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BSDEs with a random terminal time driven by a monotone generator and their links with PDEs

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Abstract

In this paper, we study one-dimensional backward stochastic differential equations (in short BSDEs) with a random terminal time driven by a monotone generator, and their links with elliptic partial differential equations. Firstly, we present the case of BSDEs driven by a strictly monotone generator, and next we consider BSDEs driven by a monotone generator.

Keywords: Backward stochastic differential equations, Probabilistic formulae for elliptic PDE, Viscosity solutions.

1 Introduction

Linear BSDEs appeared some years ago as the adjoint processes in the maximum principle of stochastic control and in the *Black-Scholes* formula for the pricing of options. In 1990, E. Pardoux and S. Peng introduced the notion of nonlinear BSDE in a foundational paper [17].
Since then, the interest in BSDEs has increased. Indeed, BSDEs provide connections with other domains such as mathematical finance (refer to [10]), stochastic control (see [19]) and partial differential equations. In our paper, we focus on the last link.

Let us first look at a BSDE with a deterministic terminal time T, introduced in [17].
We consider a Brownian motion $\big(W_t\big)_{t\geq 0}$ and we denote by $\big(\mathcal{F}_t\big)_{t\geq 0}$ its natural filtration. We look for a couple of processes $\big(Y_t, Z_t\big)_{t\geq 0}$, $\big(\mathcal{F}_t\big)_{t\geq 0}$-adapted and which satisfies the following equation

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)\ ds - \int_t^T Z_s\ dW_s, \qquad 0 \leq t \leq T, \tag{1}$$

where ξ is an $\mathcal{F}_T$-measurable random variable called the terminal condition and f is the generator.

Of course, we need suitable assumptions on ξ and on f in order to guarantee existence of a solution.
In [17], E. Pardoux and S. Peng stated the following theorem: in any dimension there exists a unique solution to (1) when f is Lipschitz in y and z, ξ is square-integrable.
Since then, some improvements have been obtained. For instance, S. Peng introduced for the first time monotone generators in [18]. On the other hand, in the one-dimensional case, J. San Martin and J.-P. Lepeltier described the case of the BSDEs with a continuous generator, and M. Kobylanski studied those with a generator which is quadratic in z.

It is now well-known that BSDEs with a deterministic terminal time provide probabilistic formulas for viscosity solutions to parabolic partial differential equations, whereas the BSDEs with a random terminal time are connected with elliptic PDEs.
In this paper, we aim to study a BSDE with a random terminal time τ.

We suppose that τ is a stopping time which is not necessarily bounded.
The terminal condition ξ has to be $\mathcal{F}_\tau$-measurable. We look for a pair of processes $\big(Y_t, Z_t\big)_{t\geq 0}$ adapted to the natural filtration and which satisfy $\quad \forall t \geq 0,\ \forall r \geq t,$

$$\left\{\begin{array}{lll} Y_{t\wedge\tau} & = & Y_{r\wedge\tau} \;+\; \int_{t\wedge\tau}^{r\wedge\tau} f(s,Y_s,Z_s)\;ds \;-\; \int_{t\wedge\tau}^{r\wedge\tau} Z_s\;dW_s, \\ Y_\tau & = & \xi \qquad \text{on } \{\tau<\infty\}. \end{array}\right. \tag{2}$$

We shall denote by $\big(f,\xi,\tau\big)$ such an equation.

Existence theorems still hold for random terminal times, but under stronger assumptions. In particular, the generator is required to be strictly monotone.
In [16], E. Pardoux proved that in the k-dimensional case ($k\in\mathbb{N}^*$), a unique solution to (2) exists under the following assumptions:

(**H1**) f is Lipschitz in z, with Lipschitz constant K.
(**H2**) f is continuous in y and has a controlled growth in y.
(**P3**) f is monotone in y, with a constant of monotonicity $\mu\in\mathbb{R}$.
(**P4**) $\exists\,\lambda>K^2-2\mu$ such that $\mathbb{E}\Big(\,exp\,(\lambda\tau)\;|\xi|^2\Big)<\infty$, $\mathbb{E}\Big(\int_0^\tau exp\,(\lambda s)\;|f(s,0,0)|^2\,ds\Big)<\infty$, and $\mathbb{E}\Big(\int_0^\tau exp\,(\lambda s)\;\big|f\big(s,\mathbb{E}(\xi|\mathcal{F}_s),\eta_s\big)\big|^2\;ds\Big)<\infty$ with η obtained thanks to the Brownian martingale representation theorem applied to $\mathbb{E}(\xi|\mathcal{F}_t)$.

But the last hypothesis is not satisfactory, and should be improved.

P. Briand and Y. Hu have improved this result in [4] for real valued BSDEs. They assume in their work that f is Lipschitz in y and z, **(P3)** with $\mu>0$; moreover, $f(t,0,z)$ and ξ have to be bounded.
Just note that the condition that f is strictly monotone, *i.e.* it is monotone with a positive constant μ, seems to be more restrictive than **(P3)**. Actually, both hypotheses **(P3)** and **(P4)** may often imply that $\lambda<0$, which leads that $\mu>\frac{K^2-\lambda}{2}>0$.

The first objective of our work is to improve the preceding results of existence and uniqueness.
The main result of this paper is a generalization of existence theorems about BSDEs with a random terminal time and $\mu=0$.
The paper is organized as follows. In section 2, we study BSDEs when $\mu>0$ and, in section 3, we extend the study to a null constant of monotonicity. Next, thanks to the connections between BSDEs and PDEs, we derive results for elliptic PDEs. We treat PDEs with a strictly monotone generator in section 4 while we discuss links to PDEs with a monotone generator in the last section.

2 BSDEs with a strictly monotone generator

2.1 Notation

Let $(\Omega,\mathcal{F},\mathbb{P})$ be a complete probability space and $(W_t)_{t\geq 0}$ be a d-dimensional Brownian motion defined on this space with $W_0=0$. We denote by $\big(\mathcal{F}_t\big)_{t\geq 0}$ the σ-algebra generated by W. We make the usual $\mathbb{P}$-augmentation to $\mathcal{F}_t$, so that the filtration $\big(\mathcal{F}_t\big)_{t\geq 0}$ satisfies the usual hypothesis.

Let τ be an $\big(\mathcal{F}_t\big)$-stopping time and μ be some real number.

We define the following sets of $\big(\mathcal{F}_t\big)$-progressively measurable $\mathbb{R}^d$-valued processes $\big(\psi_t\big)_{t\geq 0}$,

$$\mathcal{M}^{2,-2\mu}\big(0,\tau;\mathbb{R}^d\big)=\left\{\psi\;;\quad \mathbb{E}\Big(\int_0^\tau exp\,(-2\mu s)\;\|\psi_s\|^2\,ds\Big)<\infty\right\},$$

$$\mathcal{M}^2_{\text{loc}}\big(0,\tau;\mathbb{R}^d\big)=\left\{\psi\;;\quad \mathbb{E}\Big(\int_0^{t\wedge\tau}\|\psi_s\|^2\,ds\Big)<\infty,\quad \forall t\geq 0\right\}.$$

Consider $f:\;\Omega\times[0,\infty[\times\mathbb{R}\times\mathbb{R}^d\longrightarrow\mathbb{R}$ such that $\forall(y,z)\in\mathbb{R}\times\mathbb{R}^d$, $\big(f(t,y,z)\big)_{t\geq 0}$is progressively

measurable.

(H 1). f is uniformly Lipschitz in z with Lipschitz constany K:
$\forall t \geq 0, \forall y \in \mathbb{R}, \forall z, z' \in \mathbb{R}^d, \quad \big|f(t,y,z) - f(t,y,z')\big| \leq K||z - z'|| \quad \mathbb{P}a.s.$

(H 2). $\forall t \geq 0$, $\forall z \in \mathbb{R}^d$, $\ y \mapsto f(t,y,z)$ is continuous,
$\exists$ a continuous and increasing function $\varphi : \mathbb{R}_+ \mapsto \mathbb{R}_+$ such that
$\forall t \geq 0$, $\forall y \in \mathbb{R}$, $\forall z \in \mathbb{R}^d$, $\quad |f(t,y,z)| \leq |f(t,0,z)| + \varphi(|y|) \quad \mathbb{P}a.s.$

(H 3). f is strictly monotone in y with a constant of monotonicity $\mu > 0$:
$\forall t \geq 0, \forall y, y' \in \mathbb{R}, \forall z \in \mathbb{R}^d, \ \big(y - y'\big)\big(f(t,y,z) - f(t,y',z)\big) \leq -\mu\big|y - y'\big|^2 \quad \mathbb{P}a.s.$

(H 4). $\exists\, C > 0$ such that $\forall t \geq 0$, $|f(t,0,0)| \leq C \quad \mathbb{P}a.s.$ We shall denote $\sup_{t\geq 0} |f(t,0,0)|$ by C, and $\xi \in \mathcal{L}^\infty(\mathcal{F}_\tau)$; we denote by M some real such that $|\xi| \leq M \quad \mathbb{P}a.s.$

We denote **(H1)** to **(H4)** collectively by $\big(\boldsymbol{A_\mu}\big)$.

2.2 Existence and uniqueness

We study the following problem: we suppose that f satisfies $\big(\mathbf{A_\mu}\big)$, and we want to construct an adapted process $\big(Y_t, Z_t\big)_{t\geq 0}$ which solves the BSDE $\big(f, \xi, \tau\big)$.

We call solution of the equation $\big(f, \xi, \tau\big)$ a couple of progressively measurable processes $\big(Y_t, Z_t\big)_{t\geq 0}$ with values in $\mathbb{R} \times \mathbb{R}^d$ such that

1. $\mathbb{E}\Big(\sup_{t\geq 0} e^{-2\mu t}\,|Y_t|^2\Big) < \infty$ and $Z \in \mathcal{M}^2_{loc}\big(0,\tau;\mathbb{R}^d\big)$,
2. On the set $\{t \geq \tau\}$, we have $Y_t = \xi$ and $Z_t = 0$,
3. $\forall r \geq 0$, $\forall t \in [0,r]$, we have $\quad Y_{t\wedge\tau} \ = \ Y_{r\wedge\tau} \ + \ \int_{t\wedge\tau}^{r\wedge\tau} f(s, Y_s, Z_s)\, ds \ - \ \int_{t\wedge\tau}^{r\wedge\tau} Z_s\, dW_s.$

We can state the main result of this section, concerning the existence and uniqueness of solutions of BSDE $\big(f, \xi, \tau\big)$.

Theorem 2.1
There exists a solution $\big(Y_t, Z_t\big)_{t\geq 0}$ to the BSDE $\big(f, \xi, \tau\big)$ such that Y is a continuous process bounded by $M + \frac{C}{\mu}$ and $Z \in \mathcal{M}^2\big(0,\tau;\mathbb{R}^d\big)$.

Moreover, $\big(Y_t, Z_t\big)_{t\geq 0}$ is unique in the class of processes $\big(Y, Z\big)$ such that Y is continuous and uniformly bounded, and Z belongs to $\mathcal{M}^2_{loc}\big(0,\tau;\mathbb{R}^d\big)$.

Proof.

We adopt the same strategy as in [4], with some modifications.

First, we prove uniqueness.

Suppose that $\big(Y^1_t, Z^1_t\big)_{t\geq 0}$, $\big(Y^2_t, Z^2_t\big)_{t\geq 0}$ are both solutions of the BSDE $\big(f, \xi, \tau\big)$ such that Y^1, Y^2 are continuous and uniformly bounded and Z^1, Z^2 belong to $\mathcal{M}^2_{\text{loc}}\big(0,\tau;\mathbb{R}^d\big)$.
Let us set $\hat{Y} = Y^1 - Y^2$ and $\hat{Z} = Z^1 - Z^2$.
Then $\hat{Y}$ is a continuous process, bounded by some constant, say M'.

We define the $\mathbb{R}^d$-valued process β:

$$\beta_s \;=\; \begin{cases} \frac{f\left(s,Y_s^2,Z_s^1\right)-f\left(s,Y_s^2,Z_s^2\right)}{\left\|Z_s^1-Z_s^2\right\|^2}\,\left(Z_s^1-Z_s^2\right) & \text{if } Z_s^1-Z_s^2\neq 0\ , \\ 0 & \text{otherwise.} \end{cases} \qquad \text{Remark that } \|\beta\|\leq K.$$

Fix $t\in\mathbb{R}_+$ and pick $n\geq t$.

We denote by L the local time associated to $\widehat{Y}$ and we apply Tanaka's formula to the continuous semi-martingale $\widehat{Y}$. It yields

$$\begin{aligned} |\widehat{Y}_{n\wedge\tau}| \;&=\; |\widehat{Y}_{t\wedge\tau}| \;-\; \int_{t\wedge\tau}^{n\wedge\tau}\frac{\widehat{Y}_s}{|\widehat{Y}_s|}\,\big(\,f(s,Y_s^1,Z_s^1)-f(s,Y_s^2,Z_s^1)\,\big)\,ds \\ &\quad+\;\int_{t\wedge\tau}^{n\wedge\tau}\frac{\widehat{Y}_s}{|\widehat{Y}_s|}\,\widehat{Z}_s\,\big(dW_s-\beta_s ds\big)\;+\;(\,L^0_{n\wedge\tau}-L^0_{t\wedge\tau}\,). \end{aligned}$$

Then, we define Q_n the probability measure on $(\Omega,\mathcal{F}_n)$ whose density with respect to $\mathbb{P}_{|\mathcal{F}_n}$ is $M_n \;=\; exp\big(\int_0^n\;\beta_s\;dW_s-1/2\int_0^n\;||\beta_s||^2\;ds\big)$.
Since β is a bounded process, the probability measures Q_n and $\mathbb{P}_{|\mathcal{F}_n}$ are mutually absolutely continuous and $\big(\;W_t\;-\;\int_0^t\beta_s\;ds\;\big)_{0\leq t\leq n}$ is a Brownian motion under Q_n.
Now, applying Itô's formula to $exp\,(-\mu s)\;\left|\widehat{Y}_s\right|$ between $t\wedge\tau$ and $n\wedge\tau$, we get,
from the monotonicity of f

$$\begin{aligned} exp\left(-\mu\,(t\wedge\tau)\right)\;\left|\widehat{Y}_{t\wedge\tau}\right| \;&\leq\; exp\left(-\mu\,(n\wedge\tau)\right)\;\left|\widehat{Y}_{n\wedge\tau}\right| \;-\;\int_{t\wedge\tau}^{n\wedge\tau}exp\,(-\mu s)\;dL^0_s \\ &\quad-\;\int_{t\wedge\tau}^{n\wedge\tau}exp\,(-\mu s)\;\frac{\widehat{Y}_s}{|\widehat{Y}_s|}\;\widehat{Z}_s\,(dW_s-\beta_s ds). \end{aligned}$$

Taking the conditional expectation with respect to $\mathcal{F}_{t\wedge\tau}$ under Q_n, we obtain

$$\begin{aligned} \forall n\in\mathbb{N}\text{ such that } t\leq n,\quad \left|\widehat{Y}_{t\wedge\tau}\right| \;&\leq\; M'\,\mathbb{E}^{Q_n}\big(exp\left(\mu\,(t\wedge\tau)\right)\,|\mathcal{F}_{t\wedge\tau}\big)\,exp\,(-\mu n) \\ &\leq\; M'\,exp\left((\mu\,(t-n)\right)\qquad Q_n-a.s. \end{aligned}$$

We finally obtain, by sending n to infinity, that $\quad\forall t\geq 0,\;\widehat{Y}_{t\wedge\tau}\;=\;0\quad\mathbb{P}a.s.$
We conclude using the continuity of Y^1 and Y^2, which shows that $Y^1=Y^2\quad\mathbb{P}a.s.$

We easily deduce the uniqueness of Z in $\mathcal{M}^2_{\text{loc}}\big(0,\tau;\mathbb{R}^d\big)$.

Now, we turn to existence.

The proof of existence depends heavily on the idea of constructing a fixed terminal time so as to use well known results about such BSDEs. For example we can refer to [16] or to [10].

We define $\left(Y_t^n,Z_t^n\right)_{t\geq 0}$ as the unique solution of $\big(f,\xi\,1\!\!1_{\{\tau\leq n\}},\tau\wedge n\big)$.

