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Local monomialization conjecture of a singular foliation of Darboux type

Aymen Braghtha

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Abstract

After the nice result introduced by Belotto in [1] concerning the local monomialization of a singular foliation given by n first integrals, this work is a continuation in the same spirit. In this paper, we introduce an important conjecture about local monomialization of a singular foliation of Darboux type (see section 1). This conjecture can be used to study pseudo-abelian integrals [2,4].

1 Introduction

Let M be an analytic manifold of dimension $n + 2$. Given a families of first integrals of Darboux type H_ϵ

$$H_\epsilon(x, y) = H(x, y, \epsilon_1, \dots, \epsilon_n) = \prod_{i=1}^k P^{a_i}(x, y, \epsilon_1, \dots, \epsilon_n), \quad a_i > 0. \quad (1)$$

Let F be the foliation of codimension one in M with coordinates $(x, y, \epsilon_1, \dots, \epsilon_n)$ which is given by the analytic one form ω

$$\omega = \frac{H_x}{\phi} dx + \frac{H_y}{\phi} dy + \sum_{i=1}^n \frac{H_{\epsilon_i}}{\phi} d\epsilon_i = 0, \quad (2)$$

where $H_x = \frac{\partial H}{\partial x}$, $H_y = \frac{\partial H}{\partial y}$, $H_{\epsilon_i} = \frac{\partial H}{\partial \epsilon_i}$ and $\phi = \prod_{i=1}^k P_i^{a_i-1}(x, y, \epsilon_1, \dots, \epsilon_n)$ (integrating factor).

Let $F_i, i = 1, \dots, n$ are foliations of codimension one in M with coordinates $(x, y, \epsilon_1, \dots, \epsilon_n)$ which are given by the one forms ω_i

$$\omega_i = d\epsilon_i = 0, \quad i = 1, \dots, n. \quad (3)$$

Let $\mathcal{F} = (F, F_1, \dots, F_n)$ be the result foliation of dimension one in M where its leaves are given by the transversal intersection of leaves of F, F_1, \dots, F_n . Otherwise speaking, the singular foliation \mathcal{F} is given by

$$\Omega = \omega \wedge \omega_1 \wedge \dots \wedge \omega_n \quad (4)$$

$$= Q_1(x, y, \epsilon_1, \dots, \epsilon_n) dx \wedge d\epsilon_1 \wedge \dots \wedge d\epsilon_n + Q_2(x, y, \epsilon_1, \dots, \epsilon_n) dy \wedge d\epsilon_1 \wedge \dots \wedge d\epsilon_n = 0, \quad (5)$$

where $Q_1 = \frac{H_x}{\phi}$, $Q_2 = \frac{H_y}{\phi}$ are polynomials.

We shall say that Ω is a foliation of Darboux type with first integrals $(H, \epsilon_1, \dots, \epsilon_n)$.

Example. Let $H_\epsilon(x, y) = H(x, y, \epsilon) = (x - \epsilon)^{a_1}(x - y)^{a_2}(x + y)^{a_3}$ be a the first integral of Darboux type. The foliation F of codimension one in three dimensional space M with coordinates (x, y, ϵ) is given by the one form

$$\begin{aligned} \omega &= (a_1(x - y)(x + y) + a_2(x - \epsilon)(x + y) + a_3(x - \epsilon)(x - y))dx \\ &\quad - (a_2(x - \epsilon)(x + y) - a_3(x - \epsilon)(x - y))dy - a_1(x - y)(x + y)d\epsilon = 0 \end{aligned}$$

and the foliation F_1 of codimension one in M is given by the one form

$$\omega_1 = d\epsilon = 0.$$

The result foliation $\mathcal{F} = (F, F_1)$ is given by the two-form

$$\Omega = \omega \wedge \omega_1 = (a_1(x-y)(x+y) + a_2(x-\epsilon)(x+y) + a_3(x-\epsilon)(x-y)) dx \wedge d\epsilon - (a_2(x-\epsilon)(x+y) - a_3(x-\epsilon)(x-y)) dy \wedge d\epsilon = 0$$

Observe that the foliation $\mathcal{F} = (F, F_1)$ has a complicated singularity at the origin $(0, 0, 0) \in D_0 = \{\epsilon = 0\}$.

