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STABILIZED GALERKIN METHODS FOR MAGNETIC ADVECTION

Holger Heumann[1] and Ralf Hiptmair[2]

Abstract. Taking the cue from stabilized Galerkin methods for scalar advection problems, we adapt the technique to boundary value problems modeling the advection of magnetic fields. We provide rigorous *a priori* error estimates for both fully discontinuous piecewise polynomial trial functions and $\boldsymbol{H}(\mathbf{curl}, \Omega)$-conforming finite elements.

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1. Introduction

The behavior of electromagnetic fields in the stationary flow field of a conducting fluid can be modelled by the (non-dimensional) advection-diffusion equation [16], Section 5

$$\underbrace{\mathbf{curl}\,\nu\,\mathbf{curl}\,\mathbf{A}}_{\text{diffusion}} + \underbrace{\alpha\mathbf{A}}_{\text{dissipation}} + \underbrace{\mathbf{curl}\,\mathbf{A}\times\boldsymbol{\beta} + \mathbf{grad}\,(\mathbf{A}\cdot\boldsymbol{\beta})}_{\text{advection}} = \mathbf{f} \quad \text{in } \Omega\,. \tag{1.1}$$

Here $\Omega \subset \mathbb{R}^3$ is a bounded domain scaled such that $\operatorname{diam}(\Omega) \approx 1$, and the vector field $\mathbf{A} = \mathbf{A}(\boldsymbol{x})$ stands for the magnetic vector potential. The fluid velocity is $\boldsymbol{\beta} = \boldsymbol{\beta}(\boldsymbol{x})$, of which we assume $\boldsymbol{\beta} \in \boldsymbol{W}^{1,\infty}(\Omega)$ and a scaling that achieves $\max_{\boldsymbol{x}} |\boldsymbol{\beta}(\boldsymbol{x})| \approx 1$. The coefficient $\nu = \nu(\boldsymbol{x}) \geq 0$ controls the strength of magnetic diffusion, whereas the conductivity of the fluid enters through the bounded scalar function $\alpha = \alpha(\boldsymbol{x})$. The model underlying (1.1) is known as quasi-magneto-static with temporal gauge.

Remark 1.1. In a time-harmonic setting with linear materials $\mathbf{A}$ would represent a complex amplitude (phasor). In this case α will turn out to be purely imaginary. We are not going to deal with complex-valued fields in this article. On the other hand, we point out that our theoretical developments still apply to them. Indeed, a purely imaginary coefficient function α enhances stability of the problem and facilitates the numerical analysis of the stabilized Galerkin methods.

The focus of this article is on dominant advection that is, $\nu^{-1}|\boldsymbol{\beta}| \gg 1$. More precisely, we are keen to obtain methods that are robust with respect to the singular perturbation limit $\nu \to 0$. Necessarily, these methods must

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remain viable even if $\nu = 0$. Therefore, we confine the presentation to the pure magnetic advection boundary value problem

$$\begin{aligned} \alpha\mathbf{A} + \mathbf{curl}\,\mathbf{A}\times\boldsymbol{\beta} + \mathbf{grad}\,(\mathbf{A}\cdot\boldsymbol{\beta}) &= \mathbf{f} \quad \text{in } \Omega, \\ \mathbf{A}_{|\Gamma_{\text{in}}} &= \mathbf{g} \quad \text{on } \Gamma_{\text{in}}. \end{aligned} \tag{1.2}$$

Note that in (1.2) we impose Dirichlet boundary conditions only on the inflow boundary Γ_{in}, *i.e.* that part of the domain with $\boldsymbol{\beta}\cdot\mathbf{n} < 0$, with $\mathbf{n}$ the outward normal on $\partial\Omega$. In the case of translational symmetry, known as the transversal electric (TE) setting in computational electromagnetics, (1.2) can be reduced to the 2D boundary value problem on a cross section $\Omega \subset \mathbb{R}^2$

$$\begin{aligned} \alpha\mathbf{u} + \mathbf{grad}(\boldsymbol{\beta}\cdot\mathbf{u}) - \mathbf{R}\boldsymbol{\beta}\,\mathrm{div}(\mathbf{R}\mathbf{u}) &= \mathbf{f} \quad \text{in } \Omega, \\ \mathbf{u}_{|\Gamma_{\text{in}}} &= \mathbf{g} \quad \text{on } \Gamma_{\text{in}}, \end{aligned} \tag{1.3}$$

with the $\frac{\pi}{2}$-rotation matrix $\mathbf{R} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. The numerical experiments reported in Section 5 all rely on this boundary value problem in 2D. Yet, we stress, that it retains all crucial features of (1.2) and the derivation, analysis, and behavior of our numerical method will be very much alike for (1.2) and (1.3).

Remark 1.2. If a diffusive term as in (1.1) is present, one may simply augment our proposed stabilized Galerkin scheme with an interior penalty discontinuous Galerkin (IP-DG) discretization of the $\mathbf{curl}\,\nu\,\mathbf{curl}$-operator, as introduced in [18]. Stability results for the two methods can be combined in a straightforward way. Note that diffusion entails imposing tangential boundary conditions even at the outflow boundary, where boundary layers may emerge.

Magnetic advection-diffusion (1.1) is closely related to advection-diffusion for a scalar function $u : \Omega \to \mathbb{R}$

$$\underbrace{\mathrm{div}\,\nu\,\mathbf{grad}\,u}_{\text{diffusion}} + \underbrace{\alpha u}_{\text{dissipation}} + \underbrace{\boldsymbol{\beta}\cdot\mathbf{grad}\,u}_{\text{advection}} = f \quad \text{in } \Omega. \tag{1.4}$$

This relationship becomes apparent when stating the differential operators in the language of exterior calculus [3], Section 2; both problems turn out to be particular instances of a linear advection-diffusion problem for differential forms. We are not going to dip into details, but instead refer the reader to [16], Section 1 for further explanations. We only emphasize that the close link of (1.1) and (1.4) initially motivated the research underlying this paper, because it strongly suggests that successful numerical approaches to (1.4) can be adapted to (1.1).

Thus, let us recall some numerical methods developed for the scalar advection-diffusion problem (1.4) and its pure transport limit

$$\begin{aligned} \alpha u + \boldsymbol{\beta}\cdot\mathbf{grad}\,u &= f \quad \text{in } \Omega, \\ u_{|\Gamma_{\text{in}}} &= g \quad \text{on } \Gamma_{\text{in}}. \end{aligned} \tag{1.5}$$

The starting point is the observation that straightforward Galerkin finite element discretization of (1.4) suffers from instability. As a consequence, much effort has been devoted to devising stabilized Galerkin methods. We would like to refer to [36], Chapter 3 for a comprehensive presentation.

We can distinguish stabilized Galerkin methods based on either *discontinuous* or *continuous* approximation spaces. Stabilized Galerkin methods with discontinuous approximation spaces, the *stabilized* or *upwind discontinuous Galerkin methods*, *e.g.* [22,28,34], achieve stabilization by means of *upwind* fluxes on element interfaces. Classical stabilized Galerkin methods with continuous approximations spaces are the so-called *residual-based Galerkin methods* [36], Chapter 3.2. These methods, *e.g.* the *streamline diffusion method* [23] or the *Galerkin least-squares method* [24], augment the standard Galerkin formulation adding terms that represent the residual of the original equation. That preserves consistency of the formulation but introduces some sort of artificial diffusion with stabilizing effects.

The stationary magnetic advection problem (1.2) has received much less attention from numerical analysts. Its transient variant is studied in the context of magneto-hydrodynamics (MHD) and eddy current problems with moving conductors. MHD models often tackle the magnetic induction $\mathbf{B}$ and the focus is on divergence constraint preserving finite volume methods, see [13] and the articles cited there. In transient eddy current simulation Lagrangian methods are popular [9], beside numerous ad-hoc approaches based on upwinding [6,14,30]. To the best of our knowledge, hardly any convergence results are available.

Our derivation of a stabilized Galerkin methods for the magnetic advection boundary value problem (1.2) runs parallel to that of the discontinuous Galerkin method for scalar advection. A key tool is the Leibniz rule for transport operators and the corresponding integration by parts formulas. This is elaborated in Section 2.

Piecewise polynomial trial spaces are used for the Galerkin discretization of the resulting variational problems. In Section 3 we briefly review the convergence estimates in the cases of totally discontinuous approximations. As magnetic advection falls into the class of *Friedrichs symmetric operators* [12] we could just appeal to the abstract convergence theory for discontinuous Galerkin approximation from [10,11,26]. Yet, as a stepping stone for further developments, in Section 3 we pursue a different approach.

Our main interest is in the use of $\boldsymbol{H}(\mathbf{curl}, \Omega)$-conforming piecewise polynomial trial spaces that feature tangential continuity across interelement boundaries. Meanwhile such spaces have become well established and they are known as discrete 1-forms or (higher order) edge elements [17, 31, 32]. There are several reasons for insisting on tangential continuity: Firstly, since $\mathbf{A}$ is a magnetic vector potential we want its **curl** to be a well-defined square integrable magnetic flux field. Secondly, $\boldsymbol{H}(\mathbf{curl}, \Omega)$-conforming trial and test spaces pave the way for a stable Galerkin discretization of the magnetic diffusion operator $\mathbf{curl}\,\nu\,\mathbf{curl}$. This is important, because we always regard the discretization of the pure advection problem as a mere building block in schemes for the more general advection-diffusion problem (1.1). Of course, totally continuous $(H^1(\Omega))^3$-conforming trial spaces are an option in principle. However, they usually fail to provide stable Galerkin discretization of the diffusion operator [4, 5]. Therefore, we do not investigate this possibility.

The main result of this article is stated as Theorem 4.2 in Section 4. It reveals that the stabilized Galerkin method with $\boldsymbol{H}(\mathbf{curl}, \Omega)$-conforming approximation spaces enjoys the same rates of convergence as the stabilized Galerkin methods with globally discontinuous approximation spaces. Thus, it suffices to aim stabilization at the discontinuous normal components. In particular, we do not need introduce additional stabilization such as the residual-based techniques for stabilizing Galerkin methods with continuous approximation spaces. The final Section 5 presents various numerical experiments for the 2D boundary value problem (1.3) that confirm that the theoretical estimates are sharp and illustrate the strengths and weaknesses of the method.

Before we plunge into the discussion of discretization, we have to make sure that the boundary value problem (1.2) is well posed. This is guaranteed by the following assumption that will be made throughout the remainder of this article:

Assumption 1.3. We assume that $\alpha \in L^\infty(\Omega)$ and $\boldsymbol{\beta} \in \boldsymbol{W}^{1,\infty}(\Omega)$ are such that

$$\lambda_{\min}\left\{(2\alpha - \operatorname{div}\boldsymbol{\beta})\mathbf{I}_3 + D\boldsymbol{\beta} + (D\boldsymbol{\beta})^T\right\} \geq \alpha_0, \tag{1.6}$$

almost everywhere in Ω for some $\alpha_0 > 0$, where $D\boldsymbol{\beta}$ is the Jacobi matrix of $\boldsymbol{\beta}$.

Here, $\lambda_{\min}$ stands for the smallest eigenvalue of a symmetric matrix and $\mathbf{I}_3$ denotes the 3×3 identity matrix. Assumption 1.3 may seem awkward, but it is nothing but the counterpart of the common assumption $2\alpha - \operatorname{div}\boldsymbol{\beta} > \alpha_0$ for the case of scalar advection. Further explanations on Assumption 1.3 will be given in the beginning of Section 3.

2. Derivation of the method

The derivation of the method follows the derivation of the stabilized discontinuous Galerkin method for scalar advection in [7]. By similar arguments we get stability and consistency of the method.

