



HAL
open science

Nevanlinna counting function and pull back measure

Omar El-Fallah, Karim Kellay

► **To cite this version:**

Omar El-Fallah, Karim Kellay. Nevanlinna counting function and pull back measure. Proceedings of the American Mathematical Society, 2016, 144 (6), pp.2559-2564. hal-01107904v2

HAL Id: hal-01107904

<https://hal.science/hal-01107904v2>

Submitted on 8 Apr 2015

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

NEVANLINNA COUNTING FUNCTION AND PULL-BACK MEASURE

O. EL-FALLAH AND K. KELLAY

ABSTRACT. We give an explicit relation between the Nevanlinna counting function of an analytic self-map of the unit disk and its pull-back measure. This gives a simple proof of the results of Lefèvre, Li, Queffélec and Rodríguez-Piazza [3].

1. INTRODUCTION

In this note we study the relationship between the Laplacian of the Nevanlinna counting function of an analytic self-map of the unit disk and its pull-back measure. Our results allow us to refine asymptotic identities involving the Nevanlinna counting function and the pull-back measure given recently by Lefèvre, Li, Queffélec and Rodríguez-Piazza [3]. As consequences of their result, one easily recovers several previously known results about composition operators on the Hardy space, including the characterization of compact composition operators and composition operators belonging to Schatten classes (see [3, 4, 5, 7, 8]).

In order to state our main result, we need some notation. Let φ be a non-constant analytic map of the unit disk \mathbb{D} of the complex plane into itself. The Nevanlinna counting function is defined for every $z \in \mathbb{D} \setminus \{\varphi(0)\}$ by

$$N_\varphi(z) = \begin{cases} \sum_{w \in \varphi^{-1}(z)} \log \frac{1}{|w|} & \text{if } z \in \varphi(\mathbb{D}), \\ 0 & \text{if } z \notin \varphi(\mathbb{D}). \end{cases}$$

Let $\varphi^*(e^{it}) = \lim_{r \rightarrow 1^-} \varphi(re^{it})$ be the radial limit of φ . The function φ^* maps the unit circle \mathbb{T} into the unit disc $\overline{\mathbb{D}}$. The pull-back measure on $\overline{\mathbb{D}}$ of the Lebesgue measure induced by φ^* is given by

$$m_\varphi(B) := |\{\zeta \in \mathbb{T} : \varphi^*(\zeta) \in B\}|,$$

where B is a Borel subset of $\overline{\mathbb{D}}$ and $|E|$ denotes the normalized Lebesgue measure of a Borel subset E of the unit circle \mathbb{T} . For every $\zeta \in \mathbb{T}$ and $0 < h < 1$, the Carleson box $W(\zeta, h) \subset \overline{\mathbb{D}}$ centered at ζ and of size h is the set

$$W(\zeta, h) := \{z \in \overline{\mathbb{D}} : 1 - h \leq |z| \leq 1 \text{ and } |\arg(z\bar{\zeta})| \leq \pi h\}.$$

2010 *Mathematics Subject Classification.* 30C80, 47B33, 47B10, 47A15,

Key words and phrases. Nevanlinna counting function, pull back measure, composition operators.

First authors supported by CNRST (URAC/03) and Académie Hassan II des sciences et techniques.

We use, inside the Carleson box $W(\zeta, h)$, the set

$$S(\zeta, h) := \{z \in \overline{\mathbb{D}} : |z - \zeta| \leq h\},$$

whose size is comparable to that of $W(\zeta, h)$. Denote by $D(a, r)$ the disc of radius r centered at a and as before $S(\zeta, h) := \{z \in \overline{\mathbb{D}} : |z - \zeta| \leq h\}$, for $\zeta \in \overline{\mathbb{D}}$.

Now we are able to state the main result. This is a generalization of a theorem that was obtained by Lefèvre, Li, Queffélec and Rodríguez-Piazza in [3, Theorem 1.1] (see also Remarks (4) below). In fact, our proof of this more general result is simpler than the origin proof from [3].

