_2, a_3) &= a_1/2 + (a'_2 + a'_3)/2 + t_0/6 + \sigma/12 \\ &= a_1/2 + t_0/2 + \sigma/12 < 1/2. \end{aligned}$$

**Lemma 4.2.** If there exists at least one set of complementary partial sums  $\{a_1, a_2, a_3, \sigma\}$  of  $\{\theta_j\}$  such that at least one of conditions (4.2.1) - (4.2.3) holds, then  $\{\theta_j\} \in E(\theta)$ .

(4.2.1).  $a_1 \geq t_0, a_2 \geq t_0/2, a_3 \geq t_0/4$  and  $\sigma > 2t_0/7$ .

In this case, Lemma 2.4 is applied, we have  $a'_3 = \max\{a_3, t_0/4\} = a_3$ , then

$$\begin{aligned} h_3(a_1, a_2, a_3) &= (a_1 + a_2 + a_3)/2 + t_0/8 + \sigma/16 \\ &= 1/2 + t_0/8 - 7\sigma/16 < 1/2, \end{aligned}$$

since  $\sigma > 2t_0/7$ .

(4.2.2).  $a_1 \geq t_0, a_2 \geq a_3 \geq t_0/3$  and  $\sigma > 2t_0/5$ .

Using Lemma 2.3, we have  $a'_2 = \max\{t_0/3, a_2\} = a_2$  and  $a'_3 = \max\{t_0/3, a_3\} = a_3$ , then

$$\begin{aligned} h_3(a_1, a_2, a_3) &= (a_1 + a_2 + a_3)/2 + t_0/6 + \sigma/12 \\ &= (1 - \sigma)/2 + t_0/6 + \sigma/12 \\ &= 1/2 + t_0/6 - 5\sigma/12 < 1/2 \end{aligned}$$

since  $\sigma > 2t_0/5$ .

(4.2.3).  $a_1 \geq t_0, a_2 \geq t_0/3 \geq a_3$  and  $t_0/2 \geq \sigma > \max\{2t_0/5, 4t_0/5 - 6a_3/5\}$ .

We use Lemma 2.3 with  $a'_2 = \max\{t_0/3, a_2\} = a_2$ , and  $a'_3 = \max\{t_0/3, a_3\} = t_0/3$ , then

$$\begin{aligned} h_3(a_1, a_2, a_3) &= (a_1 + a_2)/2 + t_0/6 + t_0/6 + \sigma/12 \\ &= (1 - \sigma - a_3)/2 + t_0/3 + \sigma/12 \\ &= 1/2 + t_0/3 - 5\sigma/12 - a_3/3 < 1/2 \end{aligned}$$

since  $\sigma > 4t_0/5 - 6a_3/5$ .

**Lemma 4.3.** Suppose  $L = L_0$  (i.e. (1.9) holds). If  $\sigma > t_0/2$ ; then  $\{\theta_j\} \in E(\theta)$ .

It is covered by Lemma 3.1.

For a fixed  $\sigma$ , denote

$$M_\sigma = \sup_{\substack{\{a_1, a_2, \sigma\} \in \{\theta\} \\ a_2 \leq t_0 \leq a_1}} \{a_1\} \text{ and } m_\sigma = \inf_{\substack{\{a_1, a_2, \sigma\} \in \{\theta\} \\ a_2 \leq t_0 \leq a_1}} \{a_2\}.$$

**Lemma 4.4.** Suppose that  $\{\theta_j\}$  satisfies the condition (4.4.1) : (4.4.1) all of complementary partial sums  $\{a_1, a_2, \sigma\}$  with  $a_2 \leq a_1 \leq t_0$  satisfies  $\{\theta_j\} \in E(\theta)$ , ( $a_2 \leq a_1$  is not necessary). Then for a fixed  $\sigma$ ,  $\{\theta_j\} \in E(\theta)$  if

$$m_\sigma < a_2 < M_\sigma. \quad (4.4.2)$$

Moreover, we have that

$$m_\sigma + a_2 + M_\sigma = 1, \quad (4.4.3)$$

$$m_\sigma < t_0 < M_\sigma; \quad (4.4.4)$$

for  $t_0/j \geq \sigma \geq 2t_0/(2j+1)$ ,

$$M_\sigma \geq 1 - \frac{6j}{6j-1}t_0 - \frac{2j-1}{6j-1}\sigma, \quad (4.4.5)$$

$$m_\sigma \leq \frac{6j}{6j-1}t_0 - \frac{4j}{6j-1}\sigma; \quad (4.4.6)$$

and for  $2t_0/(2j+1) \geq \sigma \geq t_0/(j+1)$ ,

$$M_\sigma \geq 1 - \frac{2j}{2j+1}t_0 - \sigma, \quad (4.4.7)$$

$$m_\sigma \leq \frac{2j}{2j+1}t_0. \quad (4.4.8)$$

**Proof.** We use (4.13) with  $j = i + 1$  to prove (4.4.8). Then we have (4.4.7) by (4.4.3).

For  $\theta > 11/20$ , we can get much simple conditions for  $\{a_1, a_2, \sigma\} \in E(\theta)$ . We will use it to discuss the gap between consecutive primes in [3].

