



HAL
open science

Local monomialization conjecture of a singular foliation of Darboux type

Aymen Braghtha

► **To cite this version:**

Aymen Braghtha. Local monomialization conjecture of a singular foliation of Darboux type. 2015.
hal-01108702

HAL Id: hal-01108702

<https://hal.science/hal-01108702>

Preprint submitted on 23 Jan 2015

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Local monomialization conjecture of a singular foliation of Darboux type

Aymen Braghtha

January 23, 2015

Abstract

After the nice result introduced by Belotto in [1] concerning the local monomialization of a singular foliation given by n first integrals, this work is a continuation in the same spirit. In this paper, we introduce an important conjecture about local monomialization of a singular foliation of Darboux type (see section 1). This conjecture can be used to study pseudo-abelian integrals [2,4].

1 Introduction

Let M be an analytic manifold of dimension $n + 2$. Given a families of first integrals of Darboux type H_ϵ

$$H_\epsilon(x, y) = H(x, y, \epsilon_1, \dots, \epsilon_n) = \prod_{i=1}^k P^{a_i}(x, y, \epsilon_1, \dots, \epsilon_n), \quad a_i > 0. \quad (1)$$

Let F be the foliation of codimension one in M with coordinates $(x, y, \epsilon_1, \dots, \epsilon_n)$ which is given by the analytic one form ω

$$\omega = \frac{H_x}{\phi} dx + \frac{H_y}{\phi} dy + \sum_{i=1}^n \frac{H_{\epsilon_i}}{\phi} d\epsilon_i = 0, \quad (2)$$

where $H_x = \frac{\partial H}{\partial x}$, $H_y = \frac{\partial H}{\partial y}$, $H_{\epsilon_i} = \frac{\partial H}{\partial \epsilon_i}$ and $\phi = \prod_{i=1}^k P_i^{a_i-1}(x, y, \epsilon_1, \dots, \epsilon_n)$ (integrating factor).

Let $F_i, i = 1, \dots, n$ are foliations of codimension one in M with coordinates $(x, y, \epsilon_1, \dots, \epsilon_n)$ which are given by the one forms ω_i

$$\omega_i = d\epsilon_i = 0, \quad i = 1, \dots, n. \quad (3)$$

Let $\mathcal{F} = (F, F_1, \dots, F_n)$ be the result foliation of dimension one in M where its leaves are given by the transversal intersection of leaves of F, F_1, \dots, F_n . Otherwise speaking, the singular foliation \mathcal{F} is given by

$$\Omega = \omega \wedge \omega_1 \wedge \dots \wedge \omega_n \quad (4)$$

$$= Q_1(x, y, \epsilon_1, \dots, \epsilon_n) dx \wedge d\epsilon_1 \wedge \dots \wedge d\epsilon_n + Q_2(x, y, \epsilon_1, \dots, \epsilon_n) dy \wedge d\epsilon_1 \wedge \dots \wedge d\epsilon_n = 0, \quad (5)$$

where $Q_1 = \frac{H_x}{\phi}$, $Q_2 = \frac{H_y}{\phi}$ are polynomials.

We shall say that Ω is a foliation of Darboux type with first integrals $(H, \epsilon_1, \dots, \epsilon_n)$.

Example. Let $H_\epsilon(x, y) = H(x, y, \epsilon) = (x - \epsilon)^{a_1}(x - y)^{a_2}(x + y)^{a_3}$ be a the first integral of Darboux type. The foliation F of codimension one in three dimensional space M with coordinates (x, y, ϵ) is given by the one form

$$\begin{aligned} \omega &= (a_1(x - y)(x + y) + a_2(x - \epsilon)(x + y) + a_3(x - \epsilon)(x - y))dx \\ &\quad - (a_2(x - \epsilon)(x + y) - a_3(x - \epsilon)(x - y))dy - a_1(x - y)(x + y)d\epsilon = 0 \end{aligned}$$

and the foliation F_1 of codimension one in M is given by the one form

$$\omega_1 = d\epsilon = 0.$$

The result foliation $\mathcal{F} = (F, F_1)$ is given by the two-form

$$\Omega = \omega \wedge \omega_1 = (a_1(x-y)(x+y) + a_2(x-\epsilon)(x+y) + a_3(x-\epsilon)(x-y)) dx \wedge d\epsilon - (a_2(x-\epsilon)(x+y) - a_3(x-\epsilon)(x-y)) dy \wedge d\epsilon = 0$$

Observe that the foliation $\mathcal{F} = (F, F_1)$ has a complicated singularity at the origin $(0, 0, 0) \in D_0 = \{\epsilon = 0\}$.