Theorem 1.1. *For every $0 < c < 1/8$ and $0 < h < (1 - |\varphi(0)|)/8$, we have*

$$m_\varphi[S(\zeta, (1 - c)h)] \leq \frac{2}{c^2} \sup_{z \in S(\zeta, h) \cap \mathbb{D}} N_\varphi(z), \quad \zeta \in \overline{\mathbb{D}} \quad (1.1)$$

and

$$\sup_{z \in W(\zeta, h) \cap \mathbb{D}} N_\varphi(z) \leq \frac{100}{c^2} m_\varphi[W(\zeta, (1 + c)h)], \quad \zeta \in \mathbb{T}. \quad (1.2)$$

2. PROOF

Let H^2 be the Hardy space of the disc,

$$H^2 = \left\{ f = \sum_{n \geq 0} \widehat{f}(n) z^n \in \text{Hol}(\mathbb{D}) : \sum_{n \geq 0} |\widehat{f}(n)|^2 < \infty \right\}.$$

The normalized area measure on \mathbb{D} will be denoted by dA and let $\Delta = 4\partial^2/\partial z \partial \bar{z}$ be the usual Laplacian. For the proof of the above Theorem we need the following key Lemma which gives a generalization of the classical Littlewood-Paley identity.

Lemma 2.1. *For every analytic non-constant self-map $\varphi : \mathbb{D} \rightarrow \mathbb{D}$ and every $g \in \mathcal{C}^2$ on \mathbb{C} we have*

$$\int_{\mathbb{D}} g(z) dm_\varphi(z) = g(\varphi(0)) + \frac{1}{2} \int_{\mathbb{D}} \Delta g(w) N_\varphi(w) dA(w).$$

Proof. Since Δg is bounded on \mathbb{D} , by the change of variable formula, we have

$$\begin{aligned} \int_{\mathbb{D}} \Delta g(w) N_\varphi(w) dA(w) &= \int_{\mathbb{D}} \Delta g(\varphi(w)) |\varphi'(w)|^2 \log \frac{1}{|w|} dA(w) \\ &= \int_{\mathbb{D}} \Delta(g \circ \varphi)(w) \log \frac{1}{|w|} dA(w) \\ &= \lim_{r \rightarrow 1^-} \int_{r\mathbb{D}} \Delta(g \circ \varphi)(w) \log \frac{1}{|w|} dA(w). \end{aligned} \quad (2.1)$$

Now by Green's formula, for $r < 1$,

$$\begin{aligned} \frac{1}{2} \int_{r\mathbb{D}} \Delta(g \circ \varphi)(w) \log \frac{r}{r|w|} dA(w) = \\ -g(\varphi(0)) + \int_{\mathbb{T}} g(\varphi(r\zeta)) \frac{|d\zeta|}{2\pi} + \frac{1}{2} \log \frac{1}{r} \int_{r\mathbb{D}} \Delta(g \circ \varphi)(w) dA(w). \end{aligned} \quad (2.2)$$

Note that for $\rho < 1$,

$$\begin{aligned} \left| \log \frac{1}{\rho} \int_{\rho\mathbb{D}} \Delta(g \circ \varphi)(w) dA(w) \right| &= \left| \log \frac{1}{\rho} \int_{\rho\mathbb{D}} \Delta g(\varphi(w)) |\varphi'(w)|^2 dA(w) \right| \\ &\leq \|\Delta g\|_{\infty} \log \frac{1}{\rho} \int_{\rho\mathbb{D}} |\varphi'(w)|^2 dA(w) \\ &\leq \|\Delta g\|_{\infty} \int_{\mathbb{D}} |\varphi'(w)|^2 \log \frac{1}{|w|} dA(w) \\ &= \frac{1}{2} \|\Delta g\|_{\infty} (\|\varphi\|_{\mathbb{H}^2}^2 - |\varphi(0)|^2). \end{aligned}$$