**Lemma 4.5.** *Suppose that  $t_0 < 9/20$ . If there exists at least one set of complementary partial sums  $\{a_1, a_2, \sigma\}$  with  $a_1 \leq t_0$ , and  $a_2 \leq t_0$ , then  $\{\theta_j\} \in E(\theta)$ . Moreover, we have that*

$$M_\sigma - m_\sigma > \sigma, \text{ if } \sigma < 0.1; \quad (4.5.1)$$

and

$$M_\sigma - m_\sigma > 1 - 2t_0, \text{ if } \sigma \geq 0.1. \quad (4.5.2)$$

**Proof.** If  $a_2 \leq a_1$ , and  $a_2 < 4 - 8t_0$ , it is same as (5.1.8). If  $a_1 \geq a_2 \geq 4 - 8t_0$ , using (2.11) of Lemma 2.1 with  $j = 2$ , we have

$$\begin{aligned} h_0(a_1, a_2) &= t_0 + \min\{a_2/8, \sigma/6 + a_2/24\} \\ &\leq t_0 + (4 - 8t_0)/24 + (1 - 2(4 - 8t_0))/6 < 1/2. \end{aligned}$$

By (4.4.4), (4.4.5) and (4.4.6), we have that

$$M_\sigma - m_\sigma \geq 1 - \frac{28}{29}t_0 - \frac{9}{29}\sigma - \left(\frac{28}{29}t_0 - \frac{20}{29}t_0\right) \geq \sigma, \quad (4.5.3)$$

if  $\sigma < t_0/5$ . When  $t_0/5 \leq \sigma < 0.1$ , we have that

$$M_\sigma - m_\sigma \geq 1 - \frac{8}{9}t_0 - \sigma - \frac{8}{9}t_0 > 1 - 2t_0.$$

Then (4.5.1) holds. When  $t_0/(j+1) \leq \sigma \leq 2t_0/(2j+1)$ , by (4.4.4) and (4.4.6),

$$M_\sigma - m_\sigma \geq 1 - \frac{2j}{2j+1}t_0 - \sigma - \frac{2j}{2j+1}t_0 \geq 1 - 2t_0. \quad (4.5.4)$$

When  $2t_0/(2j+1) \leq \sigma \leq t_0/(j+1)$ , by (4.4.5) and (4.4.7)

$$M_\sigma - m_\sigma \geq 1 - \frac{6j}{6j-1}t_0 - \frac{2j-1}{6j-1}\sigma - \left( \frac{6j}{6j-1}t_0' - \frac{4j}{6j-1}\sigma \right) \geq 1 - 2t_0 \quad (4.5.5)$$

Thus (4.5.2) holds.

For  $\theta > 6/11$ , we can get some simple conditions for  $\{a_1, a_2, \sigma\} \in E(\theta)$ . We will use it to discuss the gap between consecutive primes in [3]. Moreover, we have that

**Corollary 4.5.1.** *Suppose that  $\theta > 6/11$ . (4.5.1) and (4.5.2) hold. If  $a_2 \leq a_1 \leq 1/2$  and  $\sigma = 1 - a_1 - a_2 < 1/2 - 8t_0/9$ , then  $\{\theta_j\} \in E(\theta)$ .*

