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BSDEs with random terminal time under enlarged filtration

American-style options hedging by an insider

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Anne EYRAUD-LOISEL ^{a *} and Manuela ROYER-CARENZI ^b

^a *Université de Lyon - Laboratoire SAF - ISFA - Université Lyon 1*

50 avenue Tony Garnier 69366 Lyon Cedex 07 - France ;

^b *LATP - UMR CNRS 6632 FR 3098 IFR 48 - Evolution Biologique et Modélisation - Université de Provence - Case 19, Pl. V. Hugo, 13331 Marseille Cedex 03 - France*

Abstract : Markets with asymmetrical information are generally studied from a wealth optimization point of view. We focus here on a hedging problem for a financial agent who has an additional information on the market. We extend the results given for hedging strategies with fixed terminal time to the case of a random terminal time. In particular, we provide tools to understand the behavior of American option hedging by an insider. To achieve this aim, we prove the existence and uniqueness of backward stochastic differential equations (BSDE) solutions, when terminal time is random, under an initially enlarged filtration.

Keywords : enlargement of filtration; BSDE; uncertain time horizon; American option hedging; insider trading; asymmetrical information.

Subject Classification : 60H10, 65C30, 91B28

*Corresponding author. Email: anne.eyraud@univ-lyon1.fr

1 Introduction

In this paper, we study existence and uniqueness of backward stochastic differential equations (BSDE) solutions when terminal time is random and when the natural filtration is initially enlarged. This mathematical modeling is motivated by the following financial problem: we study the financial hedging strategy for an American contingent claim in a market with asymmetrical information.

BSDEs were first introduced by E. Pardoux and S. Peng in 1990 [25]. As they have a large panel of applications, such equations have been developed since the 90's, especially in mathematical finance, modeling hedging problems as well as optimization and control problems. They also appear in several other fields such as stochastic control (see S. Peng [27], N. El Karoui, S. Peng and M.C. Quenez [10], and X. Zhou and J. Yong [29]) or problems linked with PDEs (see E. Pardoux [23] and G. Barles, R. Buckdahn and E. Pardoux [5]).

BSDEs are useful in our framework since these equations naturally appear when describing hedging problems. We study hedging of contingent claims with random exercise time and we model it with BSDEs with random terminal time. Such equations were introduced by S. Peng (1991) [26], and developed by R.W.R. Darling and E. Pardoux (1997) [7], P. Briand and Y. Hu (1998) [6], E. Pardoux (1999) [24], M. Royer (2004) [28] among others, and by E. Pardoux (1995) [22] for BSDEs with jumps and random terminal time.

Asymmetrical information within the market is modeled with initial enlargement of filtration. Theory has been developed by J. Jacod [15], T. Jeulin [17] and M. Yor [30]. It is often used to model insider trading (see A. Ghorud and M. Pontier [14], who used this model to construct a statistical test to detect insider traders, or J. Amendinger's work [4, 3] and his coauthors).

Existence and uniqueness of BSDEs solutions is stated in [11], when terminal time is constant and under an initial enlargement of filtration. This corresponds to the case of an insider who wants to hedge a European option and has an additional information about prices at a future time T' , that is greater than the terminal time T of the hedging problem. This information leads to an initial enlargement of the standard Brownian filtration, and is supposed to satisfy a standard hypothesis for enlargement of filtration denoted by (\mathbf{H}_3) . The main result states that such an insider has no other strategy for hedging a European option with terminal exercise time T , than if he had not the additional information.

In the present work, we consider the same framework: hedging problems for an insider trader. Instead of deterministic terminal time (European-type options) we consider contingent claims with random terminal time (American-type options), such as standard American options or other exotic options with random exercise. For this purpose, we have to consider BSDEs with a random terminal time horizon, under an initial enlargement of the Brownian filtration.

In Section 2, we detail the financial motivation and we present the convenient model.

We prove in Section 3 that A. Eyraud-Loisel's results stated in [11] remain true when the hedging terminal time is a random stopping time. We assume that standard existence and uniqueness hypotheses are fulfilled, following the approach of E. Pardoux (1999) in [24] who studied BSDEs with random terminal time without enlargement of filtration (i.e. under the natural Brownian filtration). We obtain existence and

uniqueness results for the BSDE with initial enlargement of filtration under hypothesis (\mathbf{H}_3) for a bounded random terminal time.

In the case of a deterministic time horizon, existence and uniqueness of the solution holds true when the terminal time of the hedging problem is strictly smaller than the time of the information ($T < T'$). In Subsection 3.1, usual stopping techniques are used to derive same results for a random time horizon, defined as a stopping time a.s. bounded by $T < T'$.

In Subsection 3.2 results are extended to all stopping times a.s. strictly bounded by T' . In that case, existence results do not exist for a problem with deterministic horizon T' , and more technical approximation tools are needed to prove existence of a solution and state Theorem 3.6.

Financial interpretation of results developed in Sections 3.1 and 3.2 is given in Section 4. It financially means that an agent who has an initial additional information satisfying hypothesis (\mathbf{H}_3) will have the same hedging strategy as a non informed agent, for a contingent claim with random terminal time, as it was previously proved for a constant terminal time. This result extends results from [11] to more general claims traded on the market (hedging of American standard or Lookback options for instance), and is consistent with the result for a fixed terminal time. It still differs from the conclusions obtained in a wealth optimization point of view by A. Ghorud and M. Pontier who stated that under the same initially enlarged filtration, an insider trader has a different strategy from a non informed trader.

2 Model

2.1 Financial Motivation

Let W be a standard d -dimensional Brownian motion, and let $(\Omega, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ be a filtered probability space, with $\Omega = \mathcal{C}(\mathbb{R}; \mathbb{R}^d)$. Let $(\mathcal{F}_t)_{0 \leq t \leq T}$ be the natural filtration of Brownian motion W . We consider a financial market with d risky assets, whose prices are driven by the following diffusion processes:

$$S_t^i = S_0^i + \int_0^t S_s^i b_s^i ds + \int_0^t S_s^i (\sigma_s^i, dW_s), \quad t \geq 0, \quad i = 1, \dots, d$$

and where the bond (or riskless asset) evolves according to the standard equation: $S_t^0 = 1 + \int_0^t S_s^0 r_s ds$. Parameters b, σ, r are supposed to be bounded, adapted, and to take values respectively in $\mathbb{R}^d, \mathbb{R}^{d \times d}, \mathbb{R}$. Matrix σ_t is invertible $dt \otimes d\mathbb{P}$ a.s. and the Doléans-Dade exponential $\mathcal{E}(-\sigma^{-1}(b - r\mathbf{1}) \cdot W)$ is supposed to be integrable. These are the usual conditions to have existence of a risk-neutral probability (which implies no-arbitrage).

A financial agent has a positive initial wealth Y_0 at time $t = 0$, and he wishes to hedge a contingent claim with terminal payoff ξ at uncertain time horizon τ , with maturity T .

τ is a \mathcal{F} -stopping time. Generally, τ is not given, it may depend on the option owner's strategic decision. It may in particular be the optimal stopping time determined thanks to price processes Snell envelope notion (see N. El Karoui et al. [9], N. El Karoui [8] and I. Karatzas and S. Shreve [21]). In the present study the stopping time may be any stopping time, not necessarily the optimal stopping time.

If the agent is a standard American option's seller with strike K , and pay-off $\xi =$

$(S_{\tau \wedge T} - K)_+$, τ is the stopping time representing the time when the buyer of the option decides to exercise his option. The seller wants to hedge against the risk of this option, and so wants to get ξ at random time τ : $Y_{\tau \wedge T} = \xi$. (from the seller's point of view, it is even sufficient that Y is larger or equal than ξ).

