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# Generalizations of Bounds on the Index of Convergence to Weighted Digraphs\*

Glenn Merlet<sup>1</sup>

Thomas Nowak<sup>2</sup>

Hans Schneider<sup>3</sup>

Sergeï Sergeev<sup>4</sup>

**Abstract**—Sequences of maximum-weight walks of a growing length in weighted digraphs have many applications in manufacturing and transportation systems, as they encode important performance parameters. It is well-known that they eventually enter a periodic regime if the digraph is strongly connected. The length of their transient phase depends, in general, both on the size of digraph and on the magnitude of the weights. In this paper, we show that certain bounds on the transients of unweighted digraphs, such as the bounds of Wielandt, Dulmage-Mendelsohn, Schwarz, Kim and Gregory-Kirkland-Pullman, remain true for critical nodes in weighted digraphs.

**Index Terms**—maximum walks; max algebra; nonnegative matrices; matrix powers; index of convergence; weighted digraphs

## I. INTRODUCTION

The study of the long-run behavior of maximum weight walks in weighted digraphs has numerous applications [1] in the analysis of transportation systems, production plants, network synchronizers, cyclic scheduling, as well as certain distributed algorithms for routing and resource allocation. More generally, it exactly corresponds to discrete event graphs and one-player mean payoff games. In all of these applications, knowledge of the system's long-run behavior is of utmost importance. It is well-known that all these systems enter a periodic regime after an initial transient phase if the digraph describing the system is strongly connected. The exact performance parameters of the periodic regime, including the period length and the linear defect, which is equal to the system's common limit average, are generally well-understood. Less is known about the initial transient phase, even though it encompasses important performance parameters for certain systems. For example, it is exactly equal to the termination time of the Full Reversal algorithm for message routing in computer networks [7].

The transient was studied by several authors, including Hartmann and Arguelles [12], Bouillard and Gaujal [3], Soto y Koelemeijer [23], Akian et al. [2, Section 7], and Charron-Bost et al. [6]. They gave upper-bounds on the transient of

the whole system, i.e., the maximum of the transients of all edges (transitions) of the system. The bounds established in the works mentioned above were systematized and improved in a more recent work of Merlet et al. [15]. This begs the question which parts of the system are likely to have small resp. large local transients. The main idea here is as follows. The transient of the whole system, or in other words, the transient at a general node depends on the magnitudes of the weights associated with relevant connections (being time lags in some applications), and such transient can be arbitrarily big even for the systems with just two nodes [8]. The bounds on such transient, given in the above mentioned works, can be rather complicated since they involve both the dimension of the system and the magnitudes of weights. However, this is not true for the so-called critical nodes. For these nodes, the present work (being a short version of [15]) presents new bounds that depend only on the system dimension as well as some graph-theoretic parameters.

More precisely, it is shown that six known bounds for the index of convergence, i.e., transient, of *unweighted* digraphs also apply to weighted digraphs, namely to the transients at *critical* nodes. Critical nodes are those that are included in a cycle of maximum mean weight (a more precise definition given in Section II.B).

To the authors' knowledge, this note (being a short version of [15]) presents the first genuine extensions of the transience bounds for unweighted graphs to the weighted case. That is, the known bounds for unweighted graphs are recovered when specializing the bounds of this work to unweighted digraphs.

All other known bounds on the transients of weighted digraphs apply to the general transient of a system. They usually appear as the maximum of two expressions, the first of them being somewhat similar to the bounds discussed in this paper (although in general higher than at least one of them), and the second usually more complicated and depending on the magnitude of the weights. For instance, see [16]. These general bounds are higher and hence less precise than those that will be presented in this note. In turn, the bounds presented in this note do not apply to the whole system but only to its "well-behaved" critical part.