Let $\mathcal{T}$ be a regular partition of Ω into tetrahedral elements T; h_T is the diameter of T, and $h = \max_{T\in\mathcal{T}} h_T$. The boundary of each element is decomposed into 4 triangles, called *facets*. We assume that each facet f has a distinguished normal $\mathbf{n}_f$. If a facet f is contained in the boundary of some element T then either $\mathbf{n}_f = \mathbf{n}_{\partial T|_f}$ or $\mathbf{n}_f = -\mathbf{n}_{\partial T|_f}$. Then, if $\mathbf{u}$ is a piecewise smooth vector field on $\mathcal{T}$, $\mathbf{u}^+$ and $\mathbf{u}^-$ denote the two different restrictions of $\mathbf{u}$ to f, *e.g.* $\mathbf{u}^+ := \mathbf{u}_{|_{T^+}}$ where element T^+ has outward normal $\mathbf{n}_f$. With these restrictions we define also the jump $[\mathbf{u}]_f = \mathbf{u}^+ - \mathbf{u}^-$ and the average $\{\mathbf{u}\}_f = \frac{1}{2}(\mathbf{u}^+ + \mathbf{u}^-)$. For $f \subset \partial\Omega$, we assume f to be oriented such that $\mathbf{n}_f$ points outwards. Let $\mathcal{F}^\circ$ and $\mathcal{F}^\partial$ be the set of interior and boundary facets. $\mathcal{F}^\partial_-, \mathcal{F}^\partial_+ \subset \mathcal{F}^\partial$ are the sets of facets on the inflow and outflow boundary, respectively.

We define the bilinear mapping:

$$(\mathbf{u},\mathbf{v})_{f,\boldsymbol{\beta}} := \int_f (\boldsymbol{\beta}\cdot\mathbf{n}_f)(\mathbf{u}\cdot\mathbf{v})\,\mathrm{d}S, \tag{2.1}$$

the magnetic advection operator, the Lie derivative,

$$\mathsf{L}_{\boldsymbol{\beta}}\,\mathbf{u} := \mathbf{grad}(\boldsymbol{\beta}\cdot\mathbf{u}) + \mathbf{curl}\,\mathbf{u}\times\boldsymbol{\beta} \tag{2.2}$$

and its formal adjoint

$$\mathcal{L}_{\boldsymbol{\beta}}\,\mathbf{u} := \mathbf{curl}(\boldsymbol{\beta}\times\mathbf{u}) - \boldsymbol{\beta}\,\mathrm{div}\,\mathbf{u}, \tag{2.3}$$

e.g. for smooth $\mathbf{u}$ and $\mathbf{v}$ we have

$$(\mathsf{L}_{\boldsymbol{\beta}}\,\mathbf{u},\mathbf{v})_\Omega - (\mathbf{u},\mathcal{L}_{\boldsymbol{\beta}}\,\mathbf{v})_\Omega = (\mathbf{u},\mathbf{v})_{\partial\Omega,\boldsymbol{\beta}}\,. \tag{2.4}$$

Further let $\mathbf{V}_h$ denote some finite element space of piecewise smooth vector fields. We fix some element T, test (1.2) with $\mathbf{v}$, $\mathbf{v}\in\mathbf{V}_h$, integrate the product over T and apply the partial integration rule (2.4):

$$(\alpha\mathbf{u},\mathbf{v})_T + (\mathbf{u},\mathcal{L}_{\boldsymbol{\beta}}\,\mathbf{v})_T + (\mathbf{u},\mathbf{v})_{\partial T,\boldsymbol{\beta}} = (\mathbf{f},\mathbf{v})_T\,.$$

Summing this equation over all elements yields:

$$(\alpha\mathbf{u},\mathbf{v})_\Omega + \sum_T (\mathbf{u},\mathcal{L}_{\boldsymbol{\beta}}\,\mathbf{v})_T + \sum_T (\mathbf{u},\mathbf{v})_{\partial T,\boldsymbol{\beta}} = (\mathbf{f},\mathbf{v})_\Omega\,,$$

or, if we write the sum over boundaries of elements as sum over facets:

$$(\alpha\mathbf{u},\mathbf{v})_\Omega + \sum_T (\mathbf{u},\mathcal{L}_{\boldsymbol{\beta}}\,\mathbf{v})_T + \sum_{f\in\mathcal{F}^\circ} \left(\mathbf{u}^+,\mathbf{v}^+\right)_{f,\boldsymbol{\beta}} - \left(\mathbf{u}^-,\mathbf{v}^-\right)_{f,\boldsymbol{\beta}} + \sum_{f\in\mathcal{F}^\partial} (\mathbf{u},\mathbf{v})_{f,\boldsymbol{\beta}} = (\mathbf{f},\mathbf{v})_\Omega\,.$$

The identity

$$\left(\mathbf{u}^+,\mathbf{v}^+\right)_{f,\boldsymbol{\beta}} - \left(\mathbf{u}^-,\mathbf{v}^-\right)_{f,\boldsymbol{\beta}} = \Big([\mathbf{u}]_f,\{\mathbf{v}\}_f\Big)_{f,\boldsymbol{\beta}} + \Big(\{\mathbf{u}\}_f,[\mathbf{v}]_f\Big)_{f,\boldsymbol{\beta}} \tag{2.5}$$

shows

$$\left(\mathbf{u}^+,\mathbf{v}^+\right)_{f,\boldsymbol{\beta}} - \left(\mathbf{u}^-,\mathbf{v}^-\right)_{f,\boldsymbol{\beta}} = \Big(\{\mathbf{u}\}_f,[\mathbf{v}]_f\Big)_{f,\boldsymbol{\beta}}$$

for smooth solutions of $\mathbf{u}$ of the advection problem (1.2), since $\mathbf{u}$ is non-smooth only across characteristic faces, *i.e.* those faces f with $\mathbf{n}_f \cdot \boldsymbol{\beta} = 0$. But for f with $\mathbf{n}_f \cdot \boldsymbol{\beta} = 0$. we have $(\cdot,\cdot)_{f,\boldsymbol{\beta}} = 0$, anyway. We are now in the position to define a stabilized Galerkin scheme for the advection problem (1.2):

Find $\mathbf{u} \in \mathbf{V}_h$, such that:

$$\mathsf{a}(\mathbf{u},\mathbf{v}) = \mathsf{l}(\mathbf{v}), \quad \forall \mathbf{v} \in \mathbf{V}_h, \tag{2.6}$$

with

$$\mathsf{l}(\mathbf{v}) := (\mathbf{f},\mathbf{v})_\Omega - \sum_{f \in \mathcal{F}^\partial_-} (\mathbf{g},\mathbf{v})_{f,\boldsymbol{\beta}} \tag{2.7}$$

and

$$\begin{aligned} \mathsf{a}(\mathbf{u},\mathbf{v}) := {} & (\alpha\mathbf{u},\mathbf{v})_\Omega + \sum_T (\mathbf{u}, \mathcal{L}_{\boldsymbol{\beta}}\mathbf{v})_T + \sum_{f \in \mathcal{F}^\partial \setminus \mathcal{F}^\partial_-} (\mathbf{u},\mathbf{v})_{f,\boldsymbol{\beta}} \\ & + \sum_{f \in \mathcal{F}^\circ} \left(\{\mathbf{u}\}_f, [\mathbf{v}]_f \right)_{f,\boldsymbol{\beta}} + \left(c_f [\mathbf{u}]_f, [\mathbf{v}]_f \right)_{f,\boldsymbol{\beta}}, \end{aligned} \tag{2.8}$$

where $c_f \in \mathbb{R}$ is a stabilization parameter that will be specified later. Since the stabilization terms $\left(c_f [\mathbf{u}]_f, [\mathbf{v}]_f \right)_{f,\boldsymbol{\beta}}$ vanish for $\mathbf{u}$ solution to (1.2), the derivation proves consistency.

Corollary 2.1. *The variational formulation* (2.6) *is consistent with problem* (1.2).

Remark 2.2. The choice $c_f = \frac{1}{2}\frac{\boldsymbol{\beta}\cdot\mathbf{n}_f}{|\boldsymbol{\beta}\cdot\mathbf{n}_f|}$ yields a scheme with so-called upwind fluxes:

$$\left(\{\mathbf{u}\}_f, [\mathbf{v}]_f \right)_{f,\boldsymbol{\beta}} + \left(c_f [\mathbf{u}]_f, [\mathbf{v}]_f \right)_{f,\boldsymbol{\beta}} = \left(\frac{1}{2}\left(1 + \frac{\boldsymbol{\beta}\cdot\mathbf{n}_f}{|\boldsymbol{\beta}\cdot\mathbf{n}_f|}\right)\mathbf{u}^+ + \frac{1}{2}\left(1 - \frac{\boldsymbol{\beta}\cdot\mathbf{n}_f}{|\boldsymbol{\beta}\cdot\mathbf{n}_f|}\right)\mathbf{u}^-, [\mathbf{v}]_f \right)_{f,\boldsymbol{\beta}}.$$

When we want to implement our variational formulation, we realize that the evaluation of the terms $(\mathbf{u}, \mathcal{L}_{\boldsymbol{\beta}}\mathbf{v})_T$ requires knowledge of first order derivatives of $\boldsymbol{\beta}$ due to $\mathcal{L}_{\boldsymbol{\beta}}\mathbf{v} = \mathbf{curl}(\boldsymbol{\beta}\times\mathbf{v}) + \boldsymbol{\beta}\,\mathrm{div}\,\mathbf{v}$. Therefore, the representation of $\mathsf{a}(\mathbf{u},\mathbf{v})$ in the following proposition is much more convenient for implementation.

Proposition 2.3. *The following equality holds for all* $\mathbf{u},\mathbf{v} \in \mathbf{V}_h$*:*

$$\begin{aligned} \mathsf{a}(\mathbf{u},\mathbf{v}) = {} & (\alpha\mathbf{u},\mathbf{v})_\Omega + \sum_T (\mathbf{curl}\,\mathbf{u}\times\boldsymbol{\beta},\mathbf{v})_T - (\mathbf{u},\boldsymbol{\beta}\mathrm{div}\,\mathbf{v})_T \\ & + \sum_{f\in\mathcal{F}^\circ} \int_f \boldsymbol{\beta}\cdot\{\mathbf{u}\}_f[\mathbf{v}]_f\cdot\mathbf{n}_f\,\mathrm{d}S - \int_f \left([\mathbf{u}]_f\times\mathbf{n}_f\right)\cdot\left(\{\mathbf{v}\}_f\times\boldsymbol{\beta}\right)\mathrm{d}S \\ & + \sum_{f\in\mathcal{F}^\circ} \int_f c_f\boldsymbol{\beta}\cdot[\mathbf{u}]_f[\mathbf{v}]_f\cdot\mathbf{n}_f\,\mathrm{d}S + \int_f c_f\left([\mathbf{u}]_f\times\mathbf{n}_f\right)\cdot\left([\mathbf{v}]_f\times\boldsymbol{\beta}\right)\mathrm{d}S \\ & + \sum_{f\in\mathcal{F}^\partial\setminus\mathcal{F}^\partial_-} \int_f (\boldsymbol{\beta}\cdot\mathbf{u})(\mathbf{v}\cdot\mathbf{n}_f)\,\mathrm{d}S - \sum_{f\in\mathcal{F}^\partial_-}\int_f (\mathbf{u}\times\mathbf{n}_f)\cdot(\mathbf{v}\times\boldsymbol{\beta})\,\mathrm{d}S, \end{aligned}$$