The consumption is here supposed to be zero. Wealth at time t is $Y_t = \sum_{i=0}^d \theta_t^i S_t^i$. The standard self-financing hypothesis can be written as :

$$dY_t = \sum_{i=0}^d \theta_t^i dS_t^i.$$

It means that the consumption is only financed with the profits realized by the portfolio, and not by outside benefits. Then, the agent's wealth satisfies the following equation :

$$dY_t = \theta_t^0 S_t^0 r_t dt + \sum_{i=1}^d \theta_t^i S_t^i b_t^i dt + \sum_{i=1}^d \theta_t^i S_t^i (\sigma_t^i, dW_t).$$

We denote by $\pi_t^i = \theta_t^i S_t^i$ the amount of wealth invested in the i^{th} asset for $i = 1, \dots, d$, and we notice that $\theta_t^0 S_t^0 = Y_t - \sum_{i=1}^d \pi_t^i$. We denote also by $\pi_t = (\pi_t^i, i = 1, \dots, d)$ the portfolio (or strategy), and so the total wealth can be written as a solution of the following stochastic differential equation :

$$dY_t = Y_t r_t dt + (\pi_t, b_t - r_t \mathbf{1}) dt + (\pi_t, \sigma_t dW_t),$$

where $\mathbf{1}$ is the vector with all coordinates equal to 1. The previous line can also be rewritten by integrating from $t \wedge \tau$ to $T \wedge \tau$:

$$Y_{T \wedge \tau} - Y_{t \wedge \tau} = \int_{t \wedge \tau}^{T \wedge \tau} Y_s r_s ds + \int_{t \wedge \tau}^{T \wedge \tau} (\pi_s, b_s - r_s \mathbf{1}) ds + \int_{t \wedge \tau}^{T \wedge \tau} (\pi_s, \sigma_s dW_s) \text{ a.s. ,}$$

which naturally becomes:

$$Y_{t \wedge \tau} = Y_{T \wedge \tau} - \int_{t \wedge \tau}^{T \wedge \tau} \underbrace{[Y_s r_s + (\pi_s, b_s - r_s \mathbf{1})]}_{-f(s, Y_s, Z_s)} ds - \int_{t \wedge \tau}^{T \wedge \tau} \underbrace{(\sigma_s^* \pi_s)}_{Z_s} dW_s \text{ a.s.}$$

The wealth equation has been written this way as a BSDE with random terminal time. A first interest of writing the problem as a BSDE is to model the previous hedging problem with a unique equation. Another interesting aspect is that such a tool does not use the notion of equivalent martingale measure to solve the hedging problem (see N. El Karoui, S. Peng and M.-C. Quenez [10]).

We consider a market with asymmetrical information. There may be two different types of agents in this market: normally informed agents, whose information at time t is modeled by the standard Brownian filtration \mathcal{F}_t , and informed agents, who have access to an additional information on the market. More precisely, we suppose that an informed agent knows, at time $t = 0$, the realization of a random variable $L \in \mathcal{F}_{T'}$, concerning prices at time T' , where $T' \geq T$ (in general $T' > T$). The filtration representing his information at time t is obtained by enlarging the natural filtration of the Brownian motion.

$$\mathcal{Y}_t = \bigcap_{s > t} (\mathcal{F}_s \vee \sigma(L)).$$

Given that the actualized assets prices are martingales in the initial probability space under a risk-neutral probability, it would be interesting and natural to check if they still have similar properties in the larger space. So we wonder under which condition we have the following useful property :

Hypothesis (H') *If $(M_t)_{0 \leq t \leq T}$ is a given $(\mathcal{F}, \mathbb{P})$ -martingale (or semi-martingale), then $(M_t)_{0 \leq t \leq T}$ is a $(\mathcal{Y}, \mathbb{P})$ -semi-martingale.*

This well-known problem has been introduced and studied by T. Jeulin and M. Yor [19, 18, 20], next by J. Jacod [15], and later by J. Amendinger [2] and A. Grorud and M. Pontier [13]. In the current paper, we will work with the stronger following assumption :

Hypothesis (H₃) *There exists a probability \mathbb{Q} equivalent to \mathbb{P} under which \mathcal{F}_t and $\sigma(L)$ are independent, $\forall t < T'$.*

Remark that (H₃) implies (H') (see for example J. Amendinger [2] or A. Grorud and M. Pontier [13]). Among the remarkable consequences of this hypothesis, we can notice that W is a $(\mathcal{Y}, \mathbb{Q})$ -Brownian motion (see J. Jacod [15]). Another very important tool that holds under hypothesis (H₃) is a martingale representation theorem for $(\mathcal{Y}, \mathbb{Q})$ -martingales with respect to Brownian motion W , stated in J. Jacod and A.N. Shiryaev [16] (Theorem III.4.33 p. 189), and used as a key tool in the proof of the main result in [11].

Questions that naturally arise when dealing with additional information are : will informed agents hedge the option as non informed agents or will they have a different hedging strategy?

The case of European contingent claims was studied in [11], where the existence and uniqueness problem of solution was solved for a BSDE with fixed horizon, under an initial enlargement of filtration. Under hypothesis (H₃), it was proved that an informed agent will have a unique hedging strategy for a European option, which is the same as a non informed agent. The main financial question is: is it still the case for American-type options, where the hedging horizon is random? This turns out to be mathematically: does the BSDE with random terminal time under enlarged filtration have a unique solution? Is this solution adapted to the small filtration? In other words, is this solution the same as the solution of the BSDE under the natural filtration?

2.2 Mathematical formulation

Mathematically speaking, from a more general point of view, we are looking for a solution of the following BSDE with random terminal time :

$$Y_{t \wedge \tau} = \xi + \int_{t \wedge \tau}^{T \wedge \tau} f(s, Y_s, Z_s) ds - \int_{t \wedge \tau}^{T \wedge \tau} (Z_s, dW_s), \quad \forall 0 \leq t \leq T, \quad (1)$$

which belongs to the enlarged σ -algebra $(\mathcal{Y}_t)_{0 \leq t \leq T}$, and where

- $\xi \in \mathcal{L}^2(\mathcal{Y}_\tau)$ is the terminal condition,
- $f : \Omega \times [0, T] \times \mathbb{R}^k \times \mathbb{R}^{k \times d} \longrightarrow \mathbb{R}^k$ is the driver,
- $Y_t \in \mathbb{R}^k$, is the total wealth of the portfolio at time t ,

- $Z_t \in \mathbb{R}^{k \times d}$, represents the portfolio investments at time t .

One of the fundamental results on solutions of BSDEs with random terminal time is an existence and uniqueness theorem given by E. Pardoux [23] under some Lipschitz hypotheses and monotonicity conditions on the driver function. P. Briand and Y. Hu [6] improved the result in the one-dimensional case and M. Royer [28] reduced again the hypotheses on monotonicity conditions.

Remark : In these financial settings, Y_t is the total wealth of the portfolio at time t , and Z_t represents the portfolio investments at time t . Hence we have $k = 1$. So for simplicity reasons, as we deal with a financial application, we will only write results here with $Y \in \mathbb{R}$, although all the mathematical results can be generalized to the multi-dimensional case $Y \in \mathbb{R}^k$.