**Proof.** If  $a_2 \geq t_0$ , it is covered by (4.1.1); if  $a_1 < t_0$ , we have that  $\{\theta_j\} \in E(\theta)$  by (4.5.1); if  $a_2 \leq t_0 < a_1 < 1/2$  and  $\sigma \geq t_0/5$ , then

$$a_2 = 1 - a_1 - \sigma > 1 - 1/2 - (1/2 - 8t_0/9) = 8t_0/9,$$

thus  $\{\theta_j\} \in E(\theta)$  by (4.1.3). If  $a_2 \leq t_0 < a_1 < 1/2$  and  $\sigma < t_0/5$ , using (2.11) in Lemma 2.1 with  $j = 5$ , then

$$\begin{aligned} h_0(a_1, a_2) &= t_0/2 + a_1/2 + \min\{a_2/8, \sigma/6 + a_2/60\} \\ &\leq t_0/2 + a_1/2 + \sigma/6 + a_2/60 \\ &= t_0/2 + a_1/2 + \sigma/6 + (1 - a_1 - \sigma)/60 \\ &= t_0/2 + 1/60 + 29a_1/60 + 3\sigma/20 \\ &\leq t_0/2 + 1/60 + 29/120 + 3t_0/100 < 1/2, \end{aligned}$$

since  $a_1 \leq 1/2$  and  $\sigma \leq t_0/5$ .

**Lemma 4.6.** *Suppose  $t_0 < 5/11$ . If there exists at least one set of complementary partial sums  $\{a_1, a_2, \sigma\}$  (or  $\{a_1, a_2, a_3, \sigma\}$ ) of  $\{\theta_j\}$  such that at least one of conditions (4.6.1) - (4.6.5) holds, then  $\{\theta_j\} \in E(\theta)$ .*

(4.6.1)  $a_2 \leq a_1 \leq t_0$  and  $\sigma < 1 - 20t_0/11$ .

**Proof.** When  $\sigma \geq t_0/3$ , we apply Lemma 2.2 with  $i = 1$ ,  $\sigma_1 = \max\{t_0/3, 1 - a_1 - a_2\} = \max\{t_0/3, \sigma\} = \sigma$  and  $a'_1 = t_0$ , then

$$\begin{aligned} h_1^i(a_1, a_2) &= t_0/3 + 1/12 + t_0/2 + \sigma/2 - (a_1 + \sigma)/12 \\ &\leq 5t_0/6 + 1/12 + 5\sigma/12 - a_1/12 \\ &\leq 5t_0/6 + 1/24 + 11\sigma/24 \\ &< 5t_0/6 + 1/24 + 11(1 - 20t_0/11)/24 = 1/2, \end{aligned}$$

since  $a_1 \geq (1 - \sigma)/2$ . ( $a_1 \geq a_2$  and  $a_1 + a_2 + \sigma = 1$ ).

When  $\sigma < t_0/3$ , we turn to (2.10) of Lemma 2.1 with  $j = 2$

$$\begin{aligned} h_0(a_1, a_2) &\leq t_0 + \sigma/6 + a_2/24 < t_0 + \sigma + (1 - \sigma)/48 \\ &\leq t_0 + \frac{1}{48} + 7\sigma/48 \leq 1/2. \end{aligned}$$

(4.6.2).  $a_1 \geq a_2 > (2i + 2)t_0/(2i + 3)$  and  $1 - 20t_0/11 > \sigma \geq t_0/i$ .

**Proof.** If  $a_2 \geq t_0$ , it is already covered by (4.1.1). If  $a_1 \geq t_0 > a_2 > (2i + 2)t_0/(2i + 3)$ , using (4.1.3). If  $t_0 > a_1 \geq a_2 > 8t_0/9$ , using (4.2.1).

We can write (4.2.2) to be

(4.6.3).  $t_0 \geq a_1 > (2i + 2)t_0/(2i + 3)$  and  $1 - 20t_0/11 > \sigma \geq t_0/i$ . (It is not necessary  $a_2 \leq a_1$ ).

**Proof.** If  $a_2 \geq t_0$ , we are back to (4.1.3). If  $a_1 < t_0$ , it is covered by (4.2.2).

(4.6.4).  $t_0 \leq a_1 \leq 1/2$ , and  $\varepsilon < \sigma < 1/2 - 8t_0/9$ ;

**Proof.** If  $\sigma \geq t_0/5$ , then  $a_2 = 1 - \sigma - a_1 > 8t_0/9$ ,  $\{\theta_j\} \in E(\theta)$  by (4.1.3)

with  $i = 3$ . If  $\sigma < t_0/5$ , we apply (2.10) with  $j = 5$  in Lemma 2.1. We have

$$\begin{aligned} h_0(a_1, a_2) &= t_0/2 + a_1/2 + \sigma/6 + a_2/60 \\ &= t_0/2 + a_1/2 + \sigma/6 + (1 - \sigma - a_1)/60 \\ &< t_0/2 + 29/120 + (9/60)(1/2 - 8t_0/9) + 1/60 < 1/2. \end{aligned}$$

(4.6.5).  $a_2 \leq a_1 \leq 1/2$  and  $\sigma < 1/2 - 8t_0/9$ .