Proof. The proof follows from integration by parts, standard identities from vector calculus and the identity $\mathbf{u}^+\cdot\mathbf{v}^+ - \mathbf{u}^-\cdot\mathbf{v}^- = [\mathbf{u}]_f\cdot\{\mathbf{v}\}_f + \{\mathbf{u}\}_f\cdot[\mathbf{v}]_f$, *cf.* [18]

$$\begin{aligned}
\sum_T (\mathbf{u}, \mathbf{curl}(\boldsymbol{\beta}\times\mathbf{v}))_T &= \sum_T (\mathbf{curl}\,\mathbf{u}, \boldsymbol{\beta}\times\mathbf{v})_T - \int_{\partial T}(\mathbf{u}\times(\boldsymbol{\beta}\times\mathbf{v}))\cdot\mathbf{n}_{\partial T}\,\mathrm{d}S\\
&= \sum_T (\mathbf{curl}\,\mathbf{u}\times\boldsymbol{\beta}, \mathbf{v})_T - \int_{\partial T}(\mathbf{u}\times\mathbf{n}_{\partial T})\cdot(\mathbf{v}\times\boldsymbol{\beta})\,\mathrm{d}S\\
&= \sum_T (\mathbf{curl}\,\mathbf{u}\times\boldsymbol{\beta}, \mathbf{v})_T - \sum_{f\in\mathcal{F}^\partial}\int_f(\mathbf{u}\times\mathbf{n}_f)\cdot(\mathbf{v}\times\boldsymbol{\beta})\,\mathrm{d}S\\
&\quad - \sum_{f\in\mathcal{F}^\circ}\int_f(\mathbf{u}^+\times\mathbf{n}_f)\cdot(\mathbf{v}^+\times\boldsymbol{\beta})\,\mathrm{d}S - \int_f(\mathbf{u}^-\times\mathbf{n}_f)\cdot(\mathbf{v}^-\times\boldsymbol{\beta})\,\mathrm{d}S\\
&= \sum_T (\mathbf{curl}\,\mathbf{u}\times\boldsymbol{\beta}, \mathbf{v})_T - \sum_{f\in\mathcal{F}^\partial}\int_f(\mathbf{u}\times\mathbf{n}_f)\cdot(\mathbf{v}\times\boldsymbol{\beta})\,\mathrm{d}S\\
&\quad - \sum_{f\in\mathcal{F}^\circ}\int_f([\mathbf{u}]_f\times\mathbf{n}_f)\cdot(\{\mathbf{v}\}_f\times\boldsymbol{\beta})\,\mathrm{d}S + \int_f(\{\mathbf{u}\}_f\times\mathbf{n}_f)\cdot([\mathbf{v}]_f\times\boldsymbol{\beta})\,\mathrm{d}S.
\end{aligned}$$

Then we get

$$\begin{aligned}
\mathsf{a}(\mathbf{u},\mathbf{v}) = (\alpha\mathbf{u},\mathbf{v})_\Omega &+ \sum_T (\mathbf{curl}\,\mathbf{u}\times\boldsymbol{\beta},\mathbf{v})_T - (\mathbf{u},\boldsymbol{\beta}\,\mathrm{div}\,\mathbf{v})_T + \sum_{f\in\mathcal{F}^\partial\setminus\mathcal{F}^\partial_-}(\mathbf{u},\mathbf{v})_{f,\boldsymbol{\beta}}\\
&+ \sum_{f\in\mathcal{F}^\circ}\Big(\{\mathbf{u}\}_f,[\mathbf{v}]_f\Big)_{f,\boldsymbol{\beta}} + \Big(c_f\,[\mathbf{u}]_f,[\mathbf{v}]_f\Big)_{f,\boldsymbol{\beta}} - \sum_{f\in\mathcal{F}^\partial}\int_f(\mathbf{u}\times\mathbf{n}_f)\cdot(\mathbf{v}\times\boldsymbol{\beta})\,\mathrm{d}S\\
&- \sum_{f\in\mathcal{F}^\circ}\int_f([\mathbf{u}]_f\times\mathbf{n}_f)\cdot(\{\mathbf{v}\}_f\times\boldsymbol{\beta})\,\mathrm{d}S + \int_f(\{\mathbf{u}\}_f\times\mathbf{n}_f)\cdot([\mathbf{v}]_f\times\boldsymbol{\beta})\,\mathrm{d}S
\end{aligned}$$

and the assertion follows from

$$(\mathbf{B}\cdot\mathbf{N})(\mathbf{U}\cdot\mathbf{V}) - (\mathbf{B}\cdot\mathbf{U})(\mathbf{V}\cdot\mathbf{N}) = (\mathbf{U}\times\mathbf{N})\cdot(\mathbf{V}\times\mathbf{B}). \qquad \square$$

We proceed by proving stability in the mesh dependent norm:

$$\|\mathbf{u}\|_h^2 := \|\mathbf{u}\|^2_{\boldsymbol{L}^2(\Omega)} + \sum_{f\in\mathcal{F}^\circ}\Big\|[\mathbf{u}]_f\Big\|^2_{f,c_f\boldsymbol{\beta}} + \sum_{f\in\mathcal{F}^\partial\setminus\mathcal{F}^\partial_-}\|\mathbf{u}\|^2_{f,\frac{1}{2}\boldsymbol{\beta}} + \sum_{f\in\mathcal{F}^\partial_-}\|\mathbf{u}\|^2_{f,-\frac{1}{2}\boldsymbol{\beta}}, \tag{2.9}$$

with the definition $\|\cdot\|^2_{f,\boldsymbol{\beta}} := (\mathbf{u},\mathbf{u})_{f,\boldsymbol{\beta}}$. $\|\cdot\|_h$ is a norm for any choice c_f with $c_f\boldsymbol{\beta}\cdot\mathbf{n}_f \geq 0$, because then $(c_f\mathbf{u},\mathbf{u})_{f,\boldsymbol{\beta}}$ is non-negative according to the definition of $(\cdot,\cdot)_{f,\boldsymbol{\beta}}$ and $\|\mathbf{u}\|^2_{f,\frac{1}{2}\boldsymbol{\beta}}$, $f\in\mathcal{F}^\partial\setminus\mathcal{F}^\partial_-$ and $\|\mathbf{u}\|^2_{f,-\frac{1}{2}\boldsymbol{\beta}}$, $f\in\mathcal{F}^\partial_-$ are non-negative according to the definition of the inflow boundary.

In the following we will consider the Galerkin formulation (2.8) where the parameter fullfills the following positivity condition.

Assumption 2.4. Assume the parameters c_f in the definition (2.8) satisfy for all faces f the positivity condition

$$c_f\boldsymbol{\beta}\cdot\mathbf{n}_f > 0.$$

Lemma 2.5. *Let the Assumptions 1.3 and 2.4 hold. Then we have for all* $\mathbf{u}\in\mathbf{V}_h$*:*

$$\mathsf{a}(\mathbf{u},\mathbf{u}) \geq \min\left(\frac{1}{2}\alpha_0, 1\right)\|\mathbf{u}\|_h^2.$$

Proof. A short calculation verifies:

$$\begin{aligned}\mathsf{L}_{\boldsymbol{\beta}}\,\mathbf{u}+\mathcal{L}_{\boldsymbol{\beta}}\,\mathbf{u}&=\mathbf{grad}(\boldsymbol{\beta}\cdot\mathbf{u})+\mathbf{curl}\,\mathbf{u}\times\boldsymbol{\beta}+\mathbf{curl}(\boldsymbol{\beta}\times\mathbf{u})-\boldsymbol{\beta}\operatorname{div}\mathbf{u}\\&=D\boldsymbol{\beta}\mathbf{u}+(D\boldsymbol{\beta})^T\mathbf{u}-\operatorname{div}\boldsymbol{\beta}\mathbf{u}\end{aligned}\tag{2.10}$$

and the assertion follows from the partial integration formula (2.4):

$$\begin{aligned}\mathsf{a}\,(\mathbf{u},\mathbf{u})&=(\alpha\mathbf{u},\mathbf{u})_{\Omega}+\sum_T\,(\mathbf{u},\mathcal{L}_{\boldsymbol{\beta}}\,\mathbf{u})_T+\sum_{f\in\mathcal{F}^\partial\setminus\mathcal{F}^\partial_-}(\mathbf{u},\mathbf{u})_{f,\boldsymbol{\beta}}\\&\quad+\sum_{f\in\mathcal{F}^\circ}\Big(\{\mathbf{u}\}_f,[\mathbf{u}]_f\Big)_{f,\boldsymbol{\beta}}+\Big(c_f\,[\mathbf{u}]_f,[\mathbf{u}]_f\Big)_{f,\boldsymbol{\beta}}\\&=(\alpha\mathbf{u},\mathbf{u})_{\Omega}+\sum_T\frac{1}{2}\,(\mathbf{u},(\mathsf{L}_{\boldsymbol{\beta}}+\mathcal{L}_{\boldsymbol{\beta}})\mathbf{u})_T+\sum_{f\in\mathcal{F}^\partial\setminus\mathcal{F}^\partial_-}(\mathbf{u},\mathbf{u})_{f,\boldsymbol{\beta}}\\&\quad+\sum_{f\in\mathcal{F}^\circ}\Big(\{\mathbf{u}\}_f,[\mathbf{u}]_f\Big)_{f,\boldsymbol{\beta}}+\Big(c_f\,[\mathbf{u}]_f,[\mathbf{u}]_f\Big)_{f,\boldsymbol{\beta}}\\&\quad-\frac{1}{2}\sum_{f\in\mathcal{F}^\circ}\Big(\{\mathbf{u}\}_f,[\mathbf{u}]_f\Big)_{f,\boldsymbol{\beta}}+\Big([\mathbf{u}]_f,\{\mathbf{u}\}_f\Big)_{f,\boldsymbol{\beta}}-\frac{1}{2}\sum_{f\in\mathcal{F}^\partial}(\mathbf{u},\mathbf{u})_{f,\boldsymbol{\beta}}\\&=(\alpha\mathbf{u},\mathbf{u})_{\Omega}+\sum_T\frac{1}{2}\,(\mathbf{u},(\mathsf{L}_{\boldsymbol{\beta}}+\mathcal{L}_{\boldsymbol{\beta}})\mathbf{u})_T+\sum_{f\in\mathcal{F}^\circ}\Big(c_f\,[\mathbf{u}]_f,[\mathbf{u}]_f\Big)_{f,\boldsymbol{\beta}}\\&\quad+\frac{1}{2}\sum_{f\in\mathcal{F}^\partial\setminus\mathcal{F}^\partial_-}(\mathbf{u},\mathbf{u})_{f,\boldsymbol{\beta}}-\frac{1}{2}\sum_{f\in\mathcal{F}^\partial_-}(\mathbf{u},\mathbf{u})_{f,\boldsymbol{\beta}}\\&\geq\min(\frac{1}{2}\alpha_0,1)\|\mathbf{u}\|_h^2,\end{aligned}$$

since $(\mathbf{u},\mathbf{u})_{f,\boldsymbol{\beta}}\geq 0$ for $f\in\mathcal{F}^\partial\setminus\mathcal{F}^\partial_-$. □

Remark 2.6. The case $c_f=0$ corresponds to the unstabilized Galerkin formulation, where we have stability only in $\boldsymbol{L}^2\,(\Omega)$:

$$\mathsf{a}\,(\mathbf{u},\mathbf{u})\geq\min(\frac{1}{2}\alpha_0,1)\,\|\mathbf{u}\|^2_{\boldsymbol{L}^2(\Omega)}\,.$$

3. Convergence: discontinuous approximation spaces

In the case of discontinuous approximation spaces, the stabilized Galerkin methods for the magnetic advection problem could be treated in the framework of Friedrichs symmetric operators.

