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► **To cite this version:**

Lou Shituo, Yao Qi. The number of primes in a short interval.. Hardy-Ramanujan Journal, 1993, Volume 16 - 1993, pp.21-43. 10.46298/hrj.1993.127 . hal-01108692

**HAL Id: hal-01108692**

**<https://hal.science/hal-01108692>**

Submitted on 23 Jan 2015

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THE NUMBER OF PRIMES IN A SHORT INTERVAL  
SHITUO LOU AND QI YAO

§ 1. INTRODUCTION.

Let  $x$  be a sufficient large number.

We shall investigate the number of primes in the interval  $(x - y, x]$  for  $y = x^\theta$  with  $1/2 < \theta \leq 7/12$ . Hoheisel [1] was the first to give a value of  $\theta < 1$  such that

$$\pi(x) - \pi(x - y) \sim \frac{y}{\log x}, y = x^\theta. \quad (1.1)$$

Ingham [2] connected the problem with zero density estimates for  $\zeta(s)$ , and Montgomery [3] showed how a method of Halász could be used to estimate  $N(\sigma, T)$  (the number of zeros of  $\zeta(s)$  in the range  $\text{Re } s \geq \sigma, 0 < \text{Im } s \leq T$ ). Huxley [4] proved that for

$$\frac{7}{12} < \theta \leq 1$$

(1.1) holds. His work built on foundations laid by the authors mentioned above.

Heath-Brown [5] has given an alternative proof of Huxley's result : Heath-Brown has actually proved more namely

**THEOREM A** [5] *Let  $\varepsilon(x) \leq 1/12$  be a non-negative function of  $x$ . Then*

$$\pi(x) - \pi(x - y) = \frac{y}{\log x} \left\{ 1 + O(\varepsilon^4(x)) + O\left(\left(\frac{\log \log x}{\log x}\right)^4\right) \right\} \quad (1.2)$$

uniformly for

$$x^{7/12-\varepsilon(x)} \leq y \leq \frac{x}{(\log x)^4}.$$

Thus (1.1) holds for such  $y$ , providing only that  $\varepsilon(x) \rightarrow 0$  as  $x \rightarrow \infty$ . Moreover, he obtained

$$\pi(x) - \pi(x - x^{7/12}) = \frac{x^{7/12}}{\log x} \left\{ 1 + O\left(\left(\frac{\log \log x}{\log x}\right)^4\right) \right\}. \quad (1.3)$$

In [5], Heath-Brown has shown :

**THEOREM B.** Let

$$\sum(z) = \sum_{\substack{x-y < p_1 \cdots p_6 \leq x \\ p_i \geq z, i=1, \dots, 6}} 1 \quad (1.4)$$

and

$$E(x, z) = \frac{1}{\log x} + \frac{1}{6} \int \cdots \int_{\substack{t_i \geq z \\ t_1 t_2 t_3 t_4 t_5 \geq x/z}} \left( \log \frac{x}{t_1 t_2 t_3 t_4 t_5} \right)^{-1} \frac{dt_1 dt_2 dt_3 dt_4 dt_5}{t_1 t_2 t_3 t_4 t_5 \log t_1 \log t_2 \log t_3 \log t_4 \log t_5} \quad (1.5)$$

is independent of  $y$ , where  $z$  may take any value in the range

$$\begin{aligned} x^{1/7} < z \leq x^{1/6} \exp(-(\log x)^{43/44}), y \geq x^{7/12}; \\ x^{1/7} < z \leq y^{50} x^{-29} \exp(-(\log x)^{43/44}), y < x^{7/12}. \end{aligned}$$

Then

$$\pi(x) - \pi(x - y) = yE(x, z) - \frac{1}{6} \sum(z) + O(y \exp(-(\log x)^{1/7})) \quad (1.6)$$

uniformly for

$$x^{7/12-1/6000} \leq y \leq x \exp(-(\log x)^{1/6}). \quad (1.7)$$

In this paper, we shall give a generalization of Theorem B in § 2. Let the interval  $|^y = (x - y, x]$  with

$$x^{1/2} < y \leq \frac{1}{2}x$$

and the parameter  $z$  satisfying

$$x^{1/k_0} < z \leq x^{1/5}$$

where  $k_0$  is a positive integer that will be chosen later. For example, with  $y = x^\theta$ , we shall choose  $k_0 = 11$  if  $\theta = 11/20 + \varepsilon$ .

Denote  $p(d_i)$  the smallest prime factor of  $d_i$ . We write

$$S_k := \{d_1 \cdots d_k : d_1 \cdots d_k = d \in \mathcal{P}, p(d_i) \geq z, 1 \leq i \leq k\}.$$

$d_1 \cdots d_r = d'_1 \cdots d'_j \in S_k$  if and only if  $r = j$  and  $d_i = d'_i$  for  $1 \leq i \leq r$ .

Let

$$\sum_{n \in \mathcal{P}} a_n(k) = \sum_{\substack{d_1 \cdots d_k = n \\ p(d_i) \geq z, 1 \leq i \leq k \\ n \in \mathcal{P}}} 1 = \sum_{s \in S_k} 1$$

Let  $r$  be a positive integer,  $I_j, 1 \leq j \leq r$ , be a set of integers, and  $I_j \subseteq [2, x]$  and  $H$  be the "Direct Product" of sets  $I_j$ , for  $1 \leq j \leq r$ , it means

$$d \in H \text{ if and only if } d = d_1 \cdots d_r \text{ with } d_j \in I_j, 1 \leq j \leq r, \text{ and } d \in \mathcal{P}. \quad (1.8)$$

Suppose  $\theta$  be fixed in the interval  $(1/2, 1)$  and  $y \in [x^\theta, x \exp(-\log x)^{1/6}]$ . Define the conditions  $(A_1)$  and  $(A_2)$  as following :

$(A_1)$ . If there exist some sets  $H_k, 1 \leq k < k_0$ , which are collections of direct products  $H$ 's and constants  $c_H$  such that

$$\sum_{n \in \mathcal{P}} a_n(k) = \sum_{H \in H_k} c_H \sum_{d \in H} 1 + O\left(\frac{y}{\log^2 x}\right), \quad (1.9)$$

then we call  $H_k, 1 \leq k < k_0$ , satisfy  $(A_1)$ .

$(A_2)$ . If  $H_k, 1 \leq k < k_0$ , satisfy  $(A_1)$ , There exists a subset  $H'_k$  and for each  $H \in H'_k$  there exists a function  $E_k(H, z)$  independent of  $y$  such that

$$\sum_{d \in H} 1 = y E_k(H, z) + O(y \exp(-(\log x)^{1/7})), \quad (1.10)$$

uniformly for

$$x^\theta \leq y \leq x \exp(-(\log x)^{1/6}),$$

then we call  $\mathbf{H}'_k, 1 \leq k < k_0$ , satisfy  $(A_2)$ .

