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Analytical bounds for damage induced planar anisotropy

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Abstract

The damage of a planar, elastic and initially isotropic material is considered in the framework of a classical approach where the damaged elasticity tensor is ruled by a fourth-rank symmetric damage tensor. The analysis is completely carried on using the so-called polar method for the invariant representation of tensors in \mathfrak{R}^2 . The final elastic behavior, induced by damage, can be anisotropic: all the possible situations of elastic symmetries are considered, and for each one an analytical expression for the bounds on the invariants of the damaged elastic tensor and of the damage tensor is given. An admissible domain for the damage invariants and for the damaged elastic invariants is so provided, the convexity of these domains is also proved.

Keywords: Planar elasticity, Damage, Anisotropy, Polar method, Tensor invariants, Explicit bounds.

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1. Introduction

We consider in this paper the anisotropy induced by damage on an initially isotropic layer. The goal is twofold: first, if the elastic tensor of the virgin material is \mathbb{C} , determine which is the final tensor $\tilde{\mathbb{C}}$. Then, to give explicit bounds for the elastic moduli of $\tilde{\mathbb{C}}$ and for the characteristics of the

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damage tensor \mathbb{D} . To this purpose, we assume that \mathbb{D} is a fourth-rank positive semi-definite tensor of the elasticity type and that the elastic tensor of the damaged material linearly depends upon \mathbb{C} and \mathbb{D} , Chaboche (1978), Chaboche (1979), Leckie and Onat (1980), Sidoroff (1980), Chow (1987), Lemaitre and R. Desmorat (2005).

The conditions of positive semi-definiteness for \mathbb{D} and positive definiteness for $\tilde{\mathbb{C}}$, together with a condition stating the effective damage of the material, provide the conditions to determine the bounds on the values of their moduli, once those on \mathbb{C} known.

To investigate this problem, we make use of the so-called *polar formalism*, Verchery (1979); Vannucci (2005). This method gives a representation of elasticity based upon tensor invariants and the different elastic symmetries are readily identified by the values taken by some of these invariants.

We obtain the polar invariants of $\tilde{\mathbb{C}}$ as functions of those of \mathbb{C} and \mathbb{D} ; while we assume that the initial material is isotropic, we consider all the possible transformations for the damaged material, leading to a final elastic behavior that can be completely anisotropic, orthotropic, specially orthotropic or also isotropic.

Then, we pass to consider the bounds that damage process impose to the polar moduli of \mathbb{D} and $\tilde{\mathbb{C}}$; starting from the simpler case, that of an isotropic tensor $\tilde{\mathbb{C}}$, we consider all the possible cases of elastic symmetries for $\tilde{\mathbb{C}}$, until the most general case of complete anisotropy. We give an explicit expression for these bounds and show that the admissible domain for the moduli is convex in all the cases, in some of them a graphical representation is also possible.

2. Essentials of the polar formalism

In this section, we briefly recall the essentials, for the present paper, of the polar formalism and justify the choices that we have made about \mathbb{D} and $\tilde{\mathbb{C}}$. The polar method is basically a mathematical method to search for a complete set of the invariants of a given tensor in \mathfrak{R}^2 . As such, it can be applied not only to elasticity tensors, but also to any other plane tensor, see for instance Vannucci (2007) or Vannucci & Verchery (2010). The polar formalism is based upon a complex variable method, a technique once widely used in physical mathematics and which has its initiators in the pioneer works of Michell (1902) and Kolosov (1909) and which finds its completion in the treatises of Muskhelishvili (1953), Green & Zerna (1954), and Milne-Thomson (1960). As a consequence, the polar formalism can be applied only to plane problems.

Verchery makes use, just like Green and Zerna, of a complex variable transformation, interpreted as a change of frame; this transformation has some algebraic properties that allows simplifying the expressions of frame rotations and symmetries, which renders rather easy the search for tensor invariants. The mathematical details and passages, rather technical and not to be detailed here, can be found in Vannucci (2005) and Vannucci & Verchery (2010), whereto the interested reader is addressed for a complete explanation of the method.

Here, we recall just the main features of the polar method for a plane fourth-rank tensor \mathbb{T} owing the minor and major tensor symmetries, i.e. such that $\forall i, j, k, l = 1, 2$,

$$\begin{aligned} T_{ijkl} &= T_{jikl} = T_{ijlk}, \\ T_{ijkl} &= T_{klij}. \end{aligned} \quad (1)$$

Five invariants suffice to completely describe \mathbb{T} , because in any frame it is completely described by six quantities. Among them, one is needed to fix a frame, so the remaining five ones can be reduced to independent tensor invariants. In the polar formalism, four of these invariants are elastic moduli and are indicated by the symbols T_0 , T_1 , R_0 and R_1 ; the last invariant is the angular difference $\Phi_0 - \Phi_1$. One of the two angles, Φ_0 or Φ_1 , may be arbitrarily chosen to fixe a reference frame (the most usual choice is $\Phi_1 = 0$).

The basic result of the polar formalism is the expression of the Cartesian components of \mathbb{T} in terms of the polar parameters, in a frame rotated through an angle θ :

$$\begin{aligned} T_{1111}(\theta) &= T_0 + 2T_1 + R_0 \cos 4(\Phi_0 - \theta) + 4R_1 \cos 2(\Phi_1 - \theta), \\ T_{1112}(\theta) &= R_0 \sin 4(\Phi_0 - \theta) + 2R_1 \sin 2(\Phi_1 - \theta), \\ T_{1122}(\theta) &= -T_0 + 2T_1 - R_0 \cos 4(\Phi_0 - \theta), \\ T_{1212}(\theta) &= T_0 - R_0 \cos 4(\Phi_0 - \theta), \\ T_{1222}(\theta) &= -R_0 \sin 4(\Phi_0 - \theta) + 2R_1 \sin 2(\Phi_1 - \theta), \\ T_{2222}(\theta) &= T_0 + 2T_1 + R_0 \cos 4(\Phi_0 - \theta) - 4R_1 \cos 2(\Phi_1 - \theta). \end{aligned} \quad (2)$$

The above relations show that T_0 and T_1 are the isotropy invariants, while anisotropy is described by the invariants R_0 , R_1 and $\Phi_0 - \Phi_1$. In particular, it is easy to recognize that for isotropic elasticity,

$$T_0 = G, \quad T_1 = \frac{1}{2}\kappa, \quad (3)$$

where G and κ are respectively the shear and bulk moduli. The same physical meaning is preserved also for all the anisotropic cases, so we can consider T_0

and T_1 as a generalization, to any elastic behavior, of the shear and bulk moduli, respectively. We remark also that

$$T_0 + 2T_1 = G + \kappa, \quad (4)$$

a quantity often appearing in the following.

The relations giving the polar components as functions of the Cartesian ones can be obtained inverting eqs. (2):

$$\begin{aligned} 8T_0 &= T_{1111}(\theta) - 2T_{1122}(\theta) + 4T_{1212}(\theta) + T_{2222}(\theta), \\ 8T_1 &= T_{1111}(\theta) + 2T_{1122}(\theta) + T_{2222}(\theta), \\ 8R_0 e^{4i(\Phi_0 - \theta)} &= T_{1111}(\theta) - 2T_{1122}(\theta) - 4T_{1212}(\theta) + T_{2222}(\theta) + \\ &\quad + 4i[T_{1112}(\theta) - T_{1222}(\theta)], \\ 8R_1 e^{2i(\Phi_1 - \theta)} &= T_{1111}(\theta) - T_{2222}(\theta) + 2i[T_{1112}(\theta) + T_{1222}(\theta)]. \end{aligned} \quad (5)$$

Denoting by lower-case letters the polar parameters of \mathbb{T}^{-1} , it is:

$$\begin{aligned} t_0 &= \frac{2}{\Delta} (T_0 T_1 - R_1^2), \\ t_1 &= \frac{1}{2\Delta} (T_0^2 - R_0^2), \\ r_0 e^{4i\varphi_0} &= \frac{2}{\Delta} (R_1^2 e^{4i\Phi_1} - T_1 R_0 e^{4i\Phi_0}), \\ r_1 e^{2i\varphi_1} &= -\frac{1}{\Delta} R_1 e^{2i\Phi_1} [T_0 - R_0 e^{4i(\Phi_0 - \Phi_1)}], \end{aligned} \quad (6)$$

with

$$\Delta = 8T_1 (T_0^2 - R_0^2) - 16R_1^2 [T_0 - R_0 \cos 4(\Phi_0 - \Phi_1)]. \quad (7)$$

There is a close relation between the polar invariants and elastic symmetries. In particular, the polar analysis of elastic symmetries let appear an *algebraic* characterization of the elastic symmetries, for some aspects more powerful than the classical *geometrical* characterization using the symmetry of the elastic properties linked to a subjacent symmetric distribution of the matter. In fact, special values taken by one or two invariants determine an elastic symmetry, and these particular values affect and characterize the properties of the matter. Upon this consideration, it can be shown that there are five different and non equivalent types of planar elastic symmetries:

- *ordinary orthotropy*: it corresponds to the condition

$$\Phi_0 - \Phi_1 = K \frac{\pi}{4}, \quad K \in \{0, 1\}; \quad (8)$$

As a consequence, it is possible to have, for the same set of invariant polar moduli T_0 , T_1 , R_0 and R_1 , two different orthotropic materials,

one with $K = 0$, the other one with $K = 1$, whose properties are quite different. These two types of ordinary orthotropic materials correspond to those termed by Pedersen (1989) as *low*, $K = 0$, and *high*, $K = 1$, shear modulus orthotropy. The results obtained by Vannucci (2009), Vincenti & Desmorat (2011), Catapano et al. (2012) and Barsotti & Vannucci (2013) suggest that such a classification is rather restrictive: the differences between these two classes are not limited to shear, but rather concern the overall mechanical response of the material. This can be observed also for the effects of damage, as it will be clear in the following of this paper;