(i) We start by proving that Y^n is a bounded process, adopting the same tools as for uniqueness. Indeed, we construct a $\mathbb{R}^d$-valued process β^n defined as below:

$$\beta_s^n \;=\; \begin{cases} \frac{f\left(s,0,Z_s^n\right)-f\left(s,0,0\right)}{\left\|Z_s^n\right\|^2}\;Z_s^n & \text{if } Z_s^n\neq 0\ , \\ 0 & \text{otherwise.} \end{cases}$$

We fix $t\geq 0$ and $n\geq t$. We first apply Tanaka's formula as previously and next Itô's formula to $exp\,(-\mu s)\;\left|Y_s^n\right|$ between $t\wedge\tau$ and $n\wedge\tau$.

From Girsanov's theorem, taking into account that both ξ and $f(s,0,0)$ are bounded $\mathbb{P}_{|\mathcal{F}_n}$ $a.s.$, we deduce that

$$\forall t\geq 0,\qquad \left|Y_t^n\right|\;\leq\;M\;+\;\frac{C}{\mu}\qquad\mathbb{P}a.s. \tag{3}$$

Remark that we can define Y^n and Z^n on the whole time axis by setting: $\forall t \geq n, \quad Y_t^n = \xi \; 1\!\!1_{\{\tau \leq n\}}$ and $\quad Z_t^n = 0$.

(ii) We study the sequence $\left(Y_t^n\right)_{n\geq 0}$ so as to show that it converges almost surely.
Actually, we are going to show that it is an almost sure Cauchy sequence.
Let $m, n \in \mathbb{N}$ such that $n \leq m$. Once again we construct a process $\beta^{n,m}$ as below:

$$\forall i \in 1, 2, ..., d, \qquad \beta_s^{n,m} = \begin{cases} \frac{f\left(s, Y_s^n, Z_s^n\right) - f\left(s, Y_s^n, Z_s^m\right)}{\left\|Z_s^m - Z_s^n\right\|^2} \left(Z_s^m - Z_s^n\right) & \text{if } Z_s^m \neq Z_s^n \;, \\ 0 & \text{otherwise.} \end{cases}$$

We use the same strategy so as to obtain,

$$\forall t \geq 0, \qquad \left|Y_t^m - Y_t^n\right| \;\leq\; \left(M \;+\; \frac{C + \varphi(M)}{\mu} \right) exp\,(\mu t) \;\; exp\,(-\mu n) \qquad \mathbb{P}\; a.s. \tag{4}$$

Consequently, $\left(Y_t^n\right)_{n\geq 0}$ is a Cauchy sequence. Hence, it converges and we denote by Y its limit.
In addition, sending m to infinity, we obtain

$$\forall t \geq 0 \qquad \left|Y_t - Y_t^n\right| \;\leq\; \alpha \; exp\,(\mu(t-n)) \; \mathbb{P}a.s., \qquad \text{where } \alpha = M \;+\; \frac{C + \varphi(M)}{\mu}. \tag{5}$$

This inequality implies that the sequence of continuous processes $\left(Y^n\right)_{n\in\mathbb{N}}$ converges almost surely to Y uniformly on the compact sets of $\mathbb{R}_+$.
Then, the limit process Y is also continuous and from (3) we get that it is bounded by $M + \frac{C}{\mu}$.

(iii) Now we are going to show that the sequence $\left(Y_t^n\right)_{n\geq 0}$ also converges in $\mathcal{M}^{2,-2\mu}\left(0, \tau; \mathbb{R}\right)$.
In order to prove that $\lim_{n\to\infty} \; \mathbb{E}\Big(\int_0^\tau exp\,(-2\mu s) \;\; |Y_s - Y_s^n|^2 \; ds \Big) \;=\; 0,$

we have to split the term into two parts and we study them successively: the integral on $[0, n\wedge\tau]$ and next the integral on $[n\wedge\tau, \tau]$.

Using the Inequality (5) for the first term, we get that

$$\mathbb{E}\Big(\int_0^{n\wedge\tau} exp\,(-2\mu s) \;\; |Y_s - Y_s^n|^2 \; ds \Big) \;\leq\; \alpha^2 \; n \; exp\,(-2\mu n).$$

In addition, from the definition of Y_t^n on $\mathbb{R}_+$, we know that $\forall t \geq n, \quad Y_t^n = \xi \; 1\!\!1_{\{\tau \leq n\}}$.

Hence, $\quad \mathbb{E}\Big(\int_{n\wedge\tau}^\tau exp\,(-2\mu s) \;\; |Y_s - Y_s^n|^2 \; ds \Big) \;=\; \mathbb{E}\Big(1\!\!1_{\{n<\tau\}} \; \int_n^\tau exp\,(-2\mu s) \;\; |Y_s - \xi \; 1\!\!1_{\{\tau\leq n\}}|^2 \; ds \Big)$

$$\leq \;\; 4 \; \mathbb{E}\Big(1\!\!1_{\{n<\tau\}} \; \big(M + \frac{C}{\mu}\big)^2 \; \int_n^\tau exp\,(-2\mu s) \;\; ds \Big) \;\leq\; \frac{2}{\mu} \; \big(M + \frac{C}{\mu}\big)^2 \; exp\,(-2\mu n).$$

Finally,

$$\mathbb{E}\Big(\int_0^\tau exp\,(-2\mu s) \;\; |Y_s - Y_s^n|^2 \; ds \Big) \;\leq\; \alpha^2 \; n \; exp\,(-2\mu n) \;\; + \; \frac{2}{\mu} \; \big(M + \frac{C}{\mu}\big)^2 \; exp\,(-2\mu n). \tag{6}$$

We derive that the sequence $\left(Y^n\right)_{n\geq 0}$ converges to Y in $\mathcal{M}^{2,-2\mu}\left(0, \tau; \mathbb{R}\right)$.

(iv) To continue, let us show that the sequence $\left(Z_t^n\right)_{n\geq 0}$ converges in $\mathcal{M}^{2,-2\mu}\left(0, \tau; \mathbb{R}^d\right)$
We still consider $m, n \in \mathbb{N}$ such that $n \leq m$ and we keep the same notations as in (ii).
We apply Itô's formula to $exp\,(-2\mu s) \; |Y_s^m - Y_s^n|^2$ between 0 and $m\wedge\tau$, and we take the expectation of the expression. Then we get, using the monotonicity of f,

$$\mathbb{E}\Big(\int_0^{m\wedge\tau} exp\,(-2\mu s)\ \|Z_s^m - Z_s^n\|^2\ ds\Big) \ \le\ \mathbb{E}\Big(exp\,\big(-2\mu(m\wedge\tau)\big)\ |Y_m^m - Y_m^n|^2\Big) \ -\ \mathbb{E}\big(|Y_0^m - Y_0^n|^2\big)$$

$$+\ 2\,\mathbb{E}\Big(\int_0^{m\wedge\tau} exp\,(-2\mu s)\,(Y_s^m - Y_s^n)\ \Big(\beta_s^{m,n}.(Z_s^m - Z_s^n)\ +\ 1\!\!1_{\{s>n\}}\ \ f(s,\xi 1\!\!1_{\{\tau\le n\}},0)\Big)\ ds\ \Big)$$

$$\le\ \ \mathbb{E}\Big(exp\,\big(-2\mu(m\wedge\tau)\big)\,|\xi|^2 1\!\!1_{\{n<\tau\le m\}}\Big)\ +\ 2K\,\mathbb{E}\Big(\ \int_0^{m\wedge\tau} exp\,(-2\mu s)\,|Y_s^m - Y_s^n|\ \|Z_s^m - Z_s^n\|\,ds\Big)$$

$$+\ 2\,\mathbb{E}\Big(\ \int_n^{m\wedge\tau} exp\,(-2\mu s)\,|Y_s^m - Y_s^n|\,\big|\,f(s,\xi 1\!\!1_{\{\tau\le n\}},0)\,\big|\,ds\Big)$$

$$\le\ \ M^2 exp\,(-2\mu n)\ \ +\ 1/2\ \mathbb{E}\Big(\int_0^{m\wedge\tau} exp\,(-2\mu s)\ \|Z_s^m - Z_s^n\|^2\ ds\Big)$$

$$+\ 2\ \Big(K^2 + C + \varphi(M)\Big)\ \mathbb{E}\Big(\ \int_0^{m\wedge\tau} exp\,(-2\mu s)\ |Y_s^m - Y_s^n|\ ds\Big).$$

Hence, $\mathbb{E}\Big(\int_0^{m\wedge\tau}\ exp\,(-2\mu s)\ \ \|Z_s^m - Z_s^n\|^2\ ds\Big)\ \le\ \alpha'\ (1+n)\ exp\,(-2\mu n)$,
where α' depends on the constants that describe f and ξ, precisely, $M,\ K,\ \varphi,\ \mu,\ C$.

But on the other hand, $\mathbb{E}\Big(\int_{m\wedge\tau}^{\tau} exp\,(-2\mu s)\,\|Z_s^m - Z_s^n\|^2\,ds\Big)\ =\ 0.$

Consequently, $$\mathbb{E}\Big(\int_0^{\tau}\ exp\,(-2\mu s)\ \ \|Z_s^m - Z_s^n\|^2\ ds\Big)\ \le\ \alpha'\ (1+n)\ exp\,(-2\mu n). \tag{7}$$

We have just shown that $\big(Z^n\big)_{n\ge 0}$ is a Cauchy sequence in $\mathcal{M}^{2,-2\mu}\big(0,\tau;\mathbb{R}^d\big)$, which allows us to consider Z, its limit in $\mathcal{M}^{2,-2\mu}\big(0,\tau;\mathbb{R}^d\big)$.

(v) It remains to prove that the process $\big(Y,Z\big)$ is a solution for the BSDE $\big(f,\xi,\tau\big)$.
We already know that the process Y is continuous and bounded and Z belongs to $\mathcal{M}^{2,-2\mu}\big(0,\tau;\mathbb{R}^d\big)$.
By definition, $\forall n\in\mathbb{N}$, $\ \forall t,\theta$ such that $0\le\theta\le t\le n$, we have

$$Y_{\theta\wedge\tau}^n\ -\ Y_{t\wedge\tau}^n\ =\ \int_{\theta\wedge\tau}^{t\wedge\tau} f(s,Y_s^n,Z_s^n)\ ds\ -\ \int_{\theta\wedge\tau}^{t\wedge\tau} Z_s^n\ dW_s. \tag{8}$$

Let us fix t and θ. We shall pass to the limit in $\mathcal{L}^1$ in the previous equality.
As the sequence $\big(Y_{t\wedge\tau}^n\big)_{n\ge 0}$ converges almost surely to Y_t and is bounded by $M+\frac{C}{\mu}$ uniformly in n, then from Lebesgue's theorem, we get that the sequence $\big(Y_{t\wedge\tau}^n\big)_{n\ge 0}$ converges to Y_t in $\mathcal{L}^1$.
Moreover, $\Big(\int_{\theta\wedge\tau}^{t\wedge\tau} f(s,Y_s^n,Z_s^n)\,ds\Big)_{n\ge 0}$ converges to $\int_{\theta\wedge\tau}^{t\wedge\tau} f(s,Y_s,Z_s)\,ds$ in $\mathcal{L}^1$. Indeed,

$$\mathbb{E}\Bigg(\Bigg|\int_{\theta\wedge\tau}^{t\wedge\tau} f(s,Y_s^n,Z_s^n)\ ds\ -\ \int_{\theta\wedge\tau}^{t\wedge\tau} f(s,Y_s,Z_s)\,ds\Bigg|\Bigg)$$

$$\le\ \ K\ t^{1/2}\ \mathbb{E}\Big(\ \int_{\theta\wedge\tau}^{t\wedge\tau} \|Z_s^n - Z_s\|^2\ ds\ \Big)^{1/2}\ +\ \mathbb{E}\Big(\int_0^t \Big|f(s,Y_s^n,Z_s)\ -\ f(s,Y_s,Z_s)\Big|\,ds\Big).$$

Hence we derive, using Inequality (7), the convergence to zero when n tends to ∞ for the first term of the upper bound.
Concerning the second term, we use Lebesgue's theorem. On the one hand,

$$\big|f(s,Y_s^n,Z_s) - f(s,Y_s,Z_s)\ \le\ 2K\,\big\|Z_s\big\|\ +\ 2\,\varphi\Big(M+\frac{C}{\mu}\Big)\ +\ 2C.$$

On the other hand, the continuity of f in y yields the almost sure convergence, $(d\mathbb{P} \times d\lambda)$-a.s., with λ the Lebesgue's measure.

Moreover, $\Big(\int_{\theta\wedge\tau}^{t\wedge\tau} Z_s^n \, dW_s\Big)_{n\geq 0}$ converges to $\int_{\theta\wedge\tau}^{t\wedge\tau} Z_s \, dW_s$ in $\mathcal{L}^2$ since

$$\mathbb{E}\Big(\Big(\int_{\theta\wedge\tau}^{t\wedge\tau} Z_s^n \, dW_s \; - \; \int_{\theta\wedge\tau}^{t\wedge\tau} Z_s \, dW_s\Big)^2\Big) \;\leq\; exp\,(2\mu t)\; \mathbb{E}\Big(\int_0^{t\wedge\tau} exp\,(-2\mu s)\; \|Z_s^n \; - \; Z_s\|^2 \, ds\Big).$$

Hence, passing to the limit in the Equation (8), we obtain $\forall t, \theta$ such that $\theta \leq t$,

$$Y_{\theta\wedge\tau} \; - \; Y_{t\wedge\tau} \;=\; \int_{\theta\wedge\tau}^{t\wedge\tau} f(s, Y_s, Z_s) \, ds \; - \; \int_{\theta\wedge\tau}^{t\wedge\tau} Z_s \, dW_s. \tag{9}$$

So as to conclude with the proof, it only remains to check the terminal condition.

Let $\omega \in \{\tau < \infty\}$, and $n \in \mathbb{N}$ such that $n \geq \tau(\omega)$.

$$\begin{aligned} |\, Y_\tau \; - \; \xi \; 1\!\!1_{\{\tau\leq 2n\}} \,|\,(\omega) \;&=\; |\, Y_{n\wedge\tau} \; - \; \xi \; 1\!\!1_{\{\tau\leq 2n\}} \,|\,(\omega) \\ &\leq\; \alpha \; exp\,\big(\mu(n\wedge\tau)(\omega)\big) \;\; exp\,(-2\mu n) \; + \; |\, Y_{n\wedge\tau}^{2n} \; - \; \xi \; 1\!\!1_{\{\tau\leq 2n\}} \,|\,(\omega) \qquad \text{from (5).} \\ &\leq\; \alpha \; exp\,(-\mu n) \qquad \text{because } \; Y_{n\wedge\tau}^{2n} = Y_\tau^{2n} = Y_{2n}^{2n} = \xi \; 1\!\!1_{\{\tau\leq 2n\}} \; . \end{aligned}$$

Then, $Y_\tau \;=\; \xi \quad \mathbb{P}a.s.$ on the set $\{\tau < \infty\}$, and the process $\big(Y, Z\big)$ is a solution for BSDE $\big(f, \xi, \tau\big)$.

This completes the proof. □

Remark. The Theorem 2.1 proves the existence and the uniqueness of a solution for the BSDE $\big(f, \xi, \tau\big)$. But we also got a construction for the solution process $\big(Y, Z\big)$.

2.3 Comparison theorem

The purpose of this section is to show a comparison theorem for a BSDE driven by a strictly monotone generator.

The comparison theorem is essential in our study. On the one hand, it gives the existence of a solution for a monotone generator instead of strictly monotone, and on the other hand, it is necessary for proving results about PDEs.

Let $\xi_1, \; \xi_2 \in \mathcal{L}^\infty(\mathcal{F}_\tau)$.
Suppose that there exists a couple of processes $\big(Y_t^1, Z_t^1\big)_{t\geq 0}$ where Y^1 is continuous and bounded, Z^1 satisfies $\mathbb{E}\Big(\int_0^\tau \|Z_s^1\|^2 \, ds\Big) < \infty$, solving the equation $\big(f_1, \xi_1, \tau\big)$.
We are also given f_2 satisfying $\big(\mathbf{A}_\mu\big)$, then Theorem 2.1, guarantees that there exists an unique adapted process $\big(Y_t^2, Z_t^2\big)_{t\geq 0}$ solving the equation $\big(f_2, \xi_2, \tau\big)$.

