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The lexicographic degree of the first two-bridge knots

Erwan Brugallé, Pierre-Vincent Koseleff, Daniel Pecker

September 13, 2018

Abstract

We study the degree of polynomial representations of knots. We give the lexicographic degree of all two-bridge knots with 11 or fewer crossings. First, we estimate the total degree of a lexicographic parametrisation of such a knot. This allows us to transform this problem into a study of real algebraic trigonal plane curves, and in particular to use the braid theoretical method developed by Orevkov.

MSC2000: 14H50, 57M25, 11A55, 14P99

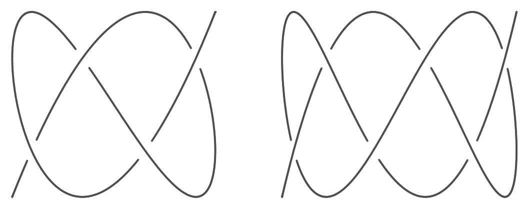
Keywords: Real pseudoholomorphic curves, polynomial knots, two-bridge knots, Chebyshev curves

Contents

1 Introduction

A polynomial parametrisation of a knot K in $\mathbf{S^3}$ is a polynomial map $\gamma : \mathbf{R} \to \mathbf{R}^3$ whose closure of the image in $\mathbf{S^3}$ is isotopic to K. Every knot admits a polynomial parametrisation, see [Sh, Va]. In this paper we are interested in determining the *lexicographic degree* of a knot $K \subset \mathbf{S^3}$, i.e. the minimal degree for the lexicographic order of a polynomial parametrisation of K.

The unknot has lexicographic degree $(-\infty, -\infty, 1)$, and it is easy to see that the lexicographic degree of any other knot is (a, b, c) with $3 \le a < b < c$. *Two-bridge* knots are precisely those with lexicographic degree $(3, b, c)$, see [KP2]; they have a xy-projection which is a trigonal curve. See Figure 1 for two examples of trigonal polynomial parametrisations of a long knot.

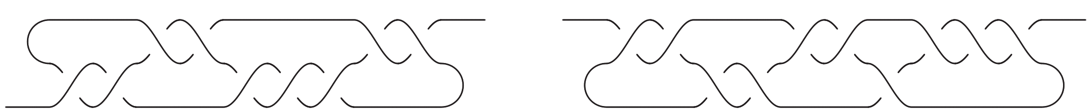


$\deg 4_1 = (3,5,7)$ $\qquad$ $\deg 5_1 = (3,7,8)$

Figure 1: Trigonal polynomial diagrams of the figure-eight knot 4_1 and the torus knot 5_1

Two-bridge knots are an important family of knots. The first 26 knots (except 8_5) are two-bridge knots. Moreover these knots are classified by their Schubert fractions, which can be easily computed from any trigonal projection, see Section 2.1.

One might expect that the lexicographic degree of a knot K is obtained for a minimal-crossing diagram of this knot. This is not true. The diagram on the left of Figure 2 is a minimal crossing diagram of the knot 9_{15}. On the right of the figure is a 10-crossing diagram of smaller degree of the same knot. This is why it is necessary to consider all the diagrams



A 9-crossing diagram of degree $\ge (3, 13, 14)$ $\qquad$ A 10-crossing diagram of degree $(3, 11, 16)$

Figure 2: Two diagrams of 9_{15}

of two-bridge knots. The enumeration of all possible diagrams of a given two-bridge knot can be efficiently done using Conway's notation.

In this paper, we show:

Theorem. *The lexicographic degree of all 186 two-bridge knots with crossing number at most $N \le 11$ is $(3, b, 3N - b)$, where the values of b are listed in Table 5, p. 28.*

We prove this result in two steps.

Proposition 2.9 *The lexicographic degree* $(3, b, c)$ *of a knot with crossing number* $N \leq 11$ *satisfies* $b + c = 3N$.

Proposition 2.9 also holds for all N when $b \leq N + 3$ or $b = \lfloor \frac{3N-1}{2} \rfloor$. We prove in Theorem 2.5 that $b + c \geq 3N$ for any polynomial parametrisation of degree $(3, b, c)$ of a knot with crossing number N. Furthermore, every two-bridge knot of crossing number N admits a parametrisation of degree $(3, b, c)$ with $b+c = 3N$, see [KP2]. We do not know if Proposition 2.9 holds for all crossing numbers $N \geq 12$.

Proposition 2.9 allows us to reduce the determination of the lexicographic degree of a two-bridge knot to the study of plane curves. For knots with 11 crossings or fewer, it is enough to determine the smallest integer b such that a plane projection admits a polynomial parametrisation of degree $(3, b)$. This reduction to plane curves enlarges the set of tools at our disposal; in particular we make an important use of Orevkov's braid theoretical approach in the study of pseudoholomorphic curves.

Hence the second step in the proof of our theorem is to focus on parametrisations of plane projections. We introduce the T-reduction in Section 3.3, that corresponds to the projection of the Lagrange isotopy on trigonal diagrams. The T-reduction allows us to remove a triangle of crossings from a diagram, and therefore to obtain an upper bound for degrees we are looking for. On the other hand, we introduce the T-augmentation in Section 3.4 that allows us to add a triangle of crossings to a given diagram D. From a polynomial parametrisation corresponding to D we deduce a parametrisation for the new diagram.

We propose an algorithm to find the lexicographic degrees of the first 186 two-bridge knots with 11 crossings or fewer. As a byproduct of our computations, we also exhibit in Table 6 the 16 two-bridge knots with 11 crossings or fewer for which the lexicographic degree is smaller than the degree of their minimal-crossing diagrams.

The paper is organised as follows. In Section 2.1 we recall Conway's notation for trigonal diagrams of two-bridge knots. Then we prove the inequality $b + c \geq 3N$ in Section 2.2 and deduce Proposition 2.9. In Section 3, we consider plane trigonal curves and we first obtain a lower bound for the lexicographic degree of a trigonal polynomial embedding in Proposition 3.2. We obtain another bound for pseudoholomorphic curves and therefore for polynomial embeddings in Proposition 3.7. In Section 4, we obtain the lexicographic degrees of the first 186 two-bridge knots with 11 crossings or fewer.

2 A lower bound for the total degree of two-bridge knots

2.1 Trigonal diagrams of two-bridge knots

A two-bridge knot admits a diagram in *Conway's open form* (or trigonal form). This diagram, denoted by $C(m_1, m_2, \ldots, m_k)$ where $m_i \in \mathbf{Z}$, is explained by Figure 3 (see [Co], [Mu, p. 187]). The number of twists is denoted by the integer $|m_i|$, and the sign of m_i is

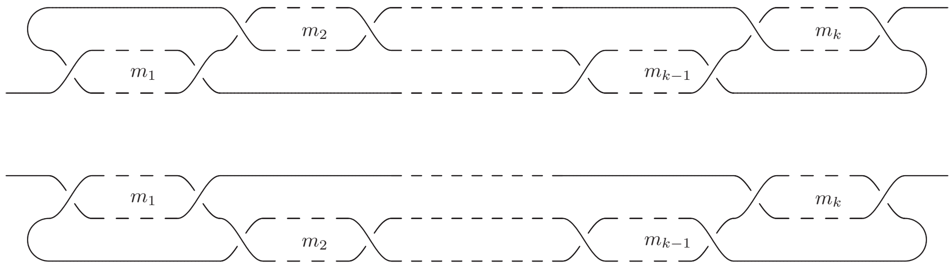
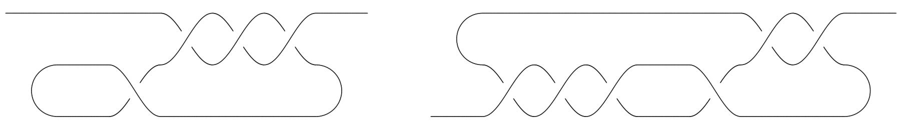


Figure 3: Conway's form for two-bridge knots (or links)

defined as follows: if i is odd, then the right twist is positive, if i is even, then the right twist is negative. In Figure 3 the integers m_i are all positive. Figure 4 shows the examples $C(0,1,3)$, $C(3,0,-1,-2)$.



$C(0,1,3)$ $C(3,0,-1,-2)$

Figure 4: Examples of trigonal diagrams

The two-bridge knots (or links) are classified by their Schubert fractions

$$\frac{\alpha}{\beta} = m_1 + \cfrac{1}{m_2 + \cfrac{1}{\cdots + \cfrac{1}{m_k}}} = [m_1, \ldots, m_k], \quad \alpha \geq 0,\ (\alpha, \beta) = 1.$$

Given $[m_1, \ldots, m_k] = \dfrac{\alpha}{\beta}$ and $[m'_1, \ldots, m'_l] = \dfrac{\alpha'}{\beta'}$, the diagrams $C(m_1, m_2, \ldots, m_k)$ and $C(m'_1, m'_2, \ldots, m'_l)$ correspond to isotopic knots (or links) if and only if $\alpha = \alpha'$ and $\beta' \equiv \beta^{\pm 1} \pmod{\alpha}$, see [Mu, Theorem 9.3.3].

Every positive fraction α/β admits a continued fraction expansion $[m_1, \ldots, m_k]$ where all the m_i are positive. Therefore every two-bridge knot K admits a diagram in *Conway's normal form*, that is an alternating diagram of the form $C(m_1, m_2, \ldots m_k)$, where the m_i are all positive or all negative.

It is classical that one can transform any trigonal diagram of a two-bridge knot into Conway's normal form using the Lagrange isotopies, see [Cr, p. 204].

Definition 2.1 *Let $C(u, m, -n, -v)$ be a trigonal diagram, where m, n are integers, and u, v are (possibly empty) sequences of integers, see Figure 5. The Lagrange isotopy on D is*

$$C(u, m, -n, -v) \to C(u, m - \varepsilon, \varepsilon, n - \varepsilon, v),\ \varepsilon = \pm 1, \tag{1}$$

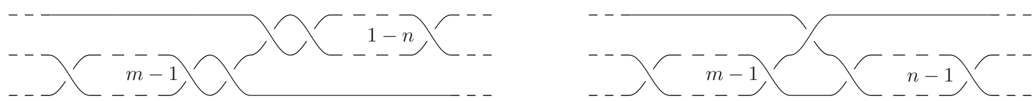


Figure 5: Lagrange isotopy: $C(u, m, -n, -v) \to C(u, m-1, 1, n-1, v)$

If $D = C(m_1, \dots, m_k)$ is not in Conway's normal form, then it may happen that $m_1 = 0$ or $m_k = 0$. In this case, the diagram $D' = C(m_3, \dots, m_k)$ or $D' = C(m_1, \dots, m_{k-2})$ respectively, is the *reduced diagram* of D. Since the diagram $C(m_1, \dots, m_i, 0, 0, m_{i+1}, m_{i+2}, \dots, m_k)$ is identical to $C(m_1, \dots, m_k)$, we can assume that if $m_i = 0$ then $m_{i-1}m_{i+1} \neq 0$.

Given a finite integer sequence $(m_1, \dots, m_k)$, we say that there is a *sign change* between m_i and m_{i+1} if $m_i m_{i+1} < 0$ or if $m_i = 0$ and $m_{i-1}m_{i+1} < 0$.

Proposition 2.2 *Let $C(m_1, \dots, m_k)$ be a diagram of a knot with crossing number N. Let $N_0 = \sum_{i=1}^{k} |m_i|$ be the number of crossings, and σ be the number of sign changes in the sequence $(m_1, \dots, m_k)$. Then we have*

$$N \leq N_0 - \sigma.$$

Proof. If $\sigma = 0$, then the inequality means that the crossing number of a knot is not greater than the number of crossings of a diagram of this knot. Consequently, we can suppose $\sigma \geq 1$. Let us prove the result by induction on $N_0 = \sum_{i=1}^{k} |m_i|$. We have to consider two cases.