The origin of the first of the bounds discussed in this note lies in Wielandt's well-known paper [24] where an upper bound for the exponent of a primitive nonnegative matrix was asserted without proof<sup>5</sup>. Dulmage and Mendelsohn [10] provided a proof of this result by interpreting it in terms of digraphs and they sharpened the result by using as additional

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<sup>1</sup>Glenn Merlet is with the Université d'Aix-Marseille, CNRS, IML, Marseille, France. Email: glenn.merlet@gmail.com

<sup>2</sup>Thomas Nowak is with the Département d'Informatique, École normale supérieure, Paris, France. Email: thomas.nowak@ens.fr

<sup>3</sup>Hans Schneider is with the University of Wisconsin-Madison, USA. Email: hans@math.wisc.edu

<sup>4</sup>Sergeï Sergeev is with the University of Birmingham, School of Mathematics, UK. Email: sergiej@gmail.com

<sup>5</sup>Wielandt's proof was published later in [18].

information in the hypotheses the length of the smallest cycle of the digraph<sup>6</sup>. Schwarz [19] generalized Wielandt's result to apply to all strongly connected digraphs by using Wielandt's bound for the cyclicity classes of the digraph, see also Shao and Li [20]. Kim's [13] bound encompasses the first three and can be proved using Dulmage and Mendelsohn's bound in the cyclicity classes. Two other bounds generalized in this paper are the one due to Kim [13], and the one established by Gregory-Kirkland-Pullman [11], both based on the concept of Boolean rank.

The six bounds mentioned above are stated in Theorem 2.2 and Theorem 2.4 after the requisite definitions. The main results of this paper and [15] are the generalizations of these bounds to weighted digraphs, as stated in Main Theorem 1 and Main Theorem 2. The proofs of these main results are sketched in Section III. For the full proofs, the reader is referred to [15]. Section IV is to give a brief summary of this work and to sketch a couple of directions for further research.

## II. PRELIMINARIES AND STATEMENT OF RESULTS

### A. Digraphs, walks, and transients

A walk in a digraph  $\mathcal{G} = (N, E)$  is a sequence  $W = (i_0, i_1, \dots, i_t)$  of successive nodes in  $\mathcal{G}$ . We denote the length of walk  $W$  by  $\ell(W)$ . A cycle is a closed walk in which no node except the start and the end node appear more than once. A path is a walk in which no node appears more than once. A walk is empty if its length is 0.

To a digraph  $\mathcal{G} = (N, E)$  with  $N = \{1, \dots, n\}$ , we associate its adjacency matrix, which is the Boolean matrix  $A = (a_{i,j}) \in \mathbb{B}^{n \times n}$  defined by

$$a_{i,j} = \begin{cases} 0 & \text{if } (i, j) \notin E \\ 1 & \text{if } (i, j) \in E \end{cases} \quad (1)$$

Conversely, one can associate a digraph to every square Boolean matrix. The connectivity in  $\mathcal{G}$  is closely related to the Boolean matrix powers of  $A$ . By the *Boolean algebra* we mean the set  $\mathbb{B} = \{0, 1\}$  equipped with the logical operations of conjunction  $a \wedge b = a \otimes b == a \cdot b$  and disjunction  $a \vee b = a \oplus b = \max(a, b)$ , for  $a, b \in \mathbb{B}$ . The Boolean multiplication of two matrices  $A \in \mathbb{B}^{m \times n}$  and  $B \in \mathbb{B}^{n \times q}$  is defined by  $(A \otimes B)_{i,j} = \bigvee_{k=1}^n (a_{i,k} \wedge b_{k,j})$ , and then we also have Boolean matrix powers  $A^{\otimes t} = \underbrace{A \otimes \dots \otimes A}_{t \text{ times}}$ . The

$(i, j)$ th entry of  $A^{\otimes t}$  is denoted by  $a_{i,j}^{(t)}$ .

The relation between Boolean powers of  $A$  and connectivity in  $\mathcal{G}$  is based on the following fact:  $a_{i,j}^{(t)} = 1$  if and only if  $\mathcal{G}$  contains a walk of length  $t$  from  $i$  to  $j$ .