- *R_0 -orthotropy*:

$$R_0 = 0; \quad (9)$$

in this case, the Cartesian components of \mathbb{T} are either constant or change, after a rotation, as those of a second- and not of a fourth-rank tensor, Vannucci (2002); the existence of this special case of orthotropy has been successively found also in \mathfrak{R}^3 , Forte (2005); a sufficient condition for having R_0 -orthotropy is to strengthen (or weaken) an isotropic layer by fibers (cracks) that are shifted of $\pi/4$;

- *r_0 -orthotropy*:

$$r_0 = 0; \quad (10)$$

as eq. (6₃) clearly shows, condition (9) does not imply the same result for \mathbb{T}^{-1} : R_0 -orthotropy does not concern the inverse tensor. Of course, by duality, a similar orthotropy exists for \mathbb{T}^{-1} and not for \mathbb{T} , and it is stated by condition (10); a very common r_0 -orthotropic material is paper, Vannucci (2010);

- *square symmetry*: it is the planar equivalent of the cubic syngony, i.e. all the properties are periodic by $\pi/2$:

$$R_1 = 0; \quad (11)$$

- *isotropy*:

$$R_0 = R_1 = 0. \quad (12)$$

It can be shown that, algebraically speaking, the invariant linked to *ordinary orthotropy* is of the third order, while *special orthotropies* (9), (10) and (11) are determined by second-order invariants. Finally, isotropy algebraically corresponds to the contemporary existence of special orthotropies (9) and (11). Also, while ordinary orthotropy with $K = 1$ is completely characterized by the knowledge of five non null independent invariants, $T_0, T_1, R_0,$

R_1 and K , and by four for the case $K = 0$, only three non null independent invariants totally determine special orthotropies: T_0, T_1 and R_1 for R_0 -orthotropy, T_0, T_1 and R_0 for square symmetry, and of course only two, T_0 and T_1 , for isotropy. Also in the case of r_0 -orthotropy there are only three non null independent invariants, because condition (10) gives

$$R_0 = \frac{R_1^2}{T_1}, \quad K = 0. \quad (13)$$

Another important feature of the polar formalism, is that it is possible to state the necessary and sufficient conditions for \mathbb{T} to be positive definite, regardless of the type of elastic symmetry, also for the completely anisotropic case. We just recall here the general result while the complete demonstration can be found in Appendix A: an elasticity-like plane tensor \mathbb{T} , i.e. satisfying conditions (1), is positive definite if and only if

$$\begin{aligned} T_0 &> R_0, \\ T_1(T_0^2 - R_0^2) &> 2R_1^2 [T_0 - R_0 \cos 4(\Phi_0 - \Phi_1)], \\ R_0 &\geq 0, \\ R_1 &\geq 0. \end{aligned} \quad (14)$$

3. Damage description and the polar formalism

The undamaged material is assumed to be isotropic, so its elastic tensor \mathbb{C} is given, in polar terms, by

$$\begin{aligned} C_{1111}(\theta) &= C_{2222}(\theta) = T_0 + 2T_1, \\ C_{1112}(\theta) &= C_{1222}(\theta) = 0, \\ C_{1122}(\theta) &= -T_0 + 2T_1, \\ C_{1212}(\theta) &= T_0. \end{aligned} \quad (15)$$

We consider in this paper a damage described by a fourth-rank tensor \mathbb{D} ; also, we assume \mathbb{D} to be positive semi-definite and to have the tensor symmetries of elasticity, i.e. $\forall i, j, k, l = 1, 2$,

$$\begin{aligned} D_{ijkl} &= D_{jikl} = D_{ijlk}, \\ D_{ijkl} &= D_{klij}. \end{aligned} \quad (16)$$

With these assumptions for \mathbb{D} , bounds (14) can be used; we will denote by $D_0, D_1, S_0, S_1, \Psi_0$ and Ψ_1 its polar parameters. Then, applying eq. (2),

the Cartesian components of \mathbb{D} are given by

$$\begin{aligned}
D_{1111}(\theta) &= D_0 + 2D_1 + S_0 \cos 4(\Psi_0 - \theta) + 4S_1 \cos 2(\Psi_1 - \theta), \\
D_{1112}(\theta) &= S_0 \sin 4(\Psi_0 - \theta) + 2S_1 \sin 2(\Psi_1 - \theta), \\
D_{1122}(\theta) &= -D_0 + 2D_1 - S_0 \cos 4(\Psi_0 - \theta), \\
D_{1212}(\theta) &= D_0 - S_0 \cos 4(\Psi_0 - \theta), \\
D_{1222}(\theta) &= -S_0 \sin 4(\Psi_0 - \theta) + 2S_1 \sin 2(\Psi_1 - \theta), \\
D_{2222}(\theta) &= D_0 + 2D_1 + S_0 \cos 4(\Psi_0 - \theta) - 4S_1 \cos 2(\Psi_1 - \theta).
\end{aligned} \tag{17}$$

No restrictions about the damage symmetries are assumed: damage can be completely anisotropic or have one of the symmetries described by equations (8) to (12).

The elastic tensor $\tilde{\mathbb{C}}$ of the damaged material is assumed to be given by, see Lemaitre et al (2009),

$$\tilde{\mathbb{C}} = [(1 - \mathbb{D})\mathbb{C}]^{Sym} \implies \tilde{\mathbb{C}} = \mathbb{C} - \frac{\mathbb{C}\mathbb{D} + \mathbb{D}\mathbb{C}}{2}, \tag{18}$$

the symbol *Sym* indicating the symmetric part of the tensor between brackets. $\tilde{\mathbb{C}}$ owns hence the same tensor symmetries (1) of \mathbb{C} and \mathbb{D} and it has to be positive definite too, being a stiffness tensor.

Equation (18) allows for expressing the elastic moduli of $\tilde{\mathbb{C}}$ as functions of those of \mathbb{C} and \mathbb{D} . The Cartesian components of $\tilde{\mathbb{C}}$ are finally expressed by

$$\begin{aligned}
\tilde{C}_{1111}(\theta) &= T_0(1 - 2D_0) + 2T_1(1 - 4D_1) - 2T_0S_0 \cos 4(\Psi_0 - \theta) - \\
&\quad - 4(T_0 + 2T_1)S_1 \cos 2(\Psi_1 - \theta), \\
\tilde{C}_{1112}(\theta) &= -2T_0S_0 \sin 4(\Psi_0 - \theta) - 2(T_0 + 2T_1)S_1 \sin 2(\Psi_1 - \theta), \\
\tilde{C}_{1122}(\theta) &= -T_0(1 - 2D_0) + 2T_1(1 - 4D_1) + 2T_0S_0 \cos 4(\Psi_0 - \theta), \\
\tilde{C}_{1212}(\theta) &= T_0(1 - 2D_0) + 2T_0S_0 \cos 4(\Psi_0 - \theta), \\
\tilde{C}_{1222}(\theta) &= 2T_0S_0 \sin 4(\Psi_0 - \theta) - 2(T_0 + 2T_1)S_1 \sin 2(\Psi_1 - \theta), \\
\tilde{C}_{2222}(\theta) &= T_0(1 - 2D_0) + 2T_1(1 - 4D_1) - 2T_0S_0 \cos 4(\Psi_0 - \theta) + \\
&\quad + 4(T_0 + 2T_1)S_1 \cos 2(\Psi_1 - \theta).
\end{aligned} \tag{19}$$

Using now the equivalent, for $\tilde{\mathbb{C}}$, of eq. (6), or simply comparing eq. (19) to eqs. (2) and (17), give the polar parameters of $\tilde{\mathbb{C}}$, denoted in the following

by a \sim :

$$\begin{aligned}
\tilde{T}_0 &= T_0(1 - 2D_0), \\
\tilde{T}_1 &= T_1(1 - 4D_1), \\
\tilde{R}_0 &= 2T_0S_0, \\
\tilde{R}_1 &= (T_0 + 2T_1)S_1, \\
\tilde{\Phi}_0 &= \Psi_0 + \frac{\pi}{4}, \\
\tilde{\Phi}_1 &= \Psi_1 + \frac{\pi}{2}.
\end{aligned} \tag{20}$$

Some remarks can be done; first of all, the polar formalism gives as a result that each one of the polar parameters of $\tilde{\mathbb{C}}$ depends exclusively upon the corresponding polar parameter of \mathbb{D} . In other words, the choice of the polar formalism allows for uncoupling the expressions of the parameters of $\tilde{\mathbb{C}}$ as functions of those of \mathbb{D} :

$$\left\{ \begin{array}{c} \tilde{T}_0 \\ \tilde{T}_1 \\ \tilde{R}_0 \\ \tilde{R}_1 \\ \tilde{\Phi}_0 \\ \tilde{\Phi}_1 \end{array} \right\} = \left[\begin{array}{cccccc} 1 - 2D_0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 - 4D_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2T_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & T_0 + 2T_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right] \left\{ \begin{array}{c} D_0 \\ D_1 \\ S_0 \\ S_1 \\ \Psi_0 + \frac{\pi}{4} \\ \Psi_1 + \frac{\pi}{2} \end{array} \right\}; \tag{21}$$

this should not be the case, generally speaking, with a Cartesian representation.

Also, eq. (20) show that the damage symmetries, eqs. (8) – (12), determine, each one, exactly the same elastic symmetry of the same type for the damaged elastic tensor $\tilde{\mathbb{C}}$ and inversely.