First, let us suppose that the diagram is of the form $C(u, m, -n, -v)$, $m, n > 0$ Then by a Lagrange isotopy we see that $C(u, m-1, 1, n-1, v)$ is another diagram of K. In this new diagram, the number of crossings and the number of sign changes are both diminished by 1. Therefore we obtain by induction:

$$N \leq (N_0 - 1) - (\sigma - 1) = N_0 - \sigma.$$

Next, let us consider a diagram of the form $C(u, m, 0, -n, v)$, $mn > 0$. In this case we consider the new diagram $C(u, m-n, v)$. If σ' is the number of sign changes of this new diagram, then a case by case inspection shows that $\sigma' \geq \sigma - 2$. As the number of crossings is diminished by at least 2, we obtain by induction:

$$N \leq (N_0 - 2) - (\sigma - 2) = N_0 - \sigma,$$

which concludes the proof. □

The proof of Proposition 2.2 also implies the following lemma.

Lemma 2.3 *In the notation of Proposition 2.2, we have:*

1. *If $\sigma = 0$, then $N < N_0$ if and only if $m_1 \cdot m_k = 0$.*
2. *If $\sigma = 1$, then we have $N < N_0 - 1$ if and only if one of the following situations occurs:*
   - *$m_1 = 0$ or $m_k = 0$,*
   - *there exists i such that $m_i = 0$ and $m_{i-1}m_{i+1} < 0$,*
   - *$|m_1| = 1$ and $m_1 m_2 < 0$ or $|m_k| = 1$ and $m_{k-1}m_k < 0$.*

Let D be a long knot diagram, and $\gamma : \mathbf{R} \to \mathbf{R}^3$ be a parametrisation of D whose crossing points corresponds to the parameters $t_1 < \cdots < t_{2m}$. Recall that the *Gauss sequence* of D is the sequence $g_1, \ldots, g_{2m}$ where $g_i = 1$ if t_i corresponds to an overpass, and $g_i = -1$ otherwise.

Proposition 2.4 *Let $C(m_1, \ldots, m_k)$, $m_i \neq 0$, be a trigonal diagram of a knot K, and $N_0 = \sum |m_i|$. Let s be the number of sign changes in the Gauss sequence of the diagram, σ be the number of sign changes in the sequence $(m_1, \ldots, m_k)$, and σ_2 be the number of consecutive sign changes in the sequence $(m_1, \ldots, m_k)$. Then, we have*

$$s = 2N_0 - 3\sigma + 2\sigma_2 - 1.$$

Proof. We proceed by induction on (σ_2, σ). If $\sigma = 0$ then $\sigma_2 = 0$ and the diagram of K is alternating. In this case we have $s = 2\sum |m_i| - 1 = 2N_0 - 1$.

If $\sigma_2 = 0$, we may assume that $m_1 > 0$. Let j be the first index just that $m_i < 0$. Then $j = k$ or $m_{j+1} < 0$, because $\sigma_2 = 0$. Let us consider the knot K' defined by $K' = C(m_1, \ldots, m_{j-1}, -m_j, -m_{j+1}, \ldots, -m_k)$. We see that the number of sign changes in the Conway sequence of K' is $\sigma' = \sigma - 1$, and that we still have $\sigma_2' = 0$. By induction we get $s' + 3\sigma' = 2\sum |m_i| - 1$. Since we have $s' = s + 3$, this completes the proof when $\sigma_2 = 0$.

Now, let us suppose that $\sigma_2 > 0$ and consider the first index j such that $m_{j-1}\, m_j < 0$ and $m_j\, m_{j+1} < 0$. Consider K' defined by $K' = C(m_1, \ldots, m_{j-1}, -m_j, -m_{j+1}, \ldots, -m_k)$. We see that the number of sign changes in the Conway sequence of K' is $\sigma' = \sigma - 1$ and also $\sigma_2' = \sigma_2 - 1$. By induction we get $s' + 3\sigma' - 2\sigma_2' = 2\sum |m_i| - 1$. Since we have $s' = s + 1$, this concludes the proof. ☐

2.2 Total degree of two-bridge knots

The next theorem provides a lower bound on the total degree of every trigonal knot diagram. It generalises [BKP2, Theorem 4.3], which proves that the lexicographic degree of a knot of crossing number N is at least $(3, N + 1, 2N - 1)$.

Theorem 2.5 *Let $\gamma : \mathbf{R} \to \mathbf{R}^3$ be a polynomial parametrisation of degree $(3, b, c)$ of a knot of crossing number N. Then we have*

$$b + c \geq 3N.$$

Proof. We shall denote our polynomial knot $\gamma(t) = (x(t), y(t), z(t))$. Without loss of generality, we may assume that b is not divisible by 3. Let $C(m_1, m_2, \ldots, m_k)$ be the corresponding xy-diagram. To simplify the exposition, we shall first suppose that $m_i \neq 0$ for $i = 2, \ldots, k-1$.

By the genus formula, the plane curve C parametrised by $C(t) = (x(t), y(t))$ has exactly $b-1$ nodes in $\mathbf{C}^2$. Let $N_0 = \sum_{i=1}^{k} |m_i|$ be the number of real crossings of C (i.e. real nodes of C which are the intersection of two real branches of C), and let $\delta = b - 1 - N_0$ be the number of other nodes of C.

The real crossings are ordered by increasing abscissae. A real crossing is called *special* if its Conway sign (for the trigonal diagram) is different from the Conway sign of the preceding crossing.

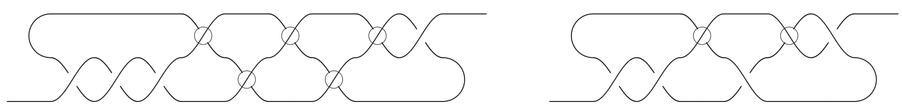


Figure 6: Special crossings of $C(3, -1, 1, -1, 1, -2)$ and $C(2, -1, -1, 2)$

The number of special crossings, denoted by σ, is the number of sign changes in the Conway sequence $(m_1, m_2, \ldots, m_k)$. By Proposition 2.2, we have $N \leq N_0 - \sigma$. Let $D(x)$ be the monic polynomial of degree $\sigma + \delta$, whose roots are the abscissas of the σ special crossings and the abscissas of the δ nodes that are not crossings. The polynomial $D(x)$ is real.

Let $\mathcal{V}$ be the vector space of polynomials $V(x, y) \in \mathbf{C}[x, y]$ such that

$$\deg\Big(V(x(t), y(t))\Big) \leq 2b - 4.$$

The monomials $x^\alpha y^\beta$ such that $3\alpha + b\beta \leq 2b - 4$ form a basis of $\mathcal{V}$, and it is not difficult to see that the number of these monomials is $b - 1$.

Let $\mathcal{F}$ be the vector space of complex functions defined on the set of nodes of C. The restriction induces a linear mapping $\iota : \mathcal{V} \to \mathcal{F}$ between spaces of the same dimension. If $U(x, y)$ is in the kernel of ι, then we have $U(x(t), y(t)) = 0$ for $2b - 2$ values of t. Since $\deg U(x(t), y(t)) \leq 2b - 4$, we see that $U(x, y) = 0$. Hence ι is an injective mapping and then it is an isomorphism.

For each non-special crossing with parameters (t_i, s_i), let h_i be a real number in the open interval $(z(t_i), z(s_i))$. Since ι is an isomorphism, there exists a unique polynomial $V(x, y)$ such that $V(x_i, y_i) = h_i D(x_i)$ for each non-special crossing (x_i, y_i), and $V(x, y) = 0$ for all other nodes of C. By uniqueness, we see that $V(x, y)$ is a real polynomial. Let us consider the rational function $h(t)$ defined by

$$h(t) = \frac{V(x(t), y(t))}{D(x(t))}.$$

Each parameter t of a special crossing (or *special parameter*) is a zero of the numerator and a simple zero of the denominator. Consequently, the function $h(t)$ is defined for all crossing parameters. Up to perturbing $z(t)$ by a constant if necessary, we can assume that $z(t_i) \neq h(t_i)$ for all crossing parameters t_i.

Now, we shall prove that the polynomial equation

$$z(t)D(x(t)) - V(x(t), y(t)) = 0 \tag{2}$$

has at least $2b-3$ distinct roots.

First, the two parameters t, s of a node such that $V(x,y) = D(x) = 0$ are roots of this equation. The number of such roots is $2(\sigma+\delta)$. The other roots are the zeroes of the rational function $\Delta(t) = z(t) - h(t)$.

An interval $[r,s] \subset \mathbf{R}$ is called *minimal* if r, s are two non-special node parameters, and if $s > r$ is minimal for this property. In other words, there is no non-special node parameter τ in (r,s). The number of minimal intervals is exactly $2(N_0 - \sigma) - 1$.

We claim that every minimal interval contains a zero of $\Delta(t)$ that is not a node parameter. Then the number of distinct roots of Equation (2) must be at least

$$2(N_0 - \sigma) - 1 + 2(\sigma + \delta) = 2(N_0 + \delta) - 1 = 2b - 3,$$

and the degree of the equation must be at least $2b-3$.

Since $\deg V(x(t), y(t)) \leq 2b - 4$, we deduce that

$$\deg(z(t)D(x(t))) = c + 3(\delta + \sigma) \geq 2b - 3,$$

and then $b + c \geq 3(b - 1 - \delta - \sigma) = 3(N_0 - \sigma) \geq 3N$, which conclude the proof in this case.

Let us prove our claim. To do so, we study the sign of the rational function $\Delta(t)$ on the minimal interval $[r,s]$. Let j be the number of special parameters contained in $[r,s]$, and let $t_0 = r$, $t_{j+1} = s$. If $j \neq 0$, then let $t_1 < t_2 < \ldots < t_j$ be the special parameters contained in $[r,s]$. The function $\Delta(t)$ is defined for each t_i, and we have $\Delta(t_i) \neq 0$. The poles occur for the parameters $\tau \in [r,s]$ such that $D(x(\tau)) = 0$ and $(x(\tau), y(\tau))$ is not a crossing, they are simple poles. Let $[t_h, t_{h+1}]$ be the interval where the function $x(t)$, $\tau \in [r,s]$ has a maximum. On this interval there is either one pole and no alternation in the Gauss sequence of the knot, or no pole and one alternation. Figures 7 and 8 shows the main cases, the interval $[t_h, t_{h+1}]$ corresponds to the rightmost sub-arc AC of the arc parametrised by $[r,s]$.

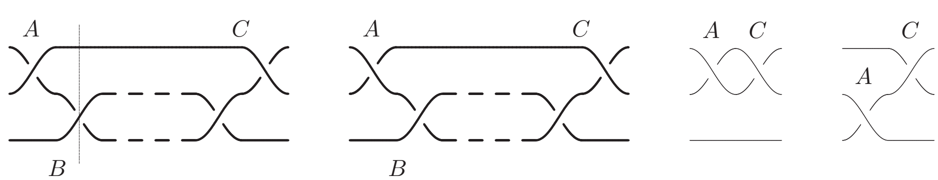


Figure 7: The rightmost sub-arc AC (ordinary cases)

On the other intervals $[t_i, t_{i+1}]$, $i \neq h$ there is either one pole and one alternation, or no pole and no alternation, see Figure 9. Consequently, we see that $\Delta(r)\Delta(s) < 0$ if and only

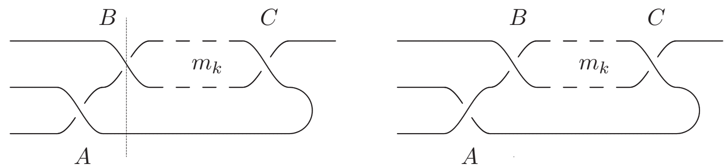


Figure 8: The rightmost sub-arc AC (exceptional case)

if the number of poles contained in $[r, s]$ is even. On the other hand, the number of sign changes in $[r, s]$ of the function $\Delta(t)$ is odd if and only if $\Delta(r)\Delta(s) < 0$. Consequently, whatever the sign of $\Delta(r)\Delta(s)$ may be, there must be at least one $u \in [r, s]$ which is not a pole, and where sign $\big(\Delta(t)\big)$ changes. Hence, u is a root of Equation (2), which proves the claim.

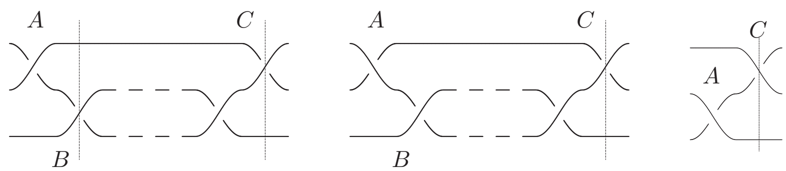


Figure 9: The other sub-arcs AC

In the general case, there may be some $m_i = 0$ in the diagram $C(m_1, m_2, \ldots, m_k)$, where $2 \le i \le k-1$. We shall inductively select some relevant crossings, and ignore the others.