Let  $\mathcal{G}$  be a digraph with associated matrix  $A \in \mathbb{B}^{n \times n}$ . The sequence of Boolean matrix powers  $A^{\otimes t}$  is eventually periodic, that is, there exists a positive  $p$  such that

$$A^{\otimes(t+p)} = A^{\otimes t} \quad (2)$$

for all  $t$  large enough. Call each such  $p$  an *eventual period*. The set of nonnegative  $t$  satisfying (2) is the same for all

<sup>6</sup>Denardo [9] later rediscovered their result.

eventual periods  $p$ . We call the least such  $t$  the *transient (of periodicity) of  $\mathcal{G}$* ; we denote it by  $T(\mathcal{G})$ . See [4] for general introduction to the theory of digraphs and [14] for a survey on their transients.<sup>7</sup>

The digraph associated with  $A^{\otimes t}$  will be further denoted by  $\mathcal{G}^t$ . Such graphs will be further referred to as the *powers of  $\mathcal{G}$* .

For a strongly connected digraph  $\mathcal{G}$ , its *cyclicity* is defined as the greatest common divisor of the lengths of all cycles of  $\mathcal{G}$ . The cyclicity  $d$  of  $\mathcal{G}$  can be equivalently defined as the least eventual period  $p$  in (2). If  $d = 1$ , then  $\mathcal{G}$  is called *primitive*, otherwise it is called *imprimitive*. Let us recall the following basic observation from [4]. We denote the greatest common divisor of  $a$  and  $b$  by  $\gcd(a, b)$ .

*Theorem 2.1 ([4, Theorem 3.4.5]):* Let  $\mathcal{G}$  be a strongly connected graph with cyclicity  $d$ . For each  $k \geq 1$ , graph  $\mathcal{G}^k$  consists of  $\gcd(k, d)$  isolated strongly connected components, and every component has cyclicity  $d/\gcd(k, d)$ .

In particular,  $\mathcal{G}^d$  has exactly  $d$  strongly connected components, each of cyclicity 1. The node sets of these components are called the *cyclicity classes* of  $\mathcal{G}$ . In terms of walks, nodes  $i$  and  $j$  belong to the same cyclicity class if and only if there is a walk from  $i$  to  $j$  whose length is a multiple of  $d$ . More generally, for each  $i$  and  $j$  there is a number  $s : 0 \leq s \leq d - 1$  such that the length of every walk connecting  $i$  to  $j$  is congruent to  $s$  modulo  $d$ . This observation defines the *circuit of cyclic classes*, being crucial for the description of  $\mathcal{G}^t$  in the periodic regime.

We will be interested in the following bounds on  $T(\mathcal{G})$ . Prior to the formulation, let us introduce the *Wielandt number*

$$\text{Wi}(n) = \begin{cases} 0 & \text{if } n = 1 \\ (n-1)^2 + 1 & \text{if } n > 1 \end{cases} \quad (3)$$

in honor of the first paper on the subject by Wielandt [24].

We denote the number of nodes of a digraph  $\mathcal{G}$  by  $|\mathcal{G}|$ . We also use the *girth* of  $\mathcal{G}$ , which is the smallest length of a nonempty cycle in  $\mathcal{G}$ , and denote it by  $g(\mathcal{G})$ .

*Theorem 2.2:* Let  $\mathcal{G}$  be a strongly connected digraph with  $n$  nodes, cyclicity  $d$ , and girth  $g$ . The following upper bounds on the transient of  $\mathcal{G}$  hold:

- (i) (Wielandt [24], [18]) If  $d = 1$ , then  $T(\mathcal{G}) \leq \text{Wi}(n)$ ;
- (ii) (Dulmage-Mendelsohn [10]) If  $d = 1$ , then  $T(\mathcal{G}) \leq (n-2) \cdot g + n$ ;
- (iii) (Schwarz [19], [20])  $T(\mathcal{G}) \leq d \cdot \text{Wi}\left(\left\lfloor \frac{n}{d} \right\rfloor\right) + (n \bmod d)$ ;
- (iv) (Kim [13])  $T(\mathcal{G}) \leq \left(\left\lfloor \frac{n}{d} \right\rfloor - 2\right) \cdot g + n$ .

*Remark 2.3:* The bound of Kim can be shown to imply the other three bounds in Theorem 2.2.