A Friedrichs symmetric operator is a linear first order differential operator of the following form:

$$T\mathbf{v}:=\sum_{j=1}^n\mathbf{B}^{(j)}\partial_j\mathbf{v}+\mathbf{C}\mathbf{v},\tag{3.1}$$

with $\mathbf{B}^{(j)},\mathbf{C}:\Omega\to\mathbb{R}^{n\times n}$, $\mathbf{B}^{(j)T}=\mathbf{B}^{(j)}$ and $\lambda_{\min}\left\{\mathbf{C}+\mathbf{C}^T-\sum_{j=1}^n\partial_j\mathbf{B}^{(j)}\right\}\geq\alpha_0$, $\alpha_0>0$. The short calculation

$$\begin{aligned}\mathsf{L}_{\boldsymbol{\beta}}\,\mathbf{u}&=\mathbf{grad}(\boldsymbol{\beta}\cdot\mathbf{u})+\mathbf{curl}\,\mathbf{u}\times\boldsymbol{\beta}\\&=\begin{pmatrix}\sum_i\partial_1\boldsymbol{\beta}_i\mathbf{u}_i+\sum_i\boldsymbol{\beta}_i\partial_1\mathbf{u}_i\\\sum_i\partial_2\boldsymbol{\beta}_i\mathbf{u}_i+\sum_i\boldsymbol{\beta}_i\partial_2\mathbf{u}_i\\\sum_i\partial_3\boldsymbol{\beta}_i\mathbf{u}_i+\sum_i\boldsymbol{\beta}_i\partial_3\mathbf{u}_i\end{pmatrix}+\begin{pmatrix}\boldsymbol{\beta}_3\partial_3\mathbf{u}_1-\boldsymbol{\beta}_3\partial_1\mathbf{u}_3-\boldsymbol{\beta}_2\partial_1\mathbf{u}_2+\boldsymbol{\beta}_2\partial_2\mathbf{u}_1\\\boldsymbol{\beta}_1\partial_1\mathbf{u}_2-\boldsymbol{\beta}_1\partial_2\mathbf{u}_1-\boldsymbol{\beta}_3\partial_2\mathbf{u}_3+\boldsymbol{\beta}_3\partial_3\mathbf{u}_2\\\boldsymbol{\beta}_2\partial_2\mathbf{u}_3-\boldsymbol{\beta}_2\partial_3\mathbf{u}_2-\boldsymbol{\beta}_1\partial_3\mathbf{u}_1+\boldsymbol{\beta}_1\partial_1\mathbf{u}_3\end{pmatrix}\\&=D\boldsymbol{\beta}^T\mathbf{u}+\sum_{i=1}^3\boldsymbol{\beta}_i\partial_i\mathbf{u}\end{aligned}$$

verifies that the magnetic avection operator in (1.5) is an operator of this type if Assumption 1.3 holds. In [12], Friedrichs gives the general form of admissible boundary conditions that ensure uniqueness of boundary value problems with Friedrichs symmetric operators. Under appropriate smoothness assumptions it then follows that the magnetic advection problem (1.2) with inflow boundary condition has a unique solution if Assumption 1.3 holds. A similar result can be derived within the framework of exterior calculus and Lie derivatives [15], Section 3.4.

We refer to [10], Section 4, [25, 26], Chapter 3 for details on discontinuous Galerkin methods for Friedrichs systems. The typical convergence results for such methods (see [10], Thm. 4.6 Cor. 4.7 or [25], Thm. 50 and Cor. 12) are optimal order convergence in the norm $\|\cdot\|_h$, *i.e.* order $r+\frac{1}{2}$ if r is the polynomial degree of the approximation space and the solution is sufficiently smooth.

Nevertheless we present here a proof that is adapted to the magnetic advection problem and stresses the significance of globally discontinuous approximation spaces for obtaining optimal convergence estimates.

Theorem 3.1. *Let the Assumptions 1.3 and 2.4 hold. Let $\mathbf{V}_h$ be the finite element space of discontinuous piecewise polynomial vector fields:*

$$\mathbf{V}_h = \mathbf{V}^r_{\text{dis}} := \{\mathbf{v} \in \boldsymbol{L}^2(\Omega), \mathbf{v}_{|T} \in (P_r(T))^3, T \in \mathcal{T}\}, \tag{3.2}$$

where P_r, $r \geq 0$ is the space of polynomials of degree r or less. Let $\mathbf{u} \in \boldsymbol{H}^{r+1}(\Omega)$ and $\mathbf{u}_h \in \mathbf{V}_h$ be the solutions to the advection problem (1.2) and its variational formulation (2.6). We get with $C > 0$ depending only on α, β, the polynomial degree and the shape regularity

$$\|\mathbf{u} - \mathbf{u}_h\|_h \leq C h^{r+\frac{1}{2}} \|\mathbf{u}\|_{\boldsymbol{H}^{r+1}(\Omega)}.$$

Proof. Let $\bar{\mathbf{u}}_h$ denote the L^2-projection of $\mathbf{u}$ on $\mathbf{V}_h$. Then stability (Lem. 2.5) and consistency (Cor. 2.1) show

$$\min\left(\frac{1}{2}\alpha_0, 1\right) \|\mathbf{u}_h - \bar{\mathbf{u}}_h\|_h^2 \leq \mathsf{a}\left(\mathbf{u}_h - \bar{\mathbf{u}}_h, \mathbf{u}_h - \bar{\mathbf{u}}_h\right) = \mathsf{a}\left(\boldsymbol{\eta}, \boldsymbol{\gamma}_h\right),$$

with $\boldsymbol{\eta} := \mathbf{u} - \bar{\mathbf{u}}_h$ and $\boldsymbol{\gamma}_h = \mathbf{u}_h - \bar{\mathbf{u}}_h$. Let $\boldsymbol{\beta}_h$ denote the L^2-projection of $\boldsymbol{\beta}$ onto $\mathbf{V}^0_{\text{dis}}$, then $\mathcal{L}_{\boldsymbol{\beta}_h}\boldsymbol{\gamma}_h \in \mathbf{V}_h$, *i.e.* $\left(\boldsymbol{\eta}, \mathcal{L}_{\boldsymbol{\beta}_h}\boldsymbol{\gamma}_h\right)_T = 0$, and

$$\begin{aligned}\mathsf{a}\left(\boldsymbol{\eta}, \boldsymbol{\gamma}_h\right) = (\alpha\boldsymbol{\eta}, \boldsymbol{\gamma}_h)_\Omega + \sum_T \left(\boldsymbol{\eta}, (\mathcal{L}_{\boldsymbol{\beta}} - \mathcal{L}_{\boldsymbol{\beta}_h})\boldsymbol{\gamma}_h\right)_T + \sum_{f \in \mathcal{F}^\partial \setminus \mathcal{F}^\partial_-} (\boldsymbol{\eta}, \boldsymbol{\gamma}_h)_{f,\boldsymbol{\beta}} \\ + \sum_{f \in \mathcal{F}^\circ} \left(\{\boldsymbol{\eta}\}_f, [\boldsymbol{\gamma}_h]_f\right)_{f,\boldsymbol{\beta}} + \left(c_f [\boldsymbol{\eta}]_f, [\boldsymbol{\gamma}_h]_f\right)_{f,\boldsymbol{\beta}}.\end{aligned} \tag{3.3}$$

The pairing $\left([\boldsymbol{\eta}]_f, [\boldsymbol{\gamma}_h]_f\right)_{f,c_f\boldsymbol{\beta}} = \left(c_f [\boldsymbol{\eta}]_f, [\boldsymbol{\gamma}_h]_f\right)_{f,\boldsymbol{\beta}}$ is a semi-definite bilinear form by the assumption $c_f\boldsymbol{\beta} \cdot \mathbf{n}_f \geq 0$. Hence Cauchy−Schwarz inequalities yield:

$$\begin{aligned}(\boldsymbol{\eta}, \boldsymbol{\gamma}_h)_{f,\boldsymbol{\beta}} &\leq \|\boldsymbol{\eta}\|_{f,\boldsymbol{\beta}} \|\boldsymbol{\gamma}_h\|_{f,\boldsymbol{\beta}}, \text{ for } f \in \mathcal{F}^\partial \setminus \mathcal{F}^\partial_-, \\ \left(c_f^{-1}\{\boldsymbol{\eta}\}_f + [\boldsymbol{\eta}]_f, [\boldsymbol{\gamma}_h]_f\right)_{f,c_f\boldsymbol{\beta}} &\leq \left\|c_f^{-1}\{\boldsymbol{\eta}\}_f + [\boldsymbol{\eta}]_f\right\|_{f,c_f\boldsymbol{\beta}} \left\|[\boldsymbol{\gamma}_h]_f\right\|_{f,c_f\boldsymbol{\beta}}, \text{ for } f \in \mathcal{F}^\circ, \\ \left(\boldsymbol{\eta}, (\mathcal{L}_{\boldsymbol{\beta}} - \mathcal{L}_{\boldsymbol{\beta}_h})\boldsymbol{\gamma}_h\right)_T &\leq \|\boldsymbol{\eta}\|_{\boldsymbol{L}^2(T)} \left\|(\mathcal{L}_{\boldsymbol{\beta}} - \mathcal{L}_{\boldsymbol{\beta}_h})\boldsymbol{\gamma}_h\right\|_{\boldsymbol{L}^2(T)}, \\ (\alpha\boldsymbol{\eta}, \boldsymbol{\gamma}_h)_\Omega &\leq \|\alpha\|_{\boldsymbol{W}^{0,\infty}(\Omega)} \|\boldsymbol{\eta}\|_{\boldsymbol{L}^2(\Omega)} \|\boldsymbol{\gamma}_h\|_{\boldsymbol{L}^2(\Omega)}.\end{aligned}$$

Next we use

- the multiplicative trace inequality,

$$\|\boldsymbol{\eta}\|_{f,c_f\boldsymbol{\beta}}^2 \leq C(h_f^{-1}\,\|\boldsymbol{\eta}\|_{\boldsymbol{L}^2(T)}^2 + h_f|\boldsymbol{\eta}|_{\boldsymbol{H}^1(T)}^2),$$

  with diameter h_f of face f and $C > 0$ depending on the minimum angle of T and $\boldsymbol{\beta}$, that follows analogous to the one for scalar functions in [1], Theorem 3.10;
- the estimate

$$\begin{aligned}\left\|(\mathcal{L}_{\boldsymbol{\beta}} - \mathcal{L}_{\boldsymbol{\beta}_h})\boldsymbol{\gamma}_h\right\|_{\boldsymbol{L}^2(T)} &\leq |\boldsymbol{\beta} - \boldsymbol{\beta}_h|_{\boldsymbol{W}^{1,\infty}(T)}\,\|\boldsymbol{\gamma}_h\|_{\boldsymbol{L}^2(T)} + \|\boldsymbol{\beta} - \boldsymbol{\beta}_h\|_{\boldsymbol{L}^\infty(T)}\,|\boldsymbol{\gamma}_h|_{\boldsymbol{H}^1(T)}\\ &\leq \max\left(|\boldsymbol{\beta} - \boldsymbol{\beta}_h|_{\boldsymbol{W}^{1,\infty}(T)}\,, C\,|\boldsymbol{\beta}_h|_{\boldsymbol{W}^{1,\infty}(T)}\right)\|\boldsymbol{\gamma}_h\|_{\boldsymbol{L}^2(T)}\,,\end{aligned}$$

  that follows by the product rule;
- the inverse inequality,

$$|\boldsymbol{\gamma}_h|_{\boldsymbol{H}^1(T)} \leq Ch_T^{-1}\,\|\boldsymbol{\gamma}_h\|_{\boldsymbol{L}^2(T)}\,,$$

  with element diameter h_T and $C > 0$ independent of h_T.

In conclusion we find with $C > 0$ depending only on α and $\boldsymbol{\beta}$

$$(\alpha\boldsymbol{\eta}, \boldsymbol{\gamma}_h)_\Omega + \sum_T\,\left(\boldsymbol{\eta}, (\mathcal{L}_{\boldsymbol{\beta}} - \mathcal{L}_{\boldsymbol{\beta}_h})\boldsymbol{\gamma}_h\right)_T \leq C\,\|\boldsymbol{\eta}\|_{\boldsymbol{L}^2(\Omega)}\,\|\boldsymbol{\gamma}_h\|_{\boldsymbol{L}^2(\Omega)}$$

and with $C > 0$ depending on c_f, $\boldsymbol{\beta}$ and the minimum angle of elements of $\mathcal{T}$

$$\begin{aligned}\sum_{f\in\mathcal{F}^\partial\setminus\mathcal{F}^\partial_-}(\boldsymbol{\eta}, \boldsymbol{\gamma}_h)_{f,\boldsymbol{\beta}} + \sum_{f\in\mathcal{F}^\circ}\left(\{\boldsymbol{\eta}\}_f, [\boldsymbol{\gamma}_h]_f\right)_{f,\boldsymbol{\beta}} + \left(c_f\,[\boldsymbol{\eta}]_f, [\boldsymbol{\gamma}_h]_f\right)_{f,\boldsymbol{\beta}}&\\ \leq C\left(h^{-\frac{1}{2}}\,\|\boldsymbol{\eta}\|_{\boldsymbol{L}^2(\Omega)} + h^{\frac{1}{2}}\|\boldsymbol{\eta}\|_{\boldsymbol{H}^1(\Omega)}\right)\|\boldsymbol{\gamma}_h\|_h\,.&\end{aligned}$$

Then triangle inequality and the approximation estimates for $\mathbf{V}_h$, *e.g.*

$$\inf_{\mathbf{w}_h\in(P_k(T))^3}\|\mathbf{u} - \mathbf{w}_h\|_{\boldsymbol{L}^2(T)} \leq Ch^{r+1}\|\mathbf{u}\|_{\boldsymbol{H}^{r+1}(T)},$$

and

$$\inf_{\mathbf{w}_h\in(P_k(T))^3}\|\mathbf{u} - \mathbf{w}_h\|_{\boldsymbol{H}^1(T)} \leq Ch^{r}\|\mathbf{u}\|_{\boldsymbol{H}^{r+1}(T)},$$

yield the assertion. □

For the non-stabilized scheme, *i.e.* $c_f = 0$ in (2.6), we get a sub-optimal convergence estimate, since we have to use another inverse inequality to bound the facet integrals $\|\boldsymbol{\gamma}_h\|_{f,\boldsymbol{\beta}}$ by L^2-norms on elements [7], page 1902.