Finally, eqs. (19_{1,6}) and (20₆) show that the axis of the strongest component of $\tilde{\mathbb{C}}$, i.e. \tilde{C}_{2222} , is turned of $\pi/2$ with respect to the direction of the strongest component of \mathbb{D} , D_{1111} . This is quite natural, because the material is more severely damaged along the direction of D_{1111} , so that, finally, $\tilde{C}_{1111}(\theta = 0) < \tilde{C}_{2222}(\theta = 0)$. Also the harmonic depending upon 4θ is turned of $\pi/4$, eq. (20₅), which gives for the angular invariant of $\tilde{\mathbb{C}}$

$$\tilde{\Phi}_0 - \tilde{\Phi}_1 = \Psi_0 - \Psi_1 - \frac{\pi}{4}. \tag{22}$$

This result shows a rather surprising fact: the damaged elasticity tensor $\tilde{\mathbb{C}}$ cannot have the same form of ordinary orthotropy of the damage tensor \mathbb{D} . In fact, for \mathbb{D} orthotropic with

$$\Psi_0 - \Psi_1 = L \frac{\pi}{4}, \quad L = \{0, 1\}, \tag{23}$$

$\tilde{\mathbb{C}}$ is orthotropic with

$$\tilde{\Phi}_0 - \tilde{\Phi}_1 = \tilde{K} \frac{\pi}{4}, \quad \tilde{K} = L - 1. \quad (24)$$

So, for $L = 1$, $\tilde{K} = 0$ and for $L = 0$, $\tilde{K} = 1$ (the sign does not matter).

4. Polar bounds for the damage and damaged elastic tensors

Equations (5), written for \mathbb{D} , and (20) give respectively the polar invariants of \mathbb{D} and $\tilde{\mathbb{C}}$. A question is of primary importance: how much the value of these invariants can change? In a more general sense, what are the bounds for \mathbb{D} and $\tilde{\mathbb{C}}$?

The use of the polar formalism is advantageous to this purpose, because, on one side, the general bounds for a tensor to be positive definite are explicit, eq. (14), and on the other side each polar parameters of $\tilde{\mathbb{C}}$ depends exclusively, as already remarked, upon the corresponding polar parameter of \mathbb{D} , eq. (20), which can considerably simplify the analytical developments.

The bounds for \mathbb{D} and $\tilde{\mathbb{C}}$ are of two types (for the sake of clarity, we number the bounds like **B1**, **B2** and so on):

- *positive definiteness polar bounds*: tensor \mathbb{D} is positive semi-definite: a null damage tensor is admissible, corresponding to the case of undamaged material. Its polar invariants must satisfy conditions (14);

$$\begin{aligned} \mathbf{B1} &\longrightarrow D_0 \geq S_0, \\ \mathbf{B2} &\longrightarrow D_1(D_0^2 - S_0^2) \geq 2S_1^2 [D_0 - S_0 \cos 4(\Psi_0 - \Psi_1)], \\ \mathbf{B3} &\longrightarrow S_0 \geq 0, \\ \mathbf{B4} &\longrightarrow S_1 \geq 0. \end{aligned} \quad (25)$$

On the other side, as any other stiffness tensor, $\tilde{\mathbb{C}}$ is positive definite, so we have

$$\begin{aligned} \mathbf{B5} &\longrightarrow \tilde{T}_0 > \tilde{R}_0, \\ \mathbf{B6} &\longrightarrow \tilde{T}_1(\tilde{T}_0^2 - \tilde{R}_0^2) > 2\tilde{R}_1^2 [\tilde{T}_0 - \tilde{R}_0 \cos 4(\tilde{\Phi}_0 - \tilde{\Phi}_1)], \\ \mathbf{B7} &\longrightarrow \tilde{R}_0 \geq 0, \\ \mathbf{B8} &\longrightarrow \tilde{R}_1 \geq 0. \end{aligned} \quad (26)$$

- *polar damage bounds*: eq. (20) show that the anisotropy polar moduli \tilde{R}_0 and \tilde{R}_1 , initially null because the undamaged material is isotropic,

can increase their value during the damage process, making the material to become anisotropic as a consequence of damage. Some of the elasticity moduli of the damaged tensor can hence be increased by damage; this is also the case of other elastic quantities, at least locally, for instance $C_{1112}(\theta)$ and $C_{1222}(\theta)$. Nevertheless, an effective damage process should reduce the elastic stiffness of the damaged material. This can be stated putting a limit on the strain energy of the damaged material: *for any identical strain state, the strain energy stocked in the damaged material cannot be greater than that stocked in the undamaged one.*

Mechanically, this is motivated by the following consideration: let us consider an initially strained material which, by the application of more and more loads, increases the state of strain and of damage. If the material is unloaded, following any unloading path, just to the initial state of strain, then the strain energy stocked by the material in the final damaged state cannot be greater than that present at the beginning.

The difference of the strain energy between the initial and damaged state is, for the same strain ε ,

$$\Delta w = w - \tilde{w} = \frac{1}{2}\varepsilon \cdot \mathbb{C}\varepsilon - \frac{1}{2}\varepsilon \cdot \tilde{\mathbb{C}}\varepsilon = \frac{1}{2}\varepsilon \cdot (\mathbb{C} - \tilde{\mathbb{C}})\varepsilon; \quad (27)$$

putting

$$\hat{\mathbb{C}} = \mathbb{C} - \tilde{\mathbb{C}}, \quad (28)$$

the condition to be posed on the strain energy becomes

$$\Delta w = \frac{1}{2}\varepsilon \cdot \hat{\mathbb{C}}\varepsilon \geq 0 \quad \forall \varepsilon, \quad (29)$$

i.e. tensor $\hat{\mathbb{C}}$ must be positive semi-definite.

The positive semi-definiteness of $\hat{\mathbb{C}}$ can be interpreted also in a thermodynamical framework (Appendix B):

- it is a necessary condition for the intrinsic dissipation due to loss of stiffness to be positive;
- it is equivalent to a positive intrinsic dissipation due to linear elasticity-damage coupling (i.e. with $\hat{\mathbb{C}}$ linear in \mathbb{D}).

Applying eq. (14) to the polar components of $\hat{\mathbb{C}}$, denoted by a hat,

gives the *polar damage bounds*:

$$\begin{aligned}
\mathbf{B9} &\longrightarrow \widehat{T}_0 \geq \widehat{R}_0, \\
\mathbf{B10} &\longrightarrow \widehat{T}_1(\widehat{T}_0^2 - \widehat{R}_0^2) \geq 2\widehat{R}_1^2 \left[\widehat{T}_0 - \widehat{R}_0 \cos 4(\widehat{\Phi}_0 - \widehat{\Phi}_1) \right], \\
\mathbf{B11} &\longrightarrow \widehat{R}_0 \geq 0, \\
\mathbf{B12} &\longrightarrow \widehat{R}_1 \geq 0.
\end{aligned} \tag{30}$$

It is worth to rewrite these bounds using the polar components of \mathbb{C} and $\widetilde{\mathbb{C}}$: through eq. (5) it is readily get that

$$\begin{aligned}
\widehat{T}_0 &= T_0 - \widetilde{T}_0, \\
\widehat{T}_1 &= T_1 - \widetilde{T}_1, \\
\widehat{R}_0 &= \widetilde{R}_0, \\
\widehat{R}_1 &= \widetilde{R}_1, \\
\widehat{\Phi}_0 &= \widetilde{\Phi}_0 + \frac{\pi}{4}, \\
\widehat{\Phi}_1 &= \widetilde{\Phi}_1 + \frac{\pi}{2}.
\end{aligned} \tag{31}$$

Bounds **B9** to **B12** become hence

$$\begin{aligned}
\mathbf{B9} &\longrightarrow T_0 - \widetilde{T}_0 \geq \widetilde{R}_0, \\
\mathbf{B10} &\longrightarrow (T_1 - \widetilde{T}_1) \left[(T_0 - \widetilde{T}_0)^2 - \widetilde{R}_0^2 \right] \geq \\
&\quad 2\widetilde{R}_1^2 \left[T_0 - \widetilde{T}_0 + \widetilde{R}_0 \cos 4(\widetilde{\Phi}_0 - \widetilde{\Phi}_1) \right], \\
\mathbf{B11} &\longrightarrow \widetilde{R}_0 \geq 0, \\
\mathbf{B12} &\longrightarrow \widetilde{R}_1 \geq 0.
\end{aligned} \tag{32}$$

It is immediate to remark that imposing conditions **B9** to **B12** comprehends the fact that the direct moduli do not increase in any direction, i.e. that

$$\widetilde{C}_{1111}(\theta) \leq C_{1111}, \quad \widetilde{C}_{1212}(\theta) \leq C_{1212} \quad \forall \theta. \tag{33}$$

5. Minimal set of polar bounds in the completely anisotropic case

Bounds **B1** to **B4** are relations among the invariants of \mathbb{D} , while **B5** to **B12** concern $\widetilde{\mathbb{C}}$. Also, while the invariants of \mathbb{D} are dimensionless, those of $\widetilde{\mathbb{C}}$ have the dimensions of a stress. It is then worth, on one side to write all

the bounds as functions of the polar parameters of \mathbb{D} or of $\tilde{\mathbb{C}}$ uniquely, and on the other side to write these last in a dimensionless form. This can be done dividing all the modules of \mathbb{C} and $\tilde{\mathbb{C}}$ by a term, T_0 , representing the undamaged material and upon introduction of the ratios:

$$\tau_1 = \frac{2T_1}{T_0}, \quad \tilde{\tau}_0 = \frac{\tilde{T}_0}{T_0}, \quad \tilde{\tau}_1 = \frac{2\tilde{T}_1}{T_0}, \quad \tilde{\rho}_0 = \frac{\tilde{R}_0}{T_0}, \quad \tilde{\rho}_1 = \frac{\tilde{R}_1}{T_0}. \quad (34)$$

The ratio $\tau_1 > 0$ will hence be the only term representing the mechanical characteristics of the undamaged material.