There are improvements of Theorem 2.2 in terms of the *factor rank* of a matrix  $A \in \mathbb{B}^{n \times n}$  (also known as the Boolean rank or Schein rank). Factor rank of  $A$  is the least

<sup>7</sup>In the literature,  $T(\mathcal{G})$  is often called the index of convergence, or its exponent if  $\mathcal{G}$  is primitive.

number  $r$  such that

$$A = \bigoplus_{\alpha=1}^r x_{\alpha} \otimes y_{\alpha}^T \quad (4)$$

with Boolean vectors  $x_1, y_1, \dots, x_r, y_r \in \mathbb{B}^n$ . The factor rank of  $A$  is at most  $n$  since (4) holds when choosing  $r = n$  and the  $y_{\alpha}$  to be the unit vectors.

The following bounds involving the factor rank were established:

*Theorem 2.4:* Let  $\mathcal{G}$  be a strongly connected primitive digraph with girth  $g$ , and let the associated matrix of  $\mathcal{G}$  have factor rank  $r$ . The following upper bounds on the index of convergence of  $\mathcal{G}$  hold:

- (i) (Gregory-Kirkland-Pullman [11])  $T(\mathcal{G}) \leq \text{Wi}(r) + 1$ ;
- (ii) (Kim [13])  $T(\mathcal{G}) \leq (r - 2) \cdot g + r + 1$ .

In fact, the bounds in Theorem 2.4 also hold for non-primitive matrices and that the analogous stronger bounds of Schwarz and Kim with the factor rank instead of  $n$  are true; we prove it in Main Theorem 2.

### B. Weighted digraphs and max algebra

In a weighted digraph  $\mathcal{G}$ , every edge  $(i, j) \in E$  is weighted by some weight  $a_{i,j}$ . We consider the case of nonnegative weights  $a_{i,j} \in \mathbb{R}_+$  and define weight of a walk  $W = (i_0, i_1, \dots, i_t)$  as the product

$$p(W) = a_{i_0, i_1} \cdot a_{i_1, i_2} \cdots a_{i_{t-1}, i_t} \quad (5)$$

Another common definition is letting edge weights be arbitrary reals and the weight of walks be the sum of the weights of its edges. One can navigate between these two definitions by taking the logarithm and the exponential.

By *max algebra* we understand the set of nonnegative real numbers  $\mathbb{R}_+$  equipped with the usual multiplication  $a \times b = a \cdot b$  and tropical addition  $a \oplus b = \max(a, b)$ . This arithmetic is extended to matrices and vectors in the usual way, which leads to max-linear algebra, i.e. the theory of max-linear systems [1], [5]. The product of two matrices  $A \in \mathbb{R}_+^{m \times n}$  and  $B \in \mathbb{R}_+^{n \times q}$  is defined by  $(A \otimes B)_{i,j} = \max_{1 \leq k \leq n} a_{i,k} \cdot b_{k,j}$ , which defines the max-algebraic matrix powers  $A^{\otimes t} = \underbrace{A \otimes \dots \otimes A}_{t \text{ times}}$ . The  $(i, j)$ th entry of  $A^{\otimes t}$  will

be denoted by  $a_{i,j}^{(t)}$ . Boolean matrices are a special case of max-algebraic matrices.

The walks of maximum weight in  $\mathcal{G}$  are closely related with the entries of max-algebraic powers of the associated nonnegative matrix of weights  $A = (a_{i,j})$ . Conversely, one can associate a weighted digraph  $\mathcal{G}(A)$  to every square max-algebraic matrix  $A$ . The connection between max-algebraic powers and weights of walks is based on the following fact called the *optimal walk interpretation* of max-algebraic matrix powers:  $a_{i,j}^{(t)}$  is the maximum weight of all walks of length  $t$  from  $i$  to  $j$ , or 0 if no such walk exists.

Let us also define the maximum geometric cycle mean:

$$\lambda(A) = \max \{ p(C)^{1/\ell(C)} \mid C \text{ is a cycle in } \mathcal{G}(A) \} \quad (6)$$

Set  $\lambda(A) = 0$  if no nonempty cycle in  $\mathcal{G}(A)$  exists. The cycles at which the maximum geometric cycle mean is

attained are called *critical*, and so are all nodes and edges that belong to them. The *critical graph*, denoted by  $\mathcal{G}^c(A) = (N_c(A), E_c(A))$ , consists of all critical nodes and edges.