4. Convergence: $\boldsymbol{H}\,(\mathbf{curl}, \Omega)$-conforming approximation spaces

The crucial step in the Proof of Theorem 3.1, equation (3.3), is based on the property $\mathcal{L}_{\boldsymbol{\beta}_h}\boldsymbol{\gamma}_h \in \mathbf{V}^r_{\text{dis}}$ for $\boldsymbol{\gamma}_h \in \mathbf{V}_h$ and piecewise constant velocity fields $\boldsymbol{\beta}_h$, $\boldsymbol{\beta}_{h|_T} \in (P_0(T))^3$. For approximation spaces $\mathbf{V}_h$ with some continuity across facets this will not hold true in general. At first, a proof similar to the proof of Theorem (3.1) gives only a suboptimal estimate, since in this setting $\left(\boldsymbol{\eta}, \mathcal{L}_{\boldsymbol{\beta}_h}\boldsymbol{\gamma}_h\right) \neq 0$, hence step (3.3) fails and we need an additional inverse estimate for $\|\mathcal{L}_{\boldsymbol{\beta}}\boldsymbol{\gamma}_h\|_{\boldsymbol{L}^2(\Omega)}$.

To adapt the Proof of Theorem 3.1, we need to introduce so-called *averaging interpolation operators* mapping discontinuous piecewise polynomial vector fields to $\boldsymbol{H}\,(\mathbf{curl}, \Omega)$-conforming piecewise polynomial vector fields and discontinuous scalar piecewise polynomial scalar functions to $H^1(\Omega)$-conforming piecewise polynomial functions. Such interpolation operators have been used previously in the analysis of Discontinuous Galerkin methods ([21], Appendix, [18], Appendix, [19], Thm. 5.1, and [27]). We recall the definition (3.2) of the discontinuous finite element vector fields $\mathbf{V}^r_{\mathrm{dis}}$, and introduce $\boldsymbol{H}(\mathbf{curl},\Omega)$-conforming finite element fields

$$\mathbf{V}^r_{\mathrm{cnf}} := \{\mathbf{v} \in \boldsymbol{H}(\mathbf{curl},\Omega)\,,\, \mathbf{v}_{|T} \in (P_r(T))^3\,,\, T \in \mathcal{T}\} \tag{4.1}$$

and the corresponding counterparts for scalar functions; the finite element space of discontinuous piecewise polynomial scalar functions:

$$S^r_{\mathrm{dis}} := \{v \in L^2(\Omega)\,,\, v_{|T} \in P_r(T),\, T \in \mathcal{T}\} \tag{4.2}$$

and the finite element space of continuous piecewise polynomial scalar functions

$$S^r_{\mathrm{cnf}} := \{v \in H^1(\Omega)\,,\, v_{|T} \in P_r(T),\, T \in \mathcal{T}\}. \tag{4.3}$$

Again, P_r is the space of polynomials of degree r or less.

Proposition 4.1. *Let $\mathbf{u} \in \mathbf{V}^r_{\mathrm{dis}}$ and $u \in S^r_{\mathrm{dis}}$. Then there exist $\mathbf{u}^c \in \mathbf{V}^r_{\mathrm{cnf}}$ and $u^c \in S^r_{\mathrm{cnf}}$ such that*

$$\|\mathbf{u} - \mathbf{u}^c\|^2_{\boldsymbol{L}^2(\Omega)} \leq C_1 \sum_{f \in \mathcal{F}^\circ} h_f \int_f \left|[\mathbf{u}]_f \times \mathbf{n}_f\right|^2 \mathrm{d}S \tag{4.4}$$

and

$$\|u - u^c\|^2_{L^2(\Omega)} \leq C_2 \sum_{f \in \mathcal{F}^\circ} h_f \int_f \left|[u]_f\right|^2 \mathrm{d}S\,, \tag{4.5}$$

where h_f is the diameter of facet f and C_1 and C_2 depend only on the shape-regularity and the polynomial degree r, and, in particular, are independent of the mesh size.

Proof. The proof of (4.4) can be found in [18], Appendix. The degrees of freedom of the finite element space $\mathbf{V}^r_{\mathrm{cnf}}$ are associated to the edges, faces and elements of the mesh $\mathcal{T}$. For given $\mathbf{u} \in \mathbf{V}^r_{\mathrm{dis}}$ the degrees of freedom of $\mathbf{V}^r_{\mathrm{cnf}}$ associated to edges and faces are not well-defined. But if we use the average of all one-sided limits we can define a $\boldsymbol{H}(\mathbf{curl},\Omega)$-conforming approximation $\mathbf{u}^c$, that differs from the finite element representation of $\mathbf{u}$ only in those coefficients that are associated to edges and faces. A technical scaling argument then yields the assertion. The proof of (4.5) follows similarly. $\square$

Theorem 4.2. *Let Assumptions 1.3 and 2.4 hold. P_r, $r \geq 0$ is the space of polynomials of degree r or less. Let then $\mathbf{V}_h$ be a finite element space of $\boldsymbol{H}(\mathbf{curl},\Omega)$-conforming piecewise polynomial vector fields of degree r or less:*

$$\mathbf{V}_h = \mathbf{V}^r_{\mathrm{cnf}} := \left\{\mathbf{v} \in \boldsymbol{H}(\mathbf{curl},\Omega)\,,\, \mathbf{v}_{|T} \in (P_r(T))^3\,,\, T \in \mathcal{T}\right\},$$

such that best approximation estimates

$$\min_{\mathbf{w}_h \in \mathbf{V}_h} \|\mathbf{u} - \mathbf{w}_h\|_{\boldsymbol{H}^s(T)} \leq C h^{r+1-s} \|\mathbf{u}\|_{\boldsymbol{H}^{r+1}(T)},\ s = 0, 1, \forall \mathbf{u} \in \boldsymbol{H}^{r+1}(\Omega)$$

hold with constants depending only on shape regularity of the mesh, e.g., *$\mathbf{V}_h$ can belong to one of the two families of spaces proposed in [31] and [32]. Let $\mathbf{u}$ and $\mathbf{u}_h \in \mathbf{V}_h$ be the solutions to the advection problem (1.2) and its discrete variational formulation (2.6). Then, with $C > 0$ depending only on α, $\boldsymbol{\beta}$, the polynomial degree and shape regularity, we get*

$$\|\mathbf{u} - \mathbf{u}_h\|_h \leq C h^{r+\frac{1}{2}} \|\mathbf{u}\|_{\boldsymbol{H}^{r+1}(\Omega)},$$

provided that h is sufficiently small.

Proof. We recall a few important properties of our approximation space $\mathbf{V}_h$. The tangential trace of vector fields in $\mathbf{V}_h$ on the intersection of elements is continuous. The normal trace of $\mathbf{curl}\,\mathbf{u}$, $\mathbf{u} \in \mathbf{V}_h$, is also continuous, since $\mathbf{curl}\,\mathbf{u} \in \boldsymbol{H}(\mathrm{div}, \Omega)$ and piecewise polynomial. The gradient of an element of the $H^1(\Omega)$-conforming finite element space S^r_{cnf} is an element of $\mathbf{V}^{r-1}_{\mathrm{cnf}} \subset \mathbf{V}^r_{\mathrm{cnf}}$.

Let $\bar{\mathbf{u}}_h$ denote the global L^2-projection of $\mathbf{u}$ onto $\mathbf{V}_h$ and define $\boldsymbol{\eta} := \mathbf{u} - \bar{\mathbf{u}}_h$ and $\boldsymbol{\gamma}_h := \mathbf{u}_h - \bar{\mathbf{u}}_h$. At first we recall that by the assumptions of the theorem

$$\|\boldsymbol{\eta}\|_{\boldsymbol{L}^2(T)} \leq Ch^{r+1}\|\mathbf{u}\|_{\boldsymbol{H}^{r+1}(T)}.$$

Then by stability, consistency and $\boldsymbol{\gamma}_h \in \mathbf{V}^r_{\mathrm{cnf}}$:

$$\min(\frac{1}{2}\alpha_0, 1)\,\|\mathbf{u} - \mathbf{u}_h\|_h^2 \leq \mathsf{a}\left(\boldsymbol{\eta}, \boldsymbol{\gamma}_h\right).$$

Let $\boldsymbol{\beta}_h$ be the L^2-projection of $\boldsymbol{\beta}$ onto $\mathbf{V}^0_{\mathrm{dis}}$. As in the Proof of Theorem 3.1, we add and subtract the Lie-derivative with respect to the projected, piecewise constant velocity field $\boldsymbol{\beta}_h$:

$$\begin{aligned}\mathsf{a}\left(\boldsymbol{\eta}, \boldsymbol{\gamma}_h\right) = {}& (\alpha\boldsymbol{\eta}, \boldsymbol{\gamma}_h)_\Omega + \sum_T \left(\boldsymbol{\eta}, (\mathcal{L}_{\boldsymbol{\beta}} - \mathcal{L}_{\boldsymbol{\beta}_h})\boldsymbol{\gamma}_h\right)_T + \left(\boldsymbol{\eta}, \mathcal{L}_{\boldsymbol{\beta}_h}\boldsymbol{\gamma}_h\right)_T \\ &+ \sum_{f\in\mathcal{F}^\partial\setminus\mathcal{F}^\partial_-} (\boldsymbol{\eta}, \boldsymbol{\gamma}_h)_{f,\boldsymbol{\beta}} + \sum_{f\in\mathcal{F}^\circ} \left(\{\boldsymbol{\eta}\}_f, [\boldsymbol{\gamma}_h]_f\right)_{f,\boldsymbol{\beta}} + \left(c_f\,[\boldsymbol{\eta}]_f, [\boldsymbol{\gamma}_h]_f\right)_{f,\boldsymbol{\beta}}.\end{aligned}$$

Yet, as $\sum_T \left(\boldsymbol{\eta}, \mathcal{L}_{\boldsymbol{\beta}_h}\boldsymbol{\gamma}_h\right)_T \neq 0$, in addition we have to prove

$$\left|\sum_T \left(\boldsymbol{\eta}, \mathbf{curl}(\boldsymbol{\gamma}_h \times \boldsymbol{\beta}_h) + \boldsymbol{\beta}_h \,\mathrm{div}\,\boldsymbol{\gamma}_h\right)_T\right| \leq Ch^{-\frac{1}{2}}\,\|\boldsymbol{\eta}\|_{\boldsymbol{L}^2(\Omega)}\,\|\boldsymbol{\gamma}_h\|_h.$$

Since, by (2.10) for piecewise constant $\boldsymbol{\beta}_h$, we have the local identity $\mathsf{L}_{\boldsymbol{\beta}_h} = -\mathcal{L}_{\boldsymbol{\beta}_h}$, this is implied by

$$\left|\sum_T \left(\boldsymbol{\eta}, \boldsymbol{\beta}_h \times \mathbf{curl}\,\boldsymbol{\gamma}_h\right)_T\right| \leq Ch^{-\frac{1}{2}}\,\|\boldsymbol{\eta}\|_{\boldsymbol{L}^2(\Omega)}\,\|\boldsymbol{\gamma}_h\|_h \tag{4.6}$$

and

$$\left|\sum_T \left(\boldsymbol{\eta}, \mathbf{grad}\,(\boldsymbol{\beta}_h \cdot \boldsymbol{\gamma}_h)\right)_T\right| \leq Ch^{-\frac{1}{2}}\,\|\boldsymbol{\eta}\|_{\boldsymbol{L}^2(\Omega)}\,\|\boldsymbol{\gamma}_h\|_h. \tag{4.7}$$