3 BSDEs with random terminal time under enlarged filtration

3.1 Stopping time a.s. bounded by $T < T'$

We first consider a stopping time τ , a.s. bounded by T . In other words, $\tau \leq T < T'$. This is the usual case for financial applications such as hedging. We also require classical assumptions on the \mathcal{F} -progressively measurable generator $f(\cdot, y, z)$:

(A 1) f is Lipschitz with respect to z (Lipschitz constant denoted by K),

(A 2) f is continuous with respect to y and \exists an increasing function $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that

$$|f(t, y, 0)| \leq |f(t, 0, 0)| + \varphi(|y|), \mathbb{P} \text{ a.s. }, \forall t, y,$$

(A 3) f is monotonous with respect to y : $\exists \mu \in \mathbb{R}$ such that

$$\langle y - y', f(t, y, z) - f(t, y', z) \rangle \leq \mu |y - y'|^2, \mathbb{P} \text{ a.s. }, \forall t, y, y', z,$$

(A 4) $\mathbb{E}_{\mathbb{P}}(\int_0^T |f(t, 0, 0)|^2 dt) < \infty$.

We denote by $(\Omega, \mathbf{F}, \mathbf{P})$ a filtered probability space which stands either for the standard space $(\Omega, \mathcal{F}, \mathbb{P})$ or for the enlarged space $(\Omega, \mathcal{Y}, \mathbb{Q})$.

We define, for any time $A \in \mathbb{R}_+^*$, the following spaces of \mathbf{F} -progressively measurable processes :

$$\mathcal{M}_{\mathbf{P}, \mathbf{F}}^2(0, A; \mathbb{R}^d) = \left\{ \mathbb{R}^d\text{-valued } \mathbf{F}\text{-adapted process } \psi; \quad \mathbb{E}_{\mathbf{P}} \left(\int_0^A \|\psi_s\|^2 ds \right) < \infty \right\}$$

$$S_{\mathbf{P}, \mathbf{F}}^2(0, A; \mathbb{R}) = \left\{ \mathbb{R}\text{-valued } \mathbf{F}\text{-adapted process } \psi; \quad \mathbb{E}_{\mathbf{P}} \left(\sup_{0 \leq s \leq A} |\psi_s|^2 \right) < \infty \right\}$$

We look for a solution of the following BSDE with random terminal time:

$$Y_{t \wedge \tau} = \xi + \int_{t \wedge \tau}^{\tau} f(s, Y_s, Z_s) ds - \int_{t \wedge \tau}^{\tau} (Z_s, dW_s), \quad \forall 0 \leq t \leq T. \quad (2)$$

Definition 3.1

A $(\Omega, \mathbf{F}, \mathbf{P})$ -solution (or a solution on $(\Omega, \mathbf{F}, \mathbf{P})$) to equation (2) is a pair of \mathbf{F} -progressively measurable processes $\mathbb{R} \times \mathbb{R}^d$ -valued $(Y_t, Z_t)_{0 \leq t \leq T}$ such that

1. $Z \in \mathcal{M}_{\mathbf{P}, \mathbf{F}}^2(0, T; \mathbb{R}^d)$,
2. On the set $\{t \geq \tau\}$, we have $Y_t = \xi$ and $Z_t = 0$,
3. $\forall t \in [0, T]$, we have $Y_{t \wedge \tau} = Y_\tau + \int_{t \wedge \tau}^\tau f(s, Y_s, Z_s) ds - \int_{t \wedge \tau}^\tau (Z_s, dW_s)$.

We look for a solution successively on the standard space $(\Omega, \mathcal{F}, \mathbb{P})$ and on the enlarged space $(\Omega, \mathcal{Y}, \mathbb{Q})$. Under the natural Brownian filtration, previous hypotheses on the driver f guarantee existence and uniqueness of a $(\Omega, \mathcal{F}, \mathbb{P})$ -solution. (see E. Pardoux [23] for a proof).

The following theorem states that under the same hypotheses on the driver f , asking for an additional integration hypothesis under the new probability \mathbb{Q} , BSDE (2) with random terminal time also has a unique solution in the enlarged space $(\Omega, \mathcal{Y}, \mathbb{Q})$.

Theorem 3.2

Under hypotheses (A1) to (A4) on f , and if $\mathbb{E}_{\mathbb{Q}}(\int_0^T |f(s, 0, 0)|^2 ds) < \infty$, then for all $\xi \in \mathcal{L}_{\mathbb{Q}}^2(\mathcal{Y}_\tau)$, the BSDE

$$Y_{t \wedge \tau} = \xi + \int_{t \wedge \tau}^\tau f(s, Y_s, Z_s) ds - \int_{t \wedge \tau}^\tau (Z_s, dW_s), \quad \forall 0 \leq t \leq T$$

has a unique $(\Omega, \mathcal{Y}, \mathbb{Q})$ -solution.

Proof. We use existence and uniqueness Theorem of [11] to define $(Y_t, Z_t)_{0 \leq t \leq T} \in \mathcal{M}_{\mathbb{Q}, \mathcal{Y}}^2(0, T; \mathbb{R}) \times \mathcal{M}_{\mathbb{Q}, \mathcal{Y}}^2(0, T; \mathbb{R}^d)$ as the unique \mathcal{Y}_t -adapted solution of the following BSDE

$$\forall t \in [0, T], \quad Y_t = \xi + \int_t^T \mathbb{1}_{s \leq \tau} f(s, Y_s, Z_s) ds - \int_t^T (Z_s, dW_s),$$

Next we derive the existence result in the enlarged space.

Suppose now that there exists two $(\Omega, \mathcal{Y}, \mathbb{Q})$ -solutions to equation (2), denoted by $(Y_t^1, Z_t^1)_{t \in [0, T]}$ and $(Y_t^2, Z_t^2)_{t \in [0, T]}$. We apply Itô's formula to $e^{\lambda s} |\widehat{Y}_s|^2$, where $(\widehat{Y}_t)_{t \in [0, T]}$ stands for $Y_t^1 - Y_t^2$ and $\lambda \geq 2\mu + K^2$. We obtain

$$\mathbb{E}_{\mathbb{Q}}\left(e^{\lambda(t \wedge \tau)} |\widehat{Y}_{t \wedge \tau}|^2\right) \leq (K^2 + 2\mu - \lambda) \mathbb{E}_{\mathbb{Q}}\left(\int_0^T e^{\lambda s} |\widehat{Y}_s|^2 ds\right).$$

This implies uniqueness of Y in $\mathcal{M}_{\mathbb{Q}, \mathcal{Y}}^2(0, T; \mathbb{R})$.

After replacing \widehat{Y} by 0 in the equation satisfied by $(\widehat{Y}, \widehat{Z})$, we obtain $\forall t \in [0, T]$,

$$\int_{t \wedge \tau}^\tau (f(s, Y_s^1, Z_s^1) - f(s, Y_s^1, Z_s^2)) ds = \int_{t \wedge \tau}^\tau (\widehat{Z}_s, dW_s).$$

Given that a martingale can be equal to a finite variation process if and only if it is a null process, this yields to uniqueness of Z in $\mathcal{M}_{\mathbb{Q}, \mathcal{Y}}^2(0, T; \mathbb{R}^d)$. \square

Corollary 3.3

Under the same hypotheses as in Theorem 3.2, the unique $(\Omega, \mathcal{Y}, \mathbb{Q})$ -solution $(Y_t, Z_t)_{0 \leq t \leq T}$ of BSDE (2) satisfies

$$(Y_t)_{0 \leq t \leq T} \in \mathcal{S}_{\mathbb{Q}, \mathcal{Y}}^2(0, T, \mathbb{R}).$$