In the situation where $\xi_1 \leq \xi_2$ and $f_1 \leq f_2$, we wonder if such an inequality remains available for the processes $\big(Y_t^1\big)_{t\geq 0}$ and $\big(Y_t^2\big)_{t\geq 0}$. We can get the following theorem :

Theorem 2.2
Let $\big(Y_t^i, Z_t^i\big)_{t\geq 0}$ be solutions of BSDE $\big(f_i, \xi_i, \tau\big)$, respectively for $i = 1, 2$.
If we suppose that $\xi_1 \leq \xi_2$ and $\forall t \geq 0, \; f_1(t, Y_t^t, Z_t^1) \leq f_2(t, Y_t^t, Z_t^1) \quad \mathbb{P}a.s.$

Then $\qquad \forall t \geq 0, \quad Y_t^1 \leq Y_t^2 \quad \mathbb{P}a.s.$

Proof.

Let us define $\widehat{Y}$, M', $\widehat{Z}$ and β as in the proof of the uniqueness for the Theorem 2.1.
We fix $t \in \mathbb{R}_+$ and pick $p \in \mathbb{N}$ such that $p \geq t$.

We denote by L the local time associated to $\widehat{Y}$ and we apply Tanaka's formula to the continuous semi-martingale $\widehat{Y}$. It yields

$$\begin{aligned}(\widehat{Y}_{p\wedge\tau})^+ &= (\widehat{Y}_{t\wedge\tau})^+ + \int_{t\wedge\tau}^{p\wedge\tau} 1\!\!1_{\widehat{Y}_s>0}\, \widehat{Z}_s\,(dW_s - \beta_s ds) + \tfrac{1}{2}(\,L^0_{p\wedge\tau} - L^0_{t\wedge\tau}\,) \\ &\quad - \int_{t\wedge\tau}^{p\wedge\tau} 1\!\!1_{\widehat{Y}_s>0}\,\big(\,f_1(s,Y^1_s,Z^1_s) - f_2(s,Y^1_s,Z^1_s)\,\big)\,ds \\ &\quad - \int_{t\wedge\tau}^{p\wedge\tau} 1\!\!1_{\widehat{Y}_s>0}\,\big(\,f_2(s,Y^1_s,Z^2_s) - f_2(s,Y^2_s,Z^2_s)\,\big)\,ds.\end{aligned}$$

Then, we set Q_p the probability measure on $(\Omega, \mathcal{F}_p)$ whose density with respect to $\mathbb{P}_{|\mathcal{F}_p}$ is $M_p = exp\big(\int_0^p \beta_s\, dW_s - 1/2 \int_0^p ||\beta_s||^2\, ds\big)$.
Since β is a bounded process, the probability measures Q_p and $\mathbb{P}_{|\mathcal{F}_p}$ are mutually absolutely continuous and $\big(\,W_t - \int_0^t \beta_s\, ds\,\big)_{0\leq t\leq p}$ is a Brownian motion under Q_p.
Now, applying Itô's formula to $exp\,(-\mu s)\ \big|\widehat{Y}_s\big|$ between $t\wedge\tau$ and $p\wedge\tau$, we get,

$$\begin{aligned}e^{-\mu\, t\wedge\tau}\,\big(\widehat{Y}_{t\wedge\tau}\big)^+ &\leq e^{-\mu\, p}\,\big(\widehat{Y}_p\big)^+\, 1\!\!1_{p\leq\tau} + e^{-\mu\,\tau}\,\big(\widehat{Y}_\tau\big)^+\, 1\!\!1_{p>\tau} + \mu \int_{t\wedge\tau}^{p\wedge\tau} e^{-\mu s}\big(\widehat{Y}_s\big)^+\, ds \\ &\quad + \int_{t\wedge\tau}^{p\wedge\tau} e^{-\mu s}\, \tfrac{\widehat{Y}_s}{|\widehat{Y}_s|}\, 1\!\!1_{\widehat{Y}_s>0}\,\big(\,f_2(s,Y^1_s,Z^2_s) - f_2(s,Y^2_s,Z^2_s)\,\big)\,ds \\ &\quad - \int_{t\wedge\tau}^{p\wedge\tau} e^{-\mu s}\, 1\!\!1_{\widehat{Y}_s>0}\,\widehat{Z}_s\, d\widehat{W}_s.\end{aligned}$$

Taking the conditional expectation with respect to $\mathcal{F}_{t\wedge\tau}$ under Q_p, since f_2 is monotone, we obtain

$$\forall p \in \mathbb{N} \text{ such that } t \leq p, \quad \big(\widehat{Y}_{t\wedge\tau}\big)^+ \leq exp\,\big(\mu\,(t-p)\big)\,\big|\widehat{Y}_p\big| \leq M'\, exp\,\big(\mu\,(t-p)\big) \quad Q_p - a.s.$$

We finally obtain, by sending p to infinity, that $\quad \forall t \geq 0,\ Y^1_t \leq Y^2_t \quad \mathbb{P}a.s.$

□

3 Main results for BSDEs with a monotone generator

We showed in Section 2 that the condition $\mu > 0$ is sufficient. But we want to weaken again the condition on μ. Indeed, in this section we are going to study the case where $\mu = 0$. However, such improvement will be paid by an additional constraint on generator at origin. We will study a new list of conditions that allows us to include any generator of the type $-(u^+)^q$ $(q > 0)$.

3.1 Notation

We still consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and τ an $(\mathcal{F}_t)$-stopping time, where $(\mathcal{F}_t)_{t\geq 0}$ is the natural filtration generated by a Brownian motion $(W_t)_{t\geq 0}$.
We define the following sets of $(\mathcal{F}_t)$-progressively measurable processes $(\psi_t)_{t\geq 0}$,

$$\mathcal{S}^2_c(0,\tau;\mathbb{R}) = \left\{\ \text{continuous real processes } \psi\ ;\quad \mathbb{E}\Big(\ \sup_{t\geq 0}|\psi_t|^2\Big) < \infty\ \right\},$$

$$\mathcal{M}^{2,0}(0,\tau;\mathbb{R}^d) = \left\{\ \mathbb{R}^d\text{-valued processes } \psi\ ;\quad \mathbb{E}\Big(\int_0^\tau \|\psi_s\|^2 ds\Big) < \infty\right\},$$

$\mathcal{M}^2_{\text{loc}}(0,\tau;\mathbb{R}^d)$ is defined as previously.

Consider $f: \ \Omega \times [0,\infty[\times \mathbb{R} \times \mathbb{R}^d \longrightarrow \mathbb{R}$ such that $\forall (y,z) \in \mathbb{R} \times \mathbb{R}^d$, $\big(f(t,y,z)\big)_{t\geq 0}$is progressively measurable. Consider also the following assumption:

(H1) f is uniformly Lipschitz in z with Lipschitz constant K

(H2) $\forall t \geq 0, \forall z \in \mathbb{R}^d, \ \ y \mapsto f(t,y,z)$ is continuous,
$\exists$ a continuous and increasing function $\varphi : \mathbb{R}_+ \mapsto \mathbb{R}_+$ such that
$\forall t \geq 0, \forall y \in \mathbb{R}, \forall z \in \mathbb{R}^d, \quad |f(t,y,z)| \leq |f(t,0,z)| + \varphi(|y|) \quad \mathbb{P}a.s.$

(H3′) f is monotone in y :
$\forall t \geq 0, \forall y, y' \in \mathbb{R}, \forall z \in \mathbb{R}^d, \ \ \big(y-y'\big)\big(f(t,y,z) - f(t,y',z)\big) \leq 0 \quad \mathbb{P}a.s.$

(H4′) $\forall t \geq 0, \ f(t,0,0) = 0 \quad \mathbb{P}a.s.$
and $\xi \in \mathcal{L}^\infty(\mathcal{F}_\tau)$; we denote by M some real such that $|\xi| \leq M \quad \mathbb{P}a.s.$

We denote **(H1)** to **(H4')** collectively by $\big(\mathbf{A_0}\big)$.

Remark. The last condition is crucial in order to build a solution.

3.2 Theorem of existence

We study the following problem:
We suppose that f satisfies $\big(\mathbf{A_0}\big)$, and we want to construct an adapted process $\big(Y_t, Z_t\big)_{t\geq 0}$ that solves the BSDE $\big(f, \xi, \tau\big)$.

Theorem 3.1
There exists a solution $\big(Y_t, Z_t\big)_{t\geq 0}$ to the BSDE $\big(f,\xi,\tau\big)$ such that Y is a continuous process bounded by M and Z belongs to $\mathcal{M}^2_{loc}\big(0,\tau;\mathbb{R}^d\big)$.

Proof.

The key of the proof lies in using the results we proved previously, in other words, we introduce BSDEs driven by a strictly monotone generator.

<u>*Step 1*</u>: From generator f, we construct a sequence of functions satisfying $\big(\mathbf{A_\mu}\big)$.

Let $n, m \in \mathbb{N}^*$ such that $m \leq n$.
We set $f_{n,m}(t,y,z) \ = \ f(t,y,z) \ - \ \frac{1}{n}\, y^+ \ + \ \frac{1}{m}\, y^-$.

Taking into account that f satisfies $\big(\mathbf{A_0}\big)$, we obtain that each generator $f_{n,m}$ satisfies conditions of kind $\big(\mathbf{A_{\mu_{n,m}}}\big)$, where monotonicity constant $\mu_{n,m} > 0$ has to be explicited.

(i) It is easy to check that $f_{n,m}$ is uniformly Lipschitz in z.
Let $t \geq 0, \ y \in \mathbb{R}, \ z, z' \in \mathbb{R}^d$, then $\mid f_{n,m}(t,y,z) - f_{n,m}(t,y,z') \mid \ \leq \ K \ \|z - z'\|$.
We remark that the Lipschitz constant does not depend on indices n and m.

(ii) We can also easily check that $f_{n,m}$ is continuous in y.
Moreover, $f_{n,m}$ is bounded in the convenient way.
Let $t \geq 0, \ y \in \mathbb{R}, \ z \in \mathbb{R}^d$, we derive that
$|f_{n,m}(t,y,z)| \ \leq \ |f_{n,m}(t,0,z)| \ + \ \varphi(|y|) \ + \ \big(\frac{1}{m} + \frac{1}{n}\big)\ |y|$.
We set $\forall t \geq 0, \ \varphi^{n,m}(t) \ = \ \varphi(t) \ + \ \big(\frac{1}{m} + \frac{1}{n}\big)\ t$, note that it depends on n and m.

(iii) Now, let us show that $f_{n,m}$ is strictly monotone in y.
Let $t \geq 0, \ y, y' \in \mathbb{R}, \ z \in \mathbb{R}^d$, we recall that $m \leq n$
$(y-y')\ \big(f_{n,m}(t,y,z) \ - \ f_{n,m}(t,y',z)\big) \ \leq \ -\frac{1}{n}(y-y')^2$. Which yields that $\mu_{n,m} = \frac{1}{n}$.

(iv) Finally, it remains to check if the generator vanishes at origin.
Let $t \geq 0$. Since $f_{n,m}(t,0,0) = f(t,0,0) = 0$, then $\sup_{t\geq 0} |f_{n,m}(t,0,0)| = 0$.

Consequently, we have checked that $f_{n,m}$ satisfies $\big(\mathbf{A}_{\boldsymbol{\mu}}\big)$ with $\mu = \frac{1}{n}$ and $C = 0$.

The Theorem 2.1 allows us to consider $\big(Y^{n,m}, Z^{n,m}\big)$ the unique bounded solution of BSDE $\big(f_{n,m},\, \xi,\, \tau\big)$. Moreover we get that $Y^{n,m}$ is a continuous process bounded by M,
$\forall t \geq 0,\;\; Y^{n,m}_{t\wedge\tau} \;=\; Y^{n,m}_t \;\mathbb{P}a.s.$ and $Z^{n,m} \in \mathcal{M}^{2,-\frac{2}{n}}\big(0,\tau;\mathbb{R}^d\big)$.

Remark. Referring to [5], such functions $f_{n,m}$ have already been studied. Their interest lies in the fact that the sequence $\big(Y^{n,m}_t\big)_{n,m\geq 1}$ is monotone with respect to each indice n and m.

<u>*Step 2*</u>: Convergence with respect to indice n.

(i) We start by proving convergence of sequence $\big(Y^{n,m}_t\big)_{n\geq m}$.

Lemma 3.2
Let $m \in \mathbb{N}^*$ be fixed. Then, $\big(Y^{n,m}_t\big)_{n\geq m}$ converges to Y^m_t almost surely.
Moreover, $\forall t \geq 0,\quad \big|Y^m_t\big| \leq M$.

Proof of Lemma 3.2
According to comparison theorem 2.2, we get that $\forall t \geq 0,\quad Y^{n,m}_t \leq Y^{n+1,m}_t \quad \mathbb{P}a.s.$, because $f_{n,m}(t,y,z) \leq f_{n+1,m}(t,y,z)$. Hence, we obtain that $\big(Y^{n,m}_t\big)_{n\geq m}$ is an increasing sequence.

In order to prove the convergence, we need the sequence to be bounded independantly of m.
It is easily done since $\big|Y^{n,m}_t\big| \leq M \quad \mathbb{P}a.s.$
Then, the sequence $\big(Y^{n,m}_t\big)_{n\geq m}$ converges almost surely towards a limit that we denote by Y^m_t.
□

(ii) Now, we turn to the convergence of $\big(Z^{n,m}_t\big)_{n\geq m}$.

Lemma 3.3
For any $m \in \mathbb{N}^*$, the sequence $\big(Z^{n,m}\big)_{n\geq m}$ converges towards Z^m in $\mathcal{M}^2_{loc}\big(0,\tau;\mathbb{R}^d\big)$.

Proof of Lemma 3.3
It suffices to establish that $\big(Z^{n,m}_t\big)_{n\geq m}$ is a Cauchy sequence in the complete space $\mathcal{M}^2_{loc}\big(0,\tau;\mathbb{R}^d\big)$.
m is still assumed to be fixed and we consider $n, q \in \mathbb{N}$ such that $m \leq n \leq q$.
Remember that it implies $Y^{n,m}_s \leq Y^{q,m}_s$.