We use the approximation results of Proposition 4.1 to prove the two assertions (4.6) and (4.7). Let $\mathbf{w}^c \in \mathbf{V}^r_{\mathrm{cnf}}$ and $w^c \in S^r_{\mathrm{cnf}}$ be the conforming approximations of $\boldsymbol{\beta}_h \times \mathbf{curl}\,\boldsymbol{\gamma}_h \in \mathbf{V}^r_{\mathrm{dis}}$ and $\boldsymbol{\beta}_h \cdot \boldsymbol{\gamma}_h \in S^r_{\mathrm{dis}}$. Since $\boldsymbol{\eta} = \mathbf{u} - \bar{\mathbf{u}}_h$ and both $\mathbf{w}^c \in \mathbf{V}^r_{\mathrm{cnf}}$ and $\mathbf{grad}\,w^{c,0} \in \mathbf{V}^r_{\mathrm{cnf}}$ we find

$$\begin{aligned}\left|\left(\boldsymbol{\eta}, \boldsymbol{\beta}_h \times \mathbf{curl}\,\boldsymbol{\gamma}_h\right)_\Omega\right| &= \left|\left(\boldsymbol{\eta}, \boldsymbol{\beta}_h \times \mathbf{curl}\,\boldsymbol{\gamma}_h - \mathbf{w}^{c,1}\right)_\Omega\right| \\ &\leq \|\boldsymbol{\eta}\|_{\boldsymbol{L}^2(\Omega)}\left\|\boldsymbol{\beta}_h \times \mathbf{curl}\,\boldsymbol{\gamma}_h - \mathbf{w}^{c,1}\right\|_{\boldsymbol{L}^2(\Omega)}\end{aligned}$$

and

$$\begin{aligned}\left|\left(\boldsymbol{\eta}, \mathbf{grad}\,(\boldsymbol{\beta}_h \cdot \boldsymbol{\gamma}_h)\right)_\Omega\right| &= \left|\sum_T \left(\boldsymbol{\eta}, \mathbf{grad}\,\left(\boldsymbol{\beta}_h \cdot \boldsymbol{\gamma}_h - w^{c,0}\right)\right)_\Omega\right| \\ &\leq C_0 h^{-1}\,\|\boldsymbol{\eta}\|_{\boldsymbol{L}^2(\Omega)}\left\|\boldsymbol{\beta}_h \cdot \boldsymbol{\gamma}_h - w^{c,0}\right\|_{L^2(\Omega)}.\end{aligned}$$

The approximation results (4.4) and (4.5) give

$$\left\|\boldsymbol{\beta}_h \times \mathbf{curl}\,\boldsymbol{\gamma}_h - \mathbf{w}^{c,1}\right\|^2_{\boldsymbol{L}^2(\Omega)} \leq C_1 h \sum_{f\in\mathcal{F}^\circ}\left\|[\boldsymbol{\beta}_h \times \mathbf{curl}\,\boldsymbol{\gamma}_h]_f \times \mathbf{n}_f\right\|^2_{\boldsymbol{L}^2(f)}$$

and

$$\left\|\boldsymbol{\beta}_h \cdot \boldsymbol{\gamma}_h - w^{c,0}\right\|_{L^2(\Omega)}^2 \leq C_2 h \sum_{f \in \mathcal{F}^\circ} \left\|[\boldsymbol{\beta}_h \cdot \boldsymbol{\gamma}_h]_f\right\|_{L^2(f)}^2 .$$

Further we have by inverse inequalities, approximation properties of $\boldsymbol{\beta}_h$ and normal continuity of $\mathbf{curl}\,\boldsymbol{\gamma}_h \in \boldsymbol{H}(\mathrm{div}, \Omega)$:

$$\begin{aligned}
&\left\|[\boldsymbol{\beta}_h \times \mathbf{curl}\,\boldsymbol{\gamma}_h]_f \times \mathbf{n}_f\right\|_{\boldsymbol{L}^2(f)} \\
&\quad \leq \left\|[(\boldsymbol{\beta}_h - \boldsymbol{\beta}) \times \mathbf{curl}\,\boldsymbol{\gamma}_h]_f \times \mathbf{n}_f\right\|_{\boldsymbol{L}^2(f)} + \left\|[\boldsymbol{\beta} \times \mathbf{curl}\,\boldsymbol{\gamma}_h]_f \times \mathbf{n}_f\right\|_{\boldsymbol{L}^2(f)} \\
&\quad \leq C_3 h \left\|[\mathbf{curl}\,\boldsymbol{\gamma}_h]_f\right\|_{\boldsymbol{L}^2(f)} + \left\|[\boldsymbol{\beta} \times \mathbf{curl}\,\boldsymbol{\gamma}_h]_f \times \mathbf{n}_f\right\|_{\boldsymbol{L}^2(f)} \\
&\quad \leq C_3 h^{\frac{1}{2}} \left\|\mathbf{curl}\,\boldsymbol{\gamma}_h\right\|_{\boldsymbol{L}^2(T_1 \cup T_2)} + \left\|[\mathbf{curl}\,\boldsymbol{\gamma}_h]_f \cdot \mathbf{n}_f \boldsymbol{\beta} - \boldsymbol{\beta} \cdot \mathbf{n}_f \left[\mathbf{curl}\,\boldsymbol{\gamma}_h\right]_f\right\|_{\boldsymbol{L}^2(f)} \\
&\quad \leq C_3 h^{-\frac{1}{2}} \left\|\boldsymbol{\gamma}_h\right\|_{\boldsymbol{L}^2(T_1 \cup T_2)} + C_4 \left\|\boldsymbol{\beta} \cdot \mathbf{n}_f \left[\mathbf{curl}\,\boldsymbol{\gamma}_h\right]_f\right\|_{\boldsymbol{L}^2(f)} \\
&\quad \leq C_3 h^{-\frac{1}{2}} \left\|\boldsymbol{\gamma}_h\right\|_{\boldsymbol{L}^2(T_1 \cup T_2)} + C_4 h^{-1} \left\|\boldsymbol{\beta} \cdot \mathbf{n}_f \left[\boldsymbol{\gamma}_h\right]_f\right\|_{\boldsymbol{L}^2(f)}
\end{aligned}$$

and similar by tangential continuity of $\boldsymbol{\gamma}_h \in \boldsymbol{H}\,(\mathbf{curl}, \Omega)$:

$$\begin{aligned}
\left\|[\boldsymbol{\beta}_h \cdot \boldsymbol{\gamma}_h]_f\right\|_{L^2(f)} &\leq \left\|[(\boldsymbol{\beta}_h - \boldsymbol{\beta}) \cdot \boldsymbol{\gamma}_h]_f\right\|_{L^2(f)} + \left\|[\boldsymbol{\beta} \cdot \boldsymbol{\gamma}_h]_f\right\|_{L^2(f)} \\
&\leq C_5 h^{\frac{1}{2}} \left\|\boldsymbol{\gamma}_h\right\|_{\boldsymbol{L}^2(T_1 \cup T_2)} + \left\|\boldsymbol{\beta} \cdot \mathbf{n}_f \left[\boldsymbol{\gamma}_h\right]_f\right\|_{\boldsymbol{L}^2(f)} ,
\end{aligned}$$

with constants C_3, C_4 and C_5 independent of h, and T_1 and T_2 those elements that share f. Hence we have proved

$$\left\|\boldsymbol{\beta}_h \times \mathbf{curl}\,\boldsymbol{\gamma}_h - \mathbf{w}^{c,1}\right\|_{\boldsymbol{L}^2(\Omega)} \leq C_6 h^{-\frac{1}{2}} \left\|\boldsymbol{\gamma}_h\right\|_h$$

and

$$\left\|\boldsymbol{\beta}_h \cdot \boldsymbol{\gamma}_h - w^{c,0}\right\|_{L^2(\Omega)} \leq C_7 h^{\frac{1}{2}} \left\|\boldsymbol{\gamma}_h\right\|_h ,$$

which yields estimates (4.6) and (4.7). □

Remark 4.3. Theorems 4.2 and 3.1 show that the stabilized Galerkin method (2.6) with $\boldsymbol{H}\,(\mathbf{curl}, \Omega)$-conforming approximation spaces provides the same approximation properties as the stabilized Galerkin methods with globally discontinuous approximation spaces. The representation of $\mathsf{a}\,(\cdot, \cdot)$ in Proposition 2.3 reveals that all the terms with jumps in tangential direction, *i.e.* $[\mathbf{u}]_f \times \mathbf{n}_f$, vanish in the case of $\boldsymbol{H}\,(\mathbf{curl}, \Omega)$-conforming approximation spaces. The stabilization or upwinding affects only the normal components.

5. Numerical experiments

In this section we set $\Omega \subset \mathbb{R}^2$ and focus on the advection problem (1.3). We consider simplicial triangulations and finite element approximation spaces $\mathbf{V}_h = \mathbf{V}^r_{\mathrm{dis}} \subset \boldsymbol{L}^2\,(\Omega)$ with no global continuity, and finite element approximation spaces $\mathbf{V}_h = \mathbf{V}^r_{\mathrm{cnf}} \subset \boldsymbol{H}(\mathrm{div}\,\mathbf{R}, \Omega)$, with $\mathbf{R} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, *i.e.* spaces that contain piecewise polynomials that are globally tangential continuous. The derivations and assertions in the previous sections, in particular the Theorems 3.1 and 4.2, remain true this setting.

All calculations are based on the C++/Python finite element library FEniCS/Dolfin [2,29]. We use uniform meshes and interpolate all coefficient functions, like the velocity $\boldsymbol{\beta}$, the boundary data $\mathbf{g}$ or the source term $\mathbf{f}$ into high order Lagrangian finite element spaces. FEniCS/Dolfin automatically applies quadrature rules of sufficient accuracy to evaluate all occurring integrals exactly.

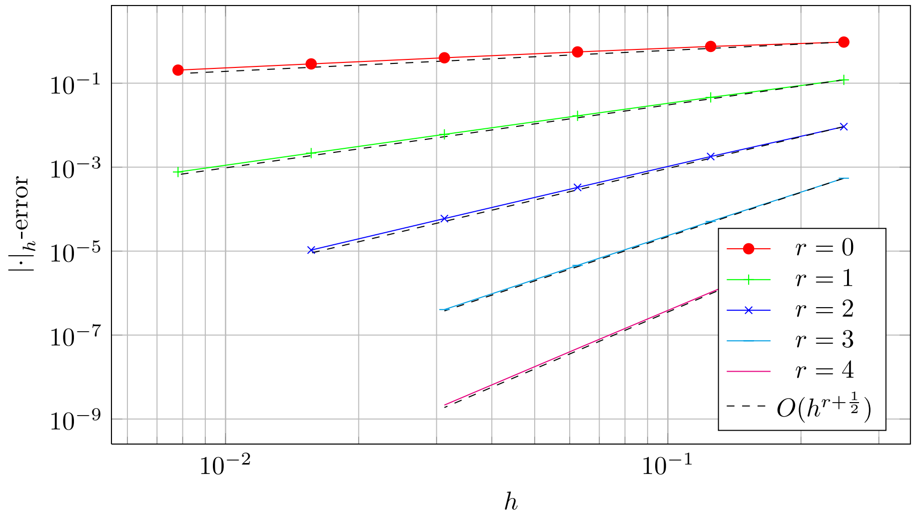

FIGURE 1. Experiment 1: fully discontinuous approximation spaces $\mathbf{V}_h = \mathbf{V}^r_{\text{dis}}$ and stabilization $c_f = \frac{1}{2}\frac{\boldsymbol{\beta}\cdot\mathbf{n}_f}{|\boldsymbol{\beta}\cdot\mathbf{n}_f|}$. The results comply with the assertions of Theorem 3.1.