If there is a subsequence of the form $(m, 0, -n), m \ge n > 0$, then we declare the last $2n$ crossings irrelevant, and we consider the new Conway sequence where $(m, 0, -n)$ has been changed to $m - n$. We iterate this selection (by elimination) until we obtain a diagram $C(m_1, \ldots, m_k)$ such that $m_i \neq 0$, for $i = 2, \ldots k-1$. Then, considering only the relevant crossings, we choose the special crossings. We also define $D(x) = V(x, y) = 0$ for the special crossings, the irrelevant crossings and the nodes that are not crossings. The rest of the proof is similar to the preceding one, except that the number of poles on each minimal interval $[r, s]$ may be increased by an even number, which does not change the sign of $\Delta(r)\Delta(s)$. □

In [KP2], it is proved that every two-bridge knot of crossing number N admits an explicit parametrisation of the form (T_3, T_b, C) where T_n is the Chebyshev polynomial of degree n defined by $T_n(\cos t) = \cos nt$, and $b + \deg C = 3N$. Moreover, the harmonic knot $\mathrm{H}(3, b, c)$: (T_3, T_b, T_c), where $b < c < 2b$, $b + c \equiv 0 \,(\mathrm{mod}\, 3)$ has crossing number $N = \frac{1}{3}(b + c)$, see [KP2, Corollary 6.6].

Combining with Theorem 2.5 we deduce the following.

Corollary 2.6 *The lexicographic degree* $(3, b, c)$ *of a two-bridge knot of crossing number* N *satisfies:*

$$3 < b < c < 2b\, b \not\equiv 0 \,(\mathrm{mod}\, 3),\ b + c \equiv 0 \,(\mathrm{mod}\, 3),\ b + c \geq 3N,$$
$$(3, N+1, 2N-1) \leq (3, b, c) \leq (3, \lfloor \tfrac{3N-1}{2} \rfloor, \lfloor \tfrac{3N}{2} \rfloor + 1).$$

Moreover, these inequalities are best possible.

Proof. The transformation $(x, y, z) \mapsto (x, y - \lambda x^u, z - \mu x^v y^w)$, where u, v, w are nonnegative integers and $\lambda, \mu \in \mathbf{R}$, does not change the nature of the knot. This ensures that $b \not\equiv 0 \,(\mathrm{mod}\, 3)$ and $b + c \equiv 0 \,(\mathrm{mod}\, 3)$. Next, it is proved in [KP2], that every two-bridge knot admits a polynomial parametrisation of lexicographic degree $(3, b, c)$, with $b + c = 3N$. This implies that $b \leq \lfloor \frac{3N-1}{2} \rfloor$. Furthermore if $b = \lfloor \frac{3N-1}{2} \rfloor$, then $c \leq 3N - \lfloor \frac{3N-1}{2} \rfloor = \lfloor \frac{3N}{2} \rfloor + 1$. If $\gamma : \mathbf{R} \to \mathbf{R}^3$ is a polynomial parametrisation of degree $(3, b, c)$ of a knot, then by forgetting the last coordinate we obtain a polynomial map $\mathbf{R} \to \mathbf{R}^2$ of degree $(3, b)$ with at least N crossings. The genus formula implies that $b \geq N + 1$. In the case $b = N + 1$, Theorem 2.5 implies that $c \geq 2N - 1$.

Let us show that these bounds are sharp. If $N \not\equiv -1 \,(\mathrm{mod}\, 3)$, then the harmonic knot $\mathrm{H}(3, N+1, 2N-1)$ is of degree $(3, N+1, 2N-1)$. If $N \equiv -1 \,(\mathrm{mod}\, 3)$, then $b \geq N+2$ and then $c \geq 2N-2$. In this case, the harmonic knot $\mathrm{H}(3, N+2, 2N-2)$ is of degree $(3, N+2, 2N-2)$. The twist knots of crossing number N are of maximal degree $(3, \lfloor \frac{3N-1}{2} \rfloor, \lfloor \frac{3N}{2} \rfloor + 1)$, see [BKP2]. □

Remark 2.7 *The degree of a harmonic knot may be smaller than the degree of its harmonic diagram. For example the knot* $\mathrm{H}(3, 11, 16) = \overline{9}_{17}$ *is of degree* $(3, 10, 17)$*, see Table 4.*

Proposition 2.8 *Let* $(3, b, c)$ *be the lexicographic degree of a two-bridge knot of crossing number* N*. If* $b \leq N + 3$ *or* $b = \lfloor \frac{3N-1}{2} \rfloor$ *then we have* $b + c = 3N$*.*

Proof. By Theorem 2.5, we have $b + c \geq 3N$, and $b + c = 3N$ if $b = \lfloor \frac{3N-1}{2} \rfloor$ by Corollary 2.6. Hence we assume now that $b \leq N + 3$. Let $\gamma(t) = (x(t), y(t), z(t))$ be a polynomial representation of our knot K of degree $(3, b, c)$, and denote by $D = C(m_1, \ldots, m_k)$ the trigonal diagram of γ. If s denote the number of sign changes in the Gauss sequence of the parametrisation γ, we clearly have $c \leq s$. Hence it remains us to obtain an upper bound for s, using Propositions 2.2 and 2.4.

Let $N_0 = \sum |m_i|$, and σ be the number of sign changes in the sequence $(m_1, \ldots, m_k)$. Combining Propositions 2.2 and the genus formula for plane curves, we obtain

$$N + \sigma \leq N_0 \leq b - 1. \qquad (3)$$

First, suppose that $b = N + 3$. In this case $N \not\equiv 0 \,(\mathrm{mod}\, 3)$, and $c \equiv -N \,(\mathrm{mod}\, 3)$, by Corollary 2.6. Consequently $c \not\equiv 2N - 1 \,(\mathrm{mod}\, 3)$ and $c \not\equiv 2N - 2 \,(\mathrm{mod}\, 3)$. Hence we only have to prove that $c \leq 2N - 1$.

1. First, suppose that $D = C(x, m, 0, -n, -y)$ with $mn > 0$, see Figure 10. Since $N_0 \leq N + 2$, we necessarily have $|m| = 1$ or $|n| = 1$. Without loss of generality, we can assume that $n = 1$ and $m > 0$. Consider the diagram $D' = C(x, m-1, -y)$ obtained by a type-II Reidemeister move on D. The diagram D' has $N_0 - 2 = N$ crossings, and then is an alternating diagram of K. Consequently the number s' of sign changes in the Gauss sequence of D' is $s' = 2N - 1$.

   If $(x, m) \neq (1)$ and $(n, y) \neq (1)$, then we have $s = s' = 2N - 1$ and consequently $c \leq 2N - 1$, see Figure 10.

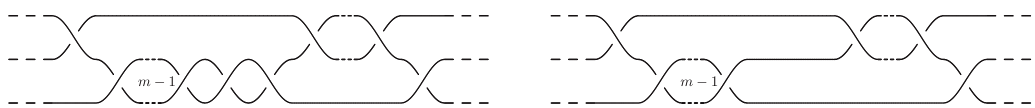


Figure 10: $C(x, m, 0, -1, -y) \mapsto C(x, m-1, -y)$

   If we have $(x, m) = (1)$ or $(n, y) = 1$, then we can suppose $(n, y) = (1)$ and $D = C(x, m, 0, -1)$. If we change the nature of the last two crossings, then we obtain another diagram $\tilde{D} = C(u, m-1, 0, -1, 0, 1)$ of K with the same xy-projection. By the previous case, we see that the number of sign changes in the Gauss sequence of $\tilde{D}$ is $\tilde{s} = 2N - 1$. Consequently $\tilde{D}$ is of degree at most $(3, N+3, 2N-3)$.

2. Then, suppose that $D = C(x, n, -1)$ (the case $D = C(1, -m, y)$ is similar). By changing the nature of the last two crossings of D, we obtain another diagram $\tilde{D} = C(x, n-1, 0, -1, 1)$ of the same knot, see Figure 11. By case 1 above, we see that the number of sign changes in the Gauss sequence of $\tilde{D}$ is $\tilde{s} = 2N - 1$ and we deduce that $\tilde{D}$ is of degree $(3, N+3, 2N-3)$.

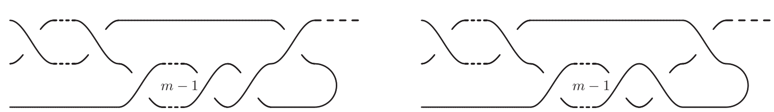


Figure 11: $C(x, m, -1) = C(x, m-1, 0, 1, -1) \mapsto \tilde{D} = C(x, m-1, 0, -1, 1)$

3. Now, suppose that D is not in the cases 1 and 2 above. If $\sigma = 2$, then $N_0 = N + 2$ and $\sigma_2 \leq 1$. By Proposition 2.4, we obtain $s = (2N_0 - 1) - 3\sigma + 2\sigma_2 \leq 2N - 1$.

   If $\sigma < 2$, then by Lemma 2.3 we have $m_1 \cdot m_k = 0$. Consider the reduced diagram D'. If $\sigma = 0$, then D' is alternating and has $N_0 = N$ crossings. Its Gauss sequence is alternating and has $s' = 2N - 1$ sign changes. If $\sigma = 1$, then D' may have $N'_0 = N$ or $N'_0 = N + 1$ crossings. If $N'_0 = N$ then D' is alternating and there are $s' = 2N - 1$ sign changes in its Gauss sequence. If $N'_0 = N + 1$, then D' is not alternating and $\sigma'_1 = 1$. We thus have $s' = 2N + 1 - 3 = 2N - 2$ by Proposition 2.4.

We then choose a polynomial of degree $c \leq s' \leq 2N-2$ as an height function for the reduced diagram D'. If $m_1 = 0$ (resp. $m_k = 0$), the signs of the $|m_2|$ (resp. $|m_{k-1}|$) crossings do not affect the nature of the knot.

At the end we find a polynomial height function $z(t)$ of degree $c \leq 2N-1$.

If $b = N+2$, then $N \not\equiv 1 \pmod 3$ and $2N-1 \in \langle 3, b \rangle$. Hence again, we only have to prove $c \leq 2N-1$. By Inequality (3), we may have $N_0 = N$ or $N_0 = N+1$.

1. If $N_0 = N$, then the diagram is alternating and $s \leq 2N-1$.

2. If $N_0 = N+1$, then $\sigma \leq 1$. If $\sigma = 1$, then $s \leq 2N-1$ by Proposition 2.4. If $\sigma = 0$, then $m_1 \cdot m_k = 0$ by Lemma 2.3. The reduced diagram is alternating and its Gauss sequence has $s' \leq 2N-1$ sign changes and so $c \leq 2N-1$.

At the end we find a polynomial function $z(t)$ of degree $c \leq 2N-1$.

If $b = N+1$ then $N_0 = N$ and the diagram is alternating. We thus have $c \leq s \leq 2N-1$. □

We deduce

Proposition 2.9 *The lexicographic degree* $(3, b, c)$ *of a knot with crossing number* $N \leq 11$ *satisfies* $b + c = 3N$.

Proof. By Proposition 2.6, we have $(3, b, c) \leq (3, \lfloor \frac{3N-1}{2} \rfloor, \lfloor \frac{3N}{2} \rfloor + 1)$. If $b \leq N+3$ or $b = \lfloor \frac{3N-1}{2} \rfloor$, we conclude using Proposition 2.8. If $b \geq N+4$ and $b < \lfloor \frac{3N-1}{2} \rfloor$, then $N = 11$, and $b = 15$ which is impossible since b is not divisible by 3. □

3 Degrees of trigonal plane diagrams

Thanks to the relation $b + c = 3N$ established in Proposition 2.9, we are now reduced to study plane trigonal curves. It is enough to determine the smallest integer b such that the xy-projection of some diagram of K admits a polynomial parametrisation of degree $(3, b)$.