Conjecture. THERE EXIST SEQUENCES OF LOCAL BLOWINGS-UP SUCH THAT THE TOTAL TRANSFORM OF THE FOLIATION $\mathcal{F} : \omega \wedge \omega_1 \wedge \dots \wedge \omega_n = 0$ HAS LOCALLY $n + 1$ MONOMIAL FIRST INTEGRALS $(z^{\gamma_0}, z^{\gamma_1}, \dots, z^{\gamma_n})$ WHERE $z^{\gamma_i} = z_1^{\gamma_{i,1}} \dots z_{n+2}^{\gamma_{i,n+2}}$ AND THE EXPONENTS MATRIX

$$m(a_1, \dots, a_k) = \begin{pmatrix} \gamma_0^1 & \dots & \gamma_0^{n+2} \\ \gamma_1^1 & \dots & \gamma_1^{n+2} \\ \vdots & \vdots & \vdots \\ \gamma_n^1 & \dots & \gamma_n^{n+2} \end{pmatrix}$$

HAS A MAXIMAL RANK.

2 Blowing-up of the foliation \mathcal{F}

In this section, we introduce the fundamental idea to prove the conjecture which is based in first step on Hironaka's reduction of singularities [3]. Let $D_0 = \{\epsilon_1 = \epsilon_2 = \dots = \epsilon_n = 0\}$ be a initial exceptional divisor.

Theorem 1. *There exist a morphism Φ such that the pull-back foliation $\tilde{\Phi}^*\mathcal{F} = \tilde{\mathcal{F}}$ is given locally in neighborhood U_1 of the divisor $\tilde{\Phi}^*(D_0)$ with coordinates $z = (z_1, \dots, z_{n+2})$ by the following system*

$$\begin{cases} \tilde{H} = z^{\gamma_0} \cdot \Delta_0, \\ \tilde{\epsilon}_1 = z^{\gamma_1} \cdot \Delta_1, \\ \vdots \\ \tilde{\epsilon}_n = z^{\gamma_n} \cdot \Delta_n, \end{cases} \quad (6)$$

where $\Delta_i, i = 0, \dots, n$ are a units.

Proof. (1) In first step, we monomialize the principal ideal $I_1 = \langle P_1 \rangle$, Hironaka theorem's guarantee the existence of a sequence of blow-ups $\tilde{\Phi}_1 = \tilde{\Phi}_{n_1}^1 \circ \tilde{\Phi}_{n_1-1}^1 \circ \dots \circ \tilde{\Phi}_1^1$ with initial center $C_0 \subset D_0$ (which is possibly a submanifold of M) such that

$$(\tilde{\Phi}_1^* P_1)^{a_1} = \delta_1 \prod_{i=1}^{n+2} z_i^{a_1 \tilde{\beta}_i^1}, \quad \delta_1(0) \neq 0.$$

(2) In the second step, we consider the principal ideal $I_2 = \langle \tilde{\Phi}_1^* P_2 \rangle$ and by Hironaka theorem's there exist a sequence of blow-ups $\tilde{\Phi}_2 = \Phi_{n_2}^2 \circ \Phi_{n_2-1}^2 \circ \dots \circ \Phi_1^2$ such that

$$(\tilde{\Phi}_2^* \circ \tilde{\Phi}_1^* P_2)^{a_2} = \delta_2 \prod_{i=1}^{n+2} z_i^{a_2 \tilde{\beta}_i^2}, \quad \delta_2(0) \neq 0.$$

In the k -th step there exist a sequence of blow-ups $\tilde{\Phi}_k = \Phi_{n_k}^k \circ \Phi_{n_k-1}^k \circ \dots \circ \Phi_1^k$ such that the principal ideal $I_k = \langle \tilde{\Phi}_{k-1}^* \circ \tilde{\Phi}_{k-2}^* \circ \dots \circ \tilde{\Phi}_1^* P_k \rangle$ has a normal crossings i.e.