Conjecture. THERE EXIST SEQUENCES OF LOCAL BLOWINGS-UP SUCH THAT THE TOTAL TRANSFORM OF THE FOLIATION $\mathcal{F} : \omega \wedge \omega_1 \wedge \dots \wedge \omega_n = 0$ HAS LOCALLY $n + 1$ MONOMIAL FIRST INTEGRALS $(z^{\gamma_0}, z^{\gamma_1}, \dots, z^{\gamma_n})$ WHERE $z^{\gamma_i} = z_1^{\gamma_{i,1}} \dots z_{n+2}^{\gamma_{i,n+2}}$ AND THE EXPONENTS MATRIX

$$m(a_1, \dots, a_k) = \begin{pmatrix} \gamma_0^1 & \dots & \gamma_0^{n+2} \\ \gamma_1^1 & \dots & \gamma_1^{n+2} \\ \vdots & \vdots & \vdots \\ \gamma_n^1 & \dots & \gamma_n^{n+2} \end{pmatrix}$$

HAS A MAXIMAL RANK.

2 Blowing-up of the foliation \mathcal{F}

In this section, we introduce the fundamental idea to prove the conjecture which is based in first step on Hironaka's reduction of singularities [3]. Let $D_0 = \{\epsilon_1 = \epsilon_2 = \dots = \epsilon_n = 0\}$ be a initial exceptional divisor.

Theorem 1. *There exist a morphism Φ such that the pull-back foliation $\tilde{\Phi}^*\mathcal{F} = \tilde{\mathcal{F}}$ is given locally in neighborhood U_1 of the divisor $\tilde{\Phi}^*(D_0)$ with coordinates $z = (z_1, \dots, z_{n+2})$ by the following system*

$$\begin{cases} \tilde{H} = z^{\gamma_0} \cdot \Delta_0, \\ \tilde{\epsilon}_1 = z^{\gamma_1} \cdot \Delta_1, \\ \vdots \\ \tilde{\epsilon}_n = z^{\gamma_n} \cdot \Delta_n, \end{cases} \quad (6)$$

where $\Delta_i, i = 0, \dots, n$ are a units.

Proof. (1) In first step, we monomialize the principal ideal $I_1 = \langle P_1 \rangle$, Hironaka theorem's guarantee the existence of a sequence of blow-ups $\tilde{\Phi}_1 = \tilde{\Phi}_{n_1}^1 \circ \tilde{\Phi}_{n_1-1}^1 \circ \dots \circ \tilde{\Phi}_1^1$ with initial center $C_0 \subset D_0$ (which is possibly a submanifold of M) such that

$$(\tilde{\Phi}_1^* P_1)^{a_1} = \delta_1 \prod_{i=1}^{n+2} z_i^{a_1 \tilde{\beta}_i^1}, \quad \delta_1(0) \neq 0.$$

(2) In the second step, we consider the principal ideal $I_2 = \langle \tilde{\Phi}_1^* P_2 \rangle$ and by Hironaka theorem's there exist a sequence of blow-ups $\tilde{\Phi}_2 = \tilde{\Phi}_{n_2}^2 \circ \tilde{\Phi}_{n_2-1}^2 \circ \dots \circ \tilde{\Phi}_1^2$ such that

$$(\tilde{\Phi}_2^* \circ \tilde{\Phi}_1^* P_2)^{a_2} = \delta_2 \prod_{i=1}^{n+2} z_i^{a_2 \tilde{\beta}_i^2}, \quad \delta_2(0) \neq 0.$$