Hence for $0 < \rho < r < 1$, we get

$$\begin{aligned} \left| \log \frac{1}{r} \int_{r\mathbb{D}} \Delta(g \circ \varphi)(w) dA(w) \right| = \\ \frac{\log(1/r)}{\log(1/\rho)} \log \frac{1}{\rho} \int_{\rho\mathbb{D}} |\Delta(g \circ \varphi)(w)| dA(w) + \log \frac{1}{r} \int_{r\mathbb{D} \setminus \rho\mathbb{D}} |\Delta(g \circ \varphi)(w)| dA(w) \leq \\ \frac{\log(1/r)}{\log(1/\rho)} \|\Delta g\|_{\infty} \|\varphi\|_{\mathbb{H}^2}^2 + \|\Delta g\|_{\infty} \int_{\mathbb{D} \setminus \rho\mathbb{D}} |\varphi'(w)|^2 \log \frac{1}{|w|} dA(w). \end{aligned} \quad (2.3)$$

Finally, if we choose $\rho = e^{-(\log 1/r)^{1/2}}$, by (2.1), (2.2) and (2.3) the result follows from $r \rightarrow 1$. \square

Proof of Theorem 1.1. Let $\zeta \in \overline{\mathbb{D}}$ and consider the function g defined on \mathbb{C} by

$$g(w) = \begin{cases} (h^2 - |w - \zeta|^2)^2 & \text{if } |w - \zeta| \leq h, \\ 0 & \text{if } |w - \zeta| \geq h. \end{cases}$$

First note that for $\epsilon > 0$, $g_{\epsilon} = g^{1+\epsilon} \in \mathcal{C}^2(\mathbb{C})$. Applying Lemma 2.1 with g_{ϵ} and letting ϵ go to 0, we obtain

$$\int_{\mathbb{D}} g(z) dm_{\varphi}(z) = g(\varphi(0)) + \frac{1}{2} \int_{\mathbb{D}} \Delta g(w) N_{\varphi}(w) dA(w). \quad (2.4)$$

We have

$$\Delta g(w) = 8(2|w - \zeta|^2 - h^2).$$

Note that if $z \in S(\zeta, (1-c)h)$, then $g(z) \geq (7^2/2^4)c^2h^4$. By (2.4), we get

$$\begin{aligned}
m_\varphi[S(\zeta, (1-c)h)] &\leq \frac{2^4}{7^2c^2h^4} \int_{S(\zeta, (1-c)h)} g(z) dm_\varphi(z) \\
&\leq \frac{2^3}{7^2c^2h^4} \int_{\mathbb{D}} \Delta g(z) N_\varphi(z) dA(z) \\
&\leq \frac{2^3}{7^2c^2h^4} \int_{h/\sqrt{2} \leq |z-\zeta| \leq h} \Delta g(z) N_\varphi(z) dA(z) \\
&\leq \frac{2^3}{7^2c^2h^4} \sup_{z \in S(\zeta, h)} N_\varphi(z) \int_{h/\sqrt{2} \leq |z-\zeta| \leq h} \Delta g(z) dA(z) \\
&\leq \frac{2^6}{7^2c^2} \sup_{z \in S(\zeta, h)} N_\varphi(z).
\end{aligned}$$

For the second estimate, let $\zeta = z/|z|$, $h = 1 - |z|$, $Z = (1 + \eta h)\zeta$ and $\rho = 1 + 2c + \eta$ where $\eta = (1 + 2c)(1 + \sqrt{2})$. Consider the function g defined by

$$g(w) = \begin{cases} (\rho^2 h^2 - |w - Z|^2)^2 & \text{if } |w - Z| < \rho h, \\ 0 & \text{if } |w - Z| \geq \rho h. \end{cases}$$