We can now rewrite all the bounds as functions of the invariants of \mathbb{D} , using eqs. (20) and (22) into eqs. (26) and (32):

$$\begin{aligned} \mathbf{B1} &\longrightarrow D_0 \geq S_0, \\ \mathbf{B2} &\longrightarrow D_1(D_0^2 - S_0^2) \geq 2S_1^2 [D_0 - S_0 \cos 4(\Psi_0 - \Psi_1)], \\ \mathbf{B3} &\longrightarrow S_0 \geq 0, \\ \mathbf{B4} &\longrightarrow S_1 \geq 0, \\ \mathbf{B5} &\longrightarrow 2(D_0 + S_0) < 1, \\ \mathbf{B6} &\longrightarrow \frac{\tau_1}{4(1+\tau_1)^2}(1 - 4D_1)[(1 - 2D_0)^2 - 4S_0^2] > S_1^2[1 - 2D_0 + \\ &\hspace{15em} 2S_0 \cos 4(\Psi_0 - \Psi_1)], \\ \mathbf{B7} &\longrightarrow S_0 \geq 0, \\ \mathbf{B8} &\longrightarrow S_1 \geq 0, \\ \mathbf{B9} &\longrightarrow D_0 \geq S_0, \\ \mathbf{B10} &\longrightarrow D_1(D_0^2 - S_0^2) \geq \frac{(1+\tau_1)^2}{2\tau_1} S_1^2 [D_0 - S_0 \cos 4(\Psi_0 - \Psi_1)], \\ \mathbf{B11} &\longrightarrow S_0 \geq 0, \\ \mathbf{B12} &\longrightarrow S_1 \geq 0. \end{aligned} \quad (35)$$

Alternatively, we can write all the bounds as functions of the dimensionless parameters of $\tilde{\mathbb{C}}$, using first eq. (34) into eq. (20) to get

$$\begin{aligned} \tilde{\tau}_0 = 1 - 2D_0 &\longrightarrow D_0 = \frac{1-\tilde{\tau}_0}{2}, \\ \tilde{\tau}_1 = \tau_1(1 - 4D_1) &\longrightarrow D_1 = \frac{\tau_1 - \tilde{\tau}_1}{4\tau_1}, \\ \tilde{\rho}_0 = 2S_0 &\longrightarrow S_0 = \frac{\tilde{\rho}_0}{2}, \\ \tilde{\rho}_1 = (1 + \tau_1)S_1 &\longrightarrow S_1 = \frac{\tilde{\rho}_1}{1+\tau_1}, \\ \Psi_0 = \tilde{\Phi}_0 - \frac{\pi}{4}, \\ \Psi_1 = \tilde{\Phi}_1 - \frac{\pi}{2}, \end{aligned} \quad (36)$$

then inserting eqs. (34) and (36) into eqs. (25), (26) and (32):

$$\begin{aligned}
\mathbf{B1} &\longrightarrow \tilde{\tau}_0 + \tilde{\rho}_0 \leq 1, \\
\mathbf{B2} &\longrightarrow (1 + \tau_1)^2(\tau_1 - \tilde{\tau}_1)[(1 - \tilde{\tau}_0)^2 - \tilde{\rho}_0^2] \geq 16 \tau_1 \tilde{\rho}_1^2 [1 - \tilde{\tau}_0 + \\
&\quad + \tilde{\rho}_0 \cos 4(\tilde{\Phi}_0 - \tilde{\Phi}_1)], \\
\mathbf{B3} &\longrightarrow \tilde{\rho}_0 \geq 0, \\
\mathbf{B4} &\longrightarrow \tilde{\rho}_1 \geq 0, \\
\mathbf{B5} &\longrightarrow \tilde{\tau}_0 > \tilde{\rho}_0, \\
\mathbf{B6} &\longrightarrow \tilde{\tau}_1(\tilde{\tau}_0^2 - \tilde{\rho}_0^2) > 4\tilde{\rho}_1^2 [\tilde{\tau}_0 - \tilde{\rho}_0 \cos 4(\tilde{\Phi}_0 - \tilde{\Phi}_1)], \\
\mathbf{B7} &\longrightarrow \tilde{\rho}_0 \geq 0, \\
\mathbf{B8} &\longrightarrow \tilde{\rho}_1 \geq 0, \\
\mathbf{B9} &\longrightarrow \tilde{\tau}_0 + \tilde{\rho}_0 \leq 1, \\
\mathbf{B10} &\longrightarrow (\tau_1 - \tilde{\tau}_1)[(1 - \tilde{\tau}_0)^2 - \tilde{\rho}_0^2] \geq 4\tilde{\rho}_1^2 [1 - \tilde{\tau}_0 + \\
&\quad + \tilde{\rho}_0 \cos 4(\tilde{\Phi}_0 - \tilde{\Phi}_1)], \\
\mathbf{B11} &\longrightarrow \tilde{\rho}_0 \geq 0, \\
\mathbf{B12} &\longrightarrow \tilde{\rho}_1 \geq 0.
\end{aligned} \tag{37}$$

It is apparent from eqs. (35) and (37) that some bounds are redundant: bounds **B3**, **B7** and **B11** are identical, just like **B4**, **B8** and **B12**. Also, it is easily recognized that bound **B10** is always more restrictive than bound **B2**, so **B2** can be discarded. There is an exception: for materials with $\tau_1 = 1$, bounds **B2** and **B10** are perfectly identical. In this case, and only in this one, the set of bounds stating the positive semi-definiteness of \mathbb{D} and $\hat{\mathbb{C}}$ are always completely coincident, regardless of the elastic symmetries of $\tilde{\mathbb{C}}$. For $\tau_1 \neq 1$, the bounds on the positive semi-definiteness of \mathbb{D} are less restrictive than those on $\hat{\mathbb{C}}$, and hence will be discarded. Nevertheless, we will prove in the next sections that for $\tilde{\mathbb{C}}$ isotropic or square-symmetric, the equivalence of bounds on \mathbb{D} and on $\hat{\mathbb{C}}$ is always true, regardless of the value of τ_1 .

It is worth noticing that

$$\tau_1 = 1 \iff T_0 = 2T_1 \iff G = \kappa, \tag{38}$$

see eqs. (3) and (34), i.e. for the materials with $\tau_1 = 1$ the shear and bulk moduli are identical.

Equations (35) and (37), resumed in Tab. 1 for the sake of convenience, have a general validity, i.e. they concern completely anisotropic materials. The redundant bounds have been discarded. A step further can be done

considering the different cases of elastic symmetry, Sec. 2; this is done in the next sections, starting from the simplest case of isotropy and going up to the more complicate case of ordinary orthotropy.

Table 1: Minimal set of dimensionless polar bounds in the completely anisotropic case

	Polar bounds for \mathbb{D}	Polar bounds for $\tilde{\mathbb{C}}$
B5	$2(D_0 + S_0) < 1$	$\tilde{\tau}_0 > \tilde{\rho}_0$
B6	$\frac{\tau_1}{4(1+\tau_1)^2}(1 - 4D_1)[(1 - 2D_0)^2 - 4S_0^2] > S_1^2[1 - 2D_0 + 2S_0 \cos 4(\Psi_0 - \Psi_1)]$	$\tilde{\tau}_1(\tilde{\tau}_0^2 - \tilde{\rho}_0^2) > 4\tilde{\rho}_1^2 \left[\tilde{\tau}_0 - \tilde{\rho}_0 \cos 4(\tilde{\Phi}_0 - \tilde{\Phi}_1) \right]$
B7	$S_0 \geq 0$	$\tilde{\rho}_0 \geq 0$
B8	$S_1 \geq 0$	$\tilde{\rho}_1 \geq 0$
B9	$D_0 \geq S_0$	$\tilde{\tau}_0 + \tilde{\rho}_0 \leq 1$
B10	$D_1(D_0^2 - S_0^2) \geq \frac{(1+\tau_1)^2}{2\tau_1} S_1^2 [D_0 - S_0 \cos 4(\Psi_0 - \Psi_1)]$	$(\tau_1 - \tilde{\tau}_1)[(1 - \tilde{\tau}_0)^2 - \tilde{\rho}_0^2] \geq 4\tilde{\rho}_1^2 [1 - \tilde{\tau}_0 + \tilde{\rho}_0 \cos 4(\tilde{\Phi}_0 - \tilde{\Phi}_1)]$

6. Polar bounds in the isotropic case

$\tilde{\mathbb{C}}$ is isotropic if and only if \mathbb{D} is isotropic, see eq. (20_{3,4}):

$$\begin{aligned} \tilde{R}_0 = \tilde{\rho}_0 = 0 &\iff S_0 = 0, \\ \tilde{R}_1 = \tilde{\rho}_1 = 0 &\iff S_1 = 0. \end{aligned} \tag{39}$$

As a consequence, bounds **B7** and **B8** are meaningless and the bounds written in terms of the components of \mathbb{D} are

$$\begin{aligned} \mathbf{B5} &\longrightarrow D_0 < \frac{1}{2}, \\ \mathbf{B6} &\longrightarrow D_1 < \frac{1}{4}, \\ \mathbf{B9} &\longrightarrow D_0 \geq 0, \\ \mathbf{B10} &\longrightarrow D_1 \geq 0, \end{aligned} \tag{40}$$

bounds that finally can be rewritten as

$$\begin{aligned} 0 &\leq D_0 < \frac{1}{2}, \\ 0 &\leq D_1 < \frac{1}{4}. \end{aligned} \tag{41}$$

The above bounds are box constraints, so the admissible domain for \mathbb{D} is of course convex, Fig. 1 a).