**Proof.** If  $a_2 \geq t_0$ , it is covered by (4.1.1). If  $a_1 < t_0$ ,  $\{\theta_j\} \in E(\theta)$  by (4.1.5) since  $\sigma < 1/2 - 8t_0/9 < 1 - 20t_0/11$ . If  $a_2 \leq t_0 < a_1 < 1/2$ , the proof is same as (4.5.6).

From Lemma 4.3 and (4.6.1), we have, for  $\sigma < 1 - 20t_0/11$ , (4.4.3) - (4.4.8) hold. Moreover, (4.5.3), (4.5.4) and (4.5.5) with  $t_0 < 5/11$  can imply that

$$M_\sigma - m_\sigma > \sigma, \text{ if } \sigma < t_0/5; \quad (4.6.6)$$

and

$$M_\sigma - m_\sigma > t_0/5, \text{ if } \sigma \geq t_0/5 \quad (4.6.7)$$

since  $1 - 2t_0 > t_0/5$ .

(4.6.6)  $t_0 \leq a_1 \leq 1/2$  and  $a_2 < 2 - 4t_0$ ;

In this case, we use Lemma 2.1. By (2.11) with  $j = 2$  and  $a'_1 = a_1$ ,

$$h_0(a_1, a_2) \leq a_1/2 + t_0/2 + a_2/8 \leq 1/4 + t_0/2 + t_0/20 < 1/2.$$

**Lemma 4.7.** Suppose that  $t_0 < 5/11$  and  $L = L_0$ . If there exists at least one set of complementary partial sums  $\{a_1, a_2, \sigma\}$  (or  $\{a_1, a_2, a_3, \sigma\}$ ) of  $\{\theta_j\}$  such that at least one of conditions (4.7.1) - (4.7.6) holds, then  $\{\theta_j\} \in E(\theta)$ .

(4.7.1).  $k_0 = 3, 1/2 \geq a_1 \geq t_0, 2t_0/5 \geq a_2 \geq t_0/3$ , and  $t_0/4 \geq a_3 \geq t_0/5$ .

**Proof.** We use Lemma 3.6 with  $a'_2 = \max\{a_2, t_0/3\} = a_2$ ; then

$$\begin{aligned} h_8(a_1, a_2, a_3) &= 13t_0/48 + (a_1 + a_2)/2 + 5a_3/16 \\ &\leq 13t_0/48 + 1/4 + t_0/5 + 5t_0/64 < 1/2. \end{aligned}$$



(4.7.2).  $k_0 = 4, a_1 \leq 8t_0/9, a_2 \leq 4t_0/9$ , and  $a_4 \leq a_3 \leq t_0/4$ .

**Proof.** Using (3.5), we have that,  $a'_3 = a'_4 = t_0/4$  and

$$h_4(a_1, a_2, a_3, a_4) = \frac{29}{64}t_0 + \max \left\{ \frac{3}{40}a_1 + \frac{a_2}{2} + \frac{7t_0}{20}, \frac{a_1}{2} + \frac{3a_2}{20} + \frac{t_0}{10} \right\} < \frac{1}{2}.$$

(4.7.3).  $k_0 = 4, a_1 + a_3 + a_4 \geq t_0, a_2 + a_3 + a_4 \leq t_0, a_1 + a_3 \leq t_0, a_1 + a_4 \leq t_0, a_4 \leq a_3 \leq t_0/5$  and  $a_2 \leq 4 - 8t_0$ .

**Proof.** Using Lemma 3.7, since  $a_2 \leq 4 - 8t_0$ , we have

$$h_7(a_1, a_2, a_3, a_4) = t_0 + a_2/8 < 1/2.$$

(4.7.4).  $k_0 = 6, 2t_0/7 \geq a_1 \geq \dots \geq a_6 \geq t_0/5$  and  $\sigma + a_1 + a_2 \geq t_0$ .

**Proof.** Using Lemma 3.8, we have that

$$h_8(a_1, \dots, a_6) = \frac{15}{16}t_0 + \frac{a_4}{12} + \frac{a_5}{12} + \frac{19}{48}a_6 \leq \frac{15}{16}t_0 + \frac{t_0}{21} + \frac{19}{168}t_0 < \frac{1}{2}.$$

(4.7.5).  $2t_0/5 \geq \theta_1 \geq \dots \geq \theta_4 \geq 1 - 20t_0/11, 2t_0/5 \geq 1 - \theta_1 - \dots - \theta_6 \geq \theta_1, \geq 1 - 20t_0/11$ , and  $\theta_5 \geq \theta_6, 2\theta_6 \leq t_0/3, \theta_1 + \theta_2 + \theta_5 < t_0 < \theta_1 + \theta_2 + \theta_5 + \theta_6$ .