We now state our Theorem here :

**THEOREM 1.** Let  $x$  be a sufficient large number,  $\theta$  be fixed in  $(1/2, 1)$ ,  $x^\theta \leq y < (1/2)x$ ,  $|y = (x - y, x]$ ,  $k_0$  be an integer which is dependent on  $\theta$ , and  $x$  be fixed in  $(x^{1/k_0}, x^{1/5}]$ . Let  $\mathbf{H}_k, 1 \leq k < k_0$ , such that  $(A_1)$ . If there exists a subset  $\mathbf{H}'_k$  of  $\mathbf{H}_k$  such that  $(A_2)$ , and writing  $\mathbf{H}''_k = \mathbf{H}_k \setminus \mathbf{H}'_k$ , then we have

$$\pi(x) - \pi(x - y) = yE(x, z) + R(y) + O(y \exp(-(\log x)^{1/7})) \quad (1.11)$$

uniformly for

$$x^\theta \leq y \leq x \exp(-(\log x)^{1/6}),$$

where  $E(x, z)$  independent of  $y$ , and

$$R(y) = \sum_{1 \leq k < k_0} (-1)^{k-1} k^{-1} \sum_{H \in \mathbf{H}''_k} c_H \sum_{d \in H} 1. \quad (1.12)$$

We call  $\mathbf{H}'_k$  a 'good set' and call  $\mathbf{H}''_k$  a 'bad set', for  $1 \leq k < k_0$ . Heath-Brown [5] prove that

$$\pi(x) - \pi(x - y) = \sum_{1 \leq k < k_0} (-1)^{k-1} k^{-1} \sum_{s \in S_k} 1 + O(yx^{-\frac{1}{2}}) \quad (1.13)$$

Comparing (1.13) and (1.6) with (1.5), Heath-Brown took  $k_0 = 7, S_1, \dots, S_5$  as good sets and only  $S_6$  as a bad set i.e.  $\mathbf{H}'_1 = S_1, \dots, \mathbf{H}'_5 = S_5, \mathbf{H}'_6 = \emptyset$ ; and  $\mathbf{H}''_1 = \dots = \mathbf{H}''_5 = \emptyset, \mathbf{H}''_6 = S_6$ . In Theorem 1, we are not limited that the good set or that the bad set should to be whole of  $S_k$ . In fact,  $R(y)$  is the contribution of all bad sets. He proved that the contribution of his bad sets is  $\sum(z)$  in (1.5). Heath-Brown applied Theorem B to improve (1.3). He obtained that if  $x$  is sufficient large,

$$\pi(x) - \pi(x - y) \geq \frac{4y}{5 \log x}, \quad (1.14)$$

where

$$x^{\frac{7}{12} - \frac{1}{6000}} \leq y \leq x. \quad (1.15)$$

In § 2 we shall prove Theorem 1. In § 2, we shall prove the following theorem also :

**THEOREM 2.** Suppose that  $\theta$  is fixed in  $(1/2, 1)$ ,  $y_0 = x \exp(-(\log x)^{1/\theta})$ ,  $H_k$ ,  $1 \leq k < k_0$ , satisfy  $(A_1)$  and  $(A_2)$ . If there exist constants  $e_1, e'_1, e_2$  and  $e'_2$  such that

$$\frac{(-e'_1 + \varepsilon)y_0}{\log x} < \sum_{1 \leq k < k_0} (-1)^{k-1} k^{-1} R_k(y_0) < \frac{(e_1 - \varepsilon)y_0}{\log x} \quad (1.16)$$

and

$$\frac{(-e'_2 + \varepsilon)y}{\log x} < \sum_{1 \leq k < k_0} (-1)^{k-1} k^{-1} R_k(y) < \frac{(e_2 - \varepsilon)y}{\log x} \quad (1.17)$$

where  $\varepsilon$  is a small positive constant. Then

$$\frac{(1 - e_1 - e'_2)y}{\log x} < \pi(x) - \pi(x - y) < \frac{(1 + e'_1 + e_2)y}{\log x} \quad (1.18)$$

uniformly for  $x^\theta \leq y \leq y_0$ .

Take an applicable form  $H_k$  with condition  $(A_1)$ , which makes it possible to extend the range of validity of

$$(1 - c) \frac{y}{\log x} < \pi(x) - \pi(x - y) < (1 + c') \frac{y}{\log x}, \quad (1.19)$$

where  $c$  and  $c'$  are constants. In this paper, we prove that (1.19) holds with  $y = x^\theta$ ,  $\theta = 11/20 + \varepsilon$  and  $c = c' = 0.01$  in § 6.

In [7], we gave some sufficient conditions that imply some kind of "direct product" be "good set". In § 3 and § 4 below, we use those conditions to prove that  $H'_k$ ,  $1 \leq k \leq k_0$ , which will be defined in (3.3) and (3.4) below be "good set".

In § 6 we will prove

**Theorem 3.** Suppose  $x$  be a large number, then

$$1.01 \frac{y}{\log x} \geq \pi(x) - \pi(x - y) \geq 0.99 \frac{y}{\log x} \quad (1.20)$$

with  $y = x^\theta$ , uniformly for

$$\frac{11}{20} < \theta \leq \frac{7}{12}. \quad (1.21)$$

A criterion for good sets is extracted. However, the technical work needed to choose good sets and to make the size of the bad sets as small

as possible, is precisely the main difference between our method and that Heath-Brown's. The new Theorem 1 will enable us to improve the results of Heath-Brown and Iwaniec [10]. Later on we shall establish one deeper results: for

$$x^\theta \leq y \leq x \exp(-(\log x)^{1/6}),$$

we have (1.19) with  $\theta = 6/11 + \varepsilon$  or  $\theta = 7/13 + \varepsilon$ .

Moreover, we can improve (1.19) further but only at the cost of much arduous computation.

## § 2 Proof of Theorem 1 and Theorem 2.

The proof of Theorem 1 is much along the method that was used by Heath-Brown [5].

Our starting point is based on a formal identity (see [5]) :

$$\log \zeta(s) \prod(s) = \sum_{1 \leq k \leq \infty} (-1)^{k-1} k^{-1} (\zeta(s) \prod(s) - 1)^k \quad (2.1)$$

$$= \sum_{1 \leq t \leq \infty} \sum_{p \geq z} \frac{1}{t p^{ts}}, \quad (2.2)$$

where

$$\prod(s) = \prod_{p < z} \left(1 - \frac{1}{p^s}\right).$$

We pick out the coefficients of  $n^{-s}$  for those terms in (2.1) and (2.2) with  $n \in |^y$ . Thus in (2.2), these coefficients total

$$\sum_{1 \leq t < \infty} \left( \pi \left(x^{\frac{1}{t}}\right) - \pi \left((x-y)^{\frac{1}{t}}\right) \right) \frac{1}{t} = \pi(x) - \pi(x-y) + O\left(yx^{-\frac{1}{2}}\right). \quad (2.3)$$

On the other hand, the Dirichlet series for  $\zeta(s) \prod(s) - 1$  is

$$\sum_{n \geq z} c_n n^{-s}, \quad (2.4)$$

where  $c_n$  is 0 or 1 according to  $n$  has a prime factor  $< z$  or not. It follows from (1.7) that there are no term of  $n^{-s}$  in (2.2) with  $n \in |^y$  corresponding to exponents  $k \geq k_0$ . Henceforth we consider only the terms with  $k < k_0$ .