5.1. Experiment 1: Smooth solution

We set $\Omega = [0,1]^2$, $\alpha = 2$ and take

$$\boldsymbol{\beta}(x,y) = \begin{pmatrix} 0.66(1-x^2) \\ 0.2 + \sin(\pi x) \end{pmatrix}.$$

We chose the data $\mathbf{f}$ and $\mathbf{g}$ such that the smooth vector field

$$\mathbf{u}(x,y) = \begin{pmatrix} \sin(\pi x) \\ (1-x^2)(1-y^2) \end{pmatrix}$$

becomes the solution of (1.3).

We first determine numerical convergence rates for stabilized schemes with stabilization $c_f = \frac{1}{2}\frac{\boldsymbol{\beta}\cdot\mathbf{n}_f}{|\boldsymbol{\beta}\cdot\mathbf{n}_f|}$. Figures 1 and 2 give the error in the semi-norm

$$|\mathbf{u}|_h^2 := \sum_{f\in\mathcal{F}^\circ} \left\| [\mathbf{u}]_f \right\|^2_{f,c_f\boldsymbol{\beta}} + \sum_{f\in\mathcal{F}^\partial\setminus\mathcal{F}^\partial_-} \|\mathbf{u}\|^2_{f,\frac{1}{2}\boldsymbol{\beta}} + \sum_{f\in\mathcal{F}^\partial_-} \|\mathbf{u}\|^2_{f,-\frac{1}{2}\boldsymbol{\beta}}.$$

The observed rates of convergence confirm that the rates of convergence found in Theorems 3.1 and 4.2 are *sharp*.

In the case of no stabilization, *e.g.* $c_f = 0$, we have stability only in $\boldsymbol{L}^2(\Omega)$ (see Rem. 2.6). For this reason the standard analysis for the unstabilized Galerkin schemes yields only suboptimal convergence rates of order r, if r is the polynomial degree of the approximation spaces. Our experiments (see Figs. 3 and 4) show, that this theory is sharp for $\boldsymbol{H}(\operatorname{div}\mathbf{R},\Omega)$-conforming approximation spaces of arbitrary polynomial degree and discontinuous approximation spaces $\mathbf{V}^r_{\text{dis}}$ of odd polynomial degree.

Finally, Figures 5 and 6 show the error for the stabilized schemes in the $\boldsymbol{L}^2(\Omega)$-norm. The rates of convergence improve by $\frac{1}{2}$ compared to the theoretical results for $\boldsymbol{L}^2(\Omega)$ in Theorems 3.1 and 4.2. This phenomenon has also been observed for stabilized Galerkin methods for scalar advection. Only on certain very special meshes, sometimes called *Peterson-meshes*, one could find that the theoretical results are also sharp for the L^2-norm [33,35,37].

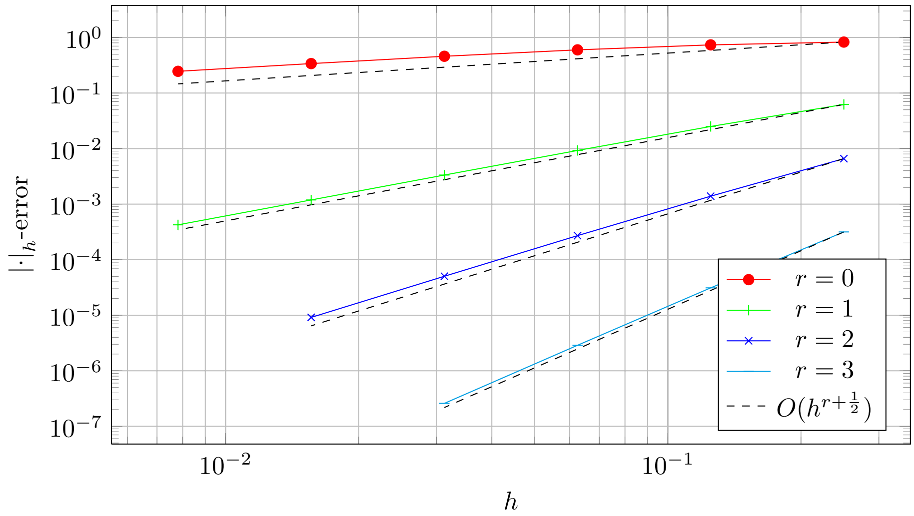

FIGURE 2. Experiment 1: $\boldsymbol{H}(\operatorname{div}\mathbf{R}, \Omega)$-conforming approximation spaces $\mathbf{V}_h = \mathbf{V}^r_{\mathrm{cnf}}$ and stabilization $c_f = \frac{1}{2}\frac{\boldsymbol{\beta}\cdot\mathbf{n}_f}{|\boldsymbol{\beta}\cdot\mathbf{n}_f|}$. The results match the assertions of Theorem 4.2.

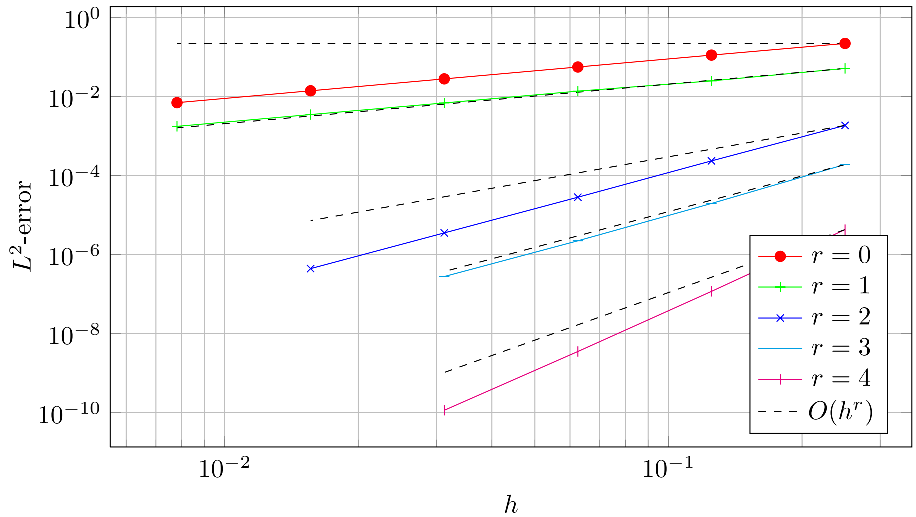

FIGURE 3. Experiment 1: fully discontinuous approximation spaces $\mathbf{V}_h = \mathbf{V}^r_{\mathrm{dis}}$ and no stabilization, *i.e.* $c_f = 0$.

5.2. Experiment 2: Non-smooth data

We set in problem (1.3) $\Omega = [-1, 1]^2$, $\alpha = 0$

$$\boldsymbol{\beta} = \begin{pmatrix} 4(4+y) \\ 4+x \end{pmatrix},$$

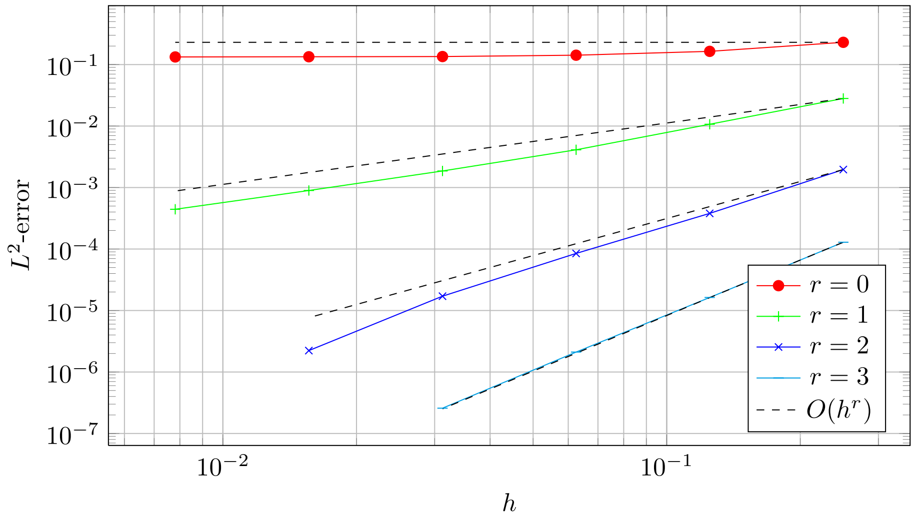

FIGURE 4. Experiment 1: $\boldsymbol{H}(\operatorname{div}\mathbf{R},\Omega)$-conforming approximation spaces $\mathbf{V}_h = \mathbf{V}^r_{\mathrm{cnf}}$ and no stabilization, *i.e.* $c_f = 0$.

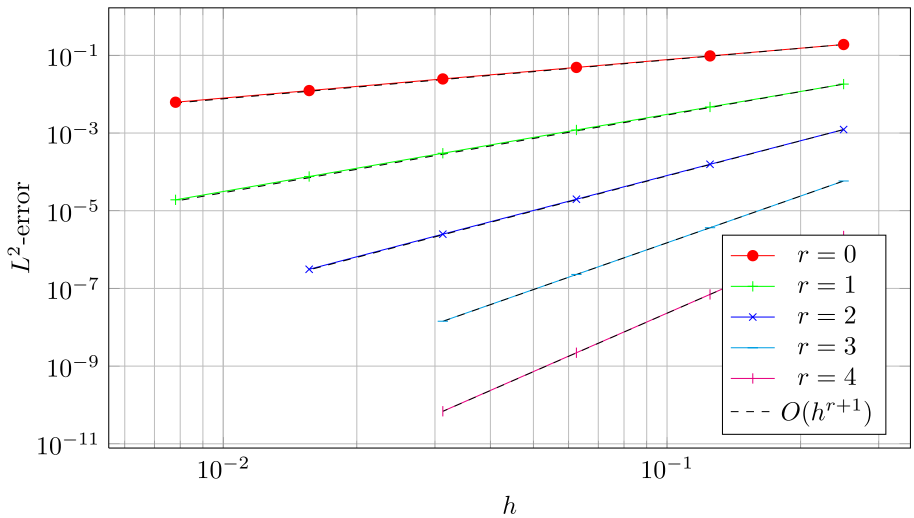

FIGURE 5. Experiment 1: Fully discontinuous approximation spaces $\mathbf{V}_h = \mathbf{V}^r_{\mathrm{dis}}$ and stabilization $c_f = \frac{1}{2}\frac{\boldsymbol{\beta}\cdot\mathbf{n}_f}{|\boldsymbol{\beta}\cdot\mathbf{n}_f|}$. As for scalar problems, on "normal" meshes we observe faster convergence of the L^2-error.

$\mathbf{f} = 0$ and

$$\mathbf{g} = \begin{pmatrix} 1 + \sin(0.5\pi x)\sin(0.5\pi y) \\ -0.5 + \cos(0.5\pi x)\cos(0.5\pi y) \end{pmatrix}.$$

Since $\boldsymbol{\beta}$ is linear we can derive a closed form expression of the solution, which fails to be smooth along the trajectory traced out by the corner point $(-1,-1)$. Figures 7 and 8 show the numerical convergence rates

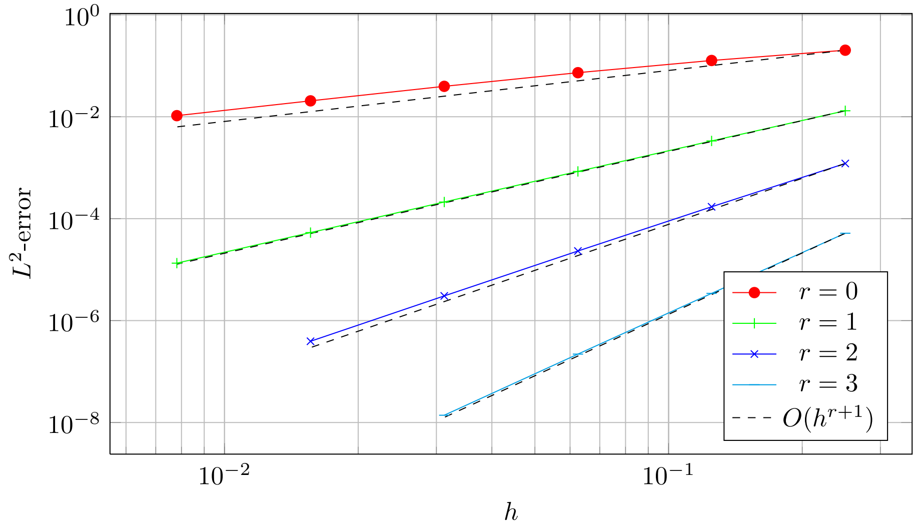

FIGURE 6. Experiment 1: $\boldsymbol{H}(\operatorname{div}\mathbf{R}, \Omega)$-conforming approximation spaces $\mathbf{V}_h = \mathbf{V}^r_{\mathrm{cnf}}$ and stabilization $c_f = \frac{1}{2}\frac{\boldsymbol{\beta}\cdot\mathbf{n}_f}{|\boldsymbol{\beta}\cdot\mathbf{n}_f|}$. As for scalar problems we observe faster convergence of the L^2-error on "normal" meshes.