In the k -th step there exist a sequence of blow-ups $\tilde{\Phi}_k = \tilde{\Phi}_{n_k}^k \circ \tilde{\Phi}_{n_k-1}^k \circ \dots \circ \tilde{\Phi}_1^k$ such that the principal ideal $I_k = \langle \tilde{\Phi}_{k-1}^* \circ \tilde{\Phi}_{k-2}^* \circ \dots \circ \tilde{\Phi}_1^* P_k \rangle$ has a normal crossings i.e.

$$(\tilde{\Phi}_{k-1}^* \circ \tilde{\Phi}_{k-2}^* \circ \dots \circ \tilde{\Phi}_1^* P_k)^{a_k} = \delta_k \prod_{i=1}^{n+2} z_i^{a_k \tilde{\beta}_i^k}, \quad \delta_k(0) \neq 0.$$

Finally, after desingularisation of each polynomial P_i of the first integral $H = \prod_{i=1}^k P_i^{a_i}$, the equations $z_1 = 0, \dots, z_{n+2} = 0$ are corresponding the irreducibles components of the exceptional divisor. For this

raison after desingularisation of $\tilde{\Phi}_{i-1}^* \circ \tilde{\Phi}_{i-2}^* \circ \dots \circ \tilde{\Phi}_1^* P_i$, the polynomial $\tilde{\Phi}_i^* \circ \tilde{\Phi}_{i-1}^* \circ \tilde{\Phi}_{i-2}^* \circ \dots \circ \tilde{\Phi}_1^* P_{i-1}$ has a normal crossings. So locally we have

$$\begin{cases} \tilde{H} = z^{\gamma_0} \cdot \Delta_0, \\ \tilde{\epsilon}_1 = z^{\gamma_1} \cdot \Delta_1, \\ \vdots \\ \tilde{\epsilon}_n = z^{\gamma_n} \cdot \Delta_n, \end{cases}$$

where $z = (z_1, \dots, z_{n+2})$, $\gamma_0 = \sum_{i=1}^k a_i \beta_i$, $\beta_i = (\beta_i^1, \dots, \beta_i^{n+2})$, $\gamma_i = (\gamma_i^1, \dots, \gamma_i^{n+2})$. □

To complete the proof its necessary to eliminate the units $\Delta_0, \Delta_1, \dots, \Delta_n$ in the system (6). Now we define the resonant locus of the foliation $(z^{\gamma_0} \cdot \Delta_0, z^{\gamma_1} \cdot \Delta_1, \dots, z^{\gamma_n} \cdot \Delta_n)$

$$\mathcal{R} := \{a = (a_1, \dots, a_k) : \gamma_0 \wedge \sum_{j=1}^n \gamma_j = 0\}.$$

To prove the conjecture, we distinguish two cases

- **generic case** $a \notin \mathcal{R}$.
- **nongeneric case** $a \in \mathcal{R}$.

3 Some examples in dimension three

To more understand the problem, we see some examples in dimension three.

Example 1: Let \mathcal{F} be the local foliation which is obtained by after k blow-ups. The foliation \mathcal{F} is given by the following system

$$\begin{cases} H_a = x^{a_1} y^{a_2} (1+z) \\ f = xy \end{cases} \quad (7)$$

In this example we have $\gamma_0 : a_1 \beta_1 + a_2 \beta_2$ where $\beta_1 = (1, 0, 0)$, $\beta_2 = (0, 1, 0)$, $\gamma_1 = (1, 1, 0)$ and $\mathcal{R} = \{a = (a_1, a_2) : \gamma_0 \wedge \gamma_1 = 0\}$. Our goal is to kill the unit $1+z$ in the first integral H_a without modifying the second monomial f in the sense to preserve its monomiality structure. For this reason, we distinguish two different cases:

(a) The generic case $a_1 \neq -a_2 \Leftrightarrow a \notin \mathcal{R}$: We take the change of variables $\tilde{x} = x(1+z)^{\frac{1}{a_1+a_2}}$, $\tilde{y} = y(1+z)^{\frac{1}{a_2+a_1}}$ and $\tilde{z} = z$. Then, we obtain the following system

$$\begin{cases} H_a = \tilde{x}^{a_1} \tilde{y}^{a_2} \\ f = \tilde{x} \tilde{y} \end{cases} \quad (8)$$

Question: How to calculate the generator vector field of the monomial foliation (7)?