Let

$$(\zeta(s) \prod (s-1)^k = \sum_{1 \leq n < \infty} a_n(k) n^{-s} \tag{2.5}$$

By (2.4),

$$(\zeta(s) \prod (s-1)^k = \left( \sum_{1 \leq n < \infty} c_n n^{-s} \right)^k \tag{2.6}$$

Then

$$\sum_{1 \leq n < \infty} a_n(k) n^{-s} = \left( \sum_{1 \leq n < \infty} c_n n^{-s} \right)^k$$

and

$$a_n(k) = \sum_{d_1 \cdots d_k = n} c_{d_1} \cdots c_{d_k}$$

Write

$$a_n(k) = | \{ (d_1, \dots, d_k) : n = d_1 \cdots d_k, p(d_i) \geq z, 1 \leq i \leq k \} |, \tag{2.7}$$

where  $(d_1, \dots, d_k) = (d'_1, \dots, d'_k)$  means  $d_i = d'_i$  for  $i = 1, \dots, k$ . Therefore

$$\sum_{n \in |P} a_n(k) = \sum_{\substack{d_1 \cdots d_k = n \\ p(d_i) \geq z, 1 \leq i \leq k \\ n \in |P}} 1, \tag{2.8}$$

and in (2.1), the coefficients total

$$\sum_{1 \leq k \leq k_0} (-1)^{k-1} k^{-1} \sum_{n \in |P} a_n(k) = \sum_{1 \leq k < \infty} (-1)^{k-1} k^{-1} \sum_{\substack{d_1 \cdots d_k = n \\ p(d_i) \geq z, 1 \leq i \leq k \\ n \in |P}} 1. \tag{2.9}$$

We have that

$$\pi(x) - \pi(x-y) = \sum_{1 \leq k < \infty} (-1)^{k-1} k^{-1} \sum_{\substack{d_1 \cdots d_k = n \\ p(d_i) \geq z, 1 \leq i \leq k \\ n \in |P}} 1 + O(yx^{-\frac{1}{2}}) \tag{2.10}$$

since (2.3) and (2.9).



By conditions  $(A_1)$  and  $(A_2)$ , we have that

$$\begin{aligned}\pi(x) - \pi(x-y) &= \sum_{1 \leq k \leq k_0} (-1)^{k-1} k^{-1} \sum_{H \in \mathbf{H}_k} c_H \sum_{d \in H} 1 + O\left(yx^{-\frac{1}{3}}\right) + O\left(\frac{y}{\log^2 x}\right) \\ &= y \sum_{1 \leq k \leq k_0} (-1)^{k-1} k^{-1} \sum_{H \in \mathbf{H}'_k} c_H E_k(H, z) + R(y) + O\left(\frac{y}{\log^2 x}\right).\end{aligned}$$

Let

$$E(x, z) = \sum_{1 \leq k \leq k_0} (-1)^{k-1} k^{-1} \sum_{H \in \mathbf{H}'_k} c_H E_k(H, z).$$

This completes the proof.

### The proof of Theorem 2

By Prime Number Theorem,

$$\pi(x) = \int_2^x \frac{dt}{\log t} + O(x \exp(-\log x)^{1/2}).$$

We have

$$\begin{aligned}\pi(x) - \pi(x-y_0) &= \int_{x-y_0}^x \left( \frac{1}{\log t} - \frac{1}{\log x} \right) dt + \frac{y_0}{\log x} + O(x \exp(-(\log x)^{1/2})) \\ &= \int_{x-y_0}^x \frac{\log \frac{x}{t}}{\log t \log x} dt + \frac{y_0}{\log x} + O(x \exp(-(\log x)^{1/2})).\end{aligned}\quad (2.11)$$

Clearly, for  $x - y_0 \leq t \leq x$ ,

$$\log \frac{x}{t} \leq \log \frac{x}{x-y_0} \leq \frac{y_0}{x-y_0} = O\left(\frac{y_0}{x}\right).$$

Therefore, (3.1) is

$$\pi(x) - \pi(x-y_0) = \frac{y_0}{\log x} + O(y_0 \exp(-(\log x)^{1/8})).\quad (2.12)$$

Using Theorem 1 with  $y = y_0$ ,  $S'_k = \mathbf{H}_k$ , and

$$R(y_0) = \sum_{1 \leq k \leq k_0} (-1)^{k-1} k^{-1} R_k(y_0) + O\left(\frac{y_0}{\log x}\right),$$

we have

$$\pi(x) - \pi(x - y_0) = y_0 E(x, z) + \sum_{1 \leq k \leq k_0} (-1)^{k-1} k^{-1} R_k(y_0) + O\left(\frac{y_0}{\log^2 x}\right). \quad (2.13)$$

Comparing (2.12) with (2.13), we have

$$\frac{y_0}{\log x} = y_0 E(x, z) + \sum_{1 \leq k \leq k_0} (-1)^{k-1} k^{-1} R_k(y_0) + O\left(\frac{y_0}{\log^2 x}\right), \quad (2.14)$$

hence

$$E(x, z) = \frac{1}{\log x} - \frac{1}{y_0} \sum_{1 \leq k \leq k_0} (-1)^{k-1} k^{-1} R_k(y_0) + O\left(\frac{y_0}{\log^2 x}\right). \quad (2.15)$$

By (2.15) and (1.16),

$$\frac{1 - e_1}{\log x} < E(x, z) < \frac{1 + e'_1}{\log x}. \quad (2.16)$$

Using Theorem 1 again,

$$yE(x, z) = \pi(x) - \pi(x - y) - \sum_{1 \leq k \leq k_0} (-1)^{k-1} k^{-1} R_k(y) + O\left(\frac{y}{\log^2 x}\right). \quad (2.17)$$

By (1.17), (2.15) and (2.17), we have

$$\frac{(1 - e_1 - e'_2)y}{\log x} < \pi(x) - \pi(x - y) < \frac{(1 + e'_1 + e_2)y}{\log x}. \quad (2.18)$$

This completes the proof.

### § 3. "Good Set"

Let  $c_0$  be a constant that will be defined later on. Let  $|_0$  be an interval  $[a_0, b_0]$  which contains in  $[1, x]$  and  $|_j (1 \leq j \leq r)$  be a subset of interval  $[a_j, b_j]$  contains in  $[x^{c_0}, x]$  also. Denote  $D = |_0 \cdots |_r$  be a direct product of  $|_j$ . Let  $i_j = \log a_j / \log x$  and  $i'_j = \log b_j / \log x$  and let  $d_j = x^{\theta_j}$  with  $i_j \leq \theta_j \leq i'_j$  and  $0 \leq j \leq r$ . For convenience, we write  $d = \{\theta_0, \theta_1, \dots, \theta_r\} \in D$ , and a set

$$D = \{ \{\theta_0, \theta_1, \dots, \theta_r\} : 1/2 \geq 1 - \theta_1 - \dots - \theta_r = \theta_0 \geq \theta_1 \geq \dots \geq \theta_r \}. \quad (3.1)$$

For short, we denote  $\{\theta_j\} = \{\theta_0, \theta_1, \dots, \theta_r\}$ .