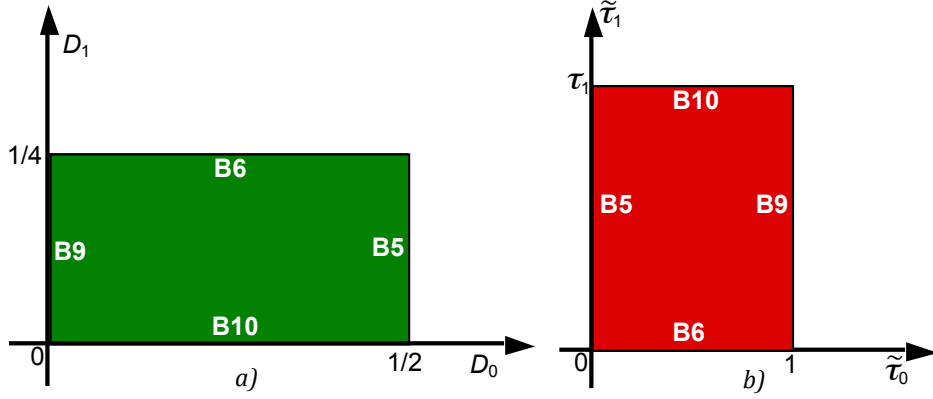


Figure 1: Admissible domain for the case of isotropic damaged material; a) domain for the invariants of \mathbb{D} ; b) domain for the invariants of $\tilde{\mathbb{C}}$.

When conditions (39) are inserted into eqs. (34) and (37) the bounds for the invariant ratios of $\tilde{\mathbb{C}}$ are easily obtained:

$$\begin{aligned}
 \mathbf{B5} &\longrightarrow \tilde{\tau}_0 > 0, \\
 \mathbf{B6} &\longrightarrow \tilde{\tau}_1 > 0, \\
 \mathbf{B9} &\longrightarrow \tilde{\tau}_0 \leq 1, \\
 \mathbf{B10} &\longrightarrow \tilde{\tau}_1 \leq \tau_1,
 \end{aligned} \tag{42}$$

or more concisely

$$\begin{aligned}
 0 < \tilde{\tau}_0 &\leq 1, \\
 0 < \tilde{\tau}_1 &\leq \tau_1.
 \end{aligned} \tag{43}$$

In the dimensional form the above bounds are

$$\begin{aligned}
 0 < \tilde{T}_0 &\leq T_0, \\
 0 < \tilde{T}_1 &\leq T_1;
 \end{aligned} \tag{44}$$

the isotropic part of \mathbb{C} is hence diminished by damage.

If conditions (39) are inserted into eqs. (35) or (37), it is immediately recognized that bounds **B2** and **B10** are identical, regardless of the value of τ_1 . Hence, for an isotropic material which is damaged isotropically, the bounds on the positive semi-definiteness of \mathbb{D} are perfectly coincident with those on $\hat{\mathbb{C}}$.

7. Polar bounds in the square symmetric case

$\tilde{\mathbb{C}}$ has the square symmetry if and only if \mathbb{D} is square symmetric, see eq. (20₄):

$$\tilde{R}_1 = \tilde{\rho}_1 = 0 \iff S_1 = 0. \quad (45)$$

This means that bound **B8** disappears, so that the conditions on the components of \mathbb{D} become:

$$\begin{aligned} \mathbf{B5} &\longrightarrow D_0 + S_0 < \frac{1}{2}, \\ \mathbf{B6} &\longrightarrow D_1 < \frac{1}{4}, \\ \mathbf{B7} &\longrightarrow S_0 \geq 0, \\ \mathbf{B9} &\longrightarrow D_0 \geq S_0, \\ \mathbf{B10} &\longrightarrow D_1 \geq 0. \end{aligned} \quad (46)$$

The above bounds can be rewritten in the condensed form

$$\begin{aligned} 0 &\leq D_1 < \frac{1}{4}, \\ 0 &\leq S_0 < \min \left\{ D_0; \frac{1}{2} - D_0 \right\}, \end{aligned} \quad (47)$$

the last condition ensuring that it is still

$$0 \leq D_0 < \frac{1}{2}, \quad (48)$$

while the highest theoretical value for S_0 is clearly

$$\max S_0 = \frac{1}{4}. \quad (49)$$

Just as in the previous case, inserting (45) into eqs. (34) and (37) gives the bounds for $\tilde{\mathbb{C}}$:

$$\begin{aligned} \mathbf{B5} &\longrightarrow \tilde{\tau}_0 > \tilde{\rho}_0, \\ \mathbf{B6} &\longrightarrow \tilde{\tau}_1 > 0, \\ \mathbf{B7} &\longrightarrow \tilde{\rho}_0 \geq 0, \\ \mathbf{B9} &\longrightarrow \tilde{\tau}_0 + \tilde{\rho}_0 \leq 1, \\ \mathbf{B10} &\longrightarrow \tilde{\tau}_1 \leq \tau_1. \end{aligned} \quad (50)$$

Finally, in this case the bounds for $\tilde{\mathbb{C}}$ are

$$\begin{aligned} 0 &< \tilde{\tau}_1 \leq \tau_1, \\ 0 &\leq \tilde{\rho}_0 < \min \{ \tilde{\tau}_0; 1 - \tilde{\tau}_0 \}, \end{aligned} \quad (51)$$

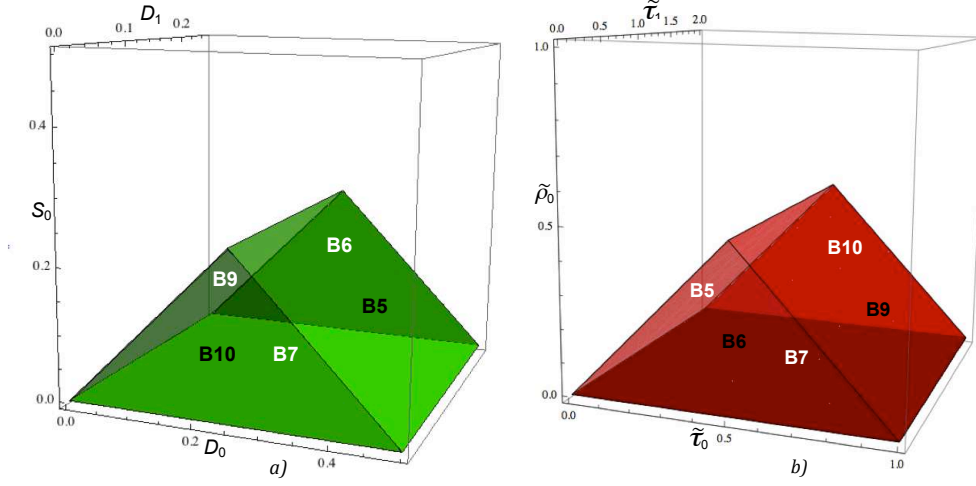


Figure 2: Admissible domain for the case of square symmetric damaged material; *a*) domain for the invariants of \mathbb{D} ; *b*) domain for the invariants of $\tilde{\mathbb{C}}$ (material with $\tau_1 = 2$; the bounds indicated by white letters are placed behind and seen in transparency).

the last condition giving

$$0 < \tilde{\tau}_0 \leq 1. \quad (52)$$

The dimensional corresponding of eqs. (51) and (52) are

$$\begin{aligned} 0 < \tilde{T}_0 &\leq T_0, \\ 0 < \tilde{T}_1 &\leq T_1, \\ 0 \leq \tilde{R}_0 &< \min \left\{ \tilde{T}_0; T_0 - \tilde{T}_0 \right\}. \end{aligned} \quad (53)$$

Also for square symmetry, the isotropic part of \mathbb{C} is decreased by damage, but at the same time the anisotropic part, here represented by the only term \tilde{R}_0 , grows from zero: damage produces hence a decrease of the averaged stiffness, the isotropic part, but at the same time an increase of the anisotropic part.

Equations (46) and (50) clearly show that the admissible domain for the invariants of \mathbb{D} and $\tilde{\mathbb{C}}$, bounded by linear conditions, is convex, see Fig. 2.

Just like in the previous case of isotropic damage, also in this case the bounds on the positive-semidefiniteness of \mathbb{D} are perfectly coincident with those given by the positive semi-definiteness of $\tilde{\mathbb{C}}$. This is easily proved inserting eq. (45) into eqs. (35) or (37): bounds **B2** and **B10** become identical.

This case and the previous one are the only two cases where bounds on \mathbb{D} and on $\widehat{\mathbb{C}}$ are perfectly equivalent regardless the material, i.e. of τ_1 . In all the following cases of elastic symmetries of the final damaged material, this equivalency exists only when $\tau_1 = 1$, as said in Sec. 4.

8. Polar bounds in the R_0 -orthotropic case

$\widetilde{\mathbb{C}}$ is R_0 -orthotropic if and only if \mathbb{D} is R_0 -orthotropic, see eq. (20₃):

$$\widetilde{R}_0 = \widetilde{\rho}_0 = 0 \iff S_0 = 0. \quad (54)$$

So now it is bound **B7** to disappear, and the conditions to be satisfied by the invariants of \mathbb{D} are:

$$\begin{aligned} \mathbf{B5} &\longrightarrow D_0 < \frac{1}{2}, \\ \mathbf{B6} &\longrightarrow S_1 < \frac{\sqrt{\tau_1(1-2D_0)(1-4D_1)}}{2(1+\tau_1)}, \\ \mathbf{B8} &\longrightarrow S_1 \geq 0, \\ \mathbf{B9} &\longrightarrow D_0 \geq 0, \\ \mathbf{B10} &\longrightarrow S_1 \leq \frac{\sqrt{2\tau_1 D_0 D_1}}{1+\tau_1}. \end{aligned} \quad (55)$$

Using conditions (55), it is not difficult to calculate that the highest theoretical value for S_1 is

$$S_{1max} = \frac{\sqrt{\tau_1}}{4(1+\tau_1)}, \quad (56)$$

quantity that gets its highest value, $1/8$, for materials with $\tau_1 = 1$.