Proof. From Itô's formula and Burckholder-Davis-Gundy's inequality, and using assumptions (A1) to (A4), we obtain

$$\begin{aligned} \frac{1}{2} \mathbb{E}_{\mathbb{Q}} \left(\sup_{0 \leq t \leq T} |Y_t|^2 \right) &\leq \mathbb{E}_{\mathbb{Q}}(|\xi|^2) + \mathbb{E}_{\mathbb{Q}} \left(\int_0^T |f(s, 0, 0)|^2 ds \right) \\ &+ (1 + 2\mu + K^2) \mathbb{E}_{\mathbb{Q}} \left(\int_0^T |Y_s|^2 ds \right) + 2C_{\text{BDG}}^2 \mathbb{E}_{\mathbb{Q}} \left(\int_0^T \|Z_s\|^2 ds \right), \end{aligned}$$

which is finite thanks to hypotheses on ξ and f , and as the unique solution (Y, Z) of the BSDE (2) is in $\mathcal{M}_{\mathbb{Q}, \mathcal{Y}}^2(0, T; \mathbb{R}) \times \mathcal{M}_{\mathbb{Q}, \mathcal{Y}}^2(0, T; \mathbb{R}^d)$. \square

3.2 Stopping time a.s. strictly bounded by T'

We extend here the results of section 3.1 to the case $\tau < T'$ but not $\tau \leq T < T'$. Here we can not use directly any existence result for a fixed deterministic horizon. Consequently, we construct a Cauchy sequence of processes defined as solutions of BSDEs with random terminal times τ_n , bounded by deterministic times $T_n < T'$. The limit process will satisfy the required properties.

Let τ be a \mathcal{F} -stopping time, a.s. strictly bounded by T' : $\tau < T'$.

We want to solve the following BSDE

$$Y_{t \wedge \tau} = \xi + \int_{t \wedge \tau}^{\tau} f(s, Y_s, Z_s) ds - \int_{t \wedge \tau}^{\tau} (Z_s, dW_s), \quad \forall 0 \leq t \leq T'. \quad (3)$$

Classical assumptions on the \mathcal{F} -progressively measurable generator $f(\cdot, y, z)$ still hold true. More precisely, assumptions (A1), (A3) and (A4) are supposed to be satisfied, as in the previous section. But (A2) is replaced by the following assumption (A2')

(A2') f is continuous with respect to y and for the same constant K :

$$|f(t, y, z)| \leq |f(t, 0, 0)| + K|y| + K \|z\|, \quad \mathbb{P} \text{ a.s. }, \forall t, y,$$

Definition 3.4

A $(\Omega, \mathbf{F}, \mathbf{P})$ -solution (or a solution on $(\Omega, \mathbf{F}, \mathbf{P})$) to equation (3) is a pair of \mathbf{F} -progressively measurable processes $\mathbb{R} \times \mathbb{R}^d$ -valued $(Y_t, Z_t)_{0 \leq t \leq T'}$ such that

1. $(Y, Z) \in S_{\mathbf{P}, \mathbf{F}}^2(0, T'; \mathbb{R}) \times \mathcal{M}_{\mathbf{P}, \mathbf{F}}^2(0, T'; \mathbb{R}^d)$,
2. On the set $\{t \geq \tau\}$, we have $Y_t = \xi$ and $Z_t = 0$,
3. $\forall t \in [0, T']$, we have $Y_{t \wedge \tau} = Y_{\tau} + \int_{t \wedge \tau}^{\tau} f(s, Y_s, Z_s) ds - \int_{t \wedge \tau}^{\tau} (Z_s, dW_s)$.

As previously, we look for a solution successively on the standard space $(\Omega, \mathcal{F}, \mathbb{P})$ and on the enlarged space $(\Omega, \mathcal{Y}, \mathbb{Q})$.

For each $n \in \mathbb{N}$, we introduce a sequence of processes, solutions of equation (3), with terminal value $\xi_n = \xi \mathbb{1}_{\tau \leq T_n}$ and random terminal time $\tau_n = T_n \wedge \tau$, instead of τ , where $T_n = T' - \frac{1}{n}$. In other words, $\{(Y_t^n, Z_t^n); t \geq 0\}$ is the solution of the following BSDE

$$Y_t^n = \xi_n + \int_{t \wedge \tau}^{\tau_n} f(s, Y_s^n, Z_s^n) ds - \int_{t \wedge \tau}^{\tau_n} (Z_s^n, dW_s), \quad 0 \leq t \leq T_n. \quad (4)$$

According to existence and uniqueness results of Theorem 3.2 proved in previous subsection, equation (4), whose random terminal time is bounded by $T' - \frac{1}{n} < T'$, has a unique solution in $\mathcal{M}_{\mathbb{Q}, \mathcal{Y}}^2(0, T_n; \mathbb{R}) \times \mathcal{M}_{\mathbb{Q}, \mathcal{Y}}^2(0, T_n; \mathbb{R}^d)$. Moreover, $\{(Y_t^n, Z_t^n); t \in [T_n, T']\}$ is defined by

$$\begin{cases} Y_t^n &= \xi_n, \quad \forall t > \tau_n, \\ Z_t^n &= 0, \quad \forall t > \tau_n. \end{cases}$$

Hence $(Z^n) \in \mathcal{M}_{\mathbb{Q}, \mathcal{Y}}^2(0, T'; \mathbb{R}^d)$, and $(Y^n) \in \mathcal{M}_{\mathbb{Q}, \mathcal{Y}}^2(0, T'; \mathbb{R})$, $\forall n \in \mathbb{N}^*$.

Lemma 3.5 (Y^n, Z^n) satisfies the following inequality, for any $\lambda \geq 2\mu + 2K^2 + \frac{3}{2}$,

$$\mathbb{E}_{\mathbb{Q}} \left(e^{\lambda(t \wedge \tau)} |Y_{t \wedge \tau}^n|^2 \right) \leq e^{\lambda T'} E_{\mathbb{Q}} (|\xi|^2) + \mathbb{E}_{\mathbb{Q}} \left(\int_0^{\tau} e^{\lambda s} |f(s, 0, 0)|^2 ds \right),$$

and also

$$\begin{aligned} & \mathbb{E}_{\mathbb{Q}} \left(\sup_{0 \leq t \leq \tau} e^{\lambda t} |Y_t^n|^2 + \int_0^{\tau} e^{\lambda s} |Y_s^n|^2 ds + \int_0^{\tau} e^{\lambda s} \|Z_s^n\|^2 ds \right) \\ & \leq C \mathbb{E}_{\mathbb{Q}} \left(e^{\lambda \tau} |\xi|^2 + \int_0^{\tau} e^{\lambda s} |f(s, 0, 0)|^2 ds \right). \end{aligned}$$

Proof. Let $(Y^n)_{n \in \mathbb{N}^*}$ denote the sequence defined in Equation (4). For the first inequality, applying Itô's formula to $e^{\lambda s} |Y_s^n|^2$ between $t \wedge \tau$ and τ_n as well as using standard inequalities lead to the following inequality, $\forall t \leq T_n$,

$$\begin{aligned} & e^{\lambda(t \wedge \tau)} |Y_{t \wedge \tau}^n|^2 + \int_{t \wedge \tau}^{\tau_n} e^{\lambda s} (\lambda |Y_s^n|^2 + \|Z_s^n\|^2) ds \\ & \leq e^{\lambda \tau_n} |\xi_n|^2 + 2 \int_{t \wedge \tau}^{\tau_n} e^{\lambda s} (\mu |Y_s^n|^2 + K |Y_s^n| \times \|Z_s^n\|) ds - 2 \int_{t \wedge \tau}^{\tau_n} e^{\lambda s} (Y_s^n, Z_s^n dW_s) \\ & \quad + 2 \int_{t \wedge \tau}^{\tau_n} e^{\lambda s} |Y_s^n| \times |f(s, 0, 0)| ds. \end{aligned} \quad (5)$$