As before for $\epsilon > 0$, $g_\epsilon = g^{1+\epsilon} \in \mathcal{C}^2(\mathbb{C})$. Applying Lemma 2.1 with g_ϵ and letting ϵ go to 0 and we get (2.4). We have

$$\Delta g(w) = 8(2|w - Z|^2 - \rho^2 h^2).$$

If $w \in \mathbb{D}$, then $|w - Z| \geq \eta h = \rho h/\sqrt{2}$ and so $\Delta g(w) \geq 0$. Note that $D(z, ch) \subset D(Z, \rho h)$. Also, if $|w - z| \leq ch$ then

$$|w - Z| \geq |Z - z| - |w - z| \geq (1 - c + \eta)h,$$

hence

$$\Delta g(w) \geq 8(2(1 - c + \eta)^2 - (1 + 2c + \eta)^2)h^2 \geq 2^3(\sqrt{2} + 2)h^2, \quad w \in D(Z, \rho h).$$

Note that $\varphi(0) \notin D(z, ch)$, by the sub-mean value property of N_φ (see Remark 3 below) and (2.4), we get

$$\begin{aligned}
N_\varphi(z) &\leq \frac{2}{c^2 h^2} \int_{D(z, ch)} N_\varphi(w) dA(w) \\
&\leq \frac{2}{2^3(\sqrt{2} + 2)c^2 h^4} \int_{D(z, ch)} N_\varphi(w) \Delta g(w) dA(w) \\
&\leq \frac{1}{2^2(\sqrt{2} + 2)c^2 h^4} \int_{\mathbb{D}} N_\varphi(w) \Delta g(w) dA(w) \\
&\leq \frac{2}{2^2(\sqrt{2} + 2)c^2 h^4} \int_{D(Z, \rho h) \cap \overline{\mathbb{D}}} g(w) dm_\varphi(w) \\
&\leq \frac{1}{2(\sqrt{2} + 2)c^2 h^4} \rho^4 h^4 m_\varphi(D(Z, \rho h) \cap \overline{\mathbb{D}}) \\
&\leq \frac{100}{c^2} m_\varphi(D(Z, \rho h) \cap \overline{\mathbb{D}}).
\end{aligned}$$

We claim that

$$D(Z, \rho h) \cap \overline{\mathbb{D}} \subset W(\zeta, (1 + 2c)h).$$

Suppose that $\zeta = 1$ and let $e^{i\theta} \in \partial D(Z, \rho h) \cap \mathbb{T}$. We have

$$\begin{aligned}
\sin^2 \theta &= \frac{1}{4(1 + \eta h)^2} [(1 + \rho h)^2 - (1 + \eta h)^2] [(1 + \eta h)^2 - (1 - \rho h)^2] \\
&\leq (3 + 2\sqrt{2})(1 + 2c)^2 h^2.
\end{aligned}$$

Note that $c < 1/8$ and $h < 1/8$, so we get $\theta \leq (1 + c)h\pi$. Now the proof is complete.

3. REMARKS

1. The following corollary, first proved by Rudin [6], (see also Bishop [1]), is an immediate consequence of the Lemma 2.1

Corollary 3.1 ((Bishop & Rudin)). *We have*

$$\Delta N_\varphi = -\delta_{\varphi(0)} + m_\varphi,$$

where Δ is the distributional Laplacian and δ_{z_0} is the Dirac mass at z_0 . Furthermore N_φ is equal to a subharmonic function almost everywhere on \mathbb{D} .

2. The formula from Lemma 2.1 was given by Stanton [2, Theorem 2] when g is a subharmonic function instead of a smooth function. The proof of Stanton's formula is based on Jensen's formula.

3. By Corollary 3.1, The Nevanlinna counting function N_φ is equal to a subharmonic function almost everywhere on \mathbb{D} . In fact N_φ fails to be subharmonic only on a set of

logarithmic capacity zero [2]. The Nevanlinna counting function satisfies the sub-mean value property

$$N_\varphi(z) \leq \frac{2}{r^2} \int_{D(z,r)} N_\varphi(w) dA(w),$$

for every disc $D(z, r)$ of center z and radius r does not contain $\varphi(0)$ see [7, 8].