Finally, the bounds for \mathbb{D} R_0 -orthotropic are

$$\begin{aligned} 0 \leq D_0 < \frac{1}{2}, \\ 0 \leq S_1 < \min \left\{ \frac{\sqrt{\tau_1(1-2D_0)(1-4D_1)}}{2(1+\tau_1)}; \frac{\sqrt{2\tau_1 D_0 D_1}}{1+\tau_1} \right\}, \end{aligned} \quad (57)$$

the above conditions ensuring also that necessarily

$$0 \leq D_1 < \frac{1}{4}. \quad (58)$$

Also in this case, inserting (54) into eqs. (34) and (37) allows for obtaining

the bounds on the invariants of $\tilde{\mathbb{C}}$:

$$\begin{aligned}
\mathbf{B5} &\longrightarrow \tilde{\tau}_0 > 0, \\
\mathbf{B6} &\longrightarrow \tilde{\rho}_1 < \frac{\sqrt{\tilde{\tau}_0 \tilde{\tau}_1}}{2}, \\
\mathbf{B8} &\longrightarrow \tilde{\rho}_1 \geq 0, \\
\mathbf{B9} &\longrightarrow \tilde{\tau}_0 \leq 1, \\
\mathbf{B10} &\longrightarrow \tilde{\rho}_1 \leq \frac{\sqrt{(1-\tilde{\tau}_0)(\tilde{\tau}_1-\tilde{\tau}_1)}}{2}.
\end{aligned} \tag{59}$$

Finally, the bounds on the invariants of $\tilde{\mathbb{C}}$ for the case of R_0 -orthotropy are

$$\begin{aligned}
&0 < \tilde{\tau}_0 \leq 1, \\
0 \leq \tilde{\rho}_1 &< \min \left\{ \frac{\sqrt{\tilde{\tau}_0 \tilde{\tau}_1}}{2}; \frac{\sqrt{(1-\tilde{\tau}_0)(\tilde{\tau}_1-\tilde{\tau}_1)}}{2} \right\},
\end{aligned} \tag{60}$$

and by consequence

$$0 < \tilde{\tau}_1 \leq \tau_1. \tag{61}$$

The dimensional corresponding of eqs. (60) and (61) are

$$\begin{aligned}
&0 < \tilde{T}_0 \leq T_0, \\
&0 < \tilde{T}_1 \leq T_1, \\
0 \leq \tilde{R}_1 &< \min \left\{ \sqrt{\frac{\tilde{T}_0 \tilde{T}_1}{2}}; \sqrt{\frac{(T_0-\tilde{T}_0)(T_1-\tilde{T}_1)}{2}} \right\}.
\end{aligned} \tag{62}$$

In this case too the isotropic part of \mathbb{C} is decreased by damage and the anisotropic part, the term \tilde{R}_1 alone, grows from zero.

Equations (55) and (59) define a convex admissible domain for the invariants of \mathbb{D} and $\tilde{\mathbb{C}}$, see Fig. 3.

9. Polar bounds in the ordinarily orthotropic case

$\tilde{\mathbb{C}}$ is ordinarily orthotropic if and only if \mathbb{D} is ordinarily orthotropic, eq. (22). In particular, ordinary orthotropy of \mathbb{D} is stated by the condition

$$\Psi_0 - \Psi_1 = L \frac{\pi}{4}, \quad L \in \{0, 1\}, \tag{63}$$

and correspondingly that of $\tilde{\mathbb{C}}$ by

$$\tilde{\Phi}_0 - \tilde{\Phi}_1 = \tilde{K} \frac{\pi}{4}, \quad \tilde{K} \in \{0, 1\}, \tag{64}$$

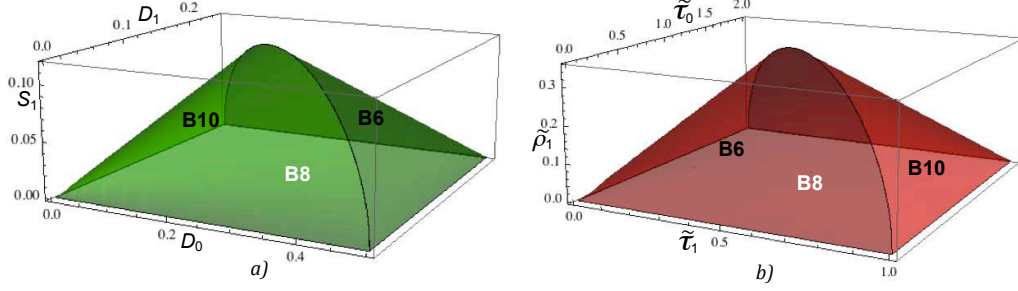


Figure 3: Admissible domain for the case of R_0 -orthotropic damaged material; a) domain for the invariants of \mathbb{D} ; b) domain for the invariants of $\tilde{\mathbb{C}}$ (material with $\tau_1 = 2$; the bounds indicated by white letters are placed behind and seen in transparency).

the orthotropies of the two tensors being linked by the relation (22) i.e. by

$$\tilde{K} = L - 1. \quad (65)$$

The conditions to be satisfied by the invariants of \mathbb{D} are then

$$\begin{aligned}
\mathbf{B5} &\longrightarrow D_0 + S_0 < \frac{1}{2}, \\
\mathbf{B6} &\longrightarrow S_1 < \frac{\sqrt{\tau_1(1-4D_1)[1-2D_0-(-1)^L S_0]}}{2(1+\tau_1)}, \\
\mathbf{B7} &\longrightarrow S_0 \geq 0, \\
\mathbf{B8} &\longrightarrow S_1 \geq 0, \\
\mathbf{B9} &\longrightarrow D_0 \geq S_0, \\
\mathbf{B10} &\longrightarrow S_1 \leq \frac{\sqrt{2\tau_1 D_1 [D_0 + (-1)^L S_0]}}{1+\tau_1}.
\end{aligned} \quad (66)$$

Finally, the bounds on the polar invariants of \mathbb{D} can be rewritten as

$$\begin{aligned}
0 \leq S_0 &< \min \left\{ D_0; \frac{1}{2} - D_0 \right\}, \\
0 \leq S_1 &< \min \left\{ \frac{\sqrt{2\tau_1 D_1 [D_0 + (-1)^L S_0]}}{1+\tau_1}; \frac{\sqrt{\tau_1(1-4D_1)[1-2D_0-(-1)^L S_0]}}{2(1+\tau_1)} \right\},
\end{aligned} \quad (67)$$

conditions that imply also

$$\begin{aligned}
0 \leq D_0 &< \frac{1}{2}, \\
0 \leq D_1 &< \frac{1}{4}.
\end{aligned} \quad (68)$$

In the case of the bounds on $\tilde{\mathbb{C}}$, following an analogous approach, we can

find

$$\begin{aligned}
\mathbf{B5} &\longrightarrow \tilde{\tau}_0 > \tilde{\rho}_0, \\
\mathbf{B6} &\longrightarrow \tilde{\rho}_1 < \frac{\sqrt{\tilde{\tau}_1[\tilde{\tau}_0+(-1)^{\tilde{K}}\tilde{\rho}_0]}}{2}, \\
\mathbf{B7} &\longrightarrow \tilde{\rho}_0 \geq 0, \\
\mathbf{B8} &\longrightarrow \tilde{\rho}_1 \geq 0, \\
\mathbf{B9} &\longrightarrow \tilde{\tau}_0 + \tilde{\rho}_0 \leq 1, \\
\mathbf{B10} &\longrightarrow \tilde{\rho}_1 \leq \frac{\sqrt{[1-\tilde{\tau}_0-(-1)^{\tilde{K}}\tilde{\rho}_0](\tau_1-\tilde{\tau}_1)}}{2}.
\end{aligned} \tag{69}$$

The bounds on the polar invariants of $\tilde{\mathbb{C}}$ can finally be rewritten as

$$\begin{aligned}
0 \leq \tilde{\rho}_0 &< \min \{ \tilde{\tau}_0; 1 - \tilde{\tau}_0 \}, \\
0 \leq \tilde{\rho}_1 &< \min \left\{ \frac{\sqrt{\tilde{\tau}_1[\tilde{\tau}_0+(-1)^{\tilde{K}}\tilde{\rho}_0]}}{2}; \frac{\sqrt{[1-\tilde{\tau}_0-(-1)^{\tilde{K}}\tilde{\rho}_0](\tau_1-\tilde{\tau}_1)}}{2} \right\},
\end{aligned} \tag{70}$$

conditions that imply also

$$\begin{aligned}
0 &< \tilde{\tau}_0 \leq 1, \\
0 &< \tilde{\tau}_1 \leq \tau_1.
\end{aligned} \tag{71}$$

The previous bounds can be given also in their dimensional form:

$$\begin{aligned}
0 &< \tilde{T}_0 \leq T_0, \\
0 &< \tilde{T}_1 \leq T_1, \\
0 \leq \tilde{R}_0 &< \min \{ \tilde{T}_0; T_0 - \tilde{T}_0 \}, \\
0 \leq \tilde{R}_1 &< \min \left\{ \sqrt{\frac{\tilde{T}_1[\tilde{T}_0+(-1)^{\tilde{K}}\tilde{R}_0]}{2}}; \sqrt{\frac{[T_0-\tilde{T}_0-(-1)^{\tilde{K}}\tilde{R}_0](T_1-\tilde{T}_1)}{2}} \right\}.
\end{aligned} \tag{72}$$

Unfortunately, it is not possible to give a graphical representation of the domains defined by eqs. (67) or (70), because they are functions of four independent quantities. Nevertheless, these domains are convex; in fact, the conditions in eqs. (67) or (70) are either linear or with a Hessian matrix whose eigenvalues are either null or negative, as it can be easily checked. So, such functions are concave and their epigraph convex. The admissible domain, intersection of convex sets, is hence convex.