Cohen et al. [8] have first proved that the sequence of max-algebraic matrix powers of an irreducible matrix  $A$  with  $\lambda(A) = 1$  is eventually periodic. Note that the case  $\lambda(A) \neq 1$  can be reduced to this case by considering the matrix  $\tilde{A} = A/\lambda(A)$ , which has  $\lambda(\tilde{A}) = 1$ . In the weighted case, the least nonnegative  $t$  satisfying (2) is called the *transient* of  $A$ .

In the present paper, we generalize all the bounds in Theorem 2.2 to the weighted case. We do this not by giving bounds on the transient of  $A$ , but by giving bounds on the transients of the critical rows and columns of  $A$ . Hereby, the transient of row  $i$  is the least  $t$  such that  $a_{i,j}^{(t+p)} = a_{i,j}^{(t)}$  for all  $j$ . The transient of a column  $j$  is defined analogously. In the Boolean case, all rows and columns are critical, hence we are really generalizing the Boolean bounds.

The following is the first main result of the paper.

*Main Theorem 1:* Let  $A \in \mathbb{R}_+^{n \times n}$  be irreducible and let  $k \in N_c(A)$  be a critical node. Denote by  $d$  the cyclicity of  $\mathcal{G}(A)$ , by  $H$  the strongly connected component of the critical graph  $\mathcal{G}^c(A)$  containing  $k$ , and by  $|H|$  the number of nodes in  $H$ . The following quantities are upper bounds on the transient of the  $k$ th row and the  $k$ th column:

- (i) (Wielandt bound)  $\text{Wi}(n)$
- (ii) (Dulmage-Mendelsohn bound)  $(n - 2) \cdot g(H) + |H|$
- (iii) (Schwarz bound)  $d \cdot \text{Wi} \left( \left\lfloor \frac{n}{d} \right\rfloor \right) + (n \bmod d)$
- (iv) (Kim bound)  $\left( \left\lfloor \frac{n}{d} \right\rfloor - 2 \right) \cdot g(H) + n$

The first two bounds also hold in the case when  $A$  is reducible.

For any  $k \in N_c(A)$ , we denote by  $T_k(A)$  the transient of the  $k$ th row, i.e. the maximum transient of the sequences  $a_{k,j}^{(t)}$  with  $j \in N$ . We will just write it as  $T_k$  if  $A$  is clear from the context.

*Remark 2.5:* Like in the Boolean case, the bound of Schwarz (resp. Kim) is tighter than the bound of Wielandt (resp. Dulmage and Mendelsohn) when the corresponding component of  $\mathcal{G}$  is imprimitive. Further, the bound of Wielandt is never tighter than that of Dulmage and Mendelsohn when  $g(H) \leq n - 1$ . Unlike for the unweighted graphs, the case  $g(H) = n$  is non-trivial and will be treated below. Likewise, the bound of Schwarz is never tighter than the bound of Kim when  $\frac{g(H)}{d} \leq \left\lfloor \frac{n}{d} \right\rfloor - 1$ , but the case  $\frac{g(H)}{d} = \left\lfloor \frac{n}{d} \right\rfloor$  has to be treated separately. Here we prefer to deduce the bound of Kim from the bound of Dulmage and Mendelsohn in the same way as the bound of Schwarz is derived from the bound of Wielandt (similar to the approach of Shao and Li [20]).

In max algebra, the factor rank of  $A \in \mathbb{R}_+^{n \times n}$  is the least number  $r$  such that (4) holds for some  $x_1, y_1, \dots, x_r, y_r \in \mathbb{R}_+^n$ . In our next main result, we show that the results of Main Theorem 1 can be improved by means of factor rank, thus obtaining a max-algebraic extension of Theorem 2.4.