$$(\tilde{\Phi}_{k-1}^* \circ \tilde{\Phi}_{k-2}^* \circ \dots \circ \tilde{\Phi}_1^* P_k)^{a_k} = \delta_k \prod_{i=1}^{n+2} z_i^{a_k \tilde{\beta}_i^k}, \quad \delta_k(0) \neq 0.$$

Finally, after desingularisation of each polynomial P_i of the first integral $H = \prod_{i=1}^k P_i^{a_i}$, the equations $z_1 = 0, \dots, z_{n+2} = 0$ are corresponding the irreducibles components of the exceptional divisor. For this

raison after desingularisation of $\tilde{\Phi}_{i-1}^* \circ \tilde{\Phi}_{i-2}^* \circ \dots \circ \tilde{\Phi}_1^* P_i$, the polynomial $\tilde{\Phi}_i^* \circ \tilde{\Phi}_{i-1}^* \circ \tilde{\Phi}_{i-2}^* \circ \dots \circ \tilde{\Phi}_1^* P_{i-1}$ has a normal crossings. So locally we have

$$\begin{cases} \tilde{H} = z^{\gamma_0} \cdot \Delta_0, \\ \tilde{\epsilon}_1 = z^{\gamma_1} \cdot \Delta_1, \\ \vdots \\ \tilde{\epsilon}_n = z^{\gamma_n} \cdot \Delta_n, \end{cases}$$

where $z = (z_1, \dots, z_{n+2})$, $\gamma_0 = \sum_{i=1}^k a_i \beta_i$, $\beta_i = (\beta_i^1, \dots, \beta_i^{n+2})$, $\gamma_i = (\gamma_i^1, \dots, \gamma_i^{n+2})$. \square

To complete the proof its necessary to eliminate the units $\Delta_0, \Delta_1, \dots, \Delta_n$ in the system (6). Now we define the resonant locus of the foliation $(z^{\gamma_0} \cdot \Delta_0, z^{\gamma_1} \cdot \Delta_1, \dots, z^{\gamma_n} \cdot \Delta_n)$

$$\mathcal{R} := \{a = (a_1, \dots, a_k) : \gamma_0 \wedge \sum_{j=1}^n \gamma_j = 0\}.$$

To prove the conjecture, we distinguish two cases

- **generic case** $a \notin \mathcal{R}$.
- **nongeneric case** $a \in \mathcal{R}$.

3 Some examples in dimension three

To more understand the problem, we see some examples in dimension three.

Example 1: Let \mathcal{F} be the local foliation which is obtained by after k blow-ups. The foliation \mathcal{F} is given by the following system

$$\begin{cases} H_a = x^{a_1} y^{a_2} (1+z) \\ f = xy \end{cases} \quad (7)$$

In this example we have $\gamma_0 : a_1 \beta_1 + a_2 \beta_2$ where $\beta_1 = (1, 0, 0)$, $\beta_2 = (0, 1, 0)$, $\gamma_1 = (1, 1, 0)$ and $\mathcal{R} = \{a = (a_1, a_2) : \gamma_0 \wedge \gamma_1 = 0\}$. Our goal is to kill the unit $1+z$ in the first integral H_a without modifying the second monomial f in the sense to preserve its monomiality structure. For this reason, we distinguish two different cases:

(a) The generic case $a_1 \neq -a_2 \Leftrightarrow a \notin \mathcal{R}$: We take the change of variables $\tilde{x} = x(1+z)^{\frac{1}{a_1+a_2}}$, $\tilde{y} = y(1+z)^{\frac{1}{a_2+a_1}}$ and $\tilde{z} = z$. Then, we obtain the following system

$$\begin{cases} H_a = \tilde{x}^{a_1} \tilde{y}^{a_2} \\ f = \tilde{x} \tilde{y} \end{cases} \quad (8)$$

Question: How to calculate the generator vector field of the monomial foliation (7)?