Let  $\mathbf{D} \cap |\nu$  be a set of integers,  $d \in \mathbf{D} \cap |\nu$  if and only if  $d \in \mathbf{D}$  and  $d \in |\nu$ .  $d = d'$  with  $d, d' \in \mathbf{D} \cap |\nu$  means  $d = d_0 \cdots d_r$  and  $d' = d'_0 \cdots d'_r$  with  $d_j = d'_j$  for  $0 \leq j < r$ . We shall show the sufficient conditions for  $\mathbf{D} \cap |\nu$  be a "good set", i.e. for a fixed  $z$  with  $x^{1/5} > z = x^c$ , there exists a function  $E_{\mathbf{D}}(x, z)$ , independent of  $y$ , which satisfies that

$$\sum_{d \in \mathbf{D} \cap |\nu} 1 = y E_{\mathbf{D}}(x, z) + O(y \exp(-\log^{1/7} x)), \quad (3.2)$$

where  $E_{\mathbf{D}}(x, z)$  and constant in "O" are uniformly for

$$x^\theta \leq y \leq x \exp(-4(\log x)^{\frac{1}{3}}(\log \log x)^{-\frac{1}{3}}).$$

Let  $\theta = 11/20 + \varepsilon$ ,  $t_0 = 1 - \theta + \varepsilon/2$  and  $z = x^c$  with  $c = 1/2 - 8t_0/9$ . Define

$$\begin{aligned} \mathbf{D}(6) = \{ \{\theta_0, \theta_1, \dots, \theta_5\} : \{\theta_0, \theta_1, \dots, \theta_5\} \in \mathbf{D}, 2t_0/5 \geq 1 - \theta_1 - \dots - \theta_5 = \\ \theta_0 \geq \theta_1 \geq \dots \geq \theta_5 \geq 1/2 - 8t_0/9 \} \end{aligned} \quad (3.3)$$

and

$$\begin{aligned} \mathbf{D}(8) = \{ \{\theta_0, \theta_1, \dots, \theta_7\} : \{\theta_0, \theta_1, \dots, \theta_7\} \in \mathbf{D}, 2t_0/7 \geq 1 - \theta_1 - \dots - \theta_7 = \\ \theta_0 \geq \theta_1 \geq \dots \geq \theta_7 \geq 1/2 - 8t_0/9 \} \end{aligned} \quad (3.4)$$

In this section, we shall prove that :

**Theorem 4.** Suppose  $\theta = 11/20 + \varepsilon$ ,  $t_0 = 1 - \theta + \varepsilon/2$ ,  $z = x^c$  with  $c = 1/2 - 8t_0/9$ ,  $\mathbf{D}'$  be a subset of  $\mathbf{D}$ , and

$$\mathbf{D}' \cap (\mathbf{D}(6) \cup \mathbf{D}(8)) = \emptyset, \quad (3.5)$$

then  $\mathbf{D}'$  satisfies (3.2), i.e.  $\mathbf{D}'$  is a good set.

Obviously, the subset of  $\mathbf{D}$  with  $r \neq 5$  or  $7$  are good sets.  $\mathbf{D}(6)$  and  $\mathbf{D}(8)$  are called exceptional sets.

We discuss those sequences  $d = \{\theta_0, \theta_1, \dots, \theta_r\} = \{\theta_j\}$  in  $D$ . For such  $\{\theta_j\}$ , we define a corresponding set  $\Theta$  of all of sequences  $\{\theta'_0, \theta_1, \dots, \theta_r, \theta_{r+1}, \dots, \theta_{r+r_1}\}$  with  $\theta'_0 \leq \theta_0, \theta_1 \geq \dots \geq \theta_r \geq \log x / \log x > \theta_{r+1} \geq \dots \geq \theta_{r+r_1}$  and

$$\theta'_0 + \theta_1 + \dots + \theta_{r+r_1} = 1. \tag{3.6}$$

By (3.6) and (3.1), we have that if  $r_1 = 0$ , then

$$\theta'_0 = \theta_0 \geq \theta_1. \tag{3.7}$$

For short, write  $\{\theta_j\}' = \{\theta'_0, \theta_1, \dots, \theta_r, \theta_{r+1}, \dots, \theta_{r+r_1}\}$ .  $\{\theta_j\}$  and  $\{\theta_j\}' \in \Theta$ . Let  $\theta'_0 = \log X / \log x, \theta_j = \log X_d^{(j)} / \log x$  ( $1 \leq j \leq r$ ) and  $\theta_{r+j} = \log Z_j / \log x$  ( $1 \leq j \leq r_1$ ). For each  $\{\theta'_0, \theta_1, \dots, \theta_r, \theta_{r+1}, \dots, \theta_{r+r_1}\}$ , we define a product of Dirichlet series :

$$W(s, \{\theta_j\}') = W(s) = X(s) \prod_{j=1}^r X_d^{(j)}(s) Y(s) \prod_{j=1}^{r_1} Z_j(s) \tag{3.8}$$

where

$$\begin{aligned} X(s) &= \sum_{X < n \leq 2X} n^{-s}; \\ X_d^{(j)}(s) &= \sum_{X_d^{(j)} < m \leq 2X_d^{(j)}} f_m^{(j)} m^{-s}, |f_m^{(j)}| \leq 1; \\ Z_j(s) &= \sum_{Z_j < t \leq 2Z_j} c_t |^{-s}, |c_t| \leq 1; \\ Y(s) &= \sum_{Y < t \leq 2Y} \mu(t) v_t t^{-s}, |v_t| \leq 1. \end{aligned}$$

with  $Y = O(x^\delta), \delta$  be a sufficient small number with  $\delta \ll \epsilon$ . Each  $\{\theta_j\} \in D$  corresponds all of  $W(s, \{\theta_j\}')$ 's for which  $\{\theta_j\}' \in \Theta$ . Define that  $W(D)$  is a set of all of such  $W(s, \{\theta_j\}')$ . For short, we write  $W(s, \{\theta_j\}') = W(s)$ . In [7], we proved that

**Theorem A.** *If  $D$  satisfies one of following conditions*

- (1)  $a_0 \geq x^{1/2}$ ;
- (2) all of  $W(s) \in W(D)$  such that

$$\int_T^{2T} |W\left(\frac{1}{2} + it\right)| dt \ll x^{\frac{1}{2}} \exp\left(-(\log x)^{\frac{1}{2}} (\log \log x)^{-\frac{2}{3}}\right) \tag{3.9}$$

for

$$T_1 \leq T \leq \frac{x^{1-\Delta}}{y},$$

where  $\Delta$  is any fixed positive constant, and

$$T_1 = \exp\left((\log x)^{\frac{1}{2}}(\log \log x)^{-\frac{1}{2}}\right).$$

Then (1.2) holds i.e.  $\mathbf{D}$  is a good set

Let  $\theta_0, \theta_1, \dots, \theta_k$  be positive numbers. In [7], we discussed the sequence  $\{\theta_0, \theta_1, \dots, \theta_k\}$  with positive number  $k$  such that

$$\theta_0 + \theta_1 + \dots + \theta_k = 1, \quad (3.10)$$

defined a set  $E(\theta)$  of some  $\{\theta_0, \theta_1, \dots, \theta_k\}$ 's and acutely proved that [7, § 5]).

**Theorem B.** Let  $\{\theta_j\} \in \mathbf{D}$ . For each  $\{\theta_j\}' \in \Theta$  define

$$W'(s) = X(s) \prod_{j=1}^r X_d^{(j)}(s) \prod_{j=1}^{r_1} Z_j(s).$$

If  $\{\theta_j\}' \in E(\theta)$ , then

$$\int_T^{2T} |W'\left(\frac{1}{2} + it\right)| dt \ll x^{1/2-\varepsilon}. \quad (3.11)$$

Moreover, (3.9) holds.