Given a long knot diagram D in $\mathbf{R}^3$, we denote by $|D|$ its projection to $\mathbf{R}^2$ (i.e. we forget about the sign of the crossings). If $D = C(m_1, \ldots, m_k)$, we use the notation $|D| = D(|m_1|, \ldots, |m_k|)$. An isotopy of $\mathbf{R}^2$ is called an *$\mathcal{L}$-isotopy* if it commutes with the projection $\mathbf{R}^2 \to \mathbf{R}$ forgetting the second coordinate.

Definition 3.1 *The* algebraic degree *of* $|D|$ *is the minimal integer* b *such that there exists a real algebraic curve* $\gamma : \mathbf{C} \to \mathbf{C}^2$ *of bidegree* $(3, b)$ *such that* $\gamma(\mathbf{R})$ *is* $\mathcal{L}$*-isotopic to* $|D|$.

We first establish a lower bound for polynomial curves in Proposition 3.2.

3.1 Lower bounds on degrees of plane trigonal diagrams

Proposition 3.2 *Let $|D|$ be the plane diagram $D(m_1, m_2, \dots, m_k)$, with $m_i \geq 2$ for $i = 1, \dots, k$. Then the algebraic degree of $|D|$ is at least $3k-1$. If in addition we have $m_i \geq 3$ for some i, then the algebraic degree of $|D|$ is at least $3k+1$.*

Proof. Let $\gamma(t) : (x(t), y(t))$ be a polynomial parametrisation of $|D|$ with $x(t)$ of degree 3, and let C be the image of γ. The complement of C contains $m_j - 1$ disks corresponding to the jth group of crossings of $|D|$. Let us choose a point P_j in one of these disks. There is a polynomial curve of equation $y = P(x)$ with $\deg P = k-1$ containing the k points P_j. Since

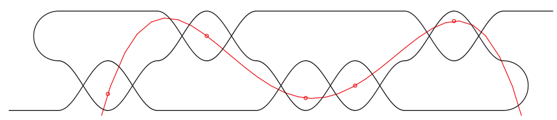


Figure 12: The plane diagram $D(2, 2, 3, 2)$

the number of intersections of this curve and C is at least $2k + (k-1) = 3k-1 > 3(k-1)$, we deduce that $\deg(y(t)) \geq 3k-1$.

If in addition some $m_i \geq 3$, we choose one more point P_{k+1} in another disk of the ith group of two-sided domains. Then we count the intersections of C with a curve $y = P(x)$ $\deg(P(x)) = k$ containing the $k+1$ points P_j, $j = 1, \dots, k+1$. Since this number is at least $2(k+1) + (k-1) = 3k+1 > 3k$, we deduce that $\deg(y(t)) \geq 3k+1$ (see Figure 12 in the case of $D(2, 2, 3, 2)$). □

3.2 Application of Orevkov's braid theoretical method

To obtain lower bounds on the algebraic degree b, it is convenient to enlarge the category of objects under interest, and to consider *real pseudoholomorphic curves* rather than real algebraic curves. Doing so, we can use the full power of the braid theoretical approach developed by Orevkov to study real curves in $\mathbf{C}^2$. Using this strategy, we determined in [BKP2] the lexicographic degree of all torus knots $C(m)$ and generalised twist-knots $C(m, n)$. We refer to [BKP2, Section 3.2] for the definition of a real pseudoholomorphic curve $\gamma : \mathbf{C} \to \mathbf{C}^2$ of bidegree $(3, b)$ where b is a positive integer. Recall that a real algebraic map $\gamma : \mathbf{C} \to \mathbf{C}^2$ of degree $(3, b)$ is an example of a real pseudoholomorphic curve of bidegree $(3, b)$. Without loss of generality, we only consider in this text nodal pseudoholomorphic curves.

Definition 3.3 *The* pseudoholomorphic *degree of $|D|$ is the minimal integer b such that there exists a real pseudoholomorphic curve $\gamma : \mathbf{C} \to \mathbf{C}^2$ of bidegree $(3, b)$ such that $\gamma(\mathbf{R})$ is $\mathcal{L}$-isotopic to $|D|$. It is not greater than the algebraic degree of $|D|$.*

Recall that the *group of braids with 3-strings* is defined as

$$B_3 = \langle \sigma_1, \sigma_2 | \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \rangle.$$

We refer to [BKP2, Sections 2 and 3] for an algorithm that associates an $\mathcal{L}$-*scheme* and a braid $\mathfrak{b}_C \in B_3$ to any real pseudoholomorphic curve $C = \gamma(\mathbf{C})$, with $\gamma : \mathbf{C} \to \mathbf{C}^2$ a real pseudoholomorphic curve of bidegree $(3, b)$. A braid $\mathfrak{b} \in B_3$ is said to be *quasipositive* if it can be written in the form

$$\mathfrak{b} = \prod_{i=1}^{l} w_i \sigma_1 w_i^{-1} \qquad \text{with } w_1, \cdots, w_l \in B_3. \tag{4}$$

Note that a braid with algebraic length 0 is quasipositive if and only if it is the trivial braid. The quasipositivity problem in B_3 has been solved by Orevkov [Or3]. We will use the following proposition in order to obtain lower bounds in lexicographic degree of knots.

Proposition 3.4 *Let $\gamma : \mathbf{C} \to \mathbf{C}^2$ be a real pseudoholomorphic curve of bidegree $(3, b)$, and let $C = \gamma(\mathbf{C})$. We denote by $\pi : \mathbf{C}^2 \to \mathbf{C}$ the projection to the first coordinate, and we assume that the two critical points of the map $\pi \circ \gamma$ are real. Then the braid $\mathfrak{b}_C$ satisfies the three following properties:*

(i) *$\mathfrak{b}_C$ is quasipositive;*

(ii) *the closure of $\mathfrak{b}_C$ is a link with three components;*

(iii) *the linking number of any two strings of $\mathfrak{b}_C$ is non-negative.*

Proof. Property (i) is a consequence of [Or1, Proposition 7.1]. Properties (ii) and (iii) are easy consequences of the Riemann–Hurwitz formula applied to the map $\pi \circ \gamma$, see [BKP2, second paragraph of the proof of Proposition 3.1]. □

Remark 3.5 *Proposition 3.4(i) can be strengthened in order to get an equivalence. Given $l \in \mathbf{Z}_{\geq 1}$ and $\varepsilon = 1$ or 2, we define $b = 3l - \varepsilon$. Let $\mathcal{L}_S$ be the trigonal $\mathcal{L}$-scheme*

$$\mathcal{L}_S = \bullet_{i_1} \cdots \bullet_{i_{\alpha_1}} \subset_* \times_{j_1} \cdots \times_{j_{N_0}} \supset_* \bullet_{i_{\alpha_1+1}} \cdots \bullet_{i_\alpha} \star, \tag{5}$$

where $\star = \downarrow$ or $\uparrow$ if $\varepsilon = 1$, and $\star = \vee$ or $\wedge$ if $\varepsilon = 2$. Then one can associate a braid $\mathfrak{b}_C$, depending on b, to the $\mathcal{L}$-scheme $\mathcal{L}_S$ using the algorithm given in [BKP2, Section 2.2]. Following [Or1, Proposition 7.1], we have that $\mathcal{L}_S$ is realised by a real pseudoholomorphic curve of bidegree $(3, b)$ in $\mathbf{C}^2$ if and only if the braid $\mathfrak{b}_C$ can be written in the form

$$\mathfrak{b}_C = \prod_{i=1}^{\ell} w_i \sigma_1^2 w_i^{-1} \qquad \textit{with } w_1, \ldots, w_\ell \in B_3. \tag{6}$$

Note that in this case, we necessarily have $\deg \mathfrak{b}_C = 2\ell = b - 1 - \alpha - N_0$.

Remark 3.6 *Proposition 3.2 also holds for the pseudoholomorphic degree of a plane trigonal diagram, and the proof is essentially the same. Nevertheless we will not need this more general version here.*

We end this section by proving a slight generalisation of [BKP2, Proposition 3.1].

Proposition 3.7 *Let $D = C(m_1, \dots, m_k)$ be a trigonal diagram of a knot K, with $m_1, \dots, m_{k-1}$ even integers. As usual, we define $N_0 = m_1 + \dots + m_k$. If $\gamma : \mathbf{C} \to \mathbf{C}^2$ is a real rational pseudoholomorphic curve of bidegree $(3, b)$ such that $\gamma(\mathbf{R})$ is $\mathcal{L}$-isotopic to $|D|$, then $2b \geq 3N_0 - 2$.*

Proof. Let us write $b = 3l - 1$ or $b = 3l - 2$, let α be the number of solitary nodes of $C = \gamma(\mathbf{C})$, and 2β be the number of complex conjugated nodes. By the genus formula, we have

$$N_0 + \alpha + 2\beta = b - 1.$$

The $\mathcal{L}$-scheme realised by C has the form

$$\bullet_{i_1} \cdots \bullet_{i_{\alpha_1}} \subset_* (\times_{j_1})^{m_1} \cdots (\times_{j_k})^{m_k} \supset_* \bullet_{i_{\alpha_1+1}} \cdots \bullet_{i_\alpha} \star,$$

where $\star = \downarrow, \uparrow, \vee$ or $\wedge$. The braid $\mathfrak{b}_C$ has 3 components L_1, L_2 and L_3, and $\mathrm{lk}\,(L_i, L_j) \geq 0$ by Proposition 3.4. Furthermore, as in [BKP2, proof of Proposition 3.1], we have $0 \leq \mathrm{lk}\,(L_i, L_j) \leq \beta$.

By the assumptions made on D, there are two strings of $\mathfrak{b}_C$, say L_1 and L_3, that do not cross at the crossing points of $\mathbf{R}C$. Each $\bullet_j \bullet_{j'}$ contributes at least -1 to $\mathrm{lk}\,(L_1, L_3)$. Hence as in [BKP2, Proof of Proposition 3.1], we obtain

$$2\beta \geq 2\mathrm{lk}\,(L_1, L_3) \geq l - \alpha - 2,$$

and thus

$$b - 1 = N_0 + \alpha + 2\beta \geq N_0 + l - 2.$$

We then deduce $3b - 3N_0 \geq 3l - 3 \geq b - 2$, and $2b \geq 3N_0 - 2$. □

3.3 The T-reduction

Definition 3.8 *Let x, y be (possibly empty) sequences of nonnegative integers and m, n be nonnegative integers. The plane diagram $D(x, m, n, y)$ is called a* T-reduction *of the diagram $D(x, m+1, 1, n+1, y)$ (see Figure 13).*

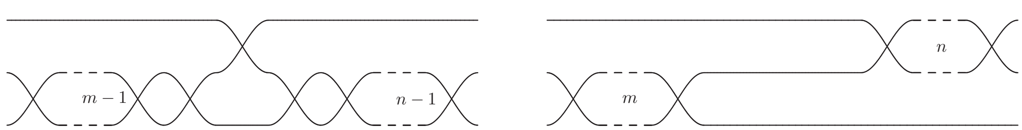


Figure 13: T-reduction

Propositions 3.9 and 3.11 below relate the pseudoholomorphic and algebraic degrees of two plane trigonal diagrams differing by a T-reduction.