10. Polar bounds in the r_0 -orthotropic case

There is, however, a special case of orthotropic $\tilde{\mathbb{C}}$ where a graphical representation is still possible: the case where $\tilde{\mathbb{C}}^{-1}$ is r_0 -orthotropic. In this case,

Sec. 2, $\tilde{\mathbb{C}}$ is orthotropic but depending upon only three non null invariants thanks to eq. (13), which inserted into eqs. (20), (34) and (65) gives

$$S_0 = \frac{(1 + \tau_1)^2}{\tau_1} \frac{S_1^2}{1 - 4D_1}, \quad \tilde{\rho}_0 = 2 \frac{\tilde{\rho}_1^2}{\tilde{\tau}_1} = 2 \frac{(1 + \tau_1)^2}{\tau_1} \frac{S_1^2}{1 - 4D_1}, \quad L = 1. \quad (73)$$

If an isotropic layer is damaged according to the previous conditions, it is transformed into a material whose behavior is of the same type of that of a sheet of paper, Vannucci (2010).

Conditions (66) become

$$\begin{aligned} \mathbf{B5} &\longrightarrow S_1 < \frac{1}{1+\tau_1} \sqrt{\frac{\tau_1(1-2D_0)(1-4D_1)}{2}}, \\ \mathbf{B6} &\longrightarrow S_1 < \frac{1}{1+\tau_1} \sqrt{\frac{\tau_1(1-2D_0)(1-4D_1)}{3}}, \\ \mathbf{B7} &\longrightarrow D_1 \leq \frac{1}{4}, \\ \mathbf{B8} &\longrightarrow S_1 \geq 0, \\ \mathbf{B9} &\longrightarrow S_1 \leq \frac{\sqrt{\tau_1 D_0 (1-4D_1)}}{1+\tau_1}, \\ \mathbf{B10} &\longrightarrow S_1 \leq \frac{1}{1+\tau_1} \sqrt{\frac{2\tau_1 D_0 D_1 (1-4D_1)}{1-2D_1}}. \end{aligned} \quad (74)$$

It is apparent that bound **B5** is less restrictive than **B6**, hence it can be discarded. Also **B9** can be neglected, because for $0 \leq D_0 < 1/4$, $\mathbf{B10} \leq \mathbf{B9} \leq \mathbf{B6}$, while $\mathbf{B6} \leq \mathbf{B10} \leq \mathbf{B9}$ for $1/4 \leq D_0 < 1/2$.

The polar bounds for \mathbb{D} in this special case can finally be summarized in

$$\begin{aligned} 0 &\leq D_0 < \frac{1}{2}, \\ 0 &\leq D_1 < \frac{1}{4}, \\ 0 &\leq S_1 < \min \left\{ \frac{1}{1+\tau_1} \sqrt{\frac{\tau_1(1-2D_0)(1-4D_1)}{3}}, \frac{1}{1+\tau_1} \sqrt{\frac{2\tau_1 D_0 D_1 (1-4D_1)}{1-2D_1}} \right\}. \end{aligned} \quad (75)$$

Going now to tensor $\tilde{\mathbb{C}}$, bounds (69) become

$$\begin{aligned} \mathbf{B5} &\longrightarrow \tilde{\rho}_1 < \sqrt{\frac{\tilde{\tau}_0 \tilde{\tau}_1}{2}}, \\ \mathbf{B6} &\longrightarrow \tilde{\rho}_1 < \sqrt{\frac{\tilde{\tau}_0 \tilde{\tau}_1}{2}}, \\ \mathbf{B7} &\longrightarrow \tilde{\tau}_1 > 0, \\ \mathbf{B8} &\longrightarrow \tilde{\rho}_1 \geq 0, \\ \mathbf{B9} &\longrightarrow \tilde{\rho}_1 \leq \sqrt{\frac{(1-\tilde{\tau}_0) \tilde{\tau}_1}{2}}, \\ \mathbf{B10} &\longrightarrow \tilde{\rho}_1 \leq \sqrt{\frac{\tilde{\tau}_1 (1-\tilde{\tau}_0) (\tau_1 - \tilde{\tau}_1)}{2(1+\tau_1)}}. \end{aligned} \quad (76)$$

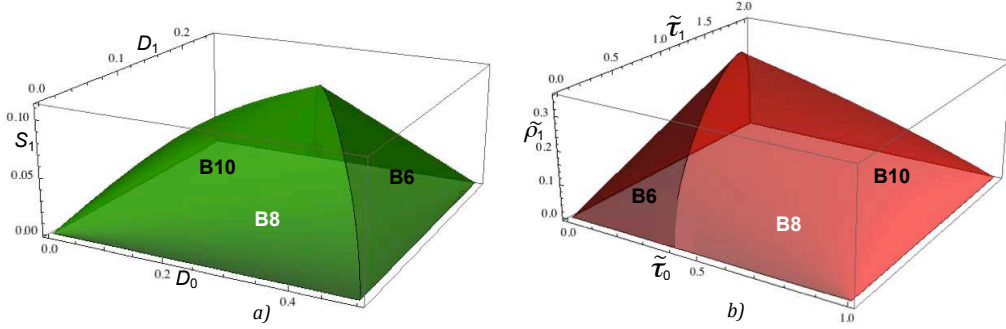


Figure 4: Admissible domain for the case of r_0 -orthotropic damaged material; a) domain for the invariants of \mathbb{D} ; b) domain for the invariants of $\tilde{\mathbb{C}}$ (material with $\tau_1 = 2$; the bounds indicated by white letters are placed behind and seen in transparency).

Once more, bound **B5** is clearly redundant, because equal to **B6**, and **B9** too: in fact, it is an easy matter to check that $\mathbf{B10} \leq \mathbf{B9}$ everywhere in the set $\{0 < \tilde{\tau}_0 \leq 1, 0 < \tilde{\tau}_1 \leq \tau_1\}$.

The dimensionless polar bounds for $\tilde{\mathbb{C}}$ resumes finally to

$$\begin{aligned}
0 < \tilde{\tau}_0 &\leq 1, \\
0 < \tilde{\tau}_1 &\leq \tau_1, \\
0 \leq \tilde{\rho}_1 &< \min \left\{ \sqrt{\frac{\tilde{\tau}_0 \tilde{\tau}_1}{2}}; \sqrt{\frac{\tilde{\tau}_1 (1 - \tilde{\tau}_0) (\tau_1 - \tilde{\tau}_1)}{2(1 + \tau_1)}} \right\},
\end{aligned} \tag{77}$$

while their dimensional corresponding is

$$\begin{aligned}
0 < \tilde{T}_0 &\leq T_0, \\
0 < \tilde{T}_1 &\leq T_1, \\
0 \leq \tilde{R}_1 &< \min \left\{ \sqrt{\tilde{T}_0 \tilde{T}_1}; \sqrt{\frac{2\tilde{T}_1 (T_0 - \tilde{T}_0) (T_1 - \tilde{T}_1)}{T_0 + 2T_1}} \right\}.
\end{aligned} \tag{78}$$

To end this section, we remark that Fig. 4 clearly confirms that the admissible domain is convex, as it has to be, being this one a particular case of ordinary orthotropy.

11. Conclusion

We have shown in this paper that it is possible, using an appropriate choice of tensor representation, the polar formalism, to give a complete analytical expression for the bounds of the damage, \mathbb{D} , and damaged elastic

tensor, $\tilde{\mathbb{C}}$. These bounds allows for $\tilde{\mathbb{C}}$ being positive definite, while \mathbb{D} and $\hat{\mathbb{C}}$, eq. (28), positive semi-definite. This last condition is related to the limited strain energy stored in the damaged material with respect to the original one, which can be interpreted as an equivalence to a positive dissipation during the damage process.

We have also shown that when damage transforms the material into an isotropic or square-symmetric material, then the bounds on the positive semi-definiteness of \mathbb{D} perfectly coincide with those of $\hat{\mathbb{C}}$. This can be interpreted as follows: for an isotropic or square-symmetric damaged material, the positive definiteness of $\tilde{\mathbb{C}}$ and positive semi-definiteness of \mathbb{D} allows for defining the bounds on $\tilde{\mathbb{C}}$ and \mathbb{D} .

For all the other types of elastic symmetries of the damaged material, the bounds on \mathbb{D} are less restrictive than those on $\hat{\mathbb{C}}$, excepted for $\tau_1 = 1$, when they perfectly coincide.

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Appendix A. Polar conditions for the positive definiteness of a planar fourth-rank anisotropic tensor of the elasticity type.

Let us consider in \mathfrak{R}^2 a fourth-rank tensor \mathbb{T} of the type of elasticity, eq. (1), and a second-rank symmetric tensor \mathbf{L} . The expressions of their independent Cartesian components by the polar method are given, for \mathbb{T} , by eq. (2), and for \mathbf{L} by:

$$\begin{aligned} L_{11}(\theta) &= t + r \cos 2(\varphi - \theta), \\ L_{12}(\theta) &= r \sin 2(\varphi - \theta), \\ L_{22}(\theta) &= t - r \cos 2(\varphi - \theta). \end{aligned} \tag{A.1}$$

The quantities t and r are the polar invariants of \mathbf{L} , the angle φ fixes the frame. It is important to recall that

$$R_0 \geq 0, \quad R_1 \geq 0, \quad r \geq 0, \tag{A.2}$$

because they mathematically represent the norm of a complex number.