We now describe the set  $E(\theta)$ .

Suppose  $\{a_1, a_2, \sigma\}$  or  $\{a_1, a_2, a_3, \sigma\}$  be a complementary partial sum (it means that each  $\theta_j$  belongs one and only one set and their sum in a set be  $\sigma$  or  $a_i$ ) of  $\{\theta_0, \theta_1, \dots, \theta_k\}$  with  $\sigma = \theta_0$  or  $\sigma \leq t_0/2$ , then

$$a_1 + a_2 + \sigma = 1. \quad (3.12)$$

or

$$a_1 + a_2 + a_3 + \sigma = 1. \quad (3.13)$$

Later on, we only define two of  $a_1, a_2$  and  $\sigma$  if (3.12) holds; or define three of  $a_1, a_2, a_3$  and  $\sigma$  if (3.13) holds. Suppose  $\theta = 11/20 + \varepsilon$  and  $t_0 = 9/20 - \varepsilon/2$ .

We define  $E(\theta)$  be a set which contains all of sequence  $\{\theta_0, \theta_1, \dots, \theta_k\}$  with (3.12) which satisfies one of following three properties :

(I) There exists at least one complementary partial sum  $\{a_1, a_2, \sigma\}$  of  $\{\theta_0, \theta_1, \dots, \theta_k\}$  which satisfies one of following conditions:

$$(3.14) \quad a_1 \leq t_0 \text{ and } a_2 \leq t_0 \text{ (see Lemma 4.4 of [7]);}$$

$$(3.15) \quad \sigma > t_0/5, a_1 > 8t_0/9 \text{ and } a_2 > 8t_0/9 \text{ (see (4.1.3) with } i = 3 \text{ of [7]);}$$

$$(3.16) \quad a_1 > 6t_0/7, a_2 > 6t_0/7 \text{ and } \sigma > t_0/4 \text{ (see (4.1.3) with } i = 2 \text{ of [7]);}$$

$$(3.17) \quad a_1 \geq t_0 \text{ and } a_2 \geq t_0 \text{ (see (4.1.1) of [7]);}$$

$$(3.18) \quad 1/2 \geq a_1 \geq t_0, \text{ and } \sigma < 1/2 - 8t_0/9 \text{ (see (4.5.6) of [7]);}$$

$$(3.19) \quad \sigma > t_0/2 \text{ (see Lemma 4.3 of [7]).}$$

(II) There exists at least one complementary partial sum  $\{a_1, a_2, a_3, \sigma\}$  of  $\{\theta_0, \theta_1, \dots, \theta_k\}$  which satisfies

$$(3.20) \quad a_1 \geq t_0, a_2 \geq t_0/3, a_3 \geq t_0/3 \text{ and } \sigma > 2t_0/5 \text{ (see (4.2.2) of [7]).}$$

For a fixed  $\sigma$ , in [7] we proved that there exists a pair of numbers  $(m_\sigma, M_\sigma)$  with the properties

$$M_\sigma - m_\sigma > 1/2 - 8t_0/9 \text{ if } \sigma \geq 1/2 - 8t_0/9; \quad (3.21)$$

$$M_\sigma - m_\sigma < \sigma \text{ if } \sigma < 1/2 - 8t_0/9; \quad (3.22)$$

$$M_\sigma > t_0 > m_\sigma; \quad (3.23)$$

and

$$M_\sigma + m_\sigma + \sigma = 1 \quad (3.24)$$

(III) Suppose  $\{a_1, a_2, \sigma\}$  or  $\{a_1, a_2, a_3, \sigma\}$  be a complementary sum of  $\{\theta_0, \theta_1, \dots, \theta_k\}$  with

$$m_\sigma < a_i < M_\sigma, \quad (i = 1 \text{ or } 2), \quad (3.25)$$

(See Lemma 4.5 of [7]).

Applying Theorem A and Theorem B, Theorem 4 follows from

**Theorem 5.** *Suppose  $\theta = 11/20 + \varepsilon$ , and  $D'$  be a subset of  $D$  such that*

$$D' \cap (D(6) \cup D(8)) = \phi.$$

*Then for every  $\{\theta_j\} \in D'$ , the all of corresponding  $\{\theta_j\}' \in \Theta$  contain in  $E(\theta)$ .*

#### § 4. LEMMAS.

Let  $\theta = 11/20 + \varepsilon$  and  $\{\theta_0, \theta_1, \dots, \theta_k\}$  with (3.11), i.e.

$$\theta_0 + \theta_1 + \dots + \theta_k = 1.$$

In this section, we shall show some sufficient conditions for  $\{\theta_0, \theta_1, \dots, \theta_k\} \in E(\theta)$ . By the definition of  $E(\theta)$  we check that  $\{\theta_j\}$  satisfies at least one of conditions (3.14) - (3.20) and (3.25).

**Lemma 4.1.** *Suppose there exist two elements  $\theta'$  and  $\theta''$  of  $\{\theta_0, \theta_1, \dots, \theta_k\}$  with  $\theta' \leq t_0/2$  and  $\theta'' < 1/2 - 8t_0/9$ . If there exists a partial sum  $s$  of  $\{\theta_0, \theta_1, \dots, \theta_k\} \setminus \{\theta', \theta''\}$  such that  $s < t_0$  and  $s + \theta' \geq t_0$ , then  $\{\theta_0, \theta_1, \dots, \theta_k\} \in E(\theta)$ .*

**Proof.** We discuss following three cases :

**Case 1.**  $t_0 \leq s + \theta' < M_{\theta''}$  and  $1 - s - \theta' \geq t_0$ .

Let  $\sigma = \theta''$  and  $a_1 = s + \theta'$ , we have that  $\{\theta_0, \theta_1, \dots, \theta_k\} \in E(\theta)$  by (3.23) and (3.25).

**Case 2.**  $s + \theta' \geq M_{\theta''}$ .

By (3.24), we have

$$1 - s - \theta' - \theta'' \leq 1 - M_{\theta''} - \theta'' = m_{\theta''}$$

and, by (3.22),

$$1 - s - \theta' \leq \theta'' + m_{\theta''} < M_{\theta''}.$$

Let  $a_1 = 1 - s - \theta'$  and  $\sigma = \theta''$ , if  $a_1 > m_{\theta''}$ , then  $\{\theta_0, \theta_1, \dots, \theta_k\} \in E(\theta)$  by (3.25). If  $a_1 \leq m_{\theta''} \leq t_0$ , then  $\{\theta_0, \theta_1, \dots, \theta_k\} \in E(\theta)$  by (3.14) (since

$$a_2 \leq m_{\theta''} \leq t_0.)$$

**Case 3.**  $t_0 \leq s + \theta' < M_{\theta''}$  and  $1 - s - \theta' < t_0$ .

Let  $a_1 = 1 - s - \theta' < t_0$ ,  $a_2 = s < t_0$  and  $\sigma = \theta'$ , then  $\{\theta_0, \theta_1, \dots, \theta_k\} \in E(\theta)$  by (3.14).