Proposition 3.9 *Let $|D_1|$ and $|D_2|$ be two plane trigonal diagrams such that $|D_2|$ is obtained from $|D_1|$ by a* T*-reduction. If $|D_1|$ has pseudoholomorphic degree b, then $|D_2|$ has pseudoholomorphic degree $b-3$.*

Proof. Let $|D_1| = D(m_1, \dots, m_k)$ and $|D_2| = D(n_1, \dots, n_l)$. Suppose that there exists a real pseudoholomorphic curve $\gamma_1 : \mathbf{C} \to \mathbf{C}^2$ of bidegree $(3, b)$ such that $\gamma_1(\mathbf{R})$ is $\mathcal{L}$-isotopic to $|D_1|$, and suppose that its associated $\mathcal{L}$-scheme is

$$\bullet_{i_1} \cdots \bullet_{i_{\alpha_1}} \subset_* (\times_{j_1})^{m_1} \cdots (\times_{j_k})^{m_k} \supset_* \bullet_{i_{\alpha_1+1}} \cdots \bullet_{i_\alpha} \star. \tag{7}$$

The braid associated to γ_1 is the same that the braid associated to the $\mathcal{L}$-scheme

$$\bullet_{i_1} \cdots \bullet_{i_{\alpha_1}} \subset_* (\times_{j_1})^{n_1} \cdots (\times_{j_l})^{n_l} \supset_* \bullet_{i_{\alpha_1+1}} \cdots \bullet_{i_\alpha} \star. \tag{8}$$

Hence according to Remark 3.5, there exists a real pseudoholomorphic curve $\gamma_2 : \mathbf{C} \to \mathbf{C}^2$ of bidegree $(3, b-3)$ such that $\gamma_2(\mathbf{R})$ is $\mathcal{L}$-isotopic to $|D_2|$. □

Corollary 3.10 *The pseudoholomorphic degree of the plane diagram $D(0, n)$ is $\lfloor \frac{3n}{2} \rfloor + 1$.*

Proof. The plane diagram $D(0, n)$ is obtained by a T-reduction from $D(1, 1, n+1)$. Since $D(1, 1, n+1)$ and $D(2, n)$ may be reduced to each other by slide isotopies, they have the same pseudoholomorphic degree by Proposition 4.3. By [BKP2, Theorem 3.9], the degrees are $\lfloor \frac{3n}{2} \rfloor + 4$, which completes the proof. □

3.4 The T-augmentation

Proposition 3.9 admits a weaker version for the algebraic degree of a plane diagram. We make use the T-augmentation that consists in adding a triangle of crossing points in a given plane diagram.

Proposition 3.11 *Let $|D_1|$ and $|D_2|$ be two plane trigonal diagrams such that $|D_2|$ is obtained from $|D_1|$ by a reduction* T*. If $|D_2|$ has algebraic degree $b-3$, then $|D_1|$ has algebraic degree at most b. Furthermore, if the pseudoholomorphic degree of $|D_2|$ is also $b-3$, then $|D_1|$ has algebraic degree exactly b.*

Proof. The last assertion simply follows from the fact that a real rational algebraic curve in $\mathbf{C}^2$ is a pseudoholomorphic curve. Let

$$\begin{array}{rccc} \gamma : & \mathbf{C} & \longrightarrow & \mathbf{C}^2 \\ & t & \longmapsto & (P(t), Q(t)) \end{array} \tag{9}$$

be a real algebraic map with $P(t)$ of degree 3 and $Q(t)$ of degree $b-3$, and such that $\gamma(\mathbf{R})$ is $\mathcal{L}$-isotopic to the plane diagram $D(x, m, n, y)$, where x, y are (possibly empty) sequences of nonnegative integers and m, n are nonnegative integers. Without loss of generality, we can suppose that the line $x = 0$ separates the m crossings from the n crossings. The curve parametrised by $t \mapsto (P(t), P(t) \cdot Q(t))$ has the same double points as $\gamma(\mathbf{R})$ and an additional ordinary triple point at $(0, 0)$. For ε small enough the curve $(P(t+\varepsilon), P(t) \cdot Q(t))$ is $\mathcal{L}$-isotopic to either $D(u, m+1, 1, n+1, v)$ or $D(u, m, 1, 1, 1, n, v)$, depending on the sign of ε (see Figure 3.12). $\square$

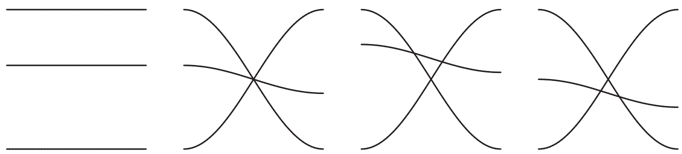


Figure 14: Perturbation of a triple point in $\mathbf{R}^2$

Example 3.12 *Let us consider the polynomial parametrisation* $(T_3(t), T_4(t))$ *of the diagram* $D(1, 1, 1)$*, where* T_n *denotes the Chebyshev polynomial of degree* n*. We choose to add a triple point in* $(-3/4, 0)$*, by considering the curve* $t \mapsto (T_3(t), Q(t))$*, where* $Q(t) = (T_3(t) + 3/4) \cdot (T_4(t) + 1)$*. Then the curve in* $(P_3(t), Q(t+\varepsilon))$ *is* $\mathcal{L}$*-isotopic to* $D(2, 1, 2, 1)$ *for* $\varepsilon > 0$ *small*

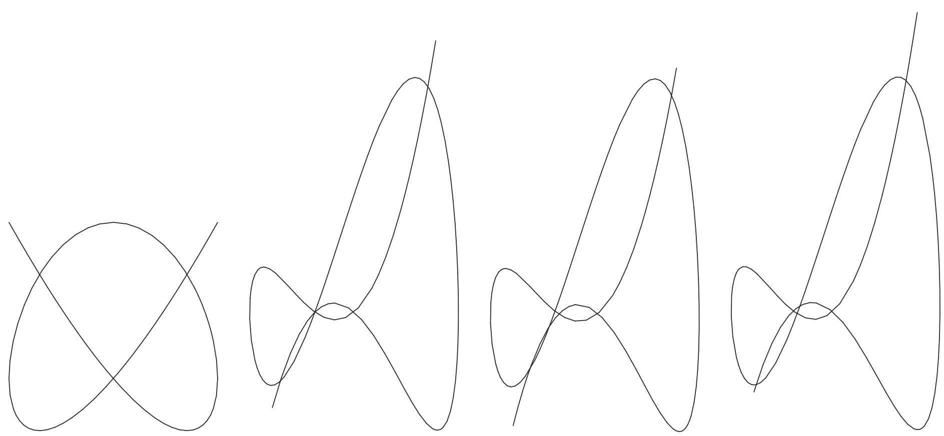


Figure 15: Adding three crossings to the trefoil

enough and is $\mathcal{L}$*-isotopic to* $D(1, 1, 1, 1, 1, 1)$ *for* $\varepsilon < 0$*, see Figure 15.*

Example 3.13 *Figure 16 shows that the algebraic degree of* $D(2, 2, 2, 1, 3)$ *is at most 11, starting from a parametrisation of the plane diagram* $D(1, 0)$ *of degree* $(3, 2)$*.*

Proposition 3.11 can be extended to spatial trigonal curves. The next result provides constructions of polynomial knot diagrams.

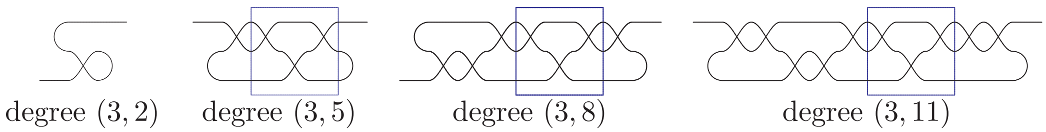


Figure 16: From $D(1,0)$ to $D(2,2,2,1,3)$

Proposition 3.14 *Let $e = \pm 1$. If the diagram $C(u,m,n,v)$ has lexicographic degree $(3, b-3, c-6)$, then the diagram $C(u, m+e, e, e+n, v)$ has lexicographic degree at most $(3,b,c)$.*

Proof. Let $t \mapsto (P(t), Q(t), R(t))$ be a parametrisation of degree $(3,b,c)$ of the diagram $C(u,m,n,v)$. Up to a change of coordinates, we may assume that the part (u,m) (resp. (n,v)) of the diagram is contained in the half-space $x < 0$ (resp. $x > 0$), and that the three points of the diagram in the plane $x = 0$ have z-coordinates of the same sign. We consider the map $\varphi(t) = (P(t), P(t)Q(t), P^2(t)R(t))$. The image of φ is a singular diagram with the three branches tangent to the plane $z = 0$ at the point $(0,0,0)$. Extending the notations of diagram in the obvious way to this particular case, we see that the image of φ realises the singular diagram $C(u,m,*,n,v)$, where $*$ stands for the triple point. By slightly perturbing the roots of the factor $P(t)$ of the polynomial $P(t)Q(t)$, we obtain a polynomial $Q_1(t)$ of degree $b+3$ such that the triple point of the curve $(P(t), P(t)Q(t))$ will be perturbed as depicted in Figure 17a or b, depending on the perturbation $Q_1(t)$. Perturbing the roots of the factor $P^2(t)$ of the polynomial $P^2(t)R(t)$ as depicted by the blue dots on Figure 17, we obtain a parametrisation of the diagram whose existence is claimed in the theorem. □

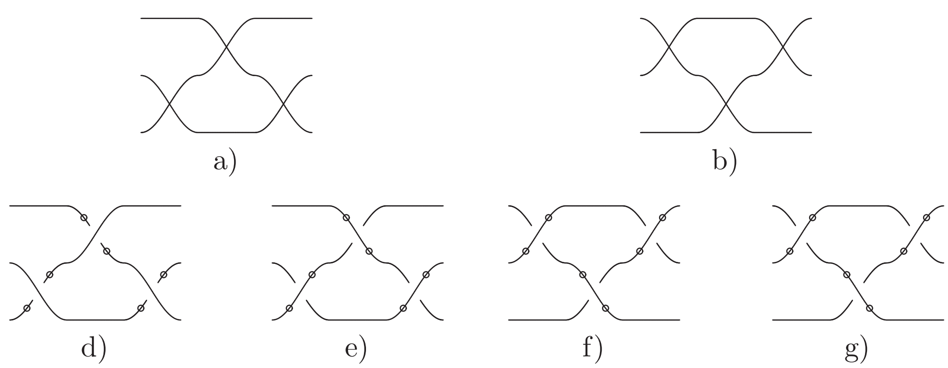


Figure 17: Perturbation of a triple point in $\mathbf{R}^3$

Example 3.15 *The trefoil admits the parametrisation (T_3, T_4, T_5) of degree $(3,4,5)$. We thus deduce that $6_2 = C(2,1,3)$ and $6_3 = C(2,1,1,2)$ admit parametrisations of degree $(3,7,11)$. By Corollary 2.6, these are the lexicographic degrees of 6_2 and 6_3.*

Thanks to Proposition 2.9, we will not need Proposition 3.14 to determine the lexicographic degrees of the first knots, but it may be useful for further results.

4 Two-bridge knots with 11 crossings or fewer

Simple diagrams of two-bridge knot have been introduced in [BKP1]. The *complexity* $c(D)$ of a trigonal diagram $D = C(m_1, \ldots, m_k)$ is defined as

$$c(D) = k + \sum_{i=1}^{k} |m_i| \, .$$

Definition 4.1 *We shall say that an isotopy of trigonal diagrams is a* slide *isotopy if the number of crossings never increases during the isotopy, and if all the intermediate diagrams remain trigonal. A trigonal diagram is called a* simple *diagram if it cannot be simplified into a diagram of lower complexity by using slide isotopies only.*

The next two propositions motivate the consideration of simple diagrams.

Proposition 4.2 ([BKP1, Corollary 3.9]) *Let D be a trigonal diagram of a two-bridge knot. Then by slide isotopies, it is possible to transform D into a simple diagram $C(m_1, \ldots, m_k)$ such that for $i = 2, \ldots, k$, either $|m_i| \neq 1$, or $m_{i-1}\, m_i > 0$.*

Proposition 4.3 ([BKP2, Corollary 3.7]) *Let D_1 and D_2 be two trigonal long knot diagrams such that D_2 is obtained from D_1 by a slide isotopy. Then the pseudoholomorphic degree of $|D_1|$ is greater than or equal to the pseudoholomorphic degree of $|D_2|$.*

In [BKP2] we proved that the lexicographic degree of the torus knot $C(n)$ or the twist knot $C(n, m)$ is precisely $(3, \lfloor \frac{3N-1}{2} \rfloor, \lfloor \frac{3N}{2} \rfloor + 1)$ by showing first that the only simple diagrams of these knots are the alternating diagrams and showing that the algebraic degrees of the corresponding plane diagrams are $\lfloor \frac{3N-1}{2} \rfloor$.

4.1 The general strategy

Given a two-bridge knot with crossing number $N \leq 11$, our strategy to determine its lexicographic degree consists in:

1. Find a first upper bound b_0 on b using constructions from [KP2] based on Chebyshev plane diagrams parametrised by (T_3, T_b), where T_n is the Chebyshev polynomial $T_n(\cos t) = \cos nt$.
2. Compute all finitely many simple diagrams of K with $b_0 - 1$ crossings or fewer. This is done by computing all continued fractions corresponding to the Schubert fractions of K.
3. For all these simple diagrams,
   (a) Compute a lower bound of their algebraic degree using Propositions 3.2 and 3.7.
   (b) Using T-reductions, try to obtain explicit constructions of these diagrams out of known constructions for diagrams with a lower number of crossings. This provides a lower bound on the lexicographic degree of the knot.