Let us assume that the foliation \mathcal{F} is generated locally by the vector field $X(\tilde{x}, \tilde{y}, z) = \alpha_1 \tilde{x} \frac{\partial}{\partial \tilde{x}} + \alpha_2 \tilde{y} \frac{\partial}{\partial \tilde{y}} + \alpha_3 z \frac{\partial}{\partial z}$ which satisfies

$$X(H) = X(\tilde{x}^{a_1} \tilde{y}^{a_2}) = 0, \quad X(f) = X(\tilde{x} \tilde{y}) = 0$$

To determine the vector $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ we use the two following relations

$$\langle \alpha, \gamma_0 \rangle = 0 \quad (\text{i.e. } X(H) = X(\tilde{x}^{a_1} \tilde{y}^{a_2}) = 0), \quad \langle \alpha, \gamma_1 \rangle = 0 \quad (\text{i.e. } X(f) = X(\tilde{x} \tilde{y}) = 0).$$

where \langle, \rangle is scalar product in \mathbb{C}^3 . Finally, the vector $(\alpha_1, \alpha_2, \alpha_3) \in \{e_3\}$ and then

$$\mathcal{F} = \left\{ z \frac{\partial}{\partial z} \right\}$$

Now, we express the vector field X in the original coordinates (x, y, z) . If we write $X(x, y, z) = Ax\frac{\partial}{\partial x} + By\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}$, to determine A, B we use the fact that

$$X(xy) = 0 \iff A = -B$$

and so $X(x, y, z) = A(x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}) + z\frac{\partial}{\partial z}$ on the other hand we have

$$Ax = X(x) = X(\tilde{x}(1+z)^{\frac{1}{a_1+a_2}}) = z\frac{\partial}{\partial z}(\tilde{x}(1+z)^{\frac{1}{a_1+a_2}}) = \tilde{x}(1+z)^{\frac{1}{a_1+a_2}-1}\frac{z}{a_1+a_2}$$

Finally, we obtain

$$X(x, y, z) = \frac{1}{a_1+a_2}\frac{z}{1+z}(x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}) + z\frac{\partial}{\partial z} \Rightarrow Y(x, y, z) = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y} + (a_1+a_2)(1+z)\frac{\partial}{\partial z}$$

Remark 1. In dimension three, if we consider the foliation \mathcal{F} which is given locally by

$$\begin{cases} f_1 = x^a y^b z^c \\ f_2 = x^{\tilde{a}} y^{\tilde{b}} z^{\tilde{c}} \end{cases}$$

where $\text{rank}\begin{pmatrix} a & b & c \\ \tilde{a} & \tilde{b} & \tilde{c} \end{pmatrix} = 2$. The generator vector field X of the form

$$X(x, y, z) = \hat{a}x\frac{\partial}{\partial x} + \hat{b}y\frac{\partial}{\partial y} + \hat{c}z\frac{\partial}{\partial z},$$

where

$$\langle (\hat{a}, \hat{b}, \hat{c}), (a, b, c) \rangle = 0, \quad \text{and} \quad \langle (\hat{a}, \hat{b}, \hat{c}), (\tilde{a}, \tilde{b}, \tilde{c}) \rangle = 0.$$

In our example we observe that in the neighborhood of the leaf $\{z = 0\}$ the vector field

$$Y \simeq x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y} + (a_1+a_2)z\frac{\partial}{\partial z}$$

is linearizable and consequently Y is transversal to the leaf $\{z = 0\}$.