**Lemma 4.2.** Suppose  $\{a_1, a_2, \sigma\}$  be a complementary partial sum of  $\{\theta_0, \theta_1, \dots, \theta_k\}$  with  $a_1 = \theta_1 + \dots + \theta_k$ ,  $a_2 = \theta'_1 + \dots + \theta'_k$ ,  $a_1 \geq a_2$ ,  $\sigma = 1 - a_1 - a_2 > 1/2 - 8t_0/9$  and

$$\max\{\theta_1, \dots, \theta_k\} - \min\{\theta'_1, \dots, \theta'_k\} < \frac{1}{2} - \frac{8t_0}{9}; \quad (4.1)$$

then  $\{\theta_0, \theta_1, \dots, \theta_k\} \in E(\theta)$ .

**Proof.** If  $a_1 \leq t_0$ , then  $\{\theta_0, \theta_1, \dots, \theta_k\} \in E(\theta)$  by (3.14); if  $a_2 \geq t_0$ , then  $\{\theta_0, \theta_1, \dots, \theta_k\} \in E(\theta)$  by (2.8); if  $m_\sigma < a_1 < M_\sigma$ ,  $\{\theta_0, \theta_1, \dots, \theta_k\} \in E(\theta)$  by (3.25).

Now we suppose  $a_1 \geq M_\sigma$ .

By (3.15) and (3.1), we have

$$\begin{aligned} \theta_1 + \dots + \theta_{k-1} + \theta'_k &= \theta_1 + \dots + \theta_k + (\theta'_k - \theta_k) \\ &> M_\sigma - \left(\frac{1}{2} - \frac{8t_0}{9}\right) \geq m_\sigma. \end{aligned}$$

If

$$\theta_1 + \dots + \theta_{k-1} + \theta'_k < M_\sigma.$$

let  $a_1 = \theta_1 + \dots + \theta_{k-1} + \theta'_k$ , then  $\{\theta_0, \theta_1, \dots, \theta_k\} \in E(\theta)$  by (3.17); if

$$\theta_1 + \dots + \theta_{k-1} + \theta'_k \geq M_\sigma.$$

and

$$\theta_1 + \dots + \theta_{k-2} + \theta'_{k-1} + \theta'_k < M_\sigma.$$

repeating above process, let  $a_1 = \theta_1 + \dots + \theta'_{k-1} + \theta'_k$  we also have  $\{\theta_0, \theta_1, \dots, \theta_k\} \in E(\theta)$ . And repeat it again, we have that, in all cases,  $\{\theta_0, \theta_1, \dots, \theta_k\} \in E(\theta)$  since

$$\theta'_1 + \dots + \theta'_k < t_0 \leq M_\sigma.$$



### § 5. Proof of Theorem 3

It is sufficient to prove Theorem 4, i.e., let  $\{\theta_j\} \in \mathbf{D}$  and  $\{\theta_j\}' \in \Theta$  that satisfies following conditions :

$$(5.1) \quad t_0 > 1 - \theta_1 - \cdots - \theta_r = \theta_0 \geq \theta'_0 \geq \theta_1 \geq \cdots \geq \theta_r > 1/2 - 8t_0/9 \geq \theta_{r+1} \geq \cdots \geq \theta_{r+r_1};$$

$$(5.2) \quad \{\theta_0, \theta_1, \dots, \theta_5\} \notin \mathbf{D}(6) \text{ if } r = 5, \text{ and } \{\theta_0, \theta_1, \dots, \theta_7\} \notin \mathbf{D}(6) \text{ if } r = 7;$$

$$(5.3) \quad \theta'_0 + \theta_1 + \cdots + \theta_{r+r_1} = 1,$$

we shall prove that  $\{\theta_j\}' \in E(\theta)$ .

We record (2.2) here : if  $r_1 = 0$ ,

$$\theta'_0 = \theta_0 \geq \theta_1.$$

Let  $k_0$  be the number such that

$$\sum_{1 \leq j \leq k_0 - 1} \theta_j \leq t_0 \tag{5.4}$$

and

$$\sum_{1 \leq j \leq k_0} \theta_j > t_0. \tag{5.5}$$

By (5.1),  $\theta_1 < t_0$ , then we have  $k_0 \geq 2$ .

If  $r + r_1 > k_0 > r$ , then  $\theta_{k_0} < 1/2 - 8t_0/9$ . In Lemma 4.1, take  $\theta' = \theta_{k_0}$ ,  $\theta'' = \theta_{r+r_1}$ , then  $\{\theta_j\}' \in E(\theta)$ . If  $k_0 = r + r_1$ , let

$$a_1 = \sum_{1 \leq j \leq k_0 - 1} \theta_j \leq t_0,$$

and

$$a_2 = \theta_{k_0} \leq 1/2 - 8t_0/9 < t_0$$

then  $\{\theta_j\}' \in E(\theta)$  by (3.14). Finally, we consider  $2 \leq k_0 \leq r$ .

**Lemma 5.1.** Suppose  $r_1 = 0$ . If  $r \leq 2(k_0 - 1)$ , or  $r \geq k_0 + 4$ , then  $\{\theta_j\}' \in E(\theta)$ .

**Proof.** If  $r \leq 2(k_0 - 1)$ , let

$$a_1 = \sum_{1 \leq j \leq k_0 - 1} \theta_j \leq t_0,$$

and

$$a_2 = \sum_{k_0 \leq j \leq r} \theta_j \leq a_1 \leq t_0.$$

Thus  $\{\theta_j\}' \in E(\theta)$  by (3.14).

If  $r \geq k_0 + 4$ , let

$$\begin{aligned} a_1 &= \sum_{k_0 + 1 \leq j \leq k_0 + 4} \theta_j > 4 \left( \frac{1}{2} - \frac{8t_0}{9} \right) > \frac{8t_0}{9}, \\ a_2 &= \sum_{1 \leq j \leq k_0} \theta_j > t_0 \end{aligned}$$

and  $\sigma = 1 - a_1 - a_2$ . Thus  $\{\theta_j\}' \in E(\theta)$  by (3.15) since  $\sigma = 1 - \theta_1 - \dots - \theta_r \geq \theta_1 > t_0/5$ . The Lemma is proved.

To prove Theorem 4, we now discuss following cases :

**Case 1.**  $k_0 \geq 3$ .

By (5.1) and (5.4), we have

$$\theta_3 \leq \theta_2 \leq \frac{1}{2}(\theta_1 + \theta_2) \leq \frac{1}{2}t_0.$$

If  $r_1 > 0$ , take  $\theta' = \theta_3$  and  $\theta'' = \theta_{r+r_1}$  in Lemma 4.1, we have that  $\{\theta_j\}' \in E(\theta)$ .

Now may suppose that

$$r_1 = 0.$$

By Lemma 5.1, we also suppose

$$2k_0 - 1 \leq r \leq k_0 + 3. \tag{5.6}$$

i.e.  $3 \leq k_0 \leq 4$ .