Itô's formula yields that

$$\begin{aligned}
\mathbb{E}\Big(\int_0^{t\wedge\tau} \|Z^{n,m}_s - Z^{q,m}_s\|^2\, ds\Big) \;=\; & \mathbb{E}\Big(\big|Y^{n,m}_{t\wedge\tau} - Y^{q,m}_{t\wedge\tau}\big|^2\Big) \;-\; \mathbb{E}\Big(\big|Y^{n,m}_0 - Y^{q,m}_0\big|^2\Big) \\
& + 2\,\mathbb{E}\Big(\int_0^{t\wedge\tau} \big(Y^{n,m}_s - Y^{q,m}_s\big)\,\Big(f\big(s,Y^{n,m}_s,Z^{n,m}_s\big) - f\big(s,Y^{n,m}_s,Z^{q,m}_s\big)\Big)\, ds\Big) \\
& + 2\,\mathbb{E}\Big(\int_0^{t\wedge\tau} \big(Y^{n,m}_s - Y^{q,m}_s\big)\,\Big(f\big(s,Y^{n,m}_s,Z^{q,m}_s\big) - f\big(s,Y^{q,m}_s,Z^{q,m}_s\big)\Big)\, ds\Big) \\
& + 2\,\mathbb{E}\Big(\int_0^{t\wedge\tau} \big|Y^{n,m}_s - Y^{q,m}_s\big|\,\Big(\frac{1}{n}\,Y^{n,m+}_s \;-\; \frac{1}{q}\,Y^{q,m+}_s\Big)\, ds\Big) \\
& - 2\,\mathbb{E}\Big(\int_0^{t\wedge\tau} \frac{1}{m}\,\big|Y^{n,m}_s - Y^{q,m}_s\big|\,\Big(Y^{n,m-}_s \;-\; Y^{q,m-}_s\Big)\, ds\Big)
\end{aligned}$$

$$\leq \quad \mathbb{E}\Big(\big|Y^{n,m}_{t\wedge\tau} - Y^{q,m}_{t\wedge\tau}\big|^2\Big) \;+\; 2\,\mathbb{E}\Big(\int_0^{t\wedge\tau} \big|Y^{n,m}_s - Y^{q,m}_s\big| \,\Big|\, \frac{1}{n}\, Y^{n,m+}_s \,\Big|\, ds\Big)$$
$$+\, 2\,\mathbb{E}\Big(\int_0^{t\wedge\tau} \big|Y^{n,m}_s - Y^{q,m}_s\big|\; K\; \|Z^{n,m}_s - Z^{q,m}_s\|\; ds\Big).$$

We finally obtain that $\quad \mathbb{E}\Big(\int_0^{t\wedge\tau} \|Z^{n,m}_s - Z^{q,m}_s\|^2\, ds\Big) \;\leq\; 2\mathbb{E}\Big(\big|Y^{n,m}_{t\wedge\tau} - Y^{q,m}_{t\wedge\tau}\big|^2\Big)$
$$+\, 4\tfrac{M^2}{n} t \;+\; 4K^2\mathbb{E}\Big(\int_0^{t\wedge\tau} \big|Y^{n,m}_s - Y^{q,m}_s\big|^2\, ds\Big).$$

By Lebesgue's dominated convergence theorem, we deduce convergence for the right hand side, which means that $\big(Z^{n,m}\big)_{n\geq m}$ is a Cauchy sequence in the complete space $\mathcal{M}^2_{loc}(0,\tau;\mathbb{R}^d)$, and we denote by Z^m its limit. $\square$

(iii) Next, we aim to identify processes $\big(Y^m_t, Z^m_t\big)_{t\geq 0}$ as a solution of a BSDE.
For this purpose, we are going to take the limit in $\mathcal{L}^1$ of the equation $\big(f_{n,m}\,,\,\xi,\,\tau\big)$ as n tends to infinity.
Lemmas 3.2 and 3.3 allow us to focus on the convergence in $\mathcal{L}^1$ of integrals
$\int_{t\wedge\tau}^{r\wedge\tau} \Big(f(s, Y^{n,m}_s, Z^{n,m}_s) \;-\; \frac{1}{n}\, Y^{n,m+}_s \;+\; \frac{1}{m}\, Y^{n,m-}_s\Big)\, ds \;-\; \int_{t\wedge\tau}^{r\wedge\tau} \Big(f(s, Y^m_s, Z^m_s) \;+\; \frac{1}{m}\, Y^{m-}_s\Big)\, ds.$
Taking into consideration assumptions on f, we obtain that
$$\mathbb{E}\Bigg(\Bigg|\int_{t\wedge\tau}^{r\wedge\tau}\Big(f(s, Y^{n,m}_s, Z^{n,m}_s) - \frac{1}{n}\, Y^{n,m+}_s + \frac{1}{m}\, Y^{n,m-}_s\Big)\, ds - \int_{t\wedge\tau}^{r\wedge\tau}\Big(f(s, Y^m_s, Z^m_s) + \frac{1}{m}\, Y^{m-}_s\Big)\, ds\Bigg|\Bigg)$$
$$\leq \quad \frac{r^{1/2}}{m}\, \mathbb{E}\Big(\int_{t\wedge\tau}^{r\wedge\tau} \big|Y^{n,m}_s - Y^m_s\big|^2\, ds\Big)^{1/2} \;+\; \frac{rM}{n} \;+\; K\, r^{1/2}\, \mathbb{E}\Big(\int_{t\wedge\tau}^{r\wedge\tau} \|Z^{n,m}_s - Z^m_s\|^2\, ds\Big)^{1/2}$$
$$+\, \mathbb{E}\Big(\int_0^r \Big|f(s, Y^{n,m}_s, Z^m_s) \;-\; f(s, Y^m_s, Z^m_s)\Big|\, ds\Big).$$

Hence we derive, using once more Lemmas 3.2 and 3.3, convergence to zero when n tends to ∞ for the first terms of the upper bound.
Concerning the last term, we use Lebesgue's dominated theorem.

It remains to identify the terminal value. The Lemma 3.2 allows us to define a negligible set $\mathcal{N}$ such that for any integer p, $\forall\omega \notin \mathcal{N}$, we have $Y^{n,m}_p(\omega) \xrightarrow[n\to+\infty]{} Y^m_p(\omega)$.
Now, we consider $\omega \in \mathcal{N}^c \cap \{\tau < \infty\}$, and we pick $l \in \mathbb{N}$ such that $\tau(\omega) \leq l$.
$$|\; Y^m_\tau \;-\; \xi \;|\;(\omega) \;\leq\; |\; Y^m_l \;-\; Y^{n,m}_l \;|\;(\omega) \xrightarrow[n\to+\infty]{} 0,$$
because $\xi(\omega) = Y^{n,m}_\tau(\omega) = Y^{n,m}_{\tau\wedge l}(\omega) = Y^{n,m}_l(\omega)$. Hence, $Y^m_\tau \;=\; \xi \quad \mathbb{P}\,p.s. \quad$ on $\{\tau < \infty\}$.
And finally, we are able to state that $\big(Y^m, Z^m\big)$ solves the equation
$$\begin{cases} Y^m_t \;=\; Y^m_r \;+\; \int_{t\wedge\tau}^{r\wedge\tau} \Big(f(s, Y^m_s, Z^m_s) \;+\; \frac{1}{m}\, Y^{m-}_s\Big)\, ds \;-\; \int_{t\wedge\tau}^{r\wedge\tau} Z^m_s\, dW_s, \\ Y^m_\tau \;=\; \xi \qquad \text{on } \{\tau < \infty\}. \end{cases} \tag{10}$$

Note that $\big(Y^m_t\big)_{t\geq 0}$ is a continuous process, with $\forall t \geq 0, \quad Y^m_{t\wedge\tau} = Y^m_t$.
Consequently, $\big(Y^m_t, Z^m_t\big)_{t\geq 0}$ is a solution of the BSDE (f_m, ξ, τ), with $f_m(t,y,z) = f(t,y,z) + \frac{1}{m}\, y^-$.

Step 3: Convergence with respect to indice m.

We adopt the same procedure as previously.

(i) We begin by proving convergence of sequence $\big(Y^m_t\big)_{m\geq 1}$.

Lemma 3.4
$\left(Y_t^m\right)_{m\geq 1}$ almost surely converges to Y_t. Moreover, $\forall t \geq 0, \quad \left|Y_t\right| \leq M$.

Proof of Lemma 3.4
Since $\left(Y_t^m\right)_{m\geq 1}$ is a bounded sequence (with uniform bound M), it suffices to show that it is monotone. Actually, it is a decreasing sequence. But the trick is that a comparison theorem does not hold for generators f_m, that is why it leads us to consider back generators $f_{n,m}$. Comparison theorem 2.2 provides that $\forall t \geq 0, \quad Y_t^{n,m+1} \leq Y_t^{n,m} \quad \mathbb{P}a.s.$ and almost sure convergence guarantees that $\left(Y_t^m\right)_{m\geq 1}$ is a nonincreasing sequence. Thus, $\left(Y_t^m\right)_{m\geq 1}$ converges almost surely to a process denoted by $\left(Y_t\right)_{m\geq 1}$. $\square$

(ii) We study the convergence of $\left(Z_t^m\right)_{m\geq 1}$ in the same way as for $\left(Z_t^{n,m}\right)_{n\geq m}$.

Lemma 3.5
The sequence $\left(Z^m\right)_{m\geq 1}$ converges towards Z in $\mathcal{M}^2_{loc}\left(0,\tau;\mathbb{R}^d\right)$.

Proof of Lemma 3.5
Since, for any fixed $m, q \in \mathbb{N}^*$,
$\mathbb{E}\Big(\int_0^{t\wedge\tau} \|Z_s^m - Z_s^q\|^2\, ds\Big) \leq 2\mathbb{E}\Big(\left|Y_{t\wedge\tau}^m - Y_{t\wedge\tau}^q\right|^2\Big) + 4\frac{M^2}{m}t + 4K^2\mathbb{E}\Big(\int_0^{t\wedge\tau} \left|Y_s^m - Y_s^q\right|^2 ds\Big)$
Then $\left(Z^m\right)_{m\geq 1}$ is a Cauchy sequence in $\mathcal{M}^2_{loc}\left(0,\tau;\mathbb{R}^d\right)$ and we denote by Z its limit. $\square$

(iii) Now, we aim to express the couple of processes $\left(Y_t, Z_t\right)_{t\geq 0}$ as a solution of a BSDE.

As before, we take the limit in $\mathcal{L}^1$ of Equation 10. Using Lemmas 3.4 and 3.5, we get the construction of a solution to the BSDE $\left(f,\, \xi,\, \tau\right)$. $\square$

We managed to prove the existence of a solution to BSDE $\left(f,\, \xi,\, \tau\right)$. We even gave its construction: it is defined as the limit of the sequence of processes $\left(Y^{n,m}, Z^{n,m}\right)_{n\geq m\geq 1}$. We built a solution but we have not treated of the problem of uniqueness. It will be the subject of the next section.

3.3 Theorem of uniqueness

Unfortunately, we have to strenghten the assumptions in order to ensure uniqueness.

3.3.1 Further assumptions

(*) Assumption on random terminal time
Let us make a fundamental remark: we cannot expect uniqueness if the terminal time τ is not almost surely finite. Note that we do not ask τ to be bounded.
Indeed, we are able to give examples where there exists several solutions to BSDE $\left(f,\, \xi,\, \tau\right)$, whence τ is not almost surely finite.

Example 1
Consider an infinite terminal time, precisely speaking, $\mathbb{P}(\tau = +\infty) = 1$. It is worth nothing to exhibit a BSDE which has several solutions.
It suffices to take $f = 0$. Note that such a generator obviously satisfies $\left(\mathbf{A_0}\right)$. Then, equation becomes $\quad Y_{t\wedge\tau} = Y_{r\wedge\tau} - \int_{t\wedge\tau}^{r\wedge\tau} Z_s\, dW_s$.
We remark that an infinite terminal time allows us to get rid of terminal condition. Thus, for any real c, the couple of constant processes $(c, 0)_{t\geq 0}$ is a solution.

Consequently, when $\mathbb{P}(\tau = +\infty) = 1$ and $f = 0$, we are able to construct an infinite number of solutions.

This first example just underlines the importance of terminal time in the context where f satisfies $(\mathbf{A_0})$. Let us now study a second example, more satisfactory.

Example 2
From Brownian motion $(W_t)_{t\geq 0}$, we can construct a stopping time τ infinite on a nonnegligible set, in other words $\mathbb{P}(\tau = +\infty) > 0$, such that the related BSDE driven by a null generator is solvable by several solutions. Let us detail this example.

Take $t_1 > 0$ and consider a set $\Omega_1 \in \mathcal{F}_{t_1}$ such that $0 < \mathbb{P}(\Omega_1) < 1$.
We set $B = \{\omega | W_{t_1}(\omega) \in \Omega_1\}$ and $\begin{cases} \tau(\omega) = +\infty & \text{if } \omega \in B, \\ \tau(\omega) = t_1 & \text{if } \omega \notin B. \end{cases}$
Then τ is a stopping time and $0 < \mathbb{P}(\tau = +\infty) = \mathbb{P}(B) < 1$.

Now, let us consider the BSDE with null generator and terminal value $\xi = 1\!\!1_B$, *i.e.* $(0,\, 1\!\!1_B,\, \tau)$.

On the one hand, Martingale Representation theorem provides a process $(\overline{Z}_t)_{0\leq t\leq T}$ such that $\forall t \geq 0,\ \ \mathbb{E}(\,\xi|\mathcal{F}_t\,) \ = \ \mathbb{E}(\,\xi\,) \ + \ \int_0^t \overline{Z}_s\, dW_s$.
Then, the couple $\big(\mathbb{E}(\,1\!\!1_B|\mathcal{F}_t\,), \overline{Z}_t\big)_{0\leq t\leq T}$ defines a solution for the equation $(0,\, 1\!\!1_B,\, \tau)$.
On the other hand, we observe that the terminal value is null on the set $\{\tau < \infty\}$.
By consequence, the null process is also a solution.
Since $\mathbb{P}\big(\ \mathbb{E}(\,1\!\!1_B|\mathcal{F}_t\,) \neq 0\ \big) > 0$, then we do not have uniqueness.

As a conclusion of these examples, we conclude that if we want to obtain uniqueness, we have to consider a finite terminal time. So that, from now on we will assume that $\tau < \infty, \quad \mathbb{P}a.s.$

(*) Assumption on generator
The proof of the uniqueness result is difficult in the general case but we can easily bypass the difficulty if we suppose that f does not depend on z anymore. We will work in this particular case.

(*) Particular framework
Let τ be an almost surely finite stopping time.
Consider $f:\ \ \Omega \times [0,\infty[\times \mathbb{R} \longrightarrow \mathbb{R}$ such that $\forall y \in \mathbb{R}\ \big(f(t,y)\big)_{t\geq 0}$ is progressively measurable and satisfies the set of following hypothesis denoted by $(\mathbf{A_0'})$.

(H 5). $\forall t \geq 0,\ \ y \mapsto f(t,y)$ is continuous,
$\exists$ a continuous and increasing function $\varphi : \mathbb{R}_+ \mapsto \mathbb{R}_+$ such that
$\forall t \geq 0, \forall y \in \mathbb{R}, \forall z \in \mathbb{R}^d, \quad |f(t,y)| \leq |f(t,0)| + \varphi(|y|)$.

(H 6). f is monotone in y.

(H 7). $\forall t \geq 0, \quad f(t,0) = 0, \quad \mathbb{P}a.s.$

Our purpose is to prove uniqueness for BSDE

$$\begin{cases} Y_{t\wedge\tau} & = & Y_{r\wedge\tau} \ + \ \int_{t\wedge\tau}^{r\wedge\tau} f(s,Y_s)\ ds \ - \ \int_{t\wedge\tau}^{r\wedge\tau} Z_s\ dW_s, \\ Y_\tau & = & \xi. \end{cases} \tag{11}$$

Theorem 3.6
There exists a solution $\big(Y_t, Z_t\big)_{t\geq 0}$ *to BSDE (11).*

$\big(Y_t, Z_t\big)_{t\geq 0}$ *is unique in the class of processes* $\big(Y, Z\big)$ *such that*
Y *is continuous and uniformly bounded, and* Z *belongs to* $\mathcal{M}^2_{loc}\big(0,\tau;\mathbb{R}^d\big)$.

3.3.2 Existence of a solution

(*) First method
As already claimed, we are able to build a solution $\big(\widetilde{Y}_t, \widetilde{Z}_t\big)_{t\geq 0}$ to BSDE (11). Process $\big(\widetilde{Y}_t\big)_{t\geq 0}$ is the monotone limit of sequence $\big(Y_t^{n,m}\big)_{n\geq m\geq 1}$.

(*) Second method
In our current context, we can adopt another point of view in order to construct a solution to (11). It deals with using bounded terminal times, in a similar way to Theorem 2.1
We denote by $\big(Y_t^q, Z_t^q\big)_{0\leq t\leq q}$ the unique solution of BSDE $\big(f,\, \xi\, 1\!\!1_{\{\tau\leq q\}},\, \tau\wedge q\big)$.
We start by proving that $\big(Y_t^q\big)_{q\geq 0}$ is an almost sure Cauchy sequence. Indeed, Itô's formula yields that for $q\leq r,\ 0\leq t\leq q,\quad \mathbb{E}\Big(\big|Y_t^q\ -\ Y_t^r\big|^2\Big)\ \leq\ M^2\ \mathbb{E}\big(\ 1\!\!1_{\{q<\tau\leq r\}}\ \big)\quad \mathbb{P}a.s.$
Hence, we can define $\overline{Y}$ as its limit.
From $\mathbb{E}\Big(\int_0^{t\wedge\tau}\ \|Z_s^q - Z_s^r\|^2\ ds\Big)\ \leq\ \mathbb{E}\big(\ |Y_{t\wedge\tau}^q - Y_{t\wedge\tau}^r|^2\big)$, we derive convergence of sequence $\big(Z^q\big)_{q\geq 0}$ in $\mathcal{M}^{2,0}\big(0,\tau;\mathbb{R}^d\big)$ towards a process denoted by $\overline{Z}$.
Such convergences enable us to prove that $\big(\overline{Y}_t, \overline{Z}_t\big)_{t\geq 0}$ is also a solution of BSDE (11).