**Proof.** In Lemma 3.9, take  $a_1 = \theta_1 + \theta_2, a_2 = \theta_3 + \theta_4, a_3 = \theta_5$ , and  $a_4 = \theta_6$ , then

$$\begin{aligned} a_4 &= \theta_6 \leq \frac{1}{2}(\theta_5 + \theta_6) \leq \frac{1}{2} \left( 1 - \theta_1 - \dots - \theta_4 - \left( 1 - \sum_{1 \leq i \leq 6} \theta_i \right) \right) \\ &\leq (1 - 5a_2)/2 \end{aligned}$$

and

$$\begin{aligned} h_9(a_1, \dots, a_4) &\leq 15t_0/16 + a_2/8 + 5(1 - 5a_2)/16 \\ &< 15t_0/16 + 5/16 - 23a_2/16 < 1/2. \end{aligned}$$

(4.7.6).  $k_0 = 4, a_1 + a_4 \leq t_0, a_2 = a_3 \leq t_0, \sigma + a_1 \geq t_0, a_3 < t_0/5, a_2 < 1/3$  and  $a_3 < t_0/5$ .

**Proof.** In Lemma 3.10,

$$h_{10}(a_1, \dots, a_4) = t_0 + a_2/8 + a_3/24 < t_0 + 1/24 + t_0/120 < 1/2.$$

(4.7.7).  $k_0 = 5, a_3 \leq a_2 \leq a_1 \leq t_0/2$ , and  $a_5 \leq a_4 \leq t_0/4$ .

**Proof.** If  $a_2 + a_1 > 8t_0/9$ , we replace  $a_1$  by  $a_2 + a_1$  and  $s$  by  $a_5$  with  $k_0 = 2$ , then  $\{\theta_j\} \in E$  by (4.6.3). If  $a_2 + a_1 \leq 8t_0/9$ , then  $a_3 > 4t_0/9$  it is covered by (4.7.2).

(4.7.8)  $k_0 = 2, a_1 \geq t_0/2, a_2 \leq t_0/5$ , and  $\sigma + a_2/2 > t_0/2$ .

See Lemma 3.2.

(4.7.9).  $k_0 = 2, a_1 \geq t_0, t_0/3 \geq a_2 \geq t_0/5$ , and  $\sigma > a_2/8 + 3t_0/8$ .

See Lemma 3.3.

### § 5. ANALYTIC FORM OF $R(x; M_1 \cdots M_k)$ .

We shall examine the remainder term  $R(x; M_1, \dots, M_k)$  (see (1.12)) which was used in [1]. Let  $x$  be a large number,  $y = x^\theta$  with  $1/2 < \theta < 7/12$ , and  $A = \{n : x - y < n < x\}$ . For convenience we define  $a_{m_i, i} = 0$ , unless  $M_i < m \leq 2M_i$ . We rewrite (1.12) :

$$R(x; M_1, \dots, M_k) = \sum_{m_1, \dots, m_k} a_{m_1, 1} \cdots a_{m_k, k} \left( \left[ \frac{x}{m_1 \cdots m_k} \right] - \left[ \frac{x - y}{m_1 \cdots m_k} \right] \right)$$

and we write

$$\begin{aligned} L &= \frac{x}{2M_1 \cdots M_k}, \\ L(s) &= \sum_{L/2^k < I \leq 3L} I^{-s}, \\ M_i(s) &= \sum_{M_i < m_i \leq 2M_i} a_{m_i, i} m_i^{-s}, \text{ for } 1 \leq i \leq k, \end{aligned}$$

and

$$g(s) = L(s) \prod_{1 \leq i \leq k} M_i(s),$$

where  $k$  is a positive integer.