4. In the proof in the first inequality (1.1), one can consider the following function $g(w) = (h^2 - \operatorname{Re}(w - \zeta)^2)(h^2 - \operatorname{Im}(w - \zeta)^2)$ if $\operatorname{Re}(w - \zeta) \leq h$ and $\operatorname{Im}(w - \zeta) \leq \pi h$ and $g(w) = 0$ otherwise. We obtain as before :

For every $0 < c < 1/8$ and $0 < h < (1 - |\varphi(0)|)/8$, we have

$$m_\varphi[W(\zeta, (1-c)h)] \leq \frac{A}{c^4} \sup_{z \in W(\zeta, h) \cap \mathbb{D}} N_\varphi(z) \quad \zeta \in \mathbb{T},$$

where A is an absolute constant.

5. The following result of Lefèvre, Li, Queffélec and Rodriguez-Piazza in [3] is based on Stanton's formula. Their proof is quite complicated. They showed that the classical Nevanlinna counting function and the pull back measure are connected as follows :

For every $\zeta \in \mathbb{T}$ and $0 < h < (1 - |\varphi(0)|)/16$, we have

$$\frac{1}{64} m_\varphi[\widetilde{W}(\zeta, h/64)] \leq \sup_{z \in W(\zeta, h)} N_\varphi(z) \leq 196 m_\varphi[\widetilde{W}(\zeta, 24h)],$$

where $\widetilde{W}(\zeta, h) := \{z \in \overline{\mathbb{D}} : 1 - h \leq |z| \leq 1 \text{ and } |\arg(z\bar{\zeta})| \leq h\}$.

6. There are several different conditions which characterize the compactness of the composition operator $C_\varphi(f) = f \circ \varphi$ on the Hardy space H^2 . For the holomorphic self-map φ of \mathbb{D} the following conditions are equivalent :

- (i) $N_\varphi(z) = o(\log 1/|z|)$ as $|z| \rightarrow 1-$,
- (ii) $m_\varphi[W(\zeta, h)] = o(h)$ as $h \rightarrow 0$,
- (iii) C_φ is compact on H^2 .

The condition (ii) means that m_φ is a vanishing Carleson measure for H^2 [5]. For Hilbert-Schmidt class membership see [4]. We refer the reader to the monograph by Shapiro for an account of these problems [7].

REFERENCES

1. C. Bishop , ‘Orthogonal functions in H^∞ ’. *Pacific J. Math.*, 220 (2005) no 1, 1–31.
2. E. Essén; D. F. Shea and C. S. Stanton, ‘A value–distribution criterion for the class $L \log L$ and some related questions.’ *Ann. Inst. Fourier (Grenoble)*, 35 (1985) 125–150.
3. P. Lefèvre; D. Li; H. Queffélec and L. Rodriguez-Piazza, ‘Nevanlinna counting function and Carleson function of analytic maps.’ *Math Ann.*, 351 (2011), 305–326.
4. P. Lefèvre; D. Li; H. Queffélec and L. Rodriguez-Piazza, ‘Some examples of compact composition operators on H^2 ’. *J. Funct. Anal.*, 255 (2008), 3098–3124 .

5. B. MacCluer, 'Compact composition operators on $H^p(\mathbb{B}_N)$ '. *Michigan Math. J.* (32) (1985) 237–248
6. W. Rudin, 'A generalization of a theorem of Frostman'. *Math Scand.*, 21 (1967) 136-143.
7. J. H. Shapiro, *Composition operators and classical function theory*, Springer Verlag, New York 1993.
8. J. H. Shapiro, 'The essential norm of a composition operator', *Annals Math.*, 125 (1987) 375–404.

LABORATOIRE ANALYSE ET APPLICATIONS URAC/03, UNIVERSITÉ MOHAMMED V -RABAT, B.P.
1014 RABAT, MOROCCO

E-mail address: elfallah@fsr.ac.ma

UNIVERSITÉ DE BORDEAUX, IMB, 351 COURS DE LA LIBERATION, 33405 TALENCE, FRANCE

E-mail address: kkellay@math.u-bordeaux1.fr