3.3.3 Proof of uniqueness

Let us consider two couples of solutions $\big(Y_t^1, Z_t^1\big)_{t\geq 0},\ \big(Y_t^2, Z_t^2\big)_{t\geq 0}$ in suitable spaces.
Then the couple of processes $\big(\ \widehat{Y}_t\ :=\ Y_t^1 - Y_t^2\ ,\ \widehat{Z}_t\ :=\ Z_t^1 - Z_t^2\ \big)_{t\geq 0}$ satisfies the following equation

$$\left\{\begin{array}{lcl} \widehat{Y}_{t\wedge\tau} & = & \widehat{Y}_{r\wedge\tau}\ +\ \int_{t\wedge\tau}^{r\wedge\tau}\Big(\ f(s,Y_s^1) - f(s,Y_s^2)\Big)\ ds\ -\ \int_{t\wedge\tau}^{r\wedge\tau}\widehat{Z}_s\ dW_s, \\ \widehat{Y}_\tau & = & 0. \end{array}\right. \tag{12}$$

$\widehat{Y}$ is continuous and uniformly bounded, and $\widehat{Z}$ belongs to $\mathcal{M}^2_{\text{loc}}\big(0,\tau;\mathbb{R}^d\big)$.

Itô's formula and assumptions $\big(\mathbf{A_0}\text{'}\big)$ yield that

$$\begin{array}{lcl} \mathbb{E}\big(|\widehat{Y}_{t\wedge\tau}|^2\big) & = & \mathbb{E}\big(|\widehat{Y}_{n\wedge\tau}|^2\big)\ +\ \mathbb{E}\Big(\displaystyle\int_{t\wedge\tau}^{n\wedge\tau} 2\ \widehat{Y}_s\ \Big(\ f(s,Y_s^1) - f(s,Y_s^2)\Big)\ ds\Big)\ -\ \mathbb{E}\Big(\displaystyle\int_{t\wedge\tau}^{n\wedge\tau}\|\widehat{Z}_s\|^2\ ds\Big) \\ & \leq & \mathbb{E}\big(|\widehat{Y}_{n\wedge\tau}|^2\big). \end{array}$$

Sending n to the infinity, we get that $\widehat{Y}_{t\wedge\tau} = 0$ and next $\mathbb{E}\big(\int_0^{t\wedge\tau}\ \big|\big|\widehat{Z}_s\big|\big|^2\ ds\big) = 0$.

3.4 Comparison theorem

Now, we state a comparison theorem for BSDEs whose generator is monotone. Such a result will be useful when studying related PDEs.

Let $\xi_1,\ \xi_2 \in \mathcal{L}^\infty(\mathcal{F}_\tau)$.
Suppose that there exists a couple of processes $\big(Y_t^1, Z_t^1\big)_{t\geq 0}$ where Y^1 is continuous and bounded,

Z^1 satisfies $\mathbb{E}\Big(\int_0^\tau \|Z_s^1\|^2\, ds\Big) < \infty$, solving the equation $\big(f_1, \xi_1, \tau\big)$.
We are also given f_2 satisfying $\big(\mathbf{A}_0'\big)$, then Theorem 3.1, guarantees that there exists an unique adapted process $\big(Y_t^2, Z_t^2\big)_{t\geq 0}$ solving the equation $\big(f_2, \xi_2, \tau\big)$.
We need terminal time τ to be almost surely finite.

Theorem 3.7
We recall that the terminal time τ is almost surely finite.
Let $\big(Y_t^i, Z_t^i\big)_{t\geq 0}$ be respective solutions for BSDE $\big(f_i,\, \xi_i,\, \tau\big)$, $\quad i = 1, 2$.

If we suppose that $\xi_1 \leq \xi_2$ and $\forall t \geq 0, \quad f_1(t, Y_t^1, Z_t^1) \leq f_2(t, Y_t^1) \quad \mathbb{P}a.s.$

Then $\quad \forall t \geq 0, \quad Y_t^1 \leq Y_t^2 \quad \mathbb{P}a.s.$

Proof.
We use the same strategy as in the proof of the Theorem 2.2. □

4 Link between BSDEs and PDEs for a strictly monotone generator

In this section, we study the link between BSDEs and elliptic PDEs. S. Peng, in [18], and recently E. Pardoux in [16] have shown that semilinear elliptic PDEs are linked to BSDEs with a random terminal time. Let the following assumptions hold

(**H1**) f is uniformly Lipschitz in z with Lipschitz constant K.

(**P2**) f is continuous on $\mathbb{R}^d \times \mathbb{R}^k \times \mathbb{R}^{k\times d}$,
and $\exists$ an increasing continuous function $\varphi : \mathbb{R}_+ \mapsto \mathbb{R}_+$
such that $|f(x, y, z)| \leq K'\,(\,1 + |x|^p + \|z\|\,) + \varphi(|y|)$ with $K', p > 0$.

(**P3**) f is monotone in y, with a constant of monotonicity $\mu \in \mathbb{R}$.

(**P4**) $\exists\, \lambda > K^2 - 2\mu$ such that $\mathbb{E}\Big(\int_0^{+\infty} exp\,(\lambda s)\ \big|f\big(X_s^x, 0, 0\big)\big|^2\, ds\Big) < \infty$,
$\mathbb{E}\Big(\ exp\,(\lambda\tau)\ \ (\,1 + |\xi|^2\,)\Big) < \infty$, and $\mathbb{E}\Big(\int_0^\tau exp\,(\lambda s)\ \big|f\big(X_s^x, \mathbb{E}(\xi|\mathcal{F}_s), \eta_s\big)\big|^2\, ds\Big) < \infty$,
where τ is the considered stopping time, and η is obtained by
the Brownian martingale representation theorem applied to $\mathbb{E}(\xi|\mathcal{F}_t)$.

Then, E. Pardoux obtained in [16] a theorem that provides a construction for a solution of the connected PDE. We are going to show the same kind of result in the one-dimensional case under weaker assumptions, that is to say when the generator f satisfies $\big(\mathbf{A}_{\boldsymbol{\mu}}\big)$.

4.1 Notation

Let $\mathcal{U}$ be an open set containing a bounded set $\mathcal{D}$ of the form $\mathcal{D} = \{x \mid \psi(x) > 0\}$, where ψ is some function in $\mathcal{C}^2\big(\mathbb{R}^d\big)$. We also require that $|\nabla\psi(x)| \neq 0$ for any x in $\partial\mathcal{D}$.

Let $b : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ and $\sigma : \mathbb{R}^d \longrightarrow \mathcal{L}(\mathbb{R}^d, \mathbb{R}^d)$ belong respectively to $\mathcal{C}_b^1\big(\mathbb{R}^d\big)$ and $\mathcal{C}_b^2\big(\mathbb{R}^d\big)$. In particular, b and σ are Lipschitz functions, which ensures the existence and uniqueness of the solution $(X_t^x)_{t\geq 0}$ to the SDE

$$X_t^x = x + \int_0^t b(X_s^x)\, ds + \int_0^t \sigma(X_s^x)\, dW_s, \qquad t \geq 0. \tag{13}$$

Remark. We study the trajectories $(X_t^x)_{t\geq 0}$ until they go out of the compact $\overline{\mathcal{D}}$ for the first time. We denote by τ_x this exit time: $\tau_x \;=\; \inf\{t \geq 0 \mid X_t^x \notin \overline{\mathcal{D}}\}$.

We suppose that $\mathbb{P}(\tau_x < \infty) \;=\; 1, \quad \forall x \in \overline{\mathcal{D}}$. In other words, if a diffusion comes from x, then it exits from the compact in a finite time.
We assume moreover that the set Γ of the points on the boundary which immediately exit is a closed set. Precisely, $\Gamma \;=\; \{\, x \in \partial\mathcal{D} \mid \mathbb{P}(\tau_x > 0) \;=\; 0 \,\}$.

We consider $g : \mathbb{R}^d \longrightarrow \mathbb{R}$ continuous on $\mathbb{R}^d$. Remark that in particular g is bounded on the compact set $\overline{\mathcal{D}}$. We denote by M an upper bound.
We are going to study BSDEs with $g\big(X_{\tau_x}^x\big)$ as a terminal condition.

Let $f : \mathbb{R}^d \times \mathbb{R} \times \mathcal{L}(\mathbb{R}^d, \mathbb{R}) \longrightarrow \mathbb{R}$ such that $\forall (y,z) \in \mathbb{R} \times \mathcal{L}(\mathbb{R}^d, \mathbb{R})$
$f(x,y,z)$ is measurable and continuous on $\mathbb{R}^d \times \mathbb{R} \times \mathcal{L}(\mathbb{R}^d; \mathbb{R})$.
Fix $x \in \overline{\mathcal{D}}$ and denote $\overline{f}(t,y,z) = f(X_t^x, y, z)$, then we assume that $\overline{f}$ satisfies $\big(\mathbf{A_\mu}\big)$ with constants that do not depend on x. For convenience, we still denote by $\big(\mathbf{A_\mu}\big)$ the condition on f.

In this context, we can construct the solution $\big(Y_t^x, Z_t^x\big)_{t\geq 0}$ of the BSDE: $\quad \forall t \geq 0,\ \forall r \geq t,$

$$\left\{ \begin{array}{lcl} Y_{t\wedge\tau_x}^x & = & Y_{r\wedge\tau_x}^x \;+\; \int_{t\wedge\tau_x}^{r\wedge\tau_x} f(X_s^x, Y_s^x, Z_s^x)\; ds \;-\; \int_{t\wedge\tau_x}^{r\wedge\tau_x} Z_s^x\; dW_s, \\ Y_{\tau_x}^x & = & g\big(X_{\tau_x}^x\big). \end{array} \right. \tag{14}$$

When we study the uniqueness, we will have to consider a further hypothesis on f called $\big(\mathbf{A^x}\big)$:
(H 8). Let $R > 0 \quad \exists\, m_R : \mathbb{R}_+ \to \mathbb{R}_+$ continuous, satisfying $m_R(0) = 0$ and such that
$\forall x, x' \in \mathbb{R}^d$ s.t $\|x\|, \|x'\| \leq R,\ \forall y \in \mathbb{R}$ s.t $|y| \leq R,\ \forall z \in \mathcal{L}(\mathbb{R}^d; \mathbb{R})$,
$|f(x,y,z) - f(x',y,z)| \leq m_R\big(\ \|x - x'\|\ (\ 1 + \|z\|\ )\ \big)$.

4.2 Existence of a viscosity solution

The aim of this section is to show that the function $x \to u(x) = Y_0^x$ is a viscosity solution for the PDE

$$\left\{ \begin{array}{lcll} Lu(x) \;+\; f\big(x, u(x), (\nabla u)\sigma(x)\big) & = & 0 & \text{if } x \in \mathcal{D}, \\ u(x) \;=\; g(x) & & & \text{if } x \in \partial\mathcal{D}, \end{array} \right. \tag{15}$$

where $\quad L \;=\; b.\nabla \;+\; 1/2 \sum_{i,j=1}^d (\sigma\sigma^*)_{i,j}\ \frac{\partial^2}{\partial x_i\ \partial x_j}$.

Before pointing out the link between the solution of (14) and a solution of (15), let us recall the notion of viscosity solution.

Definition 4.1
A continuous function $u : \overline{\mathcal{D}} \to \mathbb{R}$ is called a viscosity subsolution of the PDE (15) if for any $\phi \in \mathcal{C}^2(\overline{\mathcal{D}})$, and for any local maximum point $x \in \overline{\mathcal{D}}$ of $u - \phi$, we have

$$\begin{array}{lcll} -L\phi(x) \;-\; f\big(x, u(x), (\nabla\phi)\sigma(x)\big) & \leq & 0 & \textit{if } x \in \mathcal{D}, \\ \min\{-L\phi(x) \;-\; f\big(x, u(x), (\nabla\phi)\sigma(x)\big)\ ,\ u(x) \;-\; g(x)\} & \leq & 0 & \textit{if } x \in \partial\mathcal{D}. \end{array}$$

Remark. In the same way, we get the definition of a viscosity supersolution just reversing $\leq$ in $\geq$ and *max* in *min*.

Definition 4.2
A continuous function $u : \overline{\mathcal{D}} \to \mathbb{R}$ is called a viscosity solution of the PDE (15) if it is both a subsolution and a supersolution.

The notion of viscosity solution was introduced in 1981 by M.G. Crandall and P.-L. Lions. We can read [6] for generalities about this subject. Here, we adopt the convention that a viscosity subsolution, or supersolution, or solution is continuous whereas other authors only ask for semi-continuity, see [2] for example.
Note that the notion of viscosity solution is much weaker than the notion of classical solution. In particular, a viscosity solution does not need to be $\mathcal{C}^2$.

Theorem 4.3
Suppose that f satisfies $\big(\boldsymbol{A_\mu}\big)$.
Set $u(x) \;=\; Y_0^x \quad \forall x \in \mathcal{D}$. Then, u is continuous and bounded on $\overline{\mathcal{D}}$.

Moreover, u is a viscosity solution of the PDE (15).

Proof.

We shall start by proving that u is well-defined.

Fix $x \in \overline{\mathcal{D}}$. Theorem 2.1 states that there exists a unique solution to the BSDE (14), that we denote by $\big(Y_t^x, Z_t^x\big)_{t\geq 0}$. Hence we can set $u(x) \;=\; Y_0^x$ and u is defined on whole $\overline{\mathcal{D}}$.

Moreover, still thanks to Theorem 2.1, we know that u is bounded by $M + \frac{C}{\mu}$.

It remains to show the main point of this section, that is to say u is a viscosity solution of the PDE (15).
We split the proof of this result in two parts: first we prove that u is a continuous function and then that u satisfies the inequalities given in the definition of a viscosity solution.

<u>*Step 1*</u>: u is a continuous function.

Fix x' in $\overline{\mathcal{D}}$.
We want to prove that $|u(x) - u(x')| \xrightarrow[x\to x']{} 0$ or in other words $|Y_0^x - Y_0^{x'}| \xrightarrow[x\to x']{} 0$.

(i) First, from E. Pardoux [15, pp28-29], we know that the function $x \to \tau_x$ is continuous on $\overline{\mathcal{D}}$. We may point out that assumptions on τ_x (which has to be almost surely finite) and on Γ (which has to be a closed set) are necessary to get the continuity.
Then it also derives the continuity of the mapping $x \to X_{\tau_x}^x$ on $\overline{\mathcal{D}}$.

(ii) The key of the proof is that $\big(Y_t^x, Z_t^x\big)_{t\geq 0}$ can be approached by a sequence made of the solutions of BSDEs with a bounded terminal time. Indeed, in Section 2, we introduced for any fixed $x \in \overline{\mathcal{D}}$, $\big(Y_t^{x,n}, Z_t^{x,n}\big)_{t\geq 0}$ the solution of the BSDE $\big(\overline{f},\, g(X_{\tau_x}^x)\, 1\!\!1_{\{\tau_x \leq n\}},\, n \wedge \tau_x\big)$.
Obviously, $\forall t \geq 0, \quad |Y_t^n| \leq M + \frac{C}{\mu}$.

For more convenience, we shall denote $\tau = \tau_x,\ \xi = g(X_{\tau_x}^x),\ Y^n = Y^{x,n},\ Z^n = Z^{x,n}$
and $\tau' = \tau_{x'},\ \xi' = g(X_{\tau'}^{x'}),\ Y'^n = Y^{x',n},\ Z'^n = Z^{x',n}$.

Then $\big(Y^n, Z^n\big)$ is the unique solution of the BSDE, that we denote by $\big(f,\, \xi\, 1\!\!1_{\{\tau_x \leq n\}},\, n \wedge \tau_x\big)$:

$$Y_t^n \;=\; \xi\; 1\!\!1_{\{\tau \leq n\}} \;+\; \int_{t\wedge\tau}^{n\wedge\tau} f(X_s^x, Y_s^n, Z_s^n)ds \;-\; \int_{t\wedge\tau}^{n\wedge\tau} Z_s^n dW_s. \tag{16}$$

Similarly to the proof of the Theorem 2.1, we get that the sequence $\big(Y_t^n\big)_{n\geq 0}$ almost surely converges to Y^x and that the sequence $\big(Z_t^n\big)_{n\geq 0}$ converges to Z^x in $\mathcal{M}^{2,-2\mu}\big(0,\tau;\mathbb{R}^d\big)$.

More precisely, we get the following inequality:

$$\forall t \geq 0, \qquad \left|Y_t^n - Y_t^x\right| \;\leq\; \alpha\; exp\,(\mu t)\;\; exp\,(-\mu n) \quad \mathbb{P}a.s., \text{ where } \alpha > 0 \text{ is a constant.} \tag{17}$$

(iii) Let $\varepsilon > 0$, there exists n_0 such that $\forall n \geq n_0$ we have $2\alpha\, exp\,(-\mu n) \;\leq \varepsilon$.