When  $k_0 = 4$ , by (5.6) we have  $r = 7$ . Since  $\{\theta_j\} \notin D(8)$ , we have

$$\theta_1 + \theta_2 + \theta_3 > \frac{6}{7}t_0.$$

Let  $a_1 = \theta_1 + \theta_2 + \theta_3 > 6t_0/7$ ,  $a_2 = \theta_4 + \theta_5 + \theta_6 + \theta_7 > 8t_0/9 > 6t_0/7$ , and  $\sigma = \theta_0 > 1/8 > t_0/4$ , thus  $\{\theta_j\}' \in E(\theta)$  by (3.16). When  $k_0 = 3$ , by (5.6),  $r = 5$  or  $6$ . If  $r = 5$ , by  $\{\theta_j\} \notin \mathcal{D}(6)$ , we have

$$\theta_1 > \frac{2}{5}t_0.$$

Let  $a_1 = \theta_3 + \theta_4 + \theta_5$ , when  $\theta_3 + \theta_4 + \theta_5 \geq t_0$ , let  $a_2 = \theta_2 > t_0/3$  and  $a_2 = \theta_1 > 2t_0/5$ , then  $\{\theta_j\}' \in E(\theta)$  by (3.20). When  $\theta_3 + \theta_4 + \theta_5 < t_0$ , let  $a_2 = \theta_1 + \theta_2$ , by (5.1) we have

$$\theta_1 + \theta_2 \leq \frac{2}{3}(\theta_0 + \theta_1 + \theta_2) \leq \frac{2}{3}(1 - \theta_3 - \theta_4 - \theta_5) < t_0.$$

Thus  $\{\theta_j\}' \in E(\theta)$  by (3.14).

When  $k_0 = 3$ ,  $r = 6$  and  $r_1 = 0$ , we discuss following cases :

**Case 1.1.**  $\theta_1 + \theta_3 + \theta_5 \leq t_0$  or  $\theta_2 + \theta_4 + \theta_6 \geq t_0$ .

Let  $a_1 = \theta_1 + \theta_3 + \theta_5$  and  $a_2 = \theta_2 + \theta_4 + \theta_6$ , then  $\{\theta_j\}' \in E(\theta)$  by (3.14) or (3.17).

**Case 1.2.**  $\theta_1 + \theta_3 + \theta_5 > t_0 > \theta_2 + \theta_4 + \theta_6$ .

If  $\theta_1 - \theta_6 < 1/2 - 8t_0/9$ , take  $\sigma = 1 - a_1 - \dots - a_6 \geq a_1 > 1/2 - 8t_0/9$  in Lemma 4.2, then  $\{\theta_j\}' \in E(\theta)$ . We consider that

$$\theta_1 - \theta_6 \geq 1/2 - 8t_0/9.$$

By (4.1),  $\theta_6 \geq 1/2 - 8t_0/9$ , therefore

$$\theta_1 \geq 1 - 16t_0/9$$

and  $\theta_0 + \theta_1 \geq 2 - 32t_0/9 > 8t_0/9$ . Let  $a_1 = \theta_0 + \theta_1 > 8t_0/9$ ,  $a_2 = \theta_2 + \theta_3 + \theta_4 + \theta_5 > 8t_0/9$  and  $\sigma = \theta_6$ , thus  $\{\theta_j\}' \in E(\theta)$  by (3.15).

**Case 2.**  $k_0 = 2$ .

By (5.4), we have  $\theta_1 > t_0/2$ . If  $r_1 = 0$ , then  $\theta'_0 = \theta_0 > t_0/2$  and  $\{\theta_j\} \in E(\theta)$  by (3.19). Now may suppose that  $r_1 > 0$  and  $\theta_0 \leq t_0/2$ .

We discuss the following two cases :

**Case 2.1.**  $\theta_3 > t_0/2$ .

Let  $a_1 = \theta_2 + \theta_3 \geq t_0$ . By (5.1) and (5.3), we have that

$$\theta_2 + \theta_3 \leq 1 - \theta'_0 - \theta_1 - \theta_4 - \dots - \theta_{r+r_1} \leq 1 - \theta_2 - \theta_3;$$

then

$$\theta_2 + \theta_3 \leq 1/2. \quad (5.7)$$

Let  $\sigma = \theta_{r+r_1}$ , then  $\{\theta_j\}' \in E(\theta)$  by (3.18) (since  $r_1 > 0$  implies  $\theta_{r+r_1} < 1/2 - 8t_0/9$ ).

**Case 2.2.**  $\theta_3 \leq t_2/2$ .

We have

$$\theta_1 + \theta_0 + \theta_3 + \dots + \theta_{r+r_1} = 1 - \theta_2 > 1 - t_0$$

and  $\theta_1 + \theta_0 + \theta_3 + \dots + \theta_{r+r_1-1} > t_0$  (since  $\theta_{r+r_1} < 1/2 - 8t_0/9 < 1 - 2t_0$ ).

We can find a number  $j = 0$  or  $3$  or  $j \leq r + r_1 - 1$  with

$$\theta_1 + \theta_0 + \theta_3 + \dots + \theta_{k-1} < t_0$$

and

$$\theta_1 + \theta_0 + \theta_3 + \dots + \theta_k \geq t_0.$$

In Lemma 3.1, take  $\theta' = \theta_0$  and  $\theta'' = \theta_{r+r_1}$ , then we have that  $\{\theta_j\} \in E(\theta)$ .

The proof of Theorem 1 is complete.

### § 6. Proof of Theorem 3.

In this section we discuss that  $\theta = 11/20 + \varepsilon$ .

Let

$$S'_k = \{d = d_0 d_1 \dots d_{k-1} : d \in S_k, d_0 \geq \dots \geq d_{k-1}\}. \quad (6.1)$$

Take  $H_k = S'_k$  and  $c_H = k!$ , then  $(A_1)$  holds.

Let

$$D(6) = \left\{ d : d \in S'_6, d = d_0 \dots d_5, x^{2t_0/5} \geq d_0 \geq \dots \geq d_5 \geq x^{\frac{1}{2} - \frac{8t_0}{9}}, d_3 d_4 d_5 \geq x^{t_0} \right\} \quad (6.2)$$

and

$$D(8) = \left\{ d : d \in S'_k, d = d_0 \cdots d_7, x^{\frac{2t_0}{7}} \geq d_0 \geq \cdots \geq d_7 \geq x^{\frac{1}{2} - \frac{8t_0}{7}} \right\} \quad (6.3)$$

In [8], we proved the following lemma :

**Lemma 6.1.** *Let  $H'_6 = D(6)$ ,  $H'_8 = D(8)$  and  $H'_k = \emptyset$  for  $k \neq 6$  and  $8$ , then condition  $(A_2)$  holds, i.e. (1.10) holds for  $H'_k = H_k \setminus H'_k$ .*

**Proof of Theorem 3.** By (1.12), we have that

$$R(y) = (5!) \sum_{d \in D(6)} 1 + (7!) \sum_{d \in D(8)} 1. \quad (6.4)$$

We now estimate  $R(y)$ . By (4.2),  $d \in D(6)$  implies  $d = d_0 \cdots d_5$  with

$$p(d_j) \geq x^{\frac{1}{2} - \frac{8t_0}{7}} > (d_j)^{\frac{1}{2}}, 0 \leq j \leq 5. \quad (6.5)$$