So one has

$$\mathbb{E}_{\mathbb{Q}} \left(e^{\lambda(t \wedge \tau)} |Y_{t \wedge \tau}^n|^2 \right) \leq \mathbb{E}_{\mathbb{Q}} \left(e^{\lambda \tau_n} |\xi_n|^2 \right) + \mathbb{E}_{\mathbb{Q}} \left(\int_{t \wedge \tau}^{\tau_n} e^{\lambda s} |f(s, 0, 0)|^2 ds \right). \quad (6)$$

For the second inequality, an application of Burkholder-Davis-Gundy's inequality to equation (5) gives $\forall \eta, \varepsilon > 0$:

$$\begin{aligned} & \mathbb{E}_{\mathbb{Q}} \left[(1 - \eta) \sup_{0 \leq t \leq \tau} e^{\lambda t} |Y_t^n|^2 + \int_0^{\tau_n} e^{\lambda s} \left((\lambda - 2\mu - \varepsilon K^2 - 1) |Y_s^n|^2 + (1 - \frac{1}{\varepsilon}) \|Z_s^n\|^2 \right) ds \right] \\ & \leq \mathbb{E}_{\mathbb{Q}} \left(e^{\lambda \tau_n} |\xi_n|^2 \right) + \mathbb{E}_{\mathbb{Q}} \int_0^{\tau_n} e^{\lambda s} |f(s, 0, 0)|^2 ds + 2C_{BDG}^2 \mathbb{E}_{\mathbb{Q}} \int_0^{\tau_n} e^{\lambda s} \|Z_s^n\|^2 ds. \end{aligned}$$

Taking $\varepsilon = 2$ and $\eta = 1/2$ provides the following :

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}} \left(\sup_{0 \leq t \leq \tau} e^{\lambda t} |Y_t^n|^2 + \int_0^{\tau_n} e^{\lambda s} |Y_s^n|^2 ds + \int_0^{\tau_n} e^{\lambda s} \|Z_s^n\|^2 ds \right) \\ \leq 2\mathbb{E}_{\mathbb{Q}} \left(\left(e^{\lambda \tau_n} |\xi_n|^2 \right) + \mathbb{E}_{\mathbb{Q}} \int_0^{\tau_n} e^{\lambda s} |f(s, 0, 0)|^2 ds + 2C_{BDG}^2 \int_0^{\tau_n} e^{\lambda s} \|Z_s^n\|^2 ds \right). \end{aligned} \quad (7)$$

In order to get an upper bound on the last term, we apply Itô's formula on $e^{\lambda s} |Y_s^n|^2$ and take expectation under \mathbb{Q} :

$$\mathbb{E}_{\mathbb{Q}} \left(\int_0^{\tau_n} e^{\lambda s} \|Z_s^n\|^2 ds \right) \leq 2 \left(\mathbb{E}_{\mathbb{Q}} \left(e^{\lambda \tau_n} |\xi|^2 \right) + \mathbb{E}_{\mathbb{Q}} \left(\int_0^{\tau_n} e^{\lambda s} |f(s, 0, 0)|^2 ds \right) \right).$$

Replace it in inequality (7), and obtain finally :

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}} \left(\sup_{0 \leq t \leq \tau} e^{\lambda t} |Y_t^n|^2 + \int_0^{\tau_n} e^{\lambda s} |Y_s^n|^2 ds + \int_0^{\tau_n} e^{\lambda s} \|Z_s^n\|^2 ds \right) \\ \leq (2 + 8C_{BDG}^2) \left(\mathbb{E}_{\mathbb{Q}} \left(e^{\lambda \tau_n} |\xi_n|^2 \right) + \mathbb{E}_{\mathbb{Q}} \int_0^{\tau_n} e^{\lambda s} |f(s, 0, 0)|^2 ds \right). \end{aligned} \quad (8)$$

□

Under the Brownian filtration, previous hypotheses on the driver f guarantee existence and uniqueness of the solution for $\xi \in L_{\mathbb{P}}^2(\mathcal{F}_{\tau})$ (see E. Pardoux [23], or [24] Theorem 4.1 p. 23). As in the previous section, requiring an additional integration hypothesis under the new probability \mathbb{Q} is sufficient to insure that the BSDE with random terminal time has also a unique solution in the enlarged space $(\Omega, \mathcal{Y}, \mathbb{Q})$.

Theorem 3.6

Suppose that $\tau < T'$ a.s. and $\mathbb{E}_{\mathbb{Q}}(\int_0^T |f(s, 0, 0)|^2 ds) < \infty$. Consider $\xi \in \mathcal{L}_{\mathbb{Q}}^2(\mathcal{Y}_{\tau})$. Then, under the hypotheses (A1), (A2'), (A3) and (A4) on f ,

1. the BSDE

$$Y_{t \wedge \tau} = \xi + \int_{t \wedge \tau}^{\tau} f(s, Y_s, Z_s) ds - \int_{t \wedge \tau}^{\tau} (Z_s, dW_s), \quad \forall 0 \leq t \leq T'$$

has a $(\Omega, \mathcal{Y}, \mathbb{Q})$ -solution $(Y_t, Z_t)_{0 \leq t \leq T'}$, satisfying for any $\lambda \geq 2\mu + 2K^2 + \frac{3}{2}$

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}} \left(\sup_{0 \leq t \leq \tau} e^{\lambda t} |Y_t|^2 + \int_0^{\tau} e^{\lambda t} |Y_t|^2 dt + \int_0^{\tau} e^{\lambda t} \|Z_t\|^2 dt \right) \\ \leq C \mathbb{E}_{\mathbb{Q}} \left(e^{\lambda \tau} |\xi|^2 + \int_0^{\tau} e^{\lambda t} |f(t, 0, 0)|^2 dt \right). \end{aligned} \quad (9)$$

2. $(Y_t, Z_t)_{0 \leq t \leq T'}$ is the unique $(\Omega, \mathcal{Y}, \mathbb{Q})$ -solution of (3), satisfying (9).

Proof. Proof is given in Appendix. □

3.3 BSDEs with jumps

For simplicity reasons in the proofs, we have chosen to develop these Theorems only for BSDEs driven by a Brownian motion. Nevertheless, these results can easily be extended to the case of BSDEs driven by both a Brownian motion and a Poisson measure, as same results have been proved by A. Eyraud-Loisel [11] in the case of a fixed terminal time. Proofs are similar.

4 Financial Interpretation

4.1 Hedging by an informed agent

We supposed on the beginning a no-arbitrage market, with in particular σ invertible. This allows us to deduce from the solution (Y_t, Z_t) of the BSDE strategy π_t of the portfolio in the risky assets.

The main consequence of Theorem 3.6 is that an informed agent will have a unique hedging strategy for any contingent claim which satisfies the corresponding hypotheses. In particular, this applies to any claim for which the exercise time is a bounded stopping time (maturity strictly bounded by T' , time at which the private information is revealed). This is very often the case for most financial contingent claims traded in the market. It is presented in the following paragraph for American and Lookback options. This extends the financial conclusions of A. Eyraud-Loisel (2005) [11].

The obtained results mean that information $L \in \mathcal{F}_{T'}$ satisfying (\mathbf{H}_3) does not provide any additional hedging strategy to the informed agent. Actually, for a $\xi \in \mathcal{F}_\tau$, such asymmetrically informed agents will have a unique admissible hedging strategy, respectively \mathcal{F} and \mathcal{Y} -adapted. As the BSDE is the same for both agents, a \mathcal{F} -adapted solution for the non informed agent is also a solution of the enlarged BSDE (as it is also adapted to the enlarged filtration), so it is the unique solution. In other words, the strategy of the informed agent is adapted to the small filtration \mathcal{F} . Then both agents have the same hedging strategy.