Then, $|Y_0^x - Y_0^{x'}| \leq \varepsilon \;+\; |Y_0^{n_0} - Y'^{n_0}_0|$ $\mathbb{P}a.s.$

It remains to look for a bound $\eta > 0$ such that $|x - x'| \leq \eta$ implies $|Y_0^{n_0} - Y'^{n_0}_0| \leq \varepsilon$.
From now and on, we denote $n = n_0$.

(iv) We study the equation satisfied by $Y_t^n - Y'^n_t$. But remind that the respective BSDEs do not hold with the same terminal time ($n \wedge \tau$ and resp. $n \wedge \tau'$).
In order to compute $Y_t^n - Y'^n_t$, we consider the common terminal time $T \;=\; \max\{\tau, \tau'\}$.

Let us introduce some new notations: $\widetilde{f}(X_t^x, y, z) \;=\; \begin{cases} f(X_t^x, y, z) & \text{if } 0 \leq t \leq \tau \wedge n, \\ 0 & \text{otherwise,} \end{cases}$

and also $\widetilde{Y}_t \;=\; \begin{cases} Y_t^n & \text{if } 0 \leq t \leq \tau \wedge n, \\ \xi\; 1\!\!1_{\{\tau \leq n\}} & \text{otherwise,} \end{cases} \qquad \widetilde{Z}_t \;=\; \begin{cases} Z_t^n & \text{if } 0 \leq t \leq \tau \wedge n, \\ 0 & \text{otherwise.} \end{cases}$

Equation (16) becomes $\big(\widetilde{f},\, \xi\, 1\!\!1_{\{\tau \leq n\}},\, n \wedge T\big)$
and in the same way for x', we get $\big(\widetilde{f}',\, \xi'\, 1\!\!1_{\{\tau' \leq n\}},\, n \wedge T\big)$.

Now, we consider the processes $\widehat{Y} \;=\; \widetilde{Y} \;-\; \widetilde{Y}'$ and $\widehat{Z} \;=\; \widetilde{Z} \;-\; \widetilde{Z}'$.
We get the following equation

$$\begin{aligned} \widehat{Y}_{t\wedge T} \;&=\; \xi\; 1\!\!1_{\{\tau \leq n\}} \;-\; \xi'\; 1\!\!1_{\{\tau' \leq n\}} \;+\; \int_{t\wedge T}^{n\wedge T} \big(\widehat{\beta}_s.\widehat{Z}_s \;+\; \widehat{\gamma}_s\big)\; ds \;-\; \int_{t\wedge T}^{n\wedge T} \widehat{Z}_s\; dW_s \\ &\quad + \int_{t\wedge T}^{n\wedge T} \Big(\widetilde{f}(X_s^x, \widetilde{Y}_s, \widetilde{Z}_s) - \widetilde{f}(X_s^x, \widetilde{Y}'_s, \widetilde{Z}_s)\Big)\, ds. \end{aligned}$$

Where
$$\begin{aligned} \widehat{\beta}_s\; \widehat{Z}_s \;&=\; \widetilde{f}(X_s^x, \widetilde{Y}'_s, \widetilde{Z}_s) \;-\; \widetilde{f}(X_s^x, \widetilde{Y}'_s, \widetilde{Z}'_s), \\ \widehat{\gamma}_s \;&=\; \widetilde{f}(X_s^x, \widetilde{Y}'_s, \widetilde{Z}'_s) \;-\; \widetilde{f}'(X_s^{x'}, \widetilde{Y}'_s, \widetilde{Z}'_s). \end{aligned}$$

Note that $\widehat{\gamma}_s \;=\; \begin{cases} f(X_s^x, \xi'\, 1\!\!1_{\tau' \leq n}, 0) & \text{if } \tau' \wedge n < s \leq \tau \wedge n, \\ -f(X_s^{x'}, \widetilde{Y}'_s, \widetilde{Z}'_s) & \text{if } \tau \wedge n < s \leq \tau' \wedge n, \\ f(X_s^x, \widetilde{Y}'_s, \widetilde{Z}'_s) \;-\; f(X_s^{x'}, \widetilde{Y}'_s, \widetilde{Z}'_s) & \text{if } 0 \leq s \leq \tau \wedge \tau' \wedge n, \\ 0 & \text{if } s \geq \tau \wedge n \text{ et } s \geq \tau' \wedge n. \end{cases}$

(v) We recall that we want to prove that there exists $\eta > 0$ such that $|x - x'| \leq \eta$ implies $\left|\widehat{Y_0}\right| \leq \varepsilon$.
We consider $exp\,(\lambda t)\;\; |\widehat{Y}_t|^2$ with a constant λ which will be fixed further.

Let $\lambda > 0$.
Itô's formula yields

$$\begin{aligned} |\widehat{Y}_0|^2 \;&=\; exp\,(\lambda n)\;\; |\widehat{Y}_n|^2 \;-\; 2\int_0^n exp\,(\lambda s)\;\; \widehat{Y}_s\; \widehat{Z}_s\; dW_s \;-\; \int_0^n exp\,(\lambda s)\;\; \|\widehat{Z}_s\|^2\; ds \\ &\quad - \lambda \int_0^n exp\,(\lambda s)\; |\widehat{Y}_s|^2\; ds \;+\; 2\int_0^n exp\,(\lambda s)\; \widehat{Y}_s\; \Big(\widetilde{f}(X_s^x, \widetilde{Y}_s, \widetilde{Z}_s) \;-\; \widetilde{f}'(X_s^{x'}, \widetilde{Y}'_s, \widetilde{Z}'_s)\Big)\; ds. \end{aligned}$$

Hence, taking the expectation, we get

$$|\widehat{Y}_0|^2 \;\leq\; \mathbb{E}\Big(\; exp\,(\lambda n)\;\; |\widehat{Y}_n|^2\;\Big) \;+\; \mathbb{E}\Big(\int_0^n exp\,(\lambda s)\;\; |\widehat{\gamma}_s|^2\; ds\;\Big)$$

$$+ \ (\, 2K^2 + 1 - \lambda - 2\mu \,) \ \mathbb{E}\Big(\int_0^n \ exp\,(\lambda s) \ |\widehat{Y}_s|^2 \ ds \ \Big).$$

Consequently, we set $\lambda = 2K^2 + 1$, which provides

$$|\widehat{Y}_0|^2 \ \leq \ exp\,(\lambda n) \ \ \mathbb{E}\Big(\ |\widehat{Y}_n|^2 \ \Big) \ + \ \mathbb{E}\Big(\int_0^n \ exp\,(\lambda s) \ \ |\widehat{\gamma}_s|^2 \ ds \ \Big). \tag{18}$$

It remains to show that $\mathbb{E}\Big(\ |\widehat{Y}_n|^2 \ \Big)$ and $\mathbb{E}\Big(\int_0^n \ exp\,(\lambda s) \ \ |\widehat{\gamma}_s|^2 \ ds \ \Big)$ converge to zero when $x \to x'$.

(vi) We start by considering the first term.
Remember that $\widehat{Y}_n = g(X^x_{\tau_x}) \ 1\!\!1_{\{\tau \leq n\}} \ - \ g(X^{x'}_{\tau_{x'}}) \ 1\!\!1_{\{\tau' \leq n\}}$ and that g is supposed to be bounded by M in $\overline{\mathcal{D}}$. Then,

$$\mathbb{E}\Big(\ |\widehat{Y}_n|^2 \ \Big) \ \ \leq \ \ 3 \ \mathbb{E}\Big(\ |g(X^x_{\tau_x}) \ - \ g(X^{x'}_{\tau_{x'}})|^2 \ \Big) \ + \ 3 \ M^2 \ \mathbb{E}\Big(\ 1\!\!1_{\{\tau \leq n < \tau'\}} \ + \ 1\!\!1_{\{\tau' \leq n < \tau\}} \ \Big).$$

As the functions $x \to \tau_x$ and $x \to g\big(X^x_{\tau_x}\big)$ are continuous in $\overline{\mathcal{D}}$, we conclude thanks to Lebesgue's theorem. In other words, $\mathbb{E}\Big(\ |\widehat{Y}_n|^2 \ \Big)$ converges to zero when $x \to x'$.

(vii) Now, we turn to study $\mathbb{E}\Big(\int_{t_0}^n \ exp\,(\lambda s) \ \ |\widehat{\gamma}_s|^2 \ ds \ \Big| \ \mathcal{F}_{t_0} \ \Big)$, the quadratic increment for generators.
By definition of $\widehat{\gamma}_s$, we get

$$\begin{aligned} |\widehat{\gamma}_s| \ \ \leq \ \ & \Big(f(X^x_s, \widetilde{Y}'_s, \widetilde{Z}'_s) \ - \ f(X^{x'}_s, \widetilde{Y}'_s, \widetilde{Z}'_s) \Big) \ 1\!\!1_{\{0 \leq s \leq \tau \wedge \tau' \wedge n\}} \\ & + \Big(\varphi(M) \ + \ C \Big) \ 1\!\!1_{\{\tau' \wedge n < s \leq \tau \wedge n\}} \\ & + \Big(K \, \|\widetilde{Z}'_s\| \ + \ \varphi\Big(M + \frac{C}{\mu} \Big) \ + \ C \Big) \ 1\!\!1_{\{\tau \wedge n < s \leq \tau' \wedge n\}}. \end{aligned}$$

Hence,

$$\begin{aligned} |\widehat{\gamma}_s|^2 \ \ \leq \ \ & \gamma \ (\ 1 \ + \ \|\widetilde{Z}'_s\|^2 \ ) \ \Big(1\!\!1_{\{\tau \wedge n < s \leq \tau' \wedge n\}} \ + \ 1\!\!1_{\{\tau' \wedge n < s \leq \tau \wedge n\}} \Big) \\ & + \Big| f(X^x_s, \widetilde{Y}'_s, \widetilde{Z}'_s) \ - \ f(X^{x'}_s, \widetilde{Y}'_s, \widetilde{Z}'_s) \Big|^2 \ 1\!\!1_{\{0 \leq s \leq \tau \wedge \tau' \wedge n\}}, \end{aligned}$$

where γ is a constant. Then, we obtain

$$\begin{aligned} \mathbb{E}\Big(\int_0^n exp\,(\lambda s) \ \ |\widehat{\gamma}_s|^2 \ ds \ \Big) \ \leq \ \mathbb{E}\Big(\int_0^n exp\,(\lambda s) \ \ \Big| f(X^x_s, \widetilde{Y}'_s, \widetilde{Z}'_s) \ - \ f(X^{x'}_s, \widetilde{Y}'_s, \widetilde{Z}'_s) \Big|^2 \ ds \ \Big) \\ + \ \gamma \ exp\,(\lambda n) \ \ \mathbb{E}\Big(\int_0^n \ (\ 1 \ + \ \|\widetilde{Z}'_s\|^2 \ ) \ \Big(1\!\!1_{\{\tau \wedge n < s \leq \tau' \wedge n\}} \ + \ 1\!\!1_{\{\tau' \wedge n < s \leq \tau \wedge n\}} \Big) \ ds \ \Big). \end{aligned}$$

These two terms converge to zero when $x \to x'$ thanks to Lebesgue's theorem.

(viii) Finally, we proved that for any x' fixed, there exists $\eta > 0$ such that $\forall x \in \overline{\mathcal{D}}$ satisfying $|x - x'| \leq \eta$, we get $|\widehat{Y}_0|^2 \ \leq \ 2 \ \varepsilon$.

<u>*Step 2*</u>: u is a viscosity solution.

(i) We need a preliminary lemma in order to compute $u(X^x_{t \wedge \tau_x})$.

Lemma 4.4 $u(X^x_{t \wedge \tau_x}) = Y^x_{t \wedge \tau_x}$.

We shall explain the proof of the previous relation because it seems to be often considered as obvious (using a result of uniqueness for BSDEs) whereas it uses several results as Markov property and the concept of weak uniqueness.
The proof will be detailed in the appendix.

(ii) In order to show that u is a viscosity solution, we will only prove that it is a subsolution. Indeed, the proof that it is also a supersolution can be led in the same way. We follow the same strategy as in [7]. $\square$

4.3 Theorem of uniqueness

The previous section allows us to construct a viscosity solution for PDE (15) with a generator satisfying $(\mathbf{A}_{\boldsymbol{\mu}})$.

The notion of viscosity solution is a powerful tool. Indeed, viscosity solutions satisfy a stability result, which means that we are allowed to pass to the limit in equations. In the book [1], G. Barles gives a classical example: the method of "vanishing viscosity", whose consistency motivated the name of the notion. Moreover, the main advantage of these solutions is that uniqueness can be obtained with reasonable conditions on f.

We are interested in the second point. We assume that f satisfies $(\mathbf{A}^{\boldsymbol{x}})$, which is a property of continuity on x and we study the uniqueness of a bounded viscosity solution adopting the same strategy as in articles [15] and [2].

Theorem 4.5
We suppose that f satisfies $(\boldsymbol{A}_{\boldsymbol{\mu}})$ and $(\boldsymbol{A}^{\boldsymbol{x}})$.
Then u defined by $u(x) = Y_0^x$ is the unique bounded viscosity solution on $\overline{\mathcal{D}}$ for PDE (15).

4.4 Sign of the solution

We impose a further hypothesis: a nonnegative boundary condition and a nonnegative generator at the point $(0, 0)$.

Theorem 4.6
We suppose moreover that $\forall x \in \partial\mathcal{D},\ g(x) \geq 0$ and $\forall t \geq 0,\ f(X_t^x, 0, 0) \geq 0$. Then $u(x) \geq 0$.

Proof. It results from comparison theorem for BSDEs with a bounded terminal time applied to BSDE $\big(f,\, \xi\, 1\!\!1_{\{\tau_x \leq n\}},\, \tau_x\big)$ for each integer n, that $\forall t \geq 0, \quad Y_t^{x,n} \geq 0, \quad \mathbb{P}a.s.$. Indeed, since $X_{\tau_x}^x$ represents the position of the diffusion at its exit time from $\mathcal{D}$, we derive $g(X_{\tau_x}^x)\, 1\!\!1_{\{\tau_x \leq n\}} \geq 0$. Sending n to infinity, we get that $\forall t \geq 0, \quad Y_t^x \geq 0$ and in particular, for $t = 0$, we deduce that $u(x) \geq 0$. $\square$

5 Link between BSDEs and PDEs for a monotone generator

The purpose of this section is to state a similar result as in Theorem 4.3, under weaker assumptions, that is to say when the generator is only monotone. Nevertheless, in order to apply previous theorems, we need solutions to be unique. That is why, we restrict ourselves to the context when f does not depend on z and we suppose that the terminal time τ is almost surely finite.

5.1 Notation

We keep on the same notations as in Section 4.1 for sets $\mathcal{U}$, $\mathcal{D}$ and mappings ψ, b, σ, g.
The main difference is that we ask f to satisfy $(\mathbf{A_0'})$ instead of $(\mathbf{A_\mu})$. Here $\big(Y_t^x, Z_t^x\big)_{t\geq 0}$ will denote the unique solution of the BSDE $\Big(f,\, g\big(X_{\tau_x}^x\big),\, \tau_x\Big)$.

5.2 Existence of a viscosity solution

We aim to extend Theorem 4.3 to a framework where the generator is only monotone. In other words, we are interested in the link that exists between a solution to BSDE $\Big(f,\, g\big(X_{\tau_x}^x\big),\, \tau_x\Big)$, and a viscosity solution to the PDE

$$\begin{cases} Lu(x) \;+\; f\big(x, u(x)\big) \;=\; 0 & \text{if } x \in \mathcal{D}, \\ u(x) \;=\; g(x) & \text{if } x \in \partial\mathcal{D}, \end{cases} \tag{19}$$

where $\quad L \;=\; b.\nabla \;+\; 1/2 \;\sum_{i,j=1}^d (\sigma\sigma^*)_{i,j}\; \frac{\partial^2}{\partial x_i\; \partial x_j}$.

Theorem 5.1
Suppose that f satisfies $\big(\boldsymbol{A_0'}\big)$. Set $u(x) \;=\; Y_0^x, \quad \forall x \in \mathcal{D}$. Then, u is continuous and bounded on $\overline{\mathcal{D}}$. Moreover, u is a viscosity solution of the PDE (19).