**Theorem 2.** Suppose  $\{\theta_j\} \in E(\theta)$ , then

$$R(x; M_1, \dots, M_k) \ll x^{\theta - \epsilon}.$$

**Proof.** Lemma 3.12 of Titchmarsh [7] is applied to the function  $g(s)$ , to yield

$$\sum_{\substack{m_1, \dots, m_k \\ Im_1 \dots m_k \in A}} a_{m_1,1} \dots a_{m_k,k} = \frac{1}{2\pi i} \int_{c-iT}^{c+iT} g(s) \frac{x^s - (x-y)^s}{s} ds + O\left(\frac{x^{1-\eta}}{T}\right), \quad (5.1)$$

where  $c = 1 + (\log x)^{-1}$  and  $T > 0$ . The conditions  $Im_1 \dots m_k \in A$  and  $M_i < m_i \leq 2M_i$  imply that  $L/2^k < I \leq 3L$ , and so that the sum on the left of (5.1) becomes

$$\begin{aligned} & \sum_{m_1, \dots, m_k} a_{m_1,1} \dots a_{m_k,k} \left( \left[ \frac{x}{m_1 \dots m_k} \right] - \left[ \frac{x-y}{m_1 \dots m_k} \right] \right) \\ &= R(x; M_1, \dots, M_k) - y \sum_{m_1, \dots, m_k} \frac{a_{m_1,1} \dots a_{m_k,k}}{m_1 \dots m_k} \end{aligned}$$

For  $T_0 \leq L$ , in the range  $\{s = c + it : |t| \leq T_0 \leq L\}$ , we have

$$L(s) = \frac{(3L)^{1-s} - (L/2^k)^{1-s}}{s-1} + O(L^{-c})$$

by Theorem 4.11 of Titchmarsh [7]. Moreover if  $T_0 y \leq x$ ,

$$\frac{x^s - (x-y)^s}{s} = yx^{s-1} - O(|s| y^2 x^{c-2}) \ll yx^{c-1}.$$

Hence

$$\begin{aligned} & \frac{1}{2\pi i} \int_{c-iT_0}^{c+iT_0} g(s) \frac{x^s - (x-y)^s}{s} ds \\ &= \frac{1}{2\pi i} \int_{c-iT_0}^{c+iT_0} \frac{x^s - (x-y)^s}{s} M_1(s) \dots M_k(s) x^{s-1} ds + O(E_1) + O(E_2), \end{aligned}$$

where

$$E_1 = \int_{-T_0}^{T_0} L^{-c} \left| \prod_{1 \leq j \leq k} M_j(c-it) \right| yx^{c-1} dt \ll T_0 M_1 \dots M_k yx^{-1},$$

and

$$\begin{aligned} E_2 &= \int_{-T_0}^{T_0} L^{-c} \left| \prod_{1 \leq j \leq k} M_j(c-it) \right| y^2 x^{c-2} dt \ll T_0 L^{1-c} (M_1 \dots M_k)^{1-c} y^2 x^{c-2} \\ &\leq T_0 y^2 x^{-1}. \end{aligned}$$

Moreover, on integrating termwise as in the proof of Lemma 3.1 in Titchmarsh [7], we have

$$\begin{aligned} & \frac{1}{2\pi i} \int_{c-iT_0}^{c+iT_0} \frac{x^{-s} - (x-y)^s}{s} M_1(s) \cdots M_k(s) x^{s-1} ds \\ &= \sum_{\substack{m_1, \dots, m_k \\ 1 \leq m_1 \cdots m_k \in A}} \frac{a_{m_1,1} \cdots a_{m_k,k}}{m_1 \cdots m_k} + O\left(\frac{1}{T_0}\right). \end{aligned}$$

It follows from the above estimate, choosing  $T_0 = L^{1/2}$ , that

$$R(x; M_1, \dots, M_k) = \frac{1}{2\pi i} \left( \int_{c-iT}^{c+iT_0} + \int_{c+iT_0}^{c+iT} \right) g(s) \frac{x^s - (x-y)^s}{s} ds + O(x^{\theta-\epsilon}). \quad (5.2)$$

We divide the latter range into at most  $2 \log x$  subintervals of type  $[T_1, T_2]$ , where  $T_2 \leq 2T_1$  and bound the interval over such a range by a sum over well spaced points  $T_1 \leq t_1 < \cdots < t_k \leq T_2 (t_{r+1} - t_r \geq 1)$ . Thus, for some  $T_1$  with  $T_0 \leq T_1 \leq T$ , and some set of  $t_r$ , we have

$$R(x; M_1 \cdots M_k) \ll x^{\theta-\epsilon} + (\log x) y x^{c-1} \sum_{1 \leq r \leq k} g(c + it_r). \quad (5.3)$$

In the following sections we shall estimate (5.3).

We denote the right hand-side of (5.3) by  $R(x; N_1, \dots, N_{k_0})$ . We have

$$g(s) = N_1(s) \cdots N_{k_0}(s) L(s), \quad (5.4)$$

where the definition of  $N_i(s)$  or  $L_0(s)$  is the same as (5.1) and one of  $N_i(s)$  or  $L_0(s)$  is equal  $L(s)M_{i_1}(s) \cdots M_{i_n}(s)$ .