Let us assume that the foliation \mathcal{F} is generated locally by the vector field $X(\tilde{x}, \tilde{y}, z) = \alpha_1 \tilde{x} \frac{\partial}{\partial \tilde{x}} + \alpha_2 \tilde{y} \frac{\partial}{\partial \tilde{y}} + \alpha_3 z \frac{\partial}{\partial z}$ which satisfies

$$X(H) = X(\tilde{x}^{a_1} \tilde{y}^{a_2}) = 0, \quad X(f) = X(\tilde{x} \tilde{y}) = 0$$

To determine the vector $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ we use the two following relations

$$\langle \alpha, \gamma_0 \rangle = 0 \quad (\text{i.e. } X(H) = X(\tilde{x}^{a_1} \tilde{y}^{a_2}) = 0), \quad \langle \alpha, \gamma_1 \rangle = 0 \quad (\text{i.e. } X(f) = X(\tilde{x} \tilde{y}) = 0).$$

where \langle, \rangle is scalar product in \mathbb{C}^3 . Finally, the vector $(\alpha_1, \alpha_2, \alpha_3) \in \{e_3\}$ and then

$$\mathcal{F} = \left\{ z \frac{\partial}{\partial z} \right\}$$

Now, we express the vector field X in the original coordinates (x, y, z) . If we write $X(x, y, z) = Ax\frac{\partial}{\partial x} + By\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}$, to determine A, B we use the fact that

$$X(xy) = 0 \iff A = -B$$

and so $X(x, y, z) = A(x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}) + z\frac{\partial}{\partial z}$ on the other hand we have

$$Ax = X(x) = X(\tilde{x}(1+z)^{\frac{1}{a_1+a_2}}) = z\frac{\partial}{\partial z}(\tilde{x}(1+z)^{\frac{1}{a_1+a_2}}) = \tilde{x}(1+z)^{\frac{1}{a_1+a_2}-1}\frac{z}{a_1+a_2}$$

Finally, we obtain

$$X(x, y, z) = \frac{1}{a_1+a_2}\frac{z}{1+z}(x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}) + z\frac{\partial}{\partial z} \Rightarrow Y(x, y, z) = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y} + (a_1+a_2)(1+z)\frac{\partial}{\partial z}$$

Remark 1. In dimension three, if we consider the foliation \mathcal{F} which is given locally by

$$\begin{cases} f_1 = x^a y^b z^c \\ f_2 = x^{\tilde{a}} y^{\tilde{b}} z^{\tilde{c}} \end{cases}$$

where $\text{rank}\begin{pmatrix} a & b & c \\ \tilde{a} & \tilde{b} & \tilde{c} \end{pmatrix} = 2$. The generator vector field X of the form

$$X(x, y, z) = \hat{a}x\frac{\partial}{\partial x} + \hat{b}y\frac{\partial}{\partial y} + \hat{c}z\frac{\partial}{\partial z},$$

where

$$\langle (\hat{a}, \hat{b}, \hat{c}), (a, b, c) \rangle = 0, \quad \text{and} \quad \langle (\hat{a}, \hat{b}, \hat{c}), (\tilde{a}, \tilde{b}, \tilde{c}) \rangle = 0.$$

In our example we observe that in the neighborhood of the leaf $\{z = 0\}$ the vector field

$$Y \simeq x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y} + (a_1+a_2)z\frac{\partial}{\partial z}$$

is linearizable and consequently Y is transversal to the leaf $\{z = 0\}$.

(b) The problem suppose where $a_1 = -a_2$ i.e $a \in \mathcal{R} = \{a = (a_1, a_2) : \gamma_0 \wedge \gamma_1 = 0\}$. In this case near the leaf $\{z = 0\}$, we have

$$Y \simeq x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}.$$

It is clear that the condition of transversality of Y and the leaf $\{z = 0\}$ is not satisfied.