(b) The problem suppose where $a_1 = -a_2$ i.e $a \in \mathcal{R} = \{a = (a_1, a_2) : \gamma_0 \wedge \gamma_1 = 0\}$. In this case near the leaf $\{z = 0\}$, we have

$$Y \simeq x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}.$$

It is clear that the condition of transversality of Y and the leaf $\{z = 0\}$ is not satisfied.

Example 2: let \mathcal{F} be the local foliation which is given (after a sequence of blow-ups) by

$$\begin{cases} H = x^{a_1} y^{a_2} z^{a_3} (1 + g(x, y, z)) \\ f = xyz. \end{cases}$$

The foliation \mathcal{F} is given also

$$\begin{cases} \frac{H}{f^{a_1}} = y^{a_2-a_1} z^{a_3-a_1} (1 + g(x, y, z)) \\ f = xyz. \end{cases}$$

If $a = (a_1, a_2, a_3) \notin \mathcal{R} = \{a : (a_1\beta_1 + a_2\beta_2 + a_3\beta_3) \wedge (1, 1, 1) = 0\}$ (resonant locus), we can take the following variables change $x = \tilde{x}, y = \tilde{y}(1 + g(x, y, z))^{\frac{1}{a_2-a_3}}$ and $z = \tilde{z}(1 + g(x, y, z))^{\frac{1}{a_3-a_2}}$ and in this case the local foliation \mathcal{F} is generated by the vector field

$$X(\tilde{x}, \tilde{y}, \tilde{z}) = (a_2 - a_3)\tilde{x}\frac{\partial}{\partial \tilde{x}} + (a_3 - a_1)\tilde{y}\frac{\partial}{\partial \tilde{y}} + (a_1 - a_2)\tilde{z}\frac{\partial}{\partial \tilde{z}}$$

let us express the vector field X in the original coordinates (x, y, z) , so we have

$$X(x) = X(\tilde{x}) = (a_2 - a_3)x \frac{\partial}{\partial x}$$

$$X(\tilde{y}(1 + g(x, y, z)^{\frac{1}{a_2 - a_3}})) = (a_3 - a_1)y + \frac{1}{a_2 - a_3}y \frac{X(g(x, y, z))}{1 + g(x, y, z)}$$

and

$$X(\tilde{z}(1 + g(x, y, z)^{\frac{1}{a_3 - a_2}})) = (a_1 - a_2)z + \frac{1}{a_3 - a_2}z \frac{X(g(x, y, z))}{1 + g(x, y, z)}.$$

Finally the vector field X of the form

$$X(x, y, z) = (a_2 - a_3)x \frac{\partial}{\partial x} + (a_3 - a_1)y \frac{\partial}{\partial y} + (a_1 - a_2)z \frac{\partial}{\partial z} + \frac{1}{a_2 - a_3} \frac{X(g(x, y, z))}{1 + g(x, y, z)} (y \frac{\partial}{\partial x} - z \frac{\partial}{\partial z}).$$

Proposition 1. *For $a \notin \mathcal{R}$, there exist a local diffeomorphism $\phi : z = (z_1, \dots, z_{n+2}) \mapsto \tilde{z} = (\tilde{z}_1, \dots, \tilde{z}_{n+2})$ such that the foliation \mathcal{F} is given locally by $(\tilde{z}^{\gamma_0}, \dots, \tilde{z}^{\gamma_{n+2}})$*

Proof. We just make a convenient change of variables. □

Open question. To complete the proof of Conjecture we must solve the nongeneric case $a = (a_1, \dots, a_k) \in \mathcal{R}$ because in this case the rank of exponent matrix

$$m(a_1, \dots, a_k) = \begin{pmatrix} \gamma_0^1 & \dots & \gamma_0^{n+2} \\ \gamma_1^1 & \dots & \gamma_1^{n+2} \\ \vdots & \vdots & \vdots \\ \gamma_n^1 & \dots & \gamma_n^{n+2} \end{pmatrix}$$

is not maximal.

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