\mathbb{T} is positive definite if and only if the quadratic form f , defined on the vector space of the second rank tensors,

$$f = \mathbf{L} \cdot \mathbb{T} \mathbf{L} > 0 \quad \forall \mathbf{L}. \tag{A.3}$$

When eqs. (2) and (A.1) are inserted into eq. (A.3), after some standard passages one gets

$$f = 8T_1 t^2 + 16R_1 t r \cos 2(\Phi_1 - \varphi) + 4[R_0 \cos 4(\Phi_0 - \varphi) + T_0] r^2, \quad (\text{A.4})$$

which in matrix form reads

$$f = 4(t, r) \cdot \begin{bmatrix} 2T_1 & 2R_1 \cos 2(\Phi_1 - \varphi) \\ 2R_1 \cos 2(\Phi_1 - \varphi) & T_0 + R_0 \cos 4(\Phi_0 - \varphi) \end{bmatrix} \begin{pmatrix} t \\ r \end{pmatrix}. \quad (\text{A.5})$$

Condition (A.3) is hence satisfied if and only if $\forall \varphi$

$$\begin{cases} T_1 > 0, \\ T_0 + R_0 \cos 4(\Phi_0 - \varphi) > 0, \\ T_1 [T_0 + R_0 \cos 4(\Phi_0 - \varphi)] > 2R_1^2 \cos^2 2(\Phi_1 - \varphi). \end{cases} \quad (\text{A.6})$$

The above three conditions can be reduced to only two independent and simpler conditions: first of all, in (A.6₃) the second member is a positive quantity, so if (A.6_{2,3}) are satisfied, (A.6₁) is obviously redundant, hence it will be discarded. Condition (A.6₃) should now be transformed; to this purpose, let us introduce the angle

$$\alpha = \Phi_1 - \varphi \quad (\text{A.7})$$

which implies

$$\Phi_0 - \alpha = \Delta\Phi + \alpha, \quad (\text{A.8})$$

where

$$\Delta\Phi = \Phi_0 - \Phi_1, \quad (\text{A.9})$$

an invariant of \mathbb{T} .

The third of the (A.6) becomes hence

$$T_1 [T_0 + R_0 \cos 4(\Delta\Phi + \alpha)] > 2R_1^2 \cos^2 2\alpha \quad \forall \alpha, \quad (\text{A.10})$$

that can be transformed, using standard trigonometric identities, first to

$$T_0 T_1 - R_1^2 + \{ [T_1 R_0 \cos 4\Delta\Phi - R_1^2] \cos 4\alpha - T_1 R_0 \sin 4\Delta\Phi \sin 4\alpha \} > 0 \quad \forall \alpha, \quad (\text{A.11})$$

then to

$$T_0 T_1 - R_1^2 > \sqrt{(T_1 R_0 \cos 4\Delta\Phi - R_1^2)^2 + T_1^2 R_0^2 \sin^2 4\Delta\Phi} \cos 4(\alpha - \varpi) \quad \forall \alpha, \quad (\text{A.12})$$

where

$$\varpi = \frac{1}{4} \arctan \frac{T_1 R_0 \sin 4\Delta\Phi}{R_1^2 - T_1 R_0 \cos 4\Delta\Phi}, \quad (\text{A.13})$$

a function of only invariants of \mathbb{T} . Being the quantity under the sign of square root in (A.12) strictly positive, conditions (A.6₂) and (A.12) to be true $\forall\varphi$ with some simple manipulations resume to

$$\begin{cases} T_0 > R_0, \\ T_0 T_1 > R_1^2, \\ T_1(T_0^2 - R_0^2) > 2R_1^2 [T_0 - R_0 \cos 4\Delta\Phi]. \end{cases} \quad (\text{A.14})$$

Condition (A.14₂) is less restrictive than condition (A.14₃), and can be discarded. To show this, let us transform eq. (A.14) to a dimensionless form upon introduction of the ratios

$$\xi = \frac{T_0 T_1}{R_1^2}, \quad \eta = \frac{R_0}{T_0}. \quad (\text{A.15})$$

To remark that by the (A.2₁), (A.6₁) and (A.14₁), ξ and η are non negative quantities. Introducing eq. (A.15) into eq. (A.14) gives

$$\begin{cases} \eta < 1, \\ \xi > 1, \\ \xi > 2 \frac{1 - \eta \cos 4\Delta\Phi}{1 - \eta^2}, \end{cases} \quad (\text{A.16})$$

Then, condition (A.16₃) is more restrictive than condition (A.16₂) if

$$2 \frac{1 - \eta \cos 4\Delta\Phi}{1 - \eta^2} \geq 1, \quad (\text{A.17})$$

thanks to (A.16₁) equivalent to

$$\eta^2 - 2\eta \cos 4\Delta\Phi + 1 \geq 0, \quad (\text{A.18})$$

which is always true, as it is easily checked. Finally, condition (A.14₂) can be discarded because less restrictive than condition (A.14₃) and the only invariant conditions to be \mathbb{T} positive definite are eqs. (A.14_{1,3}), along with the two conditions (A.2_{1,2}), intrinsic to the polar method:

$$\begin{cases} T_0 > R_0, \\ T_1(T_0^2 - R_0^2) > 2R_1^2 [T_0 - R_0 \cos 4(\Phi_0 - \Phi_1)], \\ R_0 \geq 0, \\ R_1 \geq 0. \end{cases} \quad (\text{A.19})$$

To remark also that conditions (A.19) imply that the isotropic part of \mathbb{T} is strictly positive:

$$T_0 > 0, \quad T_1 > 0. \quad (\text{A.20})$$

The above four conditions are valid for a completely anisotropic planar material; the different cases of elastic symmetries can be easily derived introducing the appropriate conditions:

- ordinary orthotropy:

$$\begin{aligned} \Phi_0 - \Phi_1 = K \frac{\pi}{4}, \\ K \in \{0, 1\} \end{aligned} \longrightarrow \begin{cases} T_0 > R_0, \\ T_1 [T_0 + (-1)^K R_0] > 2R_1^2, \\ R_0 \geq 0, \\ R_1 \geq 0; \end{cases} \quad (\text{A.21})$$

- R_0 -orthotropy:

$$R_0 = 0 \longrightarrow \begin{cases} T_0 > 0, \\ T_1 T_0 > 2R_1^2, \\ R_1 \geq 0; \end{cases} \quad (\text{A.22})$$

- square symmetry:

$$R_1 = 0 \longrightarrow \begin{cases} T_0 > R_0, \\ T_1 > 0, \\ R_0 \geq 0; \end{cases} \quad (\text{A.23})$$

- isotropy:

$$R_0 = R_1 = 0 \longrightarrow \begin{cases} T_0 > 0, \\ T_1 > 0. \end{cases} \quad (\text{A.24})$$

Appendix B. On the link between a positive intrinsic dissipation and a positive semi-definite loss of stiffness tensor

Let us consider a material for which the elastic behavior is *linear*, and a process that induces a loss of stiffness (e.g. elasticity-damage coupling). The initial stiffness tensor, the actual stiffness tensor and the loss of stiffness tensor are respectively noted \mathbb{C} , $\tilde{\mathbb{C}}$ and $\hat{\mathbb{C}} = \mathbb{C} - \tilde{\mathbb{C}}$.

The intrinsic dissipation due to loss of stiffness reads (Chaboche (1978), Chaboche (1994)) :

$$\mathcal{D} = \frac{1}{2} \varepsilon^e \cdot \dot{\hat{\mathbb{C}}} \varepsilon^e \quad (\text{B.1})$$

From this last equation, it is obvious that

$$\mathcal{D} \geq 0 \quad \iff \quad \dot{\hat{\mathbb{C}}} \text{ is positive semi-definite} \quad (\text{B.2})$$

Property 1. *The positive semi-definiteness of the loss of stiffness tensor $\hat{\mathbb{C}}$ is a necessary condition for the intrinsic dissipation \mathcal{D} due to loss of stiffness to be positive.*

Proof. By definition, $\hat{\mathbb{C}}(t=0)$ is null, then

$$\hat{\mathbb{C}} = \int_0^t \dot{\hat{\mathbb{C}}} dt \quad (\text{B.3})$$

A positive intrinsic dissipation \mathcal{D} due to loss of stiffness implies that $\dot{\hat{\mathbb{C}}}(t)$ is positive semi-definite $\forall t$ (see eq (B.2)), thus eq. (B.3) implies that $\hat{\mathbb{C}}$ is positive semi-definite. \square

Let us consider now that the loss of stiffness is due to a linear elasticity-damage coupling: the loss of stiffness tensor $\hat{\mathbb{C}}$ is assumed to be a linear function of the damage tensor \mathbb{D} .

Property 2. *The positivity of the intrinsic dissipation \mathcal{D} due to linear elasticity-damage coupling is equivalent to the positive semi-definiteness of the loss of stiffness tensor $\hat{\mathbb{C}}$.*

Proof.

1. Let us first assume the positivity of \mathcal{D} : the property 1 leads to the positive semi-definiteness of $\hat{\mathbb{C}}$.
2. Let us now assume the positive semi-definiteness of $\hat{\mathbb{C}}$. $\hat{\mathbb{C}}$ being linear in \mathbb{D} , there exists a *constant* tensor \mathcal{A} such that $\hat{\mathbb{C}} = \mathcal{A}\mathbb{D}$ (with a fourth order damage tensor \mathbb{D} , \mathcal{A} is an eighth order tensor). The positive semi-definiteness of $\hat{\mathbb{C}}$ implies positive semi-definiteness of $\mathcal{A}\mathbb{D} \forall \mathbb{D}$, and thus the positive semi-definiteness of $\hat{\mathbb{C}} = \mathcal{A}\mathbb{D} \forall \mathbb{D}$. Considering eq. (B.2), this proves the positivity of the intrinsic dissipation \mathcal{D} due to linear elasticity-damage coupling.

\square

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