(c) If necessary, compute all possible braids associated to hypothetical plane curves of degree $b < b_0$ that are $\mathcal{L}$-isotopic to the diagram, and check if these braids satisfy Proposition 3.4. This may improve the lower bound obtained in step (a) above.

(d) If the lower bound and the upper bound coincide, then we have determined the lexicographic degree of the knot.

4. If the lower bound and the upper bound do not coincide, improve the upper bound by looking at non-simple diagrams on which one can perform T-reductions to reduce to knots with lower crossing number.

In Table 5, p. 28, we give the lexicographic degree of all two-bridge knots with 11 crossings or fewer. In Tables 1, 2, 3, and 4 below, we give refinements of Table 5 for two-bridge knots with crossing number at most 9. The columns 1, 2 and 3 identify the knot. The column 4 gives the lexicographic degree. The fifth column gives the upper bound on b obtained by considering Chebyshev diagrams; the sixth column gives a diagram that can be realised in the corresponding lexicographic degree; the last column gives the construction of the corresponding plane diagram, when one needs to improve the bound given by Chebyshev knots.

4.2 Some initial diagrams

Here we compute the algebraic degrees of a few trigonal plane diagrams. These computation will be used in the next sections to determine the algebraic degree of trigonal plane diagrams that reduce to the diagrams considered in this section by T-reduction. The next proposition is proved in [KP1].

Proposition 4.4 *The plane diagram $D(4n-1)$ has algebraic degree $6n-2$.*

This gives an explicit parametrisation for the plane diagrams $D(3)$ and $D(7)$.

Lemma 4.5 *We give below the algebraic degree of a few plane diagrams (see Figure 18 for the image of a polynomial parametrisation of the given degree).*

- $b = 1$: $D(0,0)$
- $b = 2$: $D(0,1)$
- $b = 4$: $D(0,2)$, $D(2,1)$
- $b = 5$: $D(0,1,1,0)$, $D(2,2)$, $D(1,1,1,1)$, $D(0,3)$, $D(1,2,0)$
- $b = 7$: $D(5)$, $D(1,4)$, $D(0,4)$

Proof. These plane diagrams are obtained with the following parametrisations – here we use the monic Chebyshev polynomials (also called Dickson polynomials) defined by $T_n(2\cos x) = 2\cos nx$:

- $D(0,0)$: (T_3, T_1)
- $D(0,1)$: $(T_3, T_2 - \frac{3}{2}T_1)$

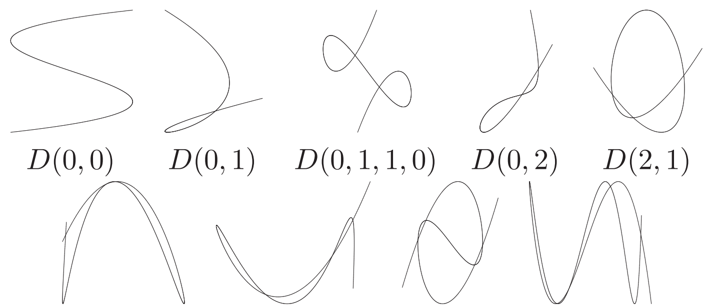


Figure 18: Algebraic degree of a few plane diagrams

- $D(0,1,1,0)$: $(T_3, t^5 - 4t^3 + 4t)$
- $D(0,2)$: $(T_3, T_4 + 3T_2 - 4T_1)$
- $D(2,1)$: $(T_3, T_4 - T_2 + \frac{1}{4}T_1)$
- $D(0,3)$: $(T_3, t^5 - \frac{9}{4}t^4 - t^3 + \frac{13}{4}t^2 + \frac{1}{8}t)$
- $D(1,2,0)$: $(T_3, T_5(6t/5 + 1/2))$
- $D(2,2)$: $(T_3, T_5 - \frac{13}{12}T_1)$
- $D(0,4)$: $(T_3, T_7(-\frac{3}{2}t + 1))$
- $D(5)$: (T_3, P_7) where $P_7 = t^7 - \frac{93659}{10000}\,t^5 - \frac{13549}{5000}\,t^4 + \frac{16453}{1000}\,t^2 + \frac{57281}{1000}\,t$
- $D(1,4)$: (T_3, Q_7) where $Q_7 = t^7 - \frac{84497}{10359}\,t^5 - \frac{47123}{18875}\,t^4 + \frac{54585}{2759}\,t^3 + \frac{85741}{7122}\,t^2 - \frac{208133}{17097}\,t - \frac{242151}{26615}$

It is shown in [BKP2] that the degree is minimal for $D(n,m)$, $n,m \geq 0$. In the case of $D(0,1,1,0)$, every line passing through the two crossing points meets the curve at 5 points at least, and therefore the degree is at least 5, which is the degree of our parametrisation. □

Remark 4.6 *One can prove using* dessins d'enfants *(see for example [Or2]) that the algebraic degree of the plane diagram $D(0,n)$ is precisely $\lfloor \frac{3n}{2} \rfloor + 1$.*

4.3 Two-bridge knots with crossing number at most 6

Proposition 4.7 *The lexicographic degrees of all two-bridge knots with crossing number at most* 6 *are given in Table 1.*

Proof. The knots 3_1, and 5_1 are torus knots, and the knots 4_1, 5_2, and 6_1 are twist knots. Hence their lexicographic degrees are computed in [BKP1]. The knots 6_2 and 6_3 admit parametrisations with $b = N + 1$, hence their lexicographic degree is $(3, 7, 11)$. □

Name	Fraction	Conway Not.	Lex. deg.	Cheb. deg.	diagram	Constr.
3_1	3	$C(3)$	$(3,4,5)$	4	$C(3)$	$D(3)$
4_1	5/2	$C(2,2)$	$(3,5,7)$	5	$C(2,2)$	$D(2,2)$
5_1	5	$C(5)$	$(3,7,8)$	7	$C(5)$	$D(5)$
5_2	7/2	$C(3,2)$	$(3,7,8)$	7	$C(3,1,1)$	$D(2,0)+\mathrm{T}$
6_1	9/2	$C(4,2)$	$(3,8,10)$	8	$C(4,2)$	$D(3,0)+\mathrm{T}$
6_2	11/3	$C(3,1,2)$	$(3,7,11)$	8	$C(3,1,2)$	$D(2,1)+\mathrm{T}$
6_3	13/5	$C(2,1,1,2)$	$(3,7,11)$	7	$C(2,1,1,2)$	$D(0,0)+2\mathrm{T}$

Table 1: Lexicographic degree of two-bridge knots with crossing number at most 6

4.4 Two-bridge knots with crossing number 7

Proposition 4.8 *The lexicographic degrees of all two-bridge knots with crossing number* 7 *are given in Table 2.*

Name	Fraction	Conway Not.	Lex. deg.	Cheb. deg.	diagram	Constr.
7_1	7	$C(7)$	$(3,10,11)$	10	$C(7)$	$D(7)$
7_2	11/2	$C(5,2)$	$(3,10,11)$	10		Cheb.
7_3	13/3	$C(4,3)$	$(3,10,11)$	10		Cheb.
7_4	15/4	$C(3,1,3)$	$(3,8,13)$	10	$C(3,1,3)$	$D(1)+2\mathrm{T}$
7_5	17/5	$C(3,2,2)$	$(3,10,11)$	10	$C(2,1,1,-4)$	$D(5)+\mathrm{T}$
7_6	19/7	$C(2,1,2,2)$	$(3,8,13)$	10		$D(1)+2\mathrm{T}$
7_7	21/8	$C(2,1,1,1,2)$	$(3,8,13)$	8		Cheb.

Table 2: Lexicographic degrees of two-bridge knots with crossing number 7

Proof. The lexicographic degree of such a knot is $(3,8,13)$ or $(3,10,11)$. The torus knot 7_1 and the twist knots 7_2 and 7_3 have lexicographic degree $(3,10,11)$, see [BKP2]. The Fibonacci knot 7_7 has degree $(3,8,13)$, see [KP2]. The knots 7_4 and 7_6 are obtained from $C(1)$ by T-augmentation. There degrees is $(3,8,13)$. The alternating diagram of the knot 7_5 is $C(3,2,2)$. By Proposition 3.2 the the degree of this diagram is at least $(3,10,11)$. Since a non-alternating diagrams of 7_5 has at least 8 crossings, we see that its degree is at least $(3,10,11)$. Hence the lexicographic degree of 7_5 is at least $(3,10,11)$. □

4.5 Two-bridge knots with crossing number 8

Proposition 4.9 *The lexicographic degrees of all two-bridge knots with crossing number* 8 *are given in Table 3.*

Name	Fraction	Conway Not.	Lex. deg.	Cheb. deg.	diagram	Constr.
8_1	13/2	$C(6,2)$	$(3,11,13)$	11		Cheb.
8_2	17/3	$C(5,1,2)$	$(3,10,14)$	11		$D(4,1)+\text{T}$
8_3	17/4	$C(4,4)$	$(3,11,13)$	11		Cheb.
8_4	19/4	$C(4,1,3)$	$(3,10,14)$	11	$C(4,1,2,1)$	$D(2,0)+2\text{T}$
8_6	23/7	$C(3,3,2)$	$(3,10,14)$	11	$C(2,2,1,-4)$	$D(1,2)+2\text{T}$
8_7	23/5	$C(4,1,1,2)$	$(3,10,14)$	10		Cheb.
8_8	25/9	$C(2,1,3,2)$	$(3,10,14)$	10		Cheb.
8_9	25/7	$C(3,1,1,3)$	$(3,10,14)$	11		$D(5)+\text{T}$
8_{11}	27/8	$C(3,2,1,2)$	$(3,10,14)$	11		$D(2,0)+2\text{T}$
8_{12}	29/12	$C(2,2,2,2)$	$(3,11,13)$	11		Cheb.
8_{13}	29/8	$C(3,1,1,1,2)$	$(3,10,14)$	10		Cheb.
8_{14}	31/12	$C(2,1,1,2,2)$	$(3,10,14)$	11		$D(2,0)+2\text{T}$

Table 3: Lexicographic degrees of two-bridge knots with crossing number 8

Proof. The lexicographic degree of such a knot is $(3,10,14)$ or $(3,11,13)$. The lexicographic degree $(3,11,13)$ of the twist knots 8_1 and 8_3 has been obtained in [BKP2]. Combining Propositions 3.2 with Chebyshev knots we obtain the following.

- The knots 8_7, 8_8, and 8_{13} have minimal lexicographic degree $(3,10,14)$, obtained as Chebyshev knots.
- The plane projection of $8_2 = C(5,1,2)$ reduces to $D(4,1)$ by T-reduction. Since $D(4,1)$ has algebraic degree 7, the diagram $D(5,1,2)$ has algebraic degree 10. Consequently, 8_2 has lexicographic degree $(3,10,14)$.
- The plane projection of $8_9 = C(3,1,1,3)$ reduces to $D(5)$ by T-reduction. Hence the algebraic degree of $D(3,1,1,3)$ is 10, and 8_9 has lexicographic degree $(3,10,14)$.
- $D(2,0)$ is obtained by two successive T-reductions from the plane projections of diagrams of 8_4, 8_{11} and 8_{14}. Consequently, 8_4, 8_{11} and 8_{14} have lexicographic degree $(3,10,14)$.
- Using two T-reductions, the plane diagram $D(2,2,1,4)$ reduces to $D(1,2)$, which has algebraic degree 4. By Proposition 3.11, the plane diagram $D(2,2,1,4)$ has algebraic degree 10, and the knot 8_6 has lexicographic degree $(3,10,14)$.
- The knot 8_{12} admits only three simple diagrams with 9 crossings or fewer: $C(2,2,2,2)$, $C(2,1,1,-3,-2)$ and $C(2,2,1,1,-3)$. By Proposition 3.2, the plane diagram $D(2,2,2,2)$ has degree at least 11. The plane diagrams $D(2,1,1,3,2)$ and $D(2,2,1,1,3)$ reduce, with two T-reductions, to $D(3,0)$ or $D(0,3)$ that have pseudoholomorphic degree 5. By Proposition 3.11, the lexicographic degree of 8_{12} is then $(3,11,13)$. □

The next result shows that the knot 8_6 is the first example of a knot for which the lexicographic degree cannot be obtained for the alternating diagram.