Proof.

We choose a strategy inspired from the proof of Theorem 4.3, especially for the beginning but we will also apply results that hold in the case of a strictly monotone generator.
Nevertheless, we still need to proceed in two steps. First, we must be sure that not only u is continuous, but also any function v defined as the initial condition of a BSDE driven by a monotone generator h and with terminal condition $g\big(X_{\tau_x}^x\big)$. Next we will prove that u is a viscosity solution thanks to the property of stability for viscosity solutions.

<u>*Step 1*</u>: Continuity.

Let h be a generator satisfying $(\mathbf{A_0'})$. We denote by $\big(Y_t^x, Z_t^x\big)_{t\geq 0}$ the unique solution of BSDE $\Big(h,\, g\big(X_{\tau_x}^x\big),\, \tau_x\Big)$.

We set $v(x) = Y_0^x$. Theorem 3.6 provides that v is well-defined and even bounded by M. It remains to check that it is continuous.
We fix x' in $\overline{\mathcal{D}}$ and we aim to prove that $|v(x) - v(x')| \xrightarrow[x\to x']{} 0$. We adopt the same procedure as in the proof of theorem 4.3, in other woords, we use BSDEs with bounded terminal times.

(i) We start with the continuity of $x \to \tau_x$, that immediatly implies continuity for $x \to X_{\tau_x}^x$ on $\overline{\mathcal{D}}$.

(ii) As in the proof of Theorem 3.6, we approach $\big(Y_t^x, Z_t^x\big)_{t\geq 0}$ by a sequence $\big(Y_t^{x,q}, Z_t^{x,q}\big)_{q\geq 0}$ defined as the solution of a BSDE with a fixed terminal time. Moreover, we obtained the following inequality

$$\forall t \text{ such that } 0 \leq t \leq q, \qquad \text{we have} \quad \big|Y_t^x \;-\; Y_t^{x,q}\big|^2 \;\leq\; M^2\; \mathbb{E}\big(\; \mathbb{1}_{\{q<\tau_x\}} \;\big|\; \mathcal{F}_{t\wedge\tau_x}\big) \quad \mathbb{P}a.s.$$

(iii) Let $\varepsilon > 0$, we aim to pick out a indice q_0 from which $|Y_0^x - Y_0^{x,q}|$ and $|Y_0^{x',q} - Y_0^{x'}|$ are small enough.

Since, for any fixed $\omega \in \Omega$, $x \to \tau_x(\omega)$ is continuous on compact $\overline{\mathcal{D}}$, then we are able to define a bound denoted by $S(\omega)$, with S a random variable almost surely finite.
Hence, $\forall x \in \overline{\mathcal{D}} \quad 1\!\!1_{\{q<\tau_x\}} \ \leq 1\!\!1_{\{q<S\}} \quad \mathbb{P}a.s.$.
Thus, we derive that there exists q_0 such that $\forall q \geq q_0$, we have $2M^2\,\mathbb{E}\big(\ 1\!\!1_{\{q<S\}}\ \big) \ \leq \varepsilon$.
Consequently, $|Y_0^x - Y_0^{x'}|^2 \leq \varepsilon \ + \ |Y_0^{q_0} - Y'^{q_0}_0|^2 \quad \mathbb{P}a.s.$

It remains to look for a bound $\eta > 0$ such that $|x - x'| \leq \eta$ implies $|Y_0^{q_0} - Y'^{q_0}_0| \leq \varepsilon$.
Through the paper, we will denote q_0 by q.

(iv) As before, we need to extend variables, so that BSDEs will hold for the same terminal time. Keeping the same notations as in the proof of Theorem 4.3,

$$\widehat{Y}_{t\wedge T} \ = \ \xi\, 1\!\!1_{\{\tau\leq q\}} \ - \ \xi'\, 1\!\!1_{\{\tau'\leq q\}} \ + \ \int_{t\wedge T}^{q\wedge T} \Big(\widetilde{h}(X_s^x, \widetilde{Y}_s) - \widetilde{h}(X_s^x, \widetilde{Y}'_s) \ + \widehat{\gamma}_s\Big)\, ds \ - \ \int_{t\wedge T}^{q\wedge T} \widehat{Z}_s\, dW_s,$$

$$\widehat{\gamma}_s \ = \ \begin{cases} h(X_s^x, \widetilde{Y}'_s) & \text{if } \tau' \wedge q < s \leq \tau \wedge q, \\ -h(X_s^{x'}, \widetilde{Y}'_s) & \text{if } \tau \wedge q < s \leq \tau' \wedge q, \\ h(X_s^x, \widetilde{Y}'_s) \ - \ h(X_s^{x'}, \widetilde{Y}'_s) & \text{if } 0 \leq s \leq \tau \wedge \tau' \wedge q, \\ 0 & \text{otherwise.} \end{cases}$$

(v) Itô's formula yields that $|\widehat{Y}_0|^2 \ \leq \ exp\,(q)\ \mathbb{E}\Big(\ |\widehat{Y}_q|^2\ \Big) \ + \ \mathbb{E}\Big(\int_0^q \ exp\,(s)\ |\widehat{\gamma}_s|^2\ ds\Big)$.
It remains to show that $\mathbb{E}\Big(\ |\widehat{Y}_q|^2\Big)$ and $\mathbb{E}\Big(\int_0^q \ exp\,(s)\ |\widehat{\gamma}_s|^2\ ds\Big)$ converge to zero when $x \to x'$.

(vi) Lebesgue's dominated convergence theorem easily provides convergence for the first expectation.

(vii) By construction of $\widehat{\gamma}$, we get that

$$\mathbb{E}\Big(\ \int_0^q exp\,(s)\ |\widehat{\gamma}_s|^2\ ds\Big) \ \leq \ \varphi(M)^2\ exp\,(q)\ \mathbb{E}\Big(\ \int_0^q \ \big(\ 1\!\!1_{\{\tau\wedge q<s\leq\tau'\wedge q\}} \ + \ 1\!\!1_{\{\tau\wedge q<s\leq\tau'\wedge q\}}\ \big)\ ds\Big)$$
$$+\ exp\,(q)\ \mathbb{E}\Big(\ \int_0^{q\wedge\tau\wedge\tau'} |h(X_s^x, \widetilde{Y}'_s) \ - \ h(X_s^{x'}, \widetilde{Y}'_s)|^2\ ds\Big).$$

Consequently, there exists $\eta > 0$ such that $\forall x \in \overline{\mathcal{D}}$ satisfying $|x - x'| \leq \eta$, $|\widehat{Y}_0|^2 \ \leq \ 2\ \varepsilon$.

Proposition 5.2
Let h be a generator satisfying $(\boldsymbol{A_0}')$, we denote by $\big(Y_t^x, Z_t^x\big)_{t\geq 0}$ the unique solution of BSDE $\Big(h,\, g\big(X_{\tau_x}^x\big),\, \tau_x\Big)$. We also set $v(x) \ = \ Y_0^x$, then v is a bounded continuous function.

In particular, we derive that u is continuous.

<u>*Step 2*</u>: u is a viscosity solution.

(i) In this part, we are going to use Theorem 4.3, that provides a link between BSDEs and PDEs for strictly monotone generators. For this purpose, we construct once again generators $f_{n,m}(t,y) \ = \ f(t,y) \ - \ \frac{1}{n}\ y^+ \ + \ \frac{1}{m}\ y^-$ and we denote by $\big(Y_t^{x,n,m}, Z_t^{x,n,m}\big)_{t\geq 0}$ the unique solution of BSDE $\Big(f_{n,m}\,,\, g\big(X_{\tau_x}^x\big),\, \tau_x\Big)$. We set $u_{n,m}(x) = Y_0^{x,n,m}$.
Since each generator $f_{n,m}$ satisfies $\big(\mathbf{A}_{\frac{1}{n}}\big)$, then Theorem 4.3 provides that $u_{n,m}$ is a viscosity solution for PDE

$$\begin{cases} L\alpha(x) \ + \ f_{n,m}\big(x, \alpha(x)\big) \ = \ 0 & \text{if } x \in \mathcal{D}, \\ \alpha(x) \ = \ g(x) & \text{if } x \in \partial\mathcal{D}. \end{cases}$$

Moreover, $u_{n,m}$ is continuous and bounded.

(ii) The proof of Theorem 3.1 states that Y^x can be constructed as the limit of sequence $\big(Y^{x,n,m}\big)_{n\geq m\geq 1}$. As a byproduct, we obtain that $u_{n,m}$ converges to u. It remains to explicit the way of convergence in order to prove that u is a viscosity solution for PDE (19). To this end, we will use stability for viscosity solutions (whence the convergence is uniform).

We begin by studying the convergence when n tends to infinity. We proved that sequences $\big(Y^{x,n,m}\big)_{n\geq m}$ and $\big(Z^{x,n,m}\big)_{n\geq m}$ respectively converge towards $Y^{x,m}$ and $Z^{x,m}$, where $\big(Y_t^{x,m}, Z_t^{x,m}\big)_{t\geq 0}$ the unique solution of BSDE $\Big(f_m,\, g\big(X^x_{\tau_x}\big),\, \tau_x\Big)$, with $\quad f_m(x,y) \;=\; f(x,y) \;+\; \frac{1}{m}\, y^-$.

We set $u_m(x) = Y_0^{x,m}$, we aim to express this function as a viscosity solution. We already know that $(u_{n,m})_{n\geq m}$ increasingly converges to u_m. In order to prove that the convergence is uniform, we may use Dini's theorem. For this purpose, it suffices to check that u_m is a continuous function. Taking into consideration that generator f_m satisfies $\big(\mathbf{A_0'}\big)$, we can use Proposition 5.2 with $h(x,y) = f_m(x,y)$ and $v = u_m$. Then, u_m is a continuous function and it is a viscosity solution for PDE

$$\begin{cases} L\alpha(x) \;+\; f_m\big(x,\alpha(x)\big) \;=\; 0 & \text{if } x\in\mathcal{D}, \\ \alpha(x) \;=\; g(x) & \text{if } x\in\partial\mathcal{D}. \end{cases}$$

(iii) We can do the same when m tends to infinity.
We know that $\big(Y^{x,m}\big)_{m\geq 1}$ and $\big(Z^{x,m}\big)_{m\geq 1}$ respectively converge towards Y^x and Z^x. In particular, we proved that $(u_m)_{m\geq 1}$ decreasingly converges to u. Moreover, Proposition 5.2 provides that u is a continuous function. Hence, applying once more Dini's theorem, we derive that the convergence is uniform.
Consequently, stability of viscosity solutions allows us to complete the proof: u is a viscosity solution for PDE (19). □

Example
Let us consider the following PDE, with $r > 0$

$$\begin{cases} Lu(x) \;-\; u(x)^r \;=\; 0 & \text{if } x\in\mathcal{D}, \\ u(x) \;=\; g(x) & \text{if } x\in\partial\mathcal{D}. \end{cases} \tag{20}$$

In other words, the generator is $h(x,y,z) = -y^r$. Note that h satisfies neither $\big(\mathbf{A_\mu}\big)$ nor $\big(\mathbf{A_0'}\big)$. It results that we are not able yet to solve this PDE. Nevertheless, we can prove the existence of a viscosity solution for a similar PDE, with generator $h_0(x,y,z) = -(y^+)^r$, since it satisfies $\big(\mathbf{A_0'}\big)$. We denote it by u_0.

5.3 Sign of the solution

Theorem 5.3
We suppose moreover that $\quad \forall x\in\partial\mathcal{D}, \quad g(x)\geq 0,$ *then u is nonnegative on $\overline{\mathcal{D}}$.*

Proof. It results from comparison Theorem 3.7 applied on the one hand to BSDE $\Big(f,\, g\big(X^x_{\tau_x}\big),\, \tau_x\Big)$ and on the other hand to the BSDE with a null terminal condition that, at any moment t, the variable Y_t^x is larger than the solution of the second equation. But note that the second equation is also driven by generator f, by consequence it is uniquely solvable (refer to Theorem 3.6). To complete the proof, it remains to identify this solution, and we denote by $\big(\overline{Y}_t, \overline{Z}_t\big)_{t\geq 0}$ the unique solution of BSDE $\big(f,\, 0,\, \tau_x\big)$.
Since $f(x,0) = 0\ \forall x\in\overline{\mathcal{D}}$, then null process $(0,0)_{t\geq 0}$ is solution of $\big(f,\, 0,\, \tau_x\big)$.
Consequently, we get that $\forall t\geq 0 \quad Y_t^x \geq 0$, and as a byproduct, for $t = 0$, we deduce that $u(x)\geq 0$.
□

Example
Let us follow with our example in the particular case when g is nonnegative.
So, we suppose that $g(x) \geq 0,\ \forall x \in \overline{\mathcal{D}}$. We consider PDE $\begin{cases} Lu(x) \ = \ \big(u(x)^+\big)^r & \text{if } x \in \mathcal{D}, \\ u(x) \ = \ g(x) & \text{if } x \in \partial\mathcal{D}. \end{cases}$

Then Theorem 5.3 yields that u_0 is nonnegative on $\overline{\mathcal{D}}$. Hence, we derive that u_0 is also a viscosity solution for the initial PDE (20). It remains to check its uniqueness thanks to the following lemma:

Lemma 5.4
Let $u,\ v\ : \mathbb{R}^d \longrightarrow \mathbb{R}$ continuous on $\overline{\mathcal{D}}$ such that u (resp. v) is a nonnegative viscosity subsolution (resp. supersolution) for PDE (20). Then $u \leq v$ on $\overline{\mathcal{D}}$.

Proof.
We set $S \ = \ \sup_{x\in\overline{\mathcal{D}}} \big(u(x) - v(x)\big)$ and we suppose that $S > 0$. Let us find a contradiction.

Let $\alpha > 0$. We set $\Psi_\alpha(x,y) \ = \ u(x) \ - \ v(y) \ - \ \frac{\alpha}{2}|x-y|^2$,
and we denote (x_α, y_α) local maximum point for Ψ_α in $\overline{\mathcal{D}} \times \overline{\mathcal{D}}$.

Then the following lemma is true. For the proof, see [15].
Lemma 5.5
$\|x_\alpha - y_\alpha\| \xrightarrow[\alpha\to+\infty]{} 0, \qquad \alpha\ \|x_\alpha - y_\alpha\|^2 \xrightarrow[\alpha\to+\infty]{} 0, \qquad u(x_\alpha) \ - \ v(y_\alpha) \xrightarrow[\alpha\to+\infty]{} S.$

Taking into consideration that on the one hand u is a subsolution for PDE (20) and x_α is a local maximum point for $u - \phi$ with $\phi(x) \ = \ v(y_\alpha) \ + \ \frac{\alpha}{2}|x-y_\alpha|^2$,
and on the other hand v is a supersolution for PDE (20) and y_α is a local minimum point for $v - \psi$ with $\psi(y) \ = \ u(x_\alpha) \ - \ \frac{\alpha}{2}|x_\alpha - y|^2$, then we obtain the inequality

$$- \big(v(y_\alpha)\big)^r \ + \ \big(u(x_\alpha)\big)^r \ \leq \ \alpha\big(b(x_\alpha) - b(y_\alpha)\big)\ .\ (x_\alpha - y_\alpha).$$

Consider subsequences such that $x_\alpha,\ y_\alpha \xrightarrow[\alpha\to+\infty]{} \overline{z}$ in $\overline{\mathcal{D}}$. Then, by continuity of u and v, we get that $S \ = \ u(\overline{z}) - v(\overline{z})$, hence $\big(u(\overline{z})\big)^r \ > \ \big(v(\overline{z})\big)^r$, whereas taking the limit of the inequality when α converges to $+\infty$, provides that $\big(u(x_\alpha)\big)^r \ \leq \ \big(v(y_\alpha)\big)^r$, which yields a contradiction. $\square$

Finally, we get that, whence g is nonnegative, PDE (20) has a unique nonnegative viscosity solution.

Our study has gives a probabilistic formula for the equation $Lu \ = \ u^r$. Note that J.F. Le Gall in [13] also considers such PDEs but he uses the superprocesses in order to represent the solution of $Lu \ = \ u^r$.