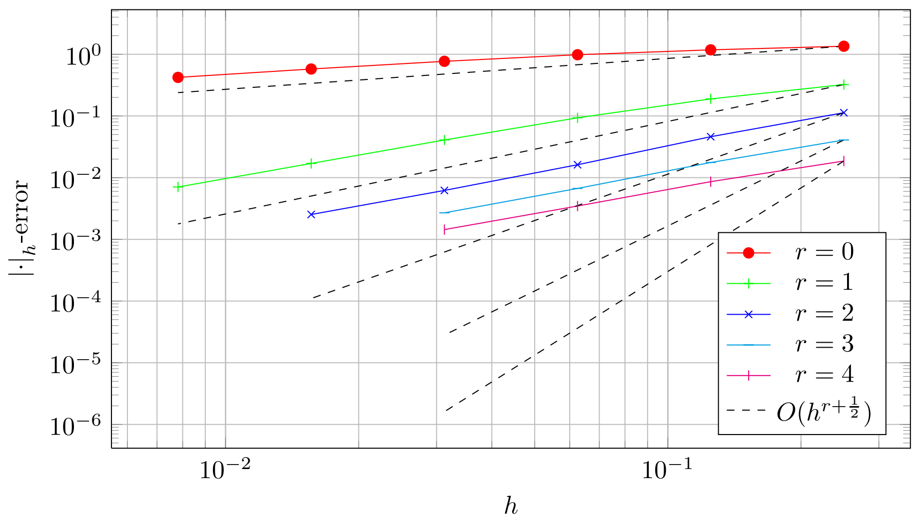

FIGURE 7. Experiment 2: Fully discontinuous approximation spaces $\mathbf{V}_h = \mathbf{V}^r_{\mathrm{dis}}$ with (upwind) stabilization $c_f = \frac{1}{2}\frac{\boldsymbol{\beta}\cdot\mathbf{n}_f}{|\boldsymbol{\beta}\cdot\mathbf{n}_f|}$.

for stabilized schemes, where $c_f = \frac{1}{2}\frac{\boldsymbol{\beta}\cdot\mathbf{n}_f}{|\boldsymbol{\beta}\cdot\mathbf{n}_f|}$. Since the analytic solution is in this case non-smooth along the trajectory of the vertex $(-1, 1)$ we observe reduced convergence rates. Figure 9 visualizes a characteristic error distribution.

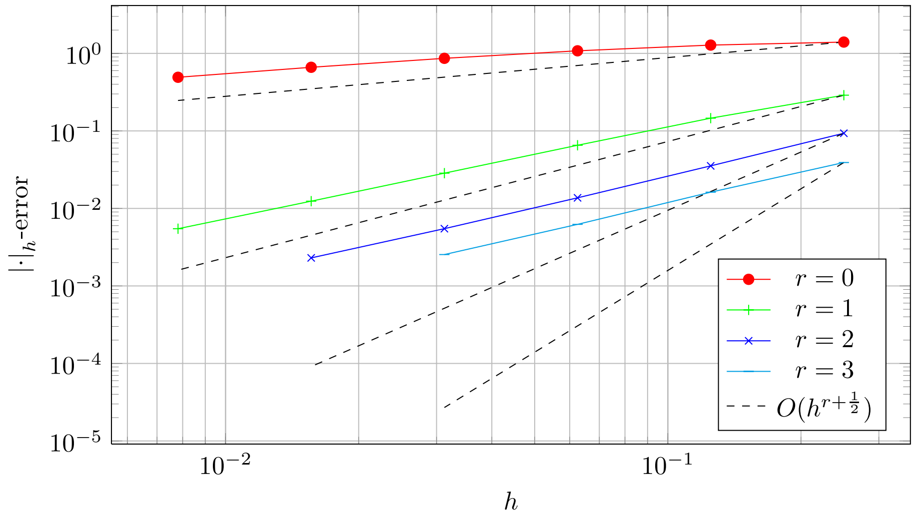

FIGURE 8. Experiment 2: $\boldsymbol{H}(\operatorname{div}\mathbf{R}, \Omega)$-conforming approximation spaces $\mathbf{V}_h = \mathbf{V}^r_{\text{cnf}}$ with (upwind) stabilization $c_f = \frac{1}{2}\frac{\boldsymbol{\beta}\cdot\mathbf{n}_f}{|\boldsymbol{\beta}\cdot\mathbf{n}_f|}$.

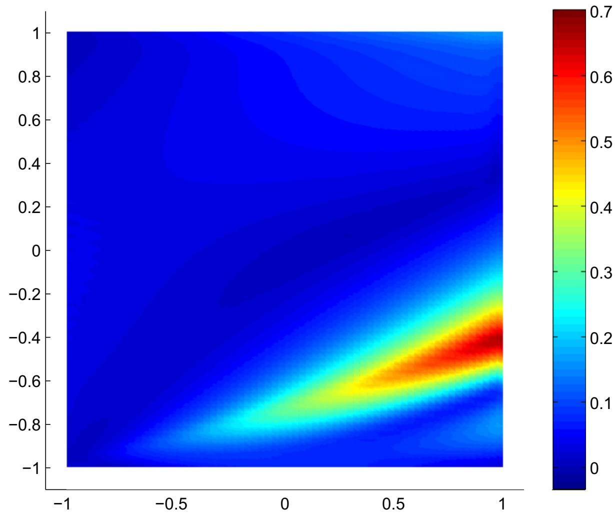

FIGURE 9. Experiment 2: typical distribution of the error.

5.3. Experiment 3: Numerical diffusivity

We set in problem (1.3) $\Omega = [0, 1]^2$, $\alpha = 0$, $\mathbf{f} = 0$,

$$\boldsymbol{\beta} = \begin{pmatrix} -y-1 \\ x+1 \end{pmatrix}, \quad \text{and} \quad \mathbf{g} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{if } x < 0.7 \text{ and else} \quad \mathbf{g} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The lower and the right boundary of the unit square are the inflow part and the discontinuity in the boundary data is advected along the circle $(x + 1)^2 + (y + 1)^2 = 0.7$. We use upwind stabilization in (2.8) and compute numerical solutions for approximation spaces with degree $r = 0, 1, 2$. As it is to be expected for approximations of linear advection problems with non-smooth solutions, we observe pronounced smearing of discontinuities

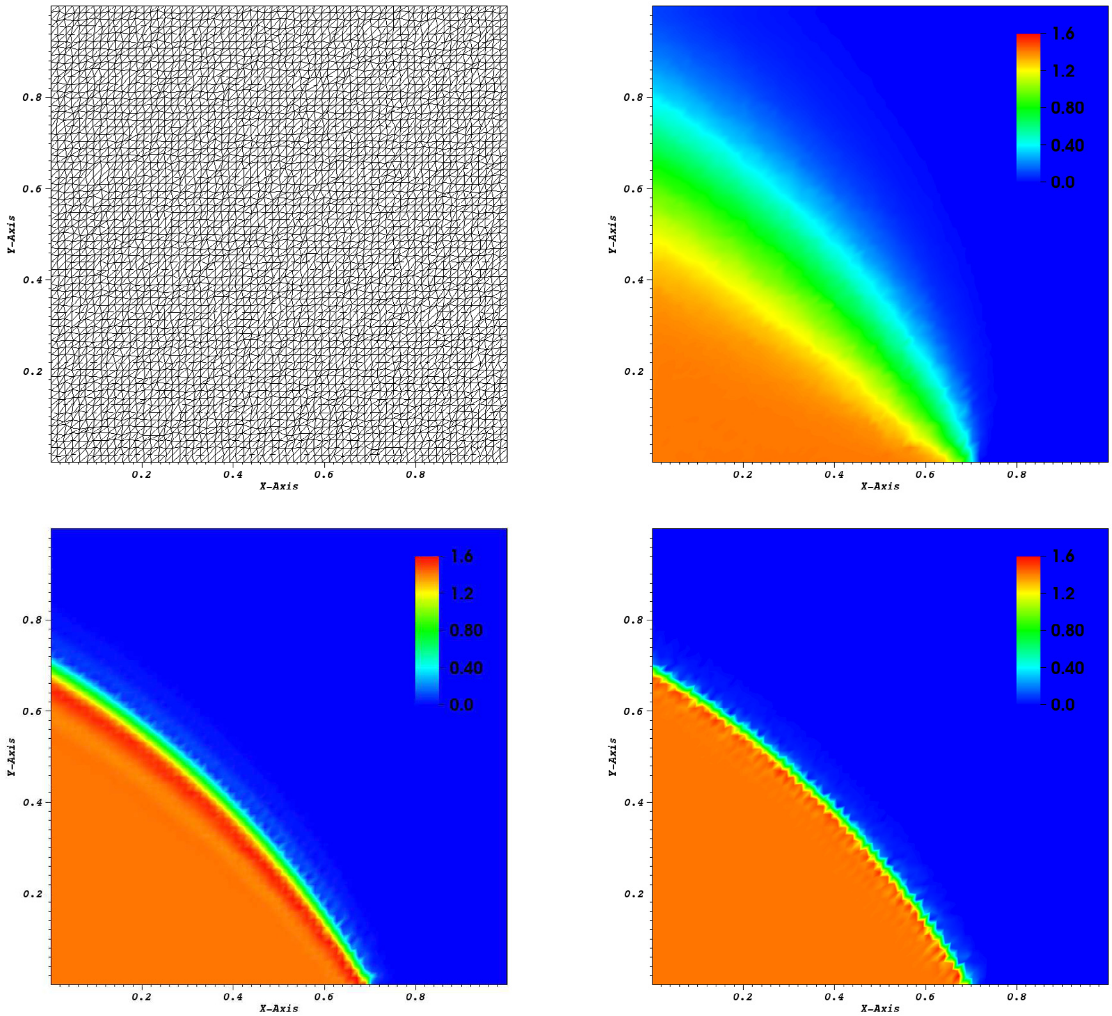

FIGURE 10. Experiment 3: $\boldsymbol{H}(\operatorname{div}\mathbf{R},\Omega)$-conforming approximation spaces $\mathbf{V}_h = \mathbf{V}^r_{\mathrm{cnf}}$ and stabilization $c_f = \frac{1}{2}\frac{\boldsymbol{\beta}\cdot\mathbf{n}_f}{|\boldsymbol{\beta}\cdot\mathbf{n}_f|}$. The magnitude of the numerical solution for polynomial degree $r=0$ (upper right), $r=1$ (lower left) and $r=2$ (lower right), calculated for the mesh shown in the upper left and upwind stabilization $c_f = \frac{1}{2}\frac{\boldsymbol{\beta}\cdot\mathbf{n}_f}{|\boldsymbol{\beta}\cdot\mathbf{n}_f|}$.

for the low order variants of (2.8), which subsides when we increase the polynomial degree (see Fig. 10). In contrast, the solutions based on higher degree polynomials are tainted by localized oscillations in the vicinity of the discontinuity. This, too, is a phenomenon observed with a wide range of methods.

6. Conclusion

We gave a comprehensive *a priori* convergence analysis of a family of stabilized Galerkin formulations of the magnetic advection boundary value problem. Optimal algebraic rates convergence in discrete norms are established rigorously. The method appears to be promising as a foundation for Eulerian discretizations for both magnetohydrodynamic equations and eddy current problems with moving conductors.

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