We trivially have

$$|L_0(c + it_r)| \leq 4, \text{ and } |N_j(c + it_r)| \leq 4 \quad (j \leq k).$$

Hence those  $t_r$  for which

$$|L_0(c + it_r)| \leq x^{-2}, \text{ or } |N_j(c + it_r)| \leq x^{-2} \quad (j \leq k_0)$$

contribute a total

$$\ll (\log x) y x^{c-1} T_2 x^{-2} \ll x^{\theta-\epsilon}$$

to  $R(x; N_1, \dots, N_{k_0})$ . Such points may therefore be neglected. We now divide the remaining  $t_r$  into at most  $(4 \log x)^{k_0+1}$  sets  $S(U_1, \dots, U_{k_0}, W)$  for which

$$\begin{aligned} U_i &< N_i^{c-1/2} |N_i(c+it_r)| < 2U_i, \quad i = 1, \dots, k_0, \\ W &\leq L_0^{c-1/2} |L_0(c+it_r)| < 2W, \end{aligned}$$

where  $x^{-2} \leq L^{1/2-c}W \leq 2^{-u} \leq 1$ , for some integer  $u$  and similarly for  $U_1, \dots, U_{k_0}$ . It follows that

$$R(x; N_1, \dots, N_{k_0}) \ll x^{\theta-\varepsilon} + (\log x)^{k_0+2} y x^{-1/2} U_1 \dots U_{k_0} W |S(U_1, \dots, U_{k_0}, W)| \quad (5.5)$$

for some  $T$ , some  $U_1, \dots, U_{k_0}, W$  and some well-spaced set  $S(U_1, \dots, U_{k_0}, W)$  having the property (1.4) - (1.7): the mean-value technique of Montgomery [8]

$$|S(U_1, \dots, U_{k_0}, W)| \ll U_i^{-2j} (N_i^j + T) \log^A x, \quad 1 \leq i \leq k_0, \quad (5.6)$$

where  $A$  is constant; Halasz's method in the form due to Huxley

$$|S(U_1, \dots, U_{k_0}, W)| \ll (U_i^{-2j} N_i + U_i^{-6j} N_i T) \log^A x; \quad (5.7)$$

while from the inequality

$$|L_0(c+it_r)|^2 \geq U^2 L_0^{1-2c},$$

we have

$$|S(U_1, \dots, U_{k_0}, W)| \ll (W^{-4} L_0^2 + W^{-12} L_0^2 T) (\log x)^A. \quad (5.8)$$

Our last estimate for  $|S(U_1, \dots, U_{k_0}, W)|$  depend on the formula

$$L(s) = \frac{1}{2\pi i} \int_{\frac{1}{2}-c-iT}^{\frac{1}{2}-c+iT} \zeta(s+z) \left( (3L)^z - \left( \frac{L}{2^k} \right)^z \right) \frac{dz}{z} + O\left(L^{\frac{1}{2}-c} \log x\right)$$

for  $s = c + it, T_1 \leq t \leq 2T_1$ . This yield, if  $L_0 = L$ ,

$$W^4 |S(U_1, \dots, U_{k_0}, W)| \leq L_0^{4c-2} S |L_0(c+it_r)|^4 \ll T (\log x)^A. \quad (5.9)$$

Deshouillers and Iwaniec [6] proved: if  $N \geq 1, T \geq 1$ , the coefficients  $d_n$  are complex numbers and  $\varepsilon > 0$ , then

$$\int_{-2T}^{2T} \left| \zeta\left(\frac{1}{2} + it\right) \right|^4 \sum_{n < n \leq 2N} |d_n n^{it}|^2 dt \ll T^\varepsilon (N_2 T^{1/2} + N^{5/4} T^{3/4} + T) \sum_{n \leq n \leq 2N} |d_n|^2 \quad (5.10)$$

From (4.1), we have

$$\begin{aligned} W^4 U_j^2 |S(U_1, \dots, U_{k_0}, W)| &\leq L^{4c-2} N^{2c-1} \sum_{\tau} |L(c + it_{\tau})|^4 |N_j(c + it_{\tau})|^2 \\ &\ll L^{4c-2} N_j^{2c-1} \sum_{\tau} \int_{-2T}^{2T} \left| \zeta\left(\frac{1}{2} + it\right) \right|^4 L^{2-4c} N_j^{1-2c} \left| N_j\left(\frac{1}{2} + it\right) \right|^2 \frac{dt}{1 + |\tau - t_{\tau}|} \\ &\quad + |S(U_1, \dots, U_{k_0}, W)| |(\log x)|^4 \ll T^\varepsilon (N_k^2 T^{1/2} + N_k^{5/4} T^{3/4} + T) \quad (5.11) \end{aligned}$$

whence, for  $1 \leq k \leq j$ ,

$$|S(U_1, \dots, U_{k_0}, W)| \ll W^{-4} U_k^{-2} (N_k^2 T^{1/2} + N_k^{5/4} T^{3/4} + T) T^\varepsilon. \quad (5.12)$$