Appendix

In order to prove Theorem 4.3, we need to complete the proof of Lemma 4.4.
Proof of Lemma 4.4.
(-) Let us introduce Markovian SDEs.
Let $v \geq 0$ be fixed and $\theta \in \mathcal{L}^2(\mathcal{F}_v)$ with values in $\overline{\mathcal{D}}$. We denote by $\big(X_t^{v,\theta}\big)_{t\geq 0}$ the unique solution of the SDE

$$X_t^{v,\theta} = \theta + \int_v^t b(X_s^{v,\theta})\, ds + \int_v^t \sigma(X_s^{v,\theta})\ dW_s, \tag{21}$$

and for $0 \leq t \leq v$, we define $X_t^{v,\theta} = \mathbb{E}(\theta|\mathcal{F}_t)$.

We set $\delta_\theta \ = \ \inf\{\ t \geq v \mid X_t^{v,\theta} \notin \overline{\mathcal{D}}\ \}$, which is a finite stopping time.
We note that for $\theta = x,\ \delta_x \ = \ \tau_x + v$ in law.

We denote by $\big(Y_t^{v,x}, Z_t^{v,x}\big)_{t\geq v}$ the unique solution for the BSDE associated to the diffusion $\big(X_t^{v,x}\big)_{t\geq 0}$ *i.e.* $\Big(f,\, g\big(X_{\delta_x}^{v,x}\big),\, \delta_x\Big)$.

(-) We also introduce a new Brownian motion : $\mathcal{B}_t \;=\; W_{v+t} - W_v$ and we denote by $\big(\mathcal{G}_t\big)_{t\geq 0}$ its natural filtration.

We define $\big(\overline{X}_t^x\big)_{t\geq 0}$ the unique solution of the following SDE, noting that it is $\big(\mathcal{G}_t\big)$-adapted.

$$\overline{X}_t^x = x + \int_0^t b(\overline{X}_s^x)\; ds + \int_0^t \sigma(\overline{X}_s^x)\; d\mathcal{B}_s. \tag{22}$$

Remark.
On the one hand, we point out that a change of variable gives $\int_0^{r-v} \sigma(\overline{X}_s^x)\; d\mathcal{B}_s \;=\; \int_v^r \sigma(\overline{X}_{s-v}^x)\; dW_s$.
And then by the result of uniqueness for SDE (21), we get that

$$\forall t \geq 0, \quad \overline{X}_t^x = X_{t+v}^{v,x} \qquad \mathbb{P}a.s. \tag{23}$$

On the other hand, we can use a result of weak uniqueness for SDEs. It ensures that the diffusions $\big(\overline{X}_t^x\big)_{t\geq 0}$ and $\big(X_t^x\big)_{t\geq 0}$ are equal in law.

We can apply the results obtained in the previous sections in the case where equations are driven by another Brownian motion, that is $\big(\mathcal{B}_t\big)_{t\geq 0}$ instead of $\big(W_t\big)_{t\geq 0}$.

We set $\overline{\tau}_x \;=\; \inf\{\, t \geq 0 \mid \overline{X}_t^x \notin \overline{\mathcal{D}}\,\}$. Remark that (23) provides the following equality: $\overline{\tau}_x = \delta_x - v$.
We denote by $\big(\overline{Y}_t^x, \overline{Z}_t^x\big)_{t\geq 0}$ the unique solution for the BSDE driven by $\big(\mathcal{B}_t\big)_{t\geq 0}$:

$$\begin{cases} \overline{Y}_{t\wedge\overline{\tau}_x}^x \;=\; \overline{Y}_{r\wedge\overline{\tau}_x}^x \;+\; \int_{t\wedge\overline{\tau}x}^{r\wedge\overline{\tau}_x} f(\overline{X}_s^x, \overline{Y}_s^x, \overline{Z}_s^x)\; ds \;-\; \int_{t\wedge\overline{\tau}_x}^{r\wedge\overline{\tau}_x} \overline{Z}_s^x\; d\mathcal{B}_s, \\ \overline{Y}_{\overline{\tau}_x}^x \;=\; g\big(\overline{X}_{\overline{\tau}_x}^x\big). \end{cases} \tag{24}$$

Since $\big(\overline{Y}_t^x, \overline{Z}_t^x\big)_{t\geq 0}$ is $\big(\mathcal{G}_t\big)$-adapted, in particular $\overline{Y}_0^x$ is $\mathcal{G}_0$-measurable, so that we can set $\forall x \in \overline{\mathcal{D}}, \quad \overline{u}(x) = \overline{Y}_0^x$. By definition, $\overline{u}$ is a continuous function on $\overline{\mathcal{D}}$.

(-) Similarly to the proof of Theorem 2.1, we still define $\big(Y_q^x, Z_q^x\big)$ the unique solution for BSDE (16).
Adapting the procedure to the construction of $\overline{u}$, we can express $\overline{u}(x)$ as the limit of the sequence $\big(\overline{Y}_{q,t}^x\big)_{q\geq 0}$ where $\big(\overline{Y}_q^x, \overline{Z}_q^x\big)$ is the unique solution of the BSDE

$$\overline{Y}_{q,t\wedge\overline{\tau}_x}^x \;=\; g\big(\overline{X}_{\overline{\tau}_x}^x\big)\; 1\!\!1_{\{\overline{\tau}_x \leq q\}} \;+\; \int_{t\wedge\overline{\tau}_x}^{q\wedge\overline{\tau}_x} f(\overline{X}_s^x, \overline{Y}_{q,s}^x, \overline{Z}_{q,s}^x) ds \;-\; \int_{t\wedge\overline{\tau}_x}^{q\wedge\overline{\tau}_x} \overline{Z}_{q,s}^x d\mathcal{B}_s. \tag{25}$$

Applying weak uniqueness for BSDEs with a bounded terminal time, we get that $Y_{q,t\wedge\tau_x}^x$ and $\overline{Y}_{q,t\wedge\overline{\tau}_x}^x$ have the same law. In particular, $Y_{q,0}^x \;=\; \overline{Y}_{q,0}^x \quad \mathbb{P}a.s.$ because they are deterministic variables. Consequently, sending q to infinity, we show that $u(x) \;=\; \overline{u}(x)$.

We are willing to extend the equality between u and $\overline{u}$ to the set of variables with values in $\overline{\mathcal{D}}$ that belong to $\mathcal{L}^2\big(\mathcal{F}_v\big)$.

As u and $\overline{u}$ are continuous, we start by defining them for simple variables.
We set $\theta_m = \sum_{i=1}^m x_i\; 1\!\!1_{A_i}$ with $x_i \in \overline{\mathcal{D}}$ and let the sets A_i constitute a $\mathcal{F}_v$-measurable partition of Ω.

We define $u(\theta_m) \;=\; \sum_{i=1}^m u(x_i)\; 1\!\!1_{A_i}$ and in the same way $\overline{u}(\theta_m) \;=\; \sum_{i=1}^m \overline{u}(x_i)\; 1\!\!1_{A_i}$.
Then, $u(\theta_m) \;=\; \overline{u}(\theta_m)$.

Taking into account the density of such functions in $\mathcal{L}^2(\mathcal{F}_v)$, we get that for any $\theta \in \mathcal{L}^2(\mathcal{F}_v)$ with values in $\overline{\mathcal{D}}$, there exists a sequence $(\theta_m)_{m\geq 0}$ of simple variables which converges to θ in $\mathcal{L}^2(\mathcal{F}_v)$, so that we can extract a subsequence $(\theta_{m_k})_{k\geq 0}$ that converges P-almost surely to θ.
Finally, we define $u(\theta) \;=\; \lim_{k\to+\infty} u(\theta_{m_k})$ and $\overline{u}(\theta) \;=\; \lim_{k\to+\infty} \overline{u}(\theta_{m_k})$, then $u(\theta) \;=\; \overline{u}(\theta)$.

(-) Now, we are going to prove that $\overline{u}(\theta) = Y_v^{v,\theta}$

We study $\big(Y_t^{v,x}, Z_t^{v,x}\big)_{t\geq v}$ and $\big(\overline{Y}_t^x, \overline{Z}_t^x\big)_{t\geq 0}$ which are solutions of BSDEs with Brownian motion $\big(W_t\big)_{t\geq 0}$ and $\big(\mathcal{B}_t\big)_{t\geq 0}$, respectively.
We begin by using the result (23), and we recall that a change of variable yields
$\int_{t'\wedge\delta_x - v}^{r'\wedge\delta_x - v} \overline{Z}_s^x \, d\mathcal{B}_s \;=\; \int_{t'\wedge\delta_x}^{r'\wedge\delta_x} \overline{Z}_{s-v}^x \, dW_s$. This provides

$$\begin{cases} \overline{Y}_{t'\wedge\delta_x - v}^x \;=\; \overline{Y}_{r'\wedge\delta_x - v}^x \;+\; \int_{t'\wedge\delta_x}^{r'\wedge\delta_x} f(X_s^{v,x}, \overline{Y}_{s-v}^x, \overline{Z}_{s-v}^x)\, ds \;-\; \int_{t'\wedge\delta_x}^{r'\wedge\delta_x} \overline{Z}_{s-v}^x \, dW_s, \\ \overline{Y}_{\delta_x - v}^x \;=\; g\big(X_{\delta_x}^{v,x}\big). \end{cases}$$

Taking into consideration the uniqueness for solution of the BSDE $\Big(f,\, g\big(X_{\delta_x}^{v,x}\big),\, \delta_x\Big)$, we obtain equality in law for the couples of processes $\big(\overline{Y}_s^x, \overline{Z}_s^x\big)$ and $\big(Y_{s+v}^{v,x}, Z_{s+v}^{v,x}\big)$.
In particular, for $s = 0$, $\overline{u}(x) = Y_v^{v,x}, \quad \forall x \in \overline{\mathcal{D}}$.

Now, we consider $\theta_m = \sum_{i=1}^m x_i \; 1\!\!1_{A_i}$, which is a simple variable.
We know that by definition, $\overline{u}(\theta_m) \;=\; \sum_{i=1}^m \overline{u}(x_i) \; 1\!\!1_{A_i} \;=\; \sum_{i=1}^m Y_v^{v,x_i} \; 1\!\!1_{A_i}$.
It remains to check that $\sum_{i=1}^m Y_v^{v,x_i} \; 1\!\!1_{A_i} \;=\; Y_v^{v,\theta_m}$.

Actually, we begin by stating that $\sum_{i=1}^m X_t^{v,x_i} \; 1\!\!1_{A_i} \;=\; X_t^{v,\theta_m}$. The key of the proof deals with the fact that $A_i \in \mathcal{F}_v$, which allows us to write $1\!\!1_{A_i} \int_v^t \sigma(X_s^{v,x_i}) \, dW_s \;=\; \int_v^t 1\!\!1_{A_i} \; \sigma(X_s^{v,x_i}) \, dW_s$.
And then $\sum_{i=1}^m X_t^{v,x_i} \; 1\!\!1_{A_i} \;=\; \theta_m \;+\; \int_v^t b(\; \sum_{i=1}^m X_s^{v,x_i} 1\!\!1_{A_i} \;)\, ds \;+\; \int_v^t \sigma(\; \sum_{i=1}^m X_s^{v,x_i} 1\!\!1_{A_i} \;)\, dW_s$.
We get the desired result thanks to uniqueness for SDEs with Lipschitz coefficients.

In the same way, we show that $\big(\sum_{i=1}^m Y_t^{v,x_i} \; 1\!\!1_{A_i} \;,\; \sum_{i=1}^m Z_t^{v,x_i} \; 1\!\!1_{A_i}\big) = \big(Y_t^{v,\theta_m}, Z_t^{v,\theta_m}\big)$,
which implies that $\overline{u}(\theta_m) = Y_v^{v,\theta_m}$.

As $x \mapsto \overline{u}(x)$ and $x \mapsto Y_v^{v,x}$ are continuous on $\overline{\mathcal{D}}$ and equal on the set of simple variables, then $\forall \theta \in \mathcal{L}^2(\mathcal{F}_v)$ we have $\overline{u}(\theta) \;=\; Y_v^{v,\theta}$.

(-) Finally, it remains to show that $Y_v^{v,X_{v\wedge\tau_x}^x} \;=\; Y_{v\wedge\tau_x}^x$.

First, note that $\delta_{X_{v\wedge\tau_x}^x} \;=\; \tau_x - v\wedge\tau_x + v$. For more convenience we will denote $\theta \;=\; X_{v\wedge\tau_x}^x$.
For $t' \geq v$ and $r' \geq t'$, we set $t = t'\wedge\tau_x - v\wedge\tau_x + v$ and $r = r'\wedge\tau_x - v\wedge\tau_x + v$.

We consider the BSDE satisfied by $\big(Y_t^{v,\theta}, Z_t^{v,\theta}\big)_{t\geq v}$ between $t\wedge\delta_\theta$ and $r\wedge\delta_\theta$,

$$\begin{aligned} Y_{t'\wedge\tau_x - v\wedge\tau_x + v}^{v,\theta} \;&=\; Y_{r'\wedge\tau_x - v\wedge\tau_x + v}^{v,\theta} \;+\; 1\!\!1_{\tau_x\geq v} \int_{t'\wedge\tau_x}^{r'\wedge\tau_x} f(X_s^{v,X_{v\wedge\tau_x}^x}, Y_s^{v,\theta}, Z_s^{v,\theta})\, ds \\ &\quad - 1\!\!1_{\tau_x\geq v} \int_{t'\wedge\tau_x}^{r'\wedge\tau_x} Z_s^{v,\theta} \, dW_s. \end{aligned}$$

Note that $1\!\!1_{\tau_x\geq v} \int_{t'\wedge\tau_x}^{r'\wedge\tau_x} f(X_s^{v,X_{v\wedge\tau_x}^x}, Y_s^{v,\theta}, Z_s^{v,\theta})\, ds \;=\; \int_{t'\wedge\tau_x}^{r'\wedge\tau_x} 1\!\!1_{\tau_x\geq v} \; f(X_s^x, Y_s^{v,\theta}, Z_s^{v,\theta})\, ds$
because the diffusion $\big(X_t^{v,\theta}\big)_{t\geq 0}$ satisfies the Markov property, *i.e.* $\forall s \geq r,\; X_s^{r,x} = X_s^{v,X_v^{r,x}}$,

and $1\!\!1_{\tau_x\geq v} \int_{t'\wedge\tau_x}^{r'\wedge\tau_x} Z_s^{v,\theta} \, dW_s \;=\; \int_{t'\wedge\tau_x}^{r'\wedge\tau_x} 1\!\!1_{\tau_x\geq v} \; Z_s^{v,\theta} \, dW_s$ because $\{\tau_x \geq v\} \in \mathcal{F}_{t'\wedge\tau_x}$, then

$$Y^{v,\theta}_{t'\wedge\tau_x - v\wedge\tau_x + v} \;=\; Y^{v,\theta}_{r'\wedge\tau_x - v\wedge\tau_x + v} + \int_{t'\wedge\tau_x}^{r'\wedge\tau_x} f(X^x_s, Y^{v,\theta}_{s-v\wedge\tau_x+v}, Z^{v,\theta}_{s-v\wedge\tau_x+v})\, ds - \int_{t'\wedge\tau_x}^{r'\wedge\tau_x} Z^{v,\theta}_{s-v\wedge\tau_x+v}\, dW_s.$$

Moreover, we can check that $Y^{v,\theta}_{\tau_x - v\wedge\tau_x + v} \;=\; g(X^x_{\tau_x})$.

Consequently, we derive from the uniqueness of the solution of BSDE (14) that $\big(Y^{v,X^x_{v\wedge\tau_x}}_{t-v\wedge\tau_x+v}, Z^{v,X^x_{v\wedge\tau_x}}_{t-v\wedge\tau_x+v}\big) \;=\; \big(Y^x_t, Z^x_t\big)_{t\geq 0}$. Hence, with $t = v\wedge\tau_x$, we get $Y^{v,X^x_{v\wedge\tau_x}}_v \;=\; Y^x_{v\wedge\tau_x}$.

(-) In conclusion, we can complete the arguments as follows:
since the variable $X^x_{v\wedge\tau_x}$ belongs to $\mathcal{L}^2(\mathcal{F}_v)$, then $u(X^x_{v\wedge\tau_x}) \;=\; \overline{u}(X^x_{v\wedge\tau_x}) \;=\; Y^{v,X^x_{v\wedge\tau_x}}_v \;=\; Y^x_{v\wedge\tau_x}$.
□

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