Proposition 4.10 *Let $t \mapsto (P(t), Q(t))$, be a parametrisation of the diagram $D(2,3,3)$, where $\deg P = 3$. Then $\deg Q \geq 11$.*

Proof. Without loss of generality, we may assume that $P(t)$ is positive for t large enough, and $\deg Q \not\equiv 0 \pmod 3$. Let us denote by C the complex algebraic curve image of the map $t \in \mathbf{C} \mapsto (P(t), Q(t)) \in \mathbf{C}^2$. The curve C has exactly $\deg Q - 1$ nodes in $\mathbf{C}^2$ and then $\deg Q \geq 10$. Let us suppose that $\deg Q = 10$. Since C has 8 real crossings, it also has a ninth solitary real point. We see that there are exactly eight possibilities for the $\mathcal{L}$-scheme

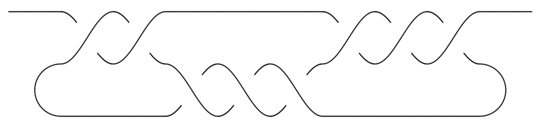


Figure 19: $C(2,3,3)$

realised by C (here we use the notations of [BKP2, Section 2.2]):

$$
\begin{array}{lll}
\supset_2 & \subset_1 \times_2 \times_2 \times_1 \times_1 \times_1 \times_2 \times_2 \times_2 \supset_1 \bullet_1 & \subset_1 \supset_1 \subset_1 \\
\supset_1 & \subset_1 \times_2 \times_2 \times_1 \times_1 \times_1 \times_2 \times_2 \times_2 \supset_1 \bullet_1 & \subset_2 \supset_2 \subset_2 \\
\supset_2 & \bullet_1 \subset_1 \times_2 \times_2 \times_1 \times_1 \times_1 \times_2 \times_2 \times_2 \supset_1 & \subset_1 \supset_1 \subset_1 \\
\supset_1 & \bullet_1 \subset_1 \times_2 \times_2 \times_1 \times_1 \times_1 \times_2 \times_2 \times_2 \supset_1 & \subset_2 \supset_2 \subset_2 \\
\supset_2 & \subset_1 \times_2 \times_2 \times_1 \times_1 \times_1 \times_2 \times_2 \times_2 \supset_1 \bullet_2 & \subset_1 \supset_1 \subset_1 \\
\supset_1 & \subset_1 \times_2 \times_2 \times_1 \times_1 \times_1 \times_2 \times_2 \times_2 \supset_1 \bullet_2 & \subset_2 \supset_2 \subset_2 \\
\supset_2 & \bullet_2 \subset_1 \times_2 \times_2 \times_1 \times_1 \times_1 \times_2 \times_2 \times_2 \supset_1 & \subset_1 \supset_1 \subset_1 \\
\supset_1 & \bullet_2 \subset_1 \times_2 \times_2 \times_1 \times_1 \times_1 \times_2 \times_2 \times_2 \supset_1 & \subset_2 \supset_2 \subset_2
\end{array}
$$

We compute all corresponding braids and obtain

$$
\begin{aligned}
\mathfrak{b}_1 &= \sigma_2^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_1^{-3}\sigma_2^{-3}\sigma_1^{-3}(\sigma_1\sigma_2\sigma_1)^4, \\
\mathfrak{b}_2 &= \sigma_1^{-1}\sigma_2^{-2}\sigma_1^{-3}\sigma_2^{-3}\sigma_1^{-2}\sigma_2^{-1}\sigma_1\sigma_2^{-1}(\sigma_1\sigma_2\sigma_1)^4, \\
\mathfrak{b}_3 &= \sigma_2^{-1}\sigma_1^{-1}\sigma_2\sigma_1^{-1}\sigma_2^{-2}\ \sigma_1^{-3}\sigma_2^{-3}\sigma_1^{-2}(\sigma_1\sigma_2\sigma_1)^4, \\
\mathfrak{b}_4 &= \sigma_1^{-2}\sigma_2^{-2}\sigma_1^{-3}\sigma_2^{-3}\sigma_1^{-1}\sigma_2^{-1}\sigma_1\sigma_2^{-1}(\sigma_1\sigma_2\sigma_1)^4, \\
\mathfrak{b}_5 &= \sigma_2^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_1^{-3}\sigma_2^{-3}\sigma_1^{-1}\sigma_2^{-1}\sigma_1\sigma_2^{-1}\sigma_1^{-1}\sigma_2\sigma_1^{-1}(\sigma_1\sigma_2\sigma_1)^4, \\
\mathfrak{b}_6 &= \sigma_1^{-1}\sigma_2^{-2}\sigma_1^{-3}\sigma_2^{-3}\sigma_1^{-1}\sigma_2^{-1}\sigma_1\sigma_2^{-2}(\sigma_1\sigma_2\sigma_1)^4, \\
\mathfrak{b}_7 &= \sigma_2^{-2}\sigma_1^{-1}\sigma_2^{-1}\sigma_1^{-3}\sigma_2^{-3}\sigma_1^{-2}(\sigma_1\sigma_2\sigma_1)^4, \\
\mathfrak{b}_8 &= \sigma_1^{-1}\sigma_2^{-1}\sigma_1\sigma_2^{-1}\sigma_1^{-1}\ \sigma_2^{-1}\sigma_1^{-3}\sigma_2^{-3}\sigma_1^{-1}\sigma_2^{-1}\sigma_1\sigma_2^{-1}(\sigma_1\sigma_2\sigma_1)^4.
\end{aligned}
$$

These 8 braids have integer length 0, and none of them is the trivial braid. Hence the result follows from Proposition 3.4. ☐

Combining Propositions 3.2 and 4.10, we obtain

Corollary 4.11 *The lexicographic degree of* $8_6 = C(2,3,3)$ *is not obtained for the alternating diagram.*

This phenomenon will appear with other knots (see Table 6).

4.6 Two-bridge knots with crossing number 9

Proposition 4.12 *The lexicographic degrees of all two-bridge knots with crossing number 9 are given in Table 4.*

Name	Fraction	Conway Not.	Lex. deg.	Cheb. deg.	diagram	Constr.
9_1	9	$C(9)$	$(3,13,14)$	13		Cheb.
9_2	15/2	$C(7,2)$	$(3,13,14)$	13		Cheb.
9_3	19/3	$C(6,3)$	$(3,13,14)$	13		Cheb.
9_4	21/4	$C(5,4)$	$(3,13,14)$	13		Cheb.
9_5	23/4	$C(5,1,3)$	$(3,11,16)$	13	$C(5,1,2,1)$	$D(3,0)+2\mathrm{T}$
9_6	27/5	$C(5,2,2)$	$(3,13,14)$	13		Cheb.
9_7	29/9	$C(3,4,2)$	$(3,13,14)$	13		Cheb.
9_8	31/11	$C(2,1,4,2)$	$(3,11,16)$	13	$C(2,1,4,1,1)$	$D(1,2,0)+2\mathrm{T}$
9_9	31/7	$C(4,2,3)$	$(3,13,14)$	13		Cheb.
9_{10}	33/10	$C(3,3,3)$	$(3,11,16)$	13	$C(3,2,1,-4)$	$D(0,1)+3\mathrm{T}$
9_{11}	33/7	$C(4,1,2,2)$	$(3,10,17)$	13		$D(3)+2\mathrm{T}$
9_{12}	35/8	$C(4,2,1,2)$	$(3,11,16)$	13		$D(3,0)+2\mathrm{T}$
9_{13}	37/10	$C(3,1,2,3)$	$(3,10,17)$	13		$D(1,2)+2\mathrm{T}$
9_{14}	37/8	$C(4,1,1,1,2)$	$(3,11,16)$	11		$D(3,0)+2\mathrm{T}$
9_{15}	39/16	$C(2,2,3,2)$	$(3,11,16)$	13	$C(2,2,2,1,-3)$	$D(1,0)+3\mathrm{T}$
9_{17}	39/14	$C(2,1,3,1,2)$	$(3,10,17)$	11		$D(3)+2\mathrm{T}$
9_{18}	41/12	$C(3,2,2,2)$	$(3,13,14)$	13		Cheb.
9_{19}	41/16	$C(2,1,1,3,2)$	$(3,11,16)$	11		$D(3,0)+2\mathrm{T}$
9_{20}	41/11	$C(3,1,2,1,2)$	$(3,10,17)$	13		$D(3)+2\mathrm{T}$
9_{21}	43/12	$C(3,1,1,2,2)$	$(3,11,16)$	13		$D(3,0)+2\mathrm{T}$
9_{23}	45/19	$C(2,2,1,2,2)$	$(3,10,17)$	13		$D(0,0)+3\mathrm{T}$
9_{26}	47/13	$C(3,1,1,1,1,2)$	$(3,11,16)$	11		$D(3)+2\mathrm{T}$
9_{27}	49/18	$C(2,1,2,1,1,2)$	$(3,10,17)$	13		$D(3)+2\mathrm{T}$
9_{31}	55/21	$C(2,1,1,1,1,1,2)$	$(3,10,17)$	10		Cheb.

Table 4: Lexicographic degree of two-bridge knots with crossing number 9

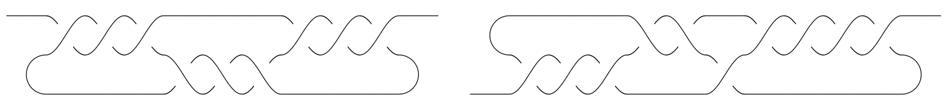
Proof. The lexicographic degree of such a knot is $(3,10,17),(3,11,16)$, or $(3,13,14)$. Furthermore, any diagram with at least 11 crossings has degree $(3,13,14)$ at least. It is proved in [KP2] that 9_{31} is the harmonic Fibonacci knot (T_3,T_{10},T_{17}). The torus knot 9_1 and the twist knots $9_2,9_3,9_4$, have lexicographic degree $(3,13,14)$, see [BKP2]. For the remaining knots, we proceed as follows.

- By T-reduction, the diagram $D(2,2,1,2,2)$ reduces to $D(0,0)$, that has algebraic degree 1. We deduce that the knot $9_{23} = C(2,2,1,2,2)$ has lexicographic degree $(3,10,17)$.
- The alternating diagrams of $9_{11}, 9_{13}$, 9_{17}, 9_{20}, 9_{26} and 9_{27} can be reduced to $D(3)$ by two T-reductions. Their lexicographic degree is then $(3,10,17)$.
- The plane alternating diagram of 9_8 is reduced to $D(1,3,2)$ by T-reduction. The algebraic degree of $D(1,3,2)$ is at most the degree of $D(4,2)$, that is 8. On the other hand, the plane projection of the diagram $C(2,1,4,1,1)$ can be reduced to $D(1,2,0)$ that has degree 8.
- The plane alternating diagrams of the knots 9_5, 9_{12}, 9_{14}, 9_{19} and 9_{21} can be reduced by two T-reductions to $D(3,0)$. Hence these diagrams have algebraic degree 11. On the other hand, any other diagram of these knots will be non-alternating with at least 10 crossing points. Hence the lexicographic degree of these knots is then $(3,11,16)$.
- The alternating diagram of 9_{15} is $C(2,2,3,2)$. From Proposition 3.2, its lexicographic degree is at least $(3,13,14)$. Any other non alternating diagram of 9_{15} will have 10 or more crossings. Consider the diagram $C(2,2,2,1,-3)$ of 9_{15}. Its projection $D(2,2,2,1,3)$ can be reduced to $D(1,0)$ by three T-reductions. Consequently 9_{15} has degree $(3,11,16)$.
- The alternating diagram of 9_6 is $C(5,2,2)$. From Proposition 3.2, its lexicographic degree is at least $(3,13,14)$. The only diagrams of 9_6 having 10 crossings are $C(2,1,1,-6)$ and $C(5,1,1,-3)$, whose plane diagrams reduce to $D(7)$ by T-reductions. Hence the lexicographic degree of 9_6 is $(3,11,16)$.
- The alternating diagram of 9_7 is $C(3,4,2)$. From Proposition 3.2, its lexicographic degree is at least $(3,13,14)$. The only diagrams of 9_7 having 10 crossings are $C(2,3,1,-4)$, and $C(3,3,1,-3)$. The plane diagrams $D(2,3,1,4)$ reduces to $D(2,2,3)$ and $D(3,3,1,3)$ to $D(3,2,2)$ by a T-reduction. Their degrees are at least 14 by Proposition 4.10.
- The alternating diagram of 9_9 is $C(4,2,3)$. From Proposition 3.2, its lexicographic degree is at least $(3,13,14)$. The only diagrams of 9_9 having 10 crossings are $C(3,1,1,-5)$ and $C(4,1,1,-4)$, whose plane diagrams reduce to $D(7)$ by T-reductions. Their lexicographic degrees are then $(3,13,14)$.
- The alternating diagram of 9_{18} is $C(3,2,2,2)$. From Proposition 3.2, its lexicographic degree is at least $(3,13,14)$. The only diagrams of 9_{18} having 10 crossings are $C(3,1,1,-3,-2)$ whose plane projection reduces to $D(5,2)$, $C(2,2,1,1,-4)$ whose plane projection reduces to $D(2,5)$, $C(2,1,1,-3,-3)$ whose plane projection reduces to $D(4,3)$, and $C(3,2,1,1,-3)$, whose plane projection reduces to $D(3,4)$. By Proposition 3.7, the degree of these four plane diagrams with seven crossings is at least 10, so the degree of the four plane diagrams with 10 crossings is at least 13 by Proposition 3.9.
- The alternating diagram of 9_{10} is $C(3,3,3)$. Suppose that there exists a polynomial parametrisation $\gamma : t \mapsto (P(t), Q(t))$ of the plane diagram $D(3,3,3)$ with $\deg(P) = 3$