Remark : Let us notice that obtaining this result only strictly until T' is not very surprising. Indeed, results can not be generalized until T' . This is mathematically due to the enlargement of filtration hypotheses, which can hold only strictly until T' . Financially, it is quite easy to find some examples of contingent claims with terminal time T' , where an information on time T' provides a different hedging strategy. Let us take for instance a digital option, $\mathbb{1}_{S_{T'} \leq K}$ and suppose that the additional information is given by $L = S_{T'}$ or even $\mathbb{1}_{S_{T'} \in [a, b]}$. Then, the insider trader will hedge this option by doing nothing if $S_{T'} > K$ and investing in the non risky asset otherwise, which is different from what would do an ordinary non informed agent.

4.2 Examples

Most of contingent claims traded in the market do not have one only fixed exercise time (as European options), but rather have an exercise period (as for American or American-type Lookback options for instance). Therefore the terminal time of the hedging problem is a random stopping time. So it is interesting to generalize existing results on BSDE with enlarged filtration for fixed terminal time to the case of random terminal time. In the particular case of standard American and American-type Lookback options, driver f satisfies the required hypotheses of Theorems 3.2 and 3.6.

4.2.1 American options

If we set the financial problem of hedging an standard American-type vanilla option with maturity $T < T'$, when the exercise time τ may occur at any time before T , the obtained BSDE is a BSDE with random terminal time. In an enlarged filtration, we can solve the BSDE as stated in the previous results (Theorem 3.2 as $\tau \leq T < T'$).

The generator is in this case the following :

$$f(s, y, z) = -r_s y - \frac{b_s - r_s}{\sigma_s} z. \quad (10)$$

And the payoff has the form $\xi = (S_\tau - K_0)_+$ for a call and $(K_0 - S_\tau)_+$ for a put. In order to satisfy the hypotheses required in Theorem 3.2, we have to check some properties on coefficients: **(A1)** needs $\left(\sup_{s \geq 0} \frac{|b_s - r_s|}{\sigma_s}\right)$ to be finite ; **(A2')** is verified with setting $K = \left(\sup_{s \geq 0} r_s\right) \vee \left(\sup_{s \geq 0} \frac{|b_s - r_s|}{\sigma_s}\right)$; for **(A3)** we set $\mu = \left(\sup_{s \geq 0} (-r_s)\right)$; and **(A4)** is satisfied since $f(t, 0, 0) = 0$.

As a conclusion, $\xi \in \mathcal{L}^2(\mathcal{F}_\tau)$, $\left(\sup_{s \geq 0} (-r_s)\right)$ and $\left(\sup_{s \geq 0} \frac{|b_s - r_s|}{\sigma_s}\right)$ finite are sufficient conditions to ensure existence and uniqueness of an hedging portfolio for an American option with or without an additional information satisfying hypothesis **(H3)**.

Remark : In this financial application, $f(t, 0, 0) = 0$. Consequently, we can write in this case (see inequality (9) for instance) :

$$\mathbb{E}_{\mathbb{Q}} \left(e^{\lambda t} |Y_t|^2 \right) \leq e^{\lambda T'} E_{\mathbb{Q}} (|\xi|^2)$$

$$\mathbb{E}_{\mathbb{Q}} \left(\sup_{0 \leq t \leq \tau} e^{\lambda t} |Y_t|^2 \right) + \mathbb{E}_{\mathbb{Q}} \left(\int_0^\tau e^{\lambda t} |Y_t|^2 dt + \int_0^\tau e^{\lambda t} \|Z_t\|^2 dt \right) \leq C \mathbb{E}_{\mathbb{Q}} \left(e^{\lambda \tau} |\xi|^2 \right),$$

which provides in particular an upper bound on the expected wealth process and the expected maximum wealth process.

4.2.2 Simulation

Let us restrict to the one-dimensional case and consider constant values for parameters r, b, σ . Then, on time interval $[0, T']$, assets are given by

$$\begin{aligned} S_t^0 &= e^{rt}, \\ S_t^1 &= S_0^1 e^{\left(b - \frac{\sigma^2}{2}\right)t} e^{\sigma W_t}. \end{aligned}$$

We study a put with exercise time τ and payoff $(K_0 - S_\tau^1)_+$. We recall that τ does not need to be the optimal time. We set for example $\tau = \inf_{0 \leq t \leq T'} \{t \mid S_t^1 \notin [a_1, a_2]\}$, where $T < T'$ and $a_1 < S_0^1 \leq K_0 < a_2$.

We consider the related BSDE

$$\begin{cases} Y_{t \wedge \tau} &= Y_{T' \wedge \tau} - \int_{t \wedge \tau}^{T' \wedge \tau} (r Y_s + \frac{b-r}{\sigma} Z_s) ds - \int_{t \wedge \tau}^{T' \wedge \tau} Z_s dW_s \\ Y_\tau &= (K_0 - S_\tau^1)_+. \end{cases}$$

Then Y_t is the market value at time t whereas $\pi_t = \frac{Z_t}{\sigma}$ represents the amount of wealth invested in the risky asset. $(\pi_t)_{0 \leq t \leq T'}$ is the strategy common to both informed and non informed agents.

[26], [24], [6] and [28] proved that solutions of BSDEs with a random terminal time are

strongly linked to the viscosity solutions of associated elliptic PDEs. In this particular case, the related PDE is an ordinary differential equation :

$$\begin{cases} 0 &= u(x) - u'(x) - \frac{\sigma^2}{2r} u''(x) & \text{when } x \in [a_1, a_2] \\ u(x) &= K_0 - x & \text{when } x = a_1 \text{ or } a_2. \end{cases}$$

Its solution can be explicitly computed :

$$u(x) = \lambda_1 e^{\alpha_1 x} + \lambda_2 e^{\alpha_2 x}, \quad \text{where}$$

$$\begin{cases} \alpha_1 &= \frac{1 + \sqrt{1 + \frac{2\sigma^2}{r}}}{2} \\ \alpha_2 &= \frac{1 - \sqrt{1 + \frac{2\sigma^2}{r}}}{2} \\ \lambda_1 &= \frac{(K_0 - a_1) e^{a_2 \alpha_2} - (K_0 - a_2) e^{a_1 \alpha_2}}{e^{a_1 \alpha_1 + a_2 \alpha_2} - e^{a_2 \alpha_1 + a_1 \alpha_2}} \\ \lambda_2 &= \frac{-(K_0 - a_1) e^{a_2 \alpha_1} + (K_0 - a_2) e^{a_1 \alpha_1}}{e^{a_1 \alpha_1 + a_2 \alpha_2} - e^{a_2 \alpha_1 + a_1 \alpha_2}}. \end{cases}$$

In Lemma 4.4 of [28], it is proved that, when τ is the exit time of a bounded set,

$$Y_{t \wedge \tau} = u(S_{t \wedge \tau}^1), \quad \text{for } 0 \leq t \leq T'.$$

Moreover, applying Itô's formula to $u(X_t)$ and identifying the solution by uniqueness, we obtain that

$$\begin{cases} Z_t &= \sigma u'(S_t^1) & \text{if } t \leq \tau, \\ Z_t &= 0 & \text{if } \tau < t \leq T'. \end{cases}$$

We performed simulations with the following parameters :

S_0^1	K_0	a_1	a_2	T	T'	b	r	σ
90	90	85	95	5	6	0.025	0.02	0.3

Time parameters are given in month, whereas prices are given in euro.

4.2.3 Lookback options

Another example is American-style Lookback option: an option with maturity $T < T'$ which pays at exercise the supremum of the price process among the last 2 months before exercise. So ξ has mathematically the form: $\xi = \sup_{t \in [\tau-2, \tau]} S_t$.