We have that

$$U_1 \cdots U_k W |S(U_1, \dots, U_{k_0}, W)| \ll x^{1/2-\varepsilon} + x^{h(a_1, \dots, a_{k_0})} \quad (5.13)$$

with  $h(a_1, \dots, a_{k_0}) < 1/2$  for a certain constant  $\varepsilon > 0$  since  $\{\theta_j\} \in E(\theta)$ . This will complete the proof of

$$R(x; M_1, \dots, M_k) \ll x^{\theta-\varepsilon}.$$

**Theorem 3.** Suppose  $\{\theta_j\} \in E(\theta)$ . then

$$\int_T^{2T} |W\left(\frac{1}{2} + it\right)| dt \ll x^{\frac{1}{2}-\varepsilon}, \quad (5.14)$$

for

$$T_1 \leq T \leq x^{1-\theta+\varepsilon}$$

where  $\theta$  is a fixed positive constant, and

$$T_1 = \exp\left((\log x)^{\frac{1}{3}} (\log \log x)^{-\frac{1}{3}}\right).$$

**Proof.** Defined  $S(U_1, \dots, U_k, W)$  be a set of  $t_r$  into for which

$$\begin{aligned} U_i &< M_i^{c-1/2} |M_i(c + it_r)| < 2U_i, \quad i = 1, \dots, k_0, \\ W &\leq L_0^{c-1/2} |L_0(c + it_r)| < 2W, \end{aligned}$$

where  $x^{-2} \leq L^{1/2-c}W \leq 2^{-u} \leq 1$ , for some integer  $u$  and similarly for  $U_1, \dots, U_{k_0}$ . In Lemma 16 of [8], Heath-Brown proved

$$\int_T^{2T} |W(\frac{1}{2} + it)| dt \ll x^{\frac{1}{2}-\epsilon} + x^{-\delta} \sum |S(U_1, \dots, U_k, W)|$$

where  $\sum$  runs over  $(U_1, \dots, U_k, W)$  such that (1.1), (1.2) and (1.3), and  $|S(U_1, \dots, U_k, W)|$  satisfies (1.4), (1.5), (1.6), (1.7) and (1.9). Since  $\{\theta_j\} \in E(\theta)$ , we have that

$$|S(U_1, \dots, U_k, W)| < x^{\frac{1}{2}-\epsilon},$$

then

$$\sum |S(U_1, \dots, U_k, W)| < x^{\frac{1}{2}-\epsilon},$$

since at most  $(4 \log x)^{k+1}$  terms in above " $\sum$ ". Thus (5.14) follows.

## References

- [1] Heath-Brown, D.R. and Iwaniec, H., *On the Difference Between Consecutive Primes*, *Invent. Math.* 55, 49-69 (1979).
- [2] Heath-Brown, D.R., *The Number of Primes in a Short Interval*, *J. Reine Angew. Math.* 389, 22-63 (1988).
- [3] Lou, S. and Yao, Q., *The Number of Primes in a Short Interval*, *Hardy-Ramanujan J.*, 16 (1993), 21-43.
- [4] Lou, S. and Yao, Q., *A Chebychev's Type of Prime Number Theorem in a Short Interval-II*, *Hardy-Ramanujan J.*, 15 (1992), 1-33.
- [5] Lou, S. and Yao, Q., *On the Difference Between Consecutive Primes*, (to be submitted).
- [6] Deshouillers J.-M. and Iwaniec H., *Power mean-value of the Riemann zeta-function*, *Mathematika*, 29 (1982), 202-212.
- [7] Titchmarsh E.C., *The Theory of the Riemann zeta-function*, Oxford (1951).
- [8] Montgomery H.L., *Topics in Multiplicative Number Theory*, *Lecture Notes in Mathematics*, Springer-Verlag, 227 (1971).

Lou, Shituo and Yao, Qi  
Department of Math.  
Dalhousie University  
Halifax, Nova Scotia B3H 3J5  
Canada