*Main Theorem 2:* Let  $A \in \mathbb{R}_+^{n \times n}$  be irreducible. Denote by  $d$  the cyclicity of  $\mathcal{G}(A)$  and by  $r$  the factor rank of  $A$ .

Let  $k \in N_c(A)$  be critical. Denote by  $H$  the component of the critical graph  $\mathcal{G}^c(A)$  containing  $k$ . The following upper bounds on the transient of the  $k$ th row and  $k$ th column hold:

- (i)  $\text{Wi}(r) + 1$ ;
- (ii)  $(r - 2) \cdot g(H) + h + 1$ .
- (iii)  $d \cdot \text{Wi}\left(\left\lfloor \frac{r}{d} \right\rfloor\right) + (r \bmod d) + 1$ ;
- (iv)  $\left(\left\lfloor \frac{r}{d} \right\rfloor - 2\right) \cdot g(H) + r + 1$ .

The first two bounds apply to reducible matrices as well.

*Remark 2.6:* All parameters appearing in the bounds of Main Theorem 1 only depend on the unweighted digraphs underlying  $\mathcal{G}(A)$  and  $\mathcal{G}^c(A)$ , but the factor rank  $r$  of Main Theorem 2 depends on the values of  $A$ , i.e., on the weights on  $\mathcal{G}(A)$ .

We prove that  $T_k(A)$  for a critical index  $k$  is less than any of the quantities in Main Theorem 1 and Main Theorem 2. Applying the result to the transposed matrix  $A^T$ , we see that the bounds also hold for the transients of the columns.

Our proofs do not use the results of Theorem 2.2 or Theorem 2.4 for the Boolean case and hence, in particular, we give new proofs for those classical results.

### III. PROOF SKETCHES

In this section, we give proof sketches for the Main Theorems 1 and 2. The detailed proofs can be found in the full version of the paper [15].

#### A. Proof of Dulmage-Mendelsohn bound

We begin by recalling a result of Nachtigall [17] concerning the transients of critical rows. The bound is formulated in terms of the shortest critical cycle that a node lies on. It shows that a stronger form of the Weighted Dulmage-Mendelsohn bound holds if  $k$  lies on a critical cycle of length  $g(H)$ . However, if  $k$  does not lie on a shortest critical cycle of  $H$ , then it is worse. Our proof of the Dulmage-Mendelsohn bound relies on transferring Nachtigall's bound for nodes on a shortest cycle of  $H$  to the remaining nodes. Denote by  $A_k$  the  $k$ th row of  $A$ .

*Lemma 3.1 (Nachtigall [17]):* Let  $k$  be a critical node on a critical cycle of length  $\ell$ . Then  $T_k \leq (n - 1) \cdot \ell$  and  $\ell$  is an eventual period of  $A_k^{\otimes t}$ .

The following result enables us to use the bound of Lemma 3.1 for nodes that do not lie on a critical cycle of minimal length. Its proof relies on the existence of a max-balancing [21] of matrix  $A$ .

*Lemma 3.2:* Let  $k$  and  $l$  be two indices of  $N_c(A)$ , and suppose that there exists a walk from  $k$  to  $l$ , of length  $r$  and with all edges critical.

- (i) If  $t \geq T_l(A)$ , then  $A_k^{\otimes(t+r)} = A_l^{\otimes t}$ .
- (ii)  $T_k(A) \leq T_l(A) + r$ .

To prove the Dulmage-Mendelsohn bound, let  $C$  be a cycle in  $H$  of length  $\ell(C) = g(H)$ . By Lemma 3.1,  $T_k \leq (n - 1) \cdot g(H)$  for all nodes  $k$  of  $C$ . Let now  $k$  be any node in  $H$ . There exist walks in  $H$  from  $k$  to  $C$  of length at most  $|H| - g(H)$ . Application of Lemma 3.2 now concludes the proof.