Applying Itô and Burkholder-Davis-Gundy on the SDE of the price process provides that $\xi \in \mathcal{L}^2(\mathcal{F}_\tau)$. Besides, the generator f is the same as in the equation (10). So with the same conditions on the parameters, we can apply Theorem 3.2 or 3.6 to this case of Lookback options also.

4.2.4 Special contingent claim

Theorem 3.6 gives a mathematical generalization to Theorem 3.2, but does not seem to have additional financial applications taking examples into existing products on the market.

Although, as options, or more generally contingent claims, are financial contracts where both parts state the conditions at the beginning, one can easily imagine a simple American-type contingent claim where the exercise period would be specified as "any time before T' , but strictly before". Then the question of having an additional information on prices at time T' does or does not give an additional hedging strategy would be more accurate. Theorem 3.6 gives the answer, stating that it does not differ from the previous case: there is no additional strategy for the informed agent.

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5 Special Remark

K. Akdim and M. El Otmani's article [1] entitled "*Studying anticipation on financial markets by BSDE with random terminal time*" is mostly inspired, and even copied from the preprint of the present article, referred here by its preprint [12] entitled "BSDE with random terminal time under Enlarged Filtration, and Financial Applications". It rewrites results contained in the last chapter of A. Eyraud-Loisel's PhD Thesis [?] defended in December 2005 constituted by an article co-written with Manuela Royer, which has been in submission processes since December 2005, but which was published in 2006 in ISFA working papers WP 2034 (2006) [12]. Their paper rewrites all of our results as if they were new, using the same arguments and the same structure, **without even citing our previous work**, whereas they knew it precisely : in fact we sent them our working paper by email several months before their submission, upon their request.

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Appendix: Proof of Theorem 3.6

Existence

Step 1. Consider $m > n$, and define $\Delta Y_t = Y_t^m - Y_t^n$, $\Delta Z_t = Z_t^m - Z_t^n$. Let $\lambda \geq 2\mu + 2K^2 + \frac{3}{2}$.

First prove that $(Y^n)_{n \in \mathbb{N}^*}$ (respectively $(Z^n)_{n \in \mathbb{N}^*}$) is a Cauchy sequence in $\mathcal{M}_{\mathbb{Q}, \mathcal{Y}}^2(0, T'; \mathbb{R})$ (resp. in $\mathcal{M}_{\mathbb{Q}, \mathcal{Y}}^2(0, T'; \mathbb{R}^d)$). They will be convergent in the same spaces.

- Consider $t \leq T_n$. Applying Itô's formula to $e^{\lambda t} |\Delta Y_t|^2$ between $t \wedge \tau$ and $\tau_n = T_n \wedge \tau$ leads to

$$\begin{aligned} & \mathbb{E}_{\mathbb{Q}} \left(e^{\lambda(t \wedge \tau)} |\Delta Y_{t \wedge \tau}|^2 \right) + \mathbb{E}_{\mathbb{Q}} \int_{t \wedge \tau}^{\tau_n} e^{\lambda s} (\lambda |\Delta Y_s|^2 + \|\Delta Z_s\|^2) ds \\ & \leq \mathbb{E}_{\mathbb{Q}} \left(e^{\lambda \tau_n} (|\xi_m - \xi_n|^2) \right) + 2\mathbb{E}_{\mathbb{Q}} \int_{t \wedge \tau}^{\tau_n} e^{\lambda s} (\mu |\Delta Y_s|^2 + K |\Delta Y_s| \times \|\Delta Z_s\|) ds . \end{aligned} \quad (11)$$

As $\lambda \geq 2\mu + K^2$, it follows

$$\mathbb{E}_{\mathbb{Q}} \left(e^{\lambda(t \wedge \tau)} |\Delta Y_{t \wedge \tau}|^2 \right) \leq e^{\lambda T'} \mathbb{E}_{\mathbb{Q}} (|\xi_m - \xi_n|^2) = \mathbb{E}_{\mathbb{Q}} (|\xi|^2 \mathbb{1}_{T_n < \tau \leq T_m}) .$$

As $\tau < T'$ a.s., the right hand term tends to 0 as n goes to infinity according to dominated convergence Theorem. Hence $\Delta Y_{t \wedge \tau}$ goes to 0 in $\mathcal{L}_{\mathbb{Q}}^2(\mathcal{Y}_{\tau})$ as n tends to infinity.

- Consider now the case when $t \in [T_n, T_m]$. Then $Y_t^n = \xi_n$, and

$$\Delta Y_t = \xi_m - \xi_n + \int_{t \wedge \tau}^{\tau_m} f(s, Y_s^m, Z_s^m) ds - \int_{t \wedge \tau}^{\tau_m} (Z_s^m, dW_s).$$

Apply Itô's formula to $e^{\lambda t} |\Delta Y_t|^2$ between $t \wedge \tau$ and τ_m .

Notice that $Z_s^n = 0 \forall s \geq T_n$, and $Z_s^m = \Delta Z_s$ for $s \geq t$.

It follows :

$$\begin{aligned} & e^{\lambda(t \wedge \tau)} |\Delta Y_{t \wedge \tau}|^2 + \int_{t \wedge \tau}^{\tau_m} e^{\lambda s} (\lambda |\Delta Y_s|^2 + \|\Delta Z_s\|^2) ds \\ & \leq e^{\lambda \tau_m} |\xi_m - \xi_n|^2 + 2 \int_{t \wedge \tau}^{\tau_m} e^{\lambda s} (\mu |\Delta Y_s|^2 + K |\Delta Y_s| \times \|\Delta Z_s\|) ds \\ & \quad + 2 \int_{t \wedge \tau}^{\tau_m} e^{\lambda s} |\Delta Y_s| \times |f(s, \xi_n, 0)| ds - 2 \int_{t \wedge \tau}^{\tau_m} e^{\lambda s} (\Delta Y_s, \Delta Z_s dW_s) . \end{aligned} \quad (12)$$

Taking \mathbb{Q} -expectation of the previous inequality, noticing that martingale term has expectation 0, and using standard inequalities gives the following :

$$\begin{aligned} & \mathbb{E}_{\mathbb{Q}} \left(e^{\lambda(t \wedge \tau)} |\Delta Y_{t \wedge \tau}|^2 \right) \\ & \leq e^{\lambda T'} \mathbb{E}_{\mathbb{Q}} (|\xi_m - \xi_n|^2) + \mathbb{E}_{\mathbb{Q}} \left(\int_{\tau_n}^{\tau_m} e^{\lambda s} |f(s, \xi_n, 0)|^2 ds \right) , \end{aligned} \quad (13)$$

as $\lambda \geq 2\mu + K^2 + 1$.

Besides, from assertion **(A2')**,

$$|f(s, \xi_n, 0)|^2 \leq 2|f(t, 0, 0)|^2 + 2K|\xi_n|^2 \leq 2|f(t, 0, 0)|^2 + 2K|\xi|^2 .$$

Consequently, hypotheses of Theorem 3.6 provide the \mathbb{Q} -integrability of $e^{\lambda s} |f(s, \xi_n, 0)|^2 ds$. Moreover, given that $\tau_m - \tau_n$ tends to zero \mathbb{Q} a.s. when n goes to infinity, the right term of (13) tends to zero as n tends to infinity. This proves that ΔY_t (respectively ΔZ_t) also tends to zero in $\mathcal{L}_{\mathbb{Q}}^2(\mathcal{Y}_\tau)$ (resp. in $\mathcal{M}_{\mathbb{Q}, \mathcal{Y}}^2(0, T'; \mathbb{R}^d)$) by taking the limit as n tends to infinity in inequality (12).