and $\deg(Q) = 10$. We denote by $C = \gamma(\mathbf{C})$. Since the curve C has 9 real crossings, it has no additional nodes. The braid associated to C is

$$b_C = \sigma_1^{-1}\sigma_2^{-1}\sigma_1^{-2}\sigma_2^{-3}\sigma_1^{-3}\sigma_2^{-2}(\sigma_1\sigma_2\sigma_1)^4.$$

Since this braid is not the trivial braid, we obtain a contradiction. Hence the alternating diagram $C(3,3,3)$ has degree at least $(3,11,16)$. On the other hand, the projection of the diagram $C(3,2,1,-4)$ of 9_{10} reduces to $D(2,2)$. Since this latter has algebraic degree 5, we deduce that 9_{10} has lexicographic degree $(3,11,16)$. □



$C(3,3,3)$ $C(3,2,1,-4)$.

Figure 20: Two diagrams of 9_{10}

4.7 Two-bridge knots with crossing number 10 or 11

The lexicographic degrees of the torus knot $C(11)$ and the twist knots $C(8,2)$, $C(9,2)$, $C(8,3)$, $C(6,4)$, $C(7,4)$ and $C(6,5)$ have been established in [BKP2]. For the 129 remaining knots with 10 or 11 crossings, we simply sketch all computations. For only 11 knots among the 186 knots with 11 crossings or fewer — 10_{11}, 10_{13}, $11a_{98}$, $11a_{166}$, $11a_{230}$, $11a_{235}$, $11a_{238}$, $11a_{311}$, $11a_{335}$, $11a_{359}$ and $11a_{365}$ — the lower bounds differ from the upper bounds in the strategy described in Section 4.1, i.e. one has to go through step 4. The projections of all the corresponding diagrams reduce by T-reduction to a finite list of eleven plane diagrams:

- $D(3,3,3)$ and $D(3,3,4)$, that have degree 13 at least,
- $D(3,3,5)$ and $(3,5,3)$, that have degree 14 at least,
- $D(3,3,6)$, $D(3,5,4)$, $D(3,2,3,4)$, $D(3,2,3,5)$, $D(3,2,5,3)$, that have degree 16 at least,
- $D(3,3,2,5)$ and $D(4,2,3,4)$ that have degree $(3,17)$ at least.

These results have been obtained by computing all possible braids associated to hypothetical plane curves of degree $b < b_0$ that are $\mathcal{L}$-isotopic to the diagram, and checking, like in Proposition 4.10, if these braids satisfy Proposition 3.4.

5 Conclusion

We list in Table 5 the lexicographic degrees of the first 186 two-bridge knots. We only write b, bearing in mind that the corresponding lexicographic degree is $(3, b, 3N - b)$. Details of our results will be available in https://.../2bk-lexdeg.html

Name	Deg.	Name	Deg.	Name	Deg.	Name	Deg.	Name	Deg.	Name	Deg.
3_1	4	4_1	5	5_1	7	5_2	7	6_1	8	6_2	7
6_3	7	7_1	10	7_2	10	7_3	10	7_4	8	7_5	10
7_6	8	7_7	8	8_1	11	8_2	10	8_3	11	8_4	10
8_6	10	8_7	10	8_8	10	8_9	10	8_{11}	10	8_{12}	11
8_{13}	10	8_{14}	10	9_1	13	9_2	13	9_3	13	9_4	13
9_5	11	9_6	13	9_7	13	9_8	11	9_9	13	9_{10}	11
9_{11}	10	9_{12}	11	9_{13}	10	9_{14}	11	9_{15}	11	9_{17}	10
9_{18}	13	9_{19}	11	9_{20}	10	9_{21}	11	9_{23}	10	9_{26}	10
9_{27}	10	9_{31}	10	10_1	14	10_2	13	10_3	14	10_4	13
10_5	13	10_6	13	10_7	13	10_8	13	10_9	13	10_{10}	13
10_{11}	13	10_{12}	13	10_{13}	14	10_{14}	13	10_{15}	13	10_{16}	11
10_{17}	13	10_{18}	13	10_{19}	13	10_{20}	13	10_{21}	13	10_{22}	13
10_{23}	13	10_{24}	13	10_{25}	13	10_{26}	13	10_{27}	13	10_{28}	11
10_{29}	11	10_{30}	11	10_{31}	13	10_{32}	13	10_{33}	11	10_{34}	13
10_{35}	14	10_{36}	13	10_{37}	13	10_{38}	11	10_{39}	13	10_{40}	13
10_{41}	11	10_{42}	11	10_{43}	11	10_{44}	11	10_{45}	11	$11a_{13}$	14
$11a_{59}$	14	$11a_{65}$	14	$11a_{75}$	13	$11a_{77}$	13	$11a_{84}$	13	$11a_{85}$	13
$11a_{89}$	13	$11a_{90}$	13	$11a_{91}$	13	$11a_{93}$	13	$11a_{95}$	13	$11a_{96}$	14
$11a_{98}$	14	$11a_{110}$	13	$11a_{111}$	13	$11a_{117}$	13	$11a_{119}$	14	$11a_{120}$	13
$11a_{121}$	14	$11a_{140}$	13	$11a_{144}$	13	$11a_{145}$	14	$11a_{154}$	14	$11a_{159}$	14
$11a_{166}$	14	$11a_{174}$	13	$11a_{175}$	13	$11a_{176}$	13	$11a_{177}$	13	$11a_{178}$	13
$11a_{179}$	13	$11a_{180}$	13	$11a_{182}$	13	$11a_{183}$	13	$11a_{184}$	13	$11a_{185}$	13
$11a_{186}$	13	$11a_{188}$	13	$11a_{190}$	13	$11a_{191}$	13	$11a_{192}$	13	$11a_{193}$	13
$11a_{195}$	14	$11a_{203}$	13	$11a_{204}$	13	$11a_{205}$	13	$11a_{206}$	13	$11a_{207}$	13
$11a_{208}$	13	$11a_{210}$	14	$11a_{211}$	14	$11a_{220}$	13	$11a_{224}$	13	$11a_{225}$	13
$11a_{226}$	14	$11a_{229}$	14	$11a_{230}$	14	$11a_{234}$	16	$11a_{235}$	16	$11a_{236}$	16
$11a_{238}$	16	$11a_{242}$	16	$11a_{243}$	16	$11a_{246}$	16	$11a_{247}$	16	$11a_{306}$	13
$11a_{307}$	13	$11a_{308}$	13	$11a_{309}$	13	$11a_{310}$	13	$11a_{311}$	14	$11a_{333}$	14
$11a_{334}$	16	$11a_{335}$	16	$11a_{336}$	13	$11a_{337}$	13	$11a_{339}$	16	$11a_{341}$	13
$11a_{342}$	16	$11a_{343}$	14	$11a_{355}$	16	$11a_{356}$	13	$11a_{357}$	13	$11a_{358}$	16
$11a_{359}$	14	$11a_{360}$	13	$11a_{363}$	14	$11a_{364}$	16	$11a_{365}$	14	$11a_{367}$	16

Table 5: Two-bridge knots with crossing number at most 11 and their y-lexicographic degree

In Table 6, we list all knots for which the algebraic degrees of their alternating diagrams are greater than their lexicographic degrees. The third column of Table 6 gives a diagram obtained by a polynomial parametrisation of lexicographic degree, the fourth column indicates a construction of the corresponding xy-plane diagram (the notation is explained in Section 3), the fifth column gives the alternating trigonal diagram of the knot, and the last column gives a lower bound on its y-degree.

Name	y-lex. degree	Lex. deg. diagram	Constr.	Alt. diagram	y-lex. degree $\geq$
8_6	10	$C(2,2,1,-4)$	$D(3)+2\mathrm{T}$	$C(3,3,2)$	11
9_{10}	11	$C(3,2,1,-4)$	$D(0,1)+3\mathrm{T}$	$C(3,3,3)$	13
9_{15}	11	$C(2,2,1,-3,-2)$	$D(1,0)+3\mathrm{T}$	$C(2,2,3,2)$	13
10_{24}	13	$C(2,2,1,-3,-3)$	$D(0,2)+3\mathrm{T}$	$C(3,2,3,2)$	14
$11a_{75}$	13	$C(2,1,3,2,1,-3)$	$D(3)+3\mathrm{T}$	$C(2,1,3,3,2)$	14
$11a_{84}$	13	$C(2,2,1,-3,-1,-1,-2)$	$D(0,0)+4\mathrm{T}$	$C(2,1,1,2,3,2)$	14
$11a_{144}$	13	$C(2,2,2,1,-5)$	$D(3)+3\mathrm{T}$	$C(4,3,2,2)$	14
$11a_{186}$	13	$C(2,2,2,1,-3,-2)$	$D(0,0)+4\mathrm{T}$	$C(2,2,3,2,2)$	16
$11a_{193}$	13	$C(2,1,1,1,2,1,-4)$	$D(3)+3\mathrm{T}$	$C(3,3,1,1,1,2)$	14
$11a_{205}$	13	$C(2,2,1,-2,-1,-1,-3)$	$D(3)+3\mathrm{T}$	$C(3,1,1,1,3,2)$	14
$11a_{208}$	13	$C(2,1,1,-2,-1,-2,-3)$	$D(3)+3\mathrm{T}$	$C(3,2,1,1,2,2)$	14
$11a_{224}$	13	$C(3,2,1,-3,-1,-2)$	$D(0,0)+4\mathrm{T}$	$C(3,3,2,1,2)$	14
$11a_{225}$	13	$C(2,3,1,-2,-4)$	$D(3)+3\mathrm{T}$	$C(4,1,4,2)$	14
$11a_{229}$	14	$C(2,2,1,-3,-4)$	$D(0,3)+3\mathrm{T}$	$C(4,2,3,2)$	16
$11a_{341}$	13	$C(3,1,3,1,-4)$	$D(3)+3\mathrm{T}$	$C(3,1,4,3)$	14
$11a_{356}$	13	$C(3,2,1,-3,-3)$	$D(3)+3\mathrm{T}$	$C(3,2,3,3)$	16

Table 6: Knots for which the alternating diagram is not of minimal degree

For $N \geq 12$ and $N+4 \leq b < \lfloor \frac{3N-1}{2} \rfloor$, it could be interesting to determine the lexicographic degree, as we do not know if $b+c = 3N$. For some knots, it could be interesting to determine explicit constructions with the lexicographic degree.

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