Then all of  $d_j$ 's be primes. Let

$$\begin{aligned} D_1(6) &= \{(p_0, \dots, p_4) : p_j \text{ primes for } 0 \leq j \leq 4, x^{\frac{2t_0}{5}} \geq p_0 \geq \dots \\ &\geq p_4 \geq x^{\frac{1}{2}(1 - \frac{8t_0}{5}), p_3 \geq x^{\frac{t_0}{3}}\}. \\ |^{(1)} &= |^{(1)}(p_0, \dots, p_4) = \left[ \frac{x-y}{p_0 \cdots p_4}, \frac{x}{p_0 \cdots p_4} \right), \end{aligned}$$

and

$$\Delta_5 = \Delta_5^1 \cup \Delta_5^2,$$

where

$$\Delta_5^1 = \left\{ (t^0, \dots, t^4) : \frac{2t_0}{5} \geq t^0 \geq \dots \geq t^4 \geq \frac{t_0}{3} \right\} \quad (6.6)$$

and

$$\Delta_5^2 = \left\{ (t^0, \dots, t^4) : \frac{2t_0}{5} \geq t^0 \geq \dots \geq t^3 \geq \frac{t_0}{3} \geq t^4 \geq \frac{1}{2} \left( 1 - \frac{8t_0}{5} \right) \right\}. \quad (6.7)$$

Then we have that

$$\sum_{d \in D(6)} 1 = \sum_{(p_0, \dots, p_4) \in D_1(6)} 1 \leq \sum_{p_3 \in |^{(1)}} 1 \leq \sum_{(p_0, \dots, p_4) \in D_1(6)} \frac{2y}{p_0 \cdots p_4 \log \frac{y}{p_0 \cdots p_4}}$$

$$\leq \frac{(2 + \varepsilon)y}{\log x} \int \dots \int_{\Delta_5} \frac{dt^0 \dots dt^4}{t^0 \dots t^4 (1 - t^0 - \dots - t^4)}. \quad (6.8)$$

By (6.6) and (6.7) we have that

$$\begin{aligned} \int \dots \int_{\Delta_5^1} \frac{dt^0 \dots dt^4}{t^0 \dots t^4 (1 - t^0 - \dots - t^4)} &\leq \frac{1}{5!} \int_{\frac{2t_0}{3}}^{\frac{2t_0}{3}} dt^0 \int_{\frac{t_0}{3}}^{\frac{2t_0}{3}} dt^1 \dots \int_{\frac{t_0}{3}}^{\frac{2t_0}{3}} \frac{dt^0 \dots dt^4}{t^0 \dots t^4 (1 - t^0 - \dots - t^4)} \\ &\leq \frac{10}{5!} \left( \log \frac{1.8}{1.5} \right)^5 < \frac{0.002015}{5!}, \end{aligned}$$

and

$$\begin{aligned} &\int \dots \int_{\Delta_5^2} \frac{dt^0 \dots dt^4}{t^0 \dots t^4 (1 - t^0 - \dots - t^4)} \leq \\ &\leq \frac{1}{4!} \int_{\frac{t_0}{3}}^{\frac{2t_0}{3}} dt^0 \int_{\frac{t_0}{3}}^{\frac{2t_0}{3}} dt^1 \dots \int_{\frac{1}{2}(1 - \frac{8t_0}{3})}^{\frac{t_0}{3}} \frac{dt^0 \dots dt^4}{t^0 \dots t^4 (1 - t^0 - \dots - t^4)} \\ &\leq \frac{1}{(4!)(1 - 0.72 - 0.15)} \left( \log \frac{0.18}{0.15} \right)^4 \left( \log \frac{0.15}{0.14} \right) < \frac{0.002933}{5!} \end{aligned}$$

Thus

$$\sum_{d \in D(6)} 1 < \frac{0.009895}{5!} \left( \frac{y}{\log x} \right). \quad (6.9)$$

Let

$$D_1(8) = \left\{ (p_0, \dots, p_6) : p_j \text{ primes for } 0 \leq j \leq 6, x^{\frac{2t_0}{7}} \geq p_0 \geq \dots \geq p_6 \geq x^{\frac{1}{2}(1 - \frac{12t_0}{7})} \right\},$$

$$|^{(2)} = |^{(2)}(p_0, \dots, p_6) = \left[ \frac{x - y}{p_0 \dots p_6}, \frac{x}{p_0 \dots p_6} \right],$$

and

$$\Delta_7 = \Delta_7^1 \cup \Delta_7^2,$$

where

$$\Delta_7^1 = \left\{ (t^0, \dots, t^6) : \frac{2t_0}{7} \geq t^0 \geq \dots \geq t^6 \geq \frac{t_0}{4} \right\} \quad (6.10)$$

and

$$\Delta_7^2 = \left\{ (t^0, \dots, t^6) : \frac{2t_0}{7} \geq t^0 \geq \dots \geq t^5 \geq \frac{t_0}{3} \geq t^6 \geq \frac{1}{2} \left( 1 - \frac{12t_0}{7} \right) \right\}. \quad (6.11)$$

Then we have that

$$\begin{aligned} \sum_{d \in \mathbf{D}(8)} 1 &= \sum_{(p_0, \dots, p_6) \in \mathbf{D}_1(8)} \sum_{p_T \in \mathcal{I}^{(2)}} 1 \leq \sum_{(p_0, \dots, p_6) \in \mathbf{D}_1(8)} \frac{2y}{p_0 \cdots p_6 \log \frac{y}{p_0 \cdots p_6}} \\ &\leq \frac{(2 + \varepsilon)y}{\log x} \int \cdots \int_{\Delta_T} \frac{dt^0 \cdots dt^6}{t^0 \cdots t^6 (1 - t^0 - \cdots - t^6)}. \end{aligned}$$

By (6.6) and (6.7) we have that

$$\begin{aligned} \int \cdots \int_{\Delta_T^1} \frac{dt^0 \cdots dt^6}{t^0 \cdots t^6 (1 - t^0 - \cdots - t^6)} &\leq \frac{1}{7!} \int_{\frac{1}{4}}^{\frac{21}{7}} dt^0 \int_{\frac{1}{4}}^{\frac{21}{7}} dt^1 \cdots \int_{\frac{1}{4}}^{\frac{21}{7}} \frac{dt^6}{t^0 \cdots t^6 (1 - t^0 - \cdots - t^6)} \\ &\leq \frac{10}{7!} \left( \log \frac{0.9}{0.45} \right)^7 < \frac{7.6(10^{-6})}{7!}, \end{aligned}$$

and

$$\begin{aligned} &\int \cdots \int_{\Delta_T^2} \frac{dt^0 \cdots dt^6}{t^0 \cdots t^6 (1 - t^0 - \cdots - t^6)} \leq \\ &\leq \frac{1}{6!} \int_{\frac{1}{4}}^{\frac{21}{7}} dt^0 \int_{\frac{1}{4}}^{\frac{21}{7}} dt^1 \cdots \int_{\frac{1}{2}(1 - \frac{121}{7})}^{\frac{1}{4}} \frac{dt^6}{t^0 \cdots t^6 (1 - t^0 - \cdots - t^6)} \\ &\leq \frac{1}{(6!) \left(1 - \frac{12(0.45)}{7} - \frac{0.45}{4}\right)} \left( \log \frac{0.9}{0.45} \right)^6 \left( \log \frac{0.45}{0.1} \right) < \frac{4.027(10^{-5})}{7!}. \end{aligned}$$

Thus

$$\sum_{d \in \mathbf{D}(8)} 1 < \frac{9.58(10)^{-5}}{7!} \left( \frac{y}{\log x} \right). \quad (6.12)$$

By (6.4), (6.9) and (6.12) we have that

$$R(y) < \frac{0.01 y}{\log x}.$$

In Theorem 2 take  $e_1 = e_2 = 0$  and  $e'_1 = e'_2 = 0.01$ , Theorem 3 follows.

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