Example 2: let \mathcal{F} be the local foliation which is given (after a sequence of blow-ups) by

$$\begin{cases} H = x^{a_1} y^{a_2} z^{a_3} (1 + g(x, y, z)) \\ f = xyz. \end{cases}$$

The foliation \mathcal{F} is given also

$$\begin{cases} \frac{H}{f^{a_1}} = y^{a_2-a_1} z^{a_3-a_1} (1 + g(x, y, z)) \\ f = xyz. \end{cases}$$

If $a = (a_1, a_2, a_3) \notin \mathcal{R} = \{a : (a_1\beta_1 + a_2\beta_2 + a_3\beta_3) \wedge (1, 1, 1) = 0\}$ (resonant locus), we can take the following variables change $x = \tilde{x}, y = \tilde{y}(1 + g(x, y, z))^{\frac{1}{a_2-a_3}}$ and $z = \tilde{z}(1 + g(x, y, z))^{\frac{1}{a_3-a_2}}$ and in this case the local foliation \mathcal{F} is generated by the vector field

$$X(\tilde{x}, \tilde{y}, \tilde{z}) = (a_2 - a_3)\tilde{x}\frac{\partial}{\partial \tilde{x}} + (a_3 - a_1)\tilde{y}\frac{\partial}{\partial \tilde{y}} + (a_1 - a_2)\tilde{z}\frac{\partial}{\partial \tilde{z}}$$

let us express the vector field X in the original coordinates (x, y, z) , so we have

$$X(x) = X(\tilde{x}) = (a_2 - a_3)x \frac{\partial}{\partial x}$$

$$X(\tilde{y}(1 + g(x, y, z)^{\frac{1}{a_2 - a_3}})) = (a_3 - a_1)y + \frac{1}{a_2 - a_3}y \frac{X(g(x, y, z))}{1 + g(x, y, z)}$$

and

$$X(\tilde{z}(1 + g(x, y, z)^{\frac{1}{a_3 - a_2}})) = (a_1 - a_2)z + \frac{1}{a_3 - a_2}z \frac{X(g(x, y, z))}{1 + g(x, y, z)}.$$

Finally the vector field X of the form

$$X(x, y, z) = (a_2 - a_3)x \frac{\partial}{\partial x} + (a_3 - a_1)y \frac{\partial}{\partial y} + (a_1 - a_2)z \frac{\partial}{\partial z} + \frac{1}{a_2 - a_3} \frac{X(g(x, y, z))}{1 + g(x, y, z)} (y \frac{\partial}{\partial x} - z \frac{\partial}{\partial z}).$$

Proposition 1. *For $a \notin \mathcal{R}$, there exist a local diffeomorphism $\phi : z = (z_1, \dots, z_{n+2}) \mapsto \tilde{z} = (\tilde{z}_1, \dots, \tilde{z}_{n+2})$ such that the foliation \mathcal{F} is given locally by $(\tilde{z}^{\gamma_0}, \dots, \tilde{z}^{\gamma_{n+2}})$*

Proof. We just make a convenient change of variables. □

Open question. To complete the proof of Conjecture we must solve the nongeneric case $a = (a_1, \dots, a_k) \in \mathcal{R}$ because in this case the rank of exponent matrix

$$m(a_1, \dots, a_k) = \begin{pmatrix} \gamma_0^1 & \dots & \gamma_0^{n+2} \\ \gamma_1^1 & \dots & \gamma_1^{n+2} \\ \vdots & \vdots & \vdots \\ \gamma_n^1 & \dots & \gamma_n^{n+2} \end{pmatrix}$$

is not maximal.

References

- [1] Belotto André, *Local monomialization of a system of first integrals*. arXiv:1411.5333v1
- [2] Bobieński, Marcin; Mardešić, Pavao *Pseudo-Abelian integrals along Darboux cycles*. *Proc. Lond. Math. Soc. (3)* 97 (2008), no. 3, 669-688.
- [3] Hironaka Heisuke, *Resolution of singularities of an algebraic variety over field of characteristic zero. I, II*. *Ann. of Math. (2)* 97 (1964), 109-203; *ibid.* (2) 97 1964 205-326
- [4] Novikov, Dmitry *On limit cycles appearing by polynomial perturbation of Darbouxian integrable systems*. *Geom. Funct. Anal.* 18 (2009), no. 5, 1750-1773.

Université de Bourgogne, Institut de Mathématiques de
Bourgogne, U.M.R. 5584 du C.N.R.S., B.P. 47870, 21078 Dijon
cedex - France.

E-mail adress: aymenbraghta@yahoo.fr