- Finally consider the case when $t \in [T_m, T']$. As said previously, $Y_t^m - Y_t^n = \xi_m - \xi_n$ converges \mathbb{Q} a.s. to zero as n tends to infinity, so in this case ΔY_t also goes to 0 in \mathbb{Q} -probability. Hence $(Y_t^n)_{n \geq 0}$ is a Cauchy sequence in \mathbb{Q} -probability.

- To conclude the first step, a-priori estimates proved in Lemma 3.5 are needed. It provides an upper bound of the expected wealth process Y under probability \mathbb{Q} , whenever $\xi \in \mathcal{L}_{\mathbb{Q}}^2(\mathcal{Y}_\tau)$. Lemma 3.5 establishes the expected inequalities only for (Y^n, Z^n) . Identical results for the solution has to be obtained by taking the limit. For this purpose, taking conditional expectation of equation (5) leads to :

$$e^{\lambda(t \wedge \tau)} |Y_{t \wedge \tau}^n|^2 \leq e^{\lambda T'} \mathbb{E}_{\mathbb{Q}} \left(|\xi|^2 + \int_0^{T'} |f(s, 0, 0)|^2 ds \middle| \mathcal{Y}_{t \wedge \tau} \right).$$

This proves that $|Y^n|^2$ is dominated by a \mathbb{Q} -integrable process independent of n , and the same is true for $\sup_{0 \leq t \leq \tau} e^{\lambda t} |Y_t^n|^2$. It yields also to a domination of $|\Delta Y|^2$. Applying dominated convergence Theorem, it implies that Y^n is a Cauchy sequence in $\mathcal{M}_{\mathbb{Q}, \mathcal{Y}}^2(0, T'; \mathbb{R})$. Taking the limit in equation (11) proves that the same holds for ΔZ_t , and that $(Z_t^n)_{n \geq 0}$ is also a Cauchy sequence in $\mathcal{M}_{\mathbb{Q}, \mathcal{Y}}^2(0, T'; \mathbb{R}^d)$. Define then (Y, Z) as the limit of the sequence $(Y^n, Z^n)_{n \geq 0}$.

Step 2. Study of the limit (Y, Z) .

First prove that it satisfies the BSDE (3).

Taking the limit in equation (6) as n tends to infinity, we obtain finally

$$\mathbb{E}_{\mathbb{Q}} \left(e^{\lambda(t \wedge \tau)} |Y_{t \wedge \tau}|^2 \right) \leq e^{\lambda T'} E_{\mathbb{Q}} (|\xi|^2) + \mathbb{E}_{\mathbb{Q}} \left(\int_0^\tau e^{\lambda s} |f(s, 0, 0)|^2 ds \right),$$

which provides the first inequality.

Taking the limit as n goes to infinity in equation (8) leads to :

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}} \left(\sup_{0 \leq t \leq \tau} e^{\lambda t} |Y_t|^2 + \int_0^\tau e^{\lambda s} |Y_s|^2 ds + \int_0^\tau e^{\lambda s} \|Z_s\|^2 ds \right) \\ \leq (2 + 8C_{BDG}^2) \left(\mathbb{E}_{\mathbb{Q}} \left(e^{\lambda \tau} |\xi|^2 \right) + \mathbb{E}_{\mathbb{Q}} \int_0^\tau e^{\lambda s} |f(s, 0, 0)|^2 ds \right). \end{aligned}$$

As process (Y, Z) is defined as the limit of the sequence $(Y^n, Z^n)_{n \geq 0}$, this process satisfies equation (3). So it is a solution of the BSDE (f, ξ, τ) , and it satisfies Equation (9): existence is proved.

Uniqueness

Let (Y, Z) and (Y', Z') be two solutions, which satisfy equations (3) and (9). Let $(\bar{Y}, \bar{Z}) = (Y - Y', Z - Z')$. It follows from Itô's formula applied between $t \wedge \tau$ and $\tau \wedge T_n = \tau_n$, and from assumptions (A1), (A2'), (A3) and (A4) that, for all $\lambda \in \mathbb{R}$

and for $t \leq T_n$,

$$\begin{aligned}
& e^{\lambda(t \wedge \tau)} |\bar{Y}_{t \wedge \tau}|^2 + \int_{t \wedge \tau}^{\tau_n} e^{\lambda s} (\lambda |\bar{Y}_s|^2 + \|\bar{Z}_s\|^2) ds \\
& \leq e^{\lambda \tau_n} |\bar{Y}_{\tau_n}|^2 + 2 \int_{t \wedge \tau}^{\tau_n} e^{\lambda s} (\mu |\bar{Y}_s|^2 + K |\bar{Y}_s| \times \|\bar{Z}_s\|) ds \\
& \quad - 2 \int_{t \wedge \tau}^{\tau_n} e^{\lambda s} (\bar{Y}_s, \bar{Z}_s dW_s). \tag{14}
\end{aligned}$$

Combining the above inequality with

$$2K |\bar{Y}_s| \times \|\bar{Z}_s\| \leq \|\bar{Z}_s\|^2 + K^2 |\bar{Y}_s|^2,$$

it implies

$$e^{\lambda(t \wedge \tau)} |\bar{Y}_{t \wedge \tau}|^2 \leq e^{\lambda \tau_n} |\bar{Y}_{\tau_n}|^2 + (2\mu + K^2 - \lambda) \int_{t \wedge \tau}^{\tau_n} e^{\lambda s} |\bar{Y}_s|^2 ds - 2 \int_{t \wedge \tau}^{\tau_n} e^{\lambda s} (\bar{Y}_s, \bar{Z}_s dW_s).$$

As $(W_s)_{0 \leq s < T'}$ is a $(\mathcal{Y}, \mathbb{Q})$ -Brownian motion, the last term is a $(\mathcal{Y}, \mathbb{Q})$ -martingale on $[0, T_n]$. Since it is null at T_n , its expectation under \mathbb{Q} is null. Hence

$$E_{\mathbb{Q}} \left(e^{\lambda(t \wedge \tau)} |\bar{Y}_{t \wedge \tau}|^2 \right) \leq E_{\mathbb{Q}} \left(e^{\lambda \tau_n} |\bar{Y}_{\tau_n}|^2 \right) \quad \forall t \in [0, T_n].$$

$e^{\lambda \tau_n} |\bar{Y}_{\tau_n}|^2$ is dominated by $e^{\lambda T'} (2 \sup_{0 \leq t \leq T'} |Y_t|^2 + 2 \sup_{0 \leq t \leq T'} |Y'_t|^2)$, which is \mathbb{Q} -integrable (by definition of a solution). By dominated convergence Theorem, since τ_n goes to τ , and T_n goes to T' as n goes to infinity, we deduce

$$E_{\mathbb{Q}} \left(e^{\lambda(t \wedge \tau)} |\bar{Y}_{t \wedge \tau}|^2 \right) \leq E_{\mathbb{Q}} \left(e^{\lambda \tau} |\bar{Y}_{\tau}|^2 \right).$$

Given that $Y_{\tau} = Y'_{\tau} = \xi$, it is obvious then that $\bar{Y}_{\tau} = 0$. This proves that $\forall t \leq \tau$, $\bar{Y}_t = 0$, \mathbb{Q} a.s. It is also true for $t \geq \tau$, as $Y_t = Y'_t = \xi$ by Definition 3.4. Then $\bar{Y}_t = 0$ $\forall t \in [0, T']$ ($Y_{T'} = Y'_{T'} = \xi$ as $\tau < T'$ a.s.; so this also holds for T').

Replace \bar{Y}_t by 0 in equation (14), and obtain $\bar{Z}_t = 0$ $\forall t \in [0, T']$, \mathbb{Q} a.s. Uniqueness is proved, and this ends the proof of Theorem 3.6.