#### B. Proof of Kim bound

Set  $D = A^{\otimes d}$ . The cyclicity classes of  $\mathcal{G}(A)$  are strongly connected components of  $\mathcal{G}(D)$ , and the corresponding principal sub matrix of  $\mathcal{G}(D)$  is completely reducible, i.e. it has no edge between two different strongly connected components. Obviously, any cycle in  $\mathcal{G}(A)$  has to go through every cyclicity class. Thus,  $d$  divides  $g(H)$  and if  $k$  belongs to  $H$  the girth of its strongly connected components in  $\mathcal{G}^c(D)$  is at most  $g(H)/d$ .

Call a cyclicity class of  $\mathcal{G}(A)$  *small* if it contains the minimal number of nodes amongst cyclicity classes. Let  $m$  be the number of nodes in any small class. By the weighted Dulmage-Mendelsohn bound, we have  $T_k(D) \leq (m - 2) \cdot g(H)/d + m$  for each critical node  $k$  of  $H$  in a small class. Because  $T_k(A) \leq d \cdot T_k(D)$ , this implies  $T_k(A) \leq (m - 2) \cdot g(H) + d \cdot m$  for all critical nodes  $k$  of  $H$  in small classes.

We distinguish the cases (A)  $m \leq \lfloor n/d \rfloor - 1$  and (B)  $m = \lfloor n/d \rfloor$ . Note that  $m \geq \lfloor n/d \rfloor + 1$  is not possible.

In case (A), a crude estimation for all critical  $k$  in small classes is  $T_k \leq (\lfloor n/d \rfloor - 2) \cdot g(H) + n - d$ . Because every critical node has paths consisting of critical edges to a small class of length at most  $d - 1$ , Lemma 3.2 proves the Kim bound in case (A).

In case (B), there are at least  $d - (n \bmod d)$  small classes. There hence is a path of length at most  $(n \bmod d)$  consisting of critical edges to a small class. Hence, again by Lemma 3.2,

$$\begin{aligned} T_k &\leq (\lfloor n/d \rfloor - 2) \cdot g(H) + d \cdot \lfloor n/d \rfloor + (n \bmod d) \\ &= (\lfloor n/d \rfloor - 2) \cdot g(H) + n. \end{aligned}$$

This concludes the proof in case (B).

#### C. Proof of Wielandt bound

If  $g(H) \leq n - 1$ , then the Wielandt bound follows from the Dulmage-Mendelsohn bound. It remains to treat the case that  $g(H) = n$ , i.e.,  $\mathcal{G}^c(A)$  is a Hamiltonian cycle. We prove a result on cycle removal and insertion (Theorem 3.4) which implies the Wielandt bound for matrices with a critical Hamiltonian cycle. It relies on the following elementary application of the pigeonhole principle.

*Lemma 3.3:* Let  $x_1, \dots, x_n$  be integers. There exists a nonempty subset  $I$  of  $\{1, \dots, n\}$  such that  $\sum_{i \in I} x_i$  is a multiple of  $n$ .

One can use this lemma for cycle decomposition arguments that lead to the following theorem.

*Theorem 3.4:* Let  $\mathcal{G}$  be a digraph with  $n$  nodes. For any Hamiltonian cycle  $C_H$  in  $\mathcal{G}$  and any walk  $W$ , there is a walk  $V$  that has the same start and end node as  $W$ , is formed by removing cycles from  $W$  and possibly inserting copies of  $C_H$ , and has a length satisfying  $(n - 1)^2 + 1 \leq \ell(V) \leq (n - 1)^2 + n$  and  $\ell(V) \equiv \ell(W) \pmod{n}$ .

Theorem 3.4 can be used to prove the Wielandt bound in the case the critical graph is a Hamiltonian cycle.

#### D. Proof of Schwarz bound

The Schwarz bound is deduced from the Wielandt bound in the same way as the Kim bound is deduced from the

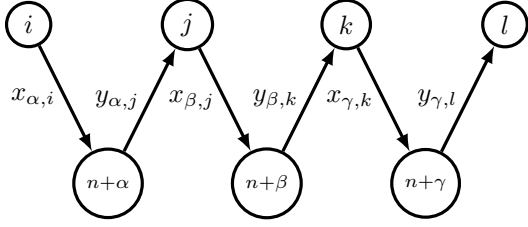


Fig. 1. A walk in  $\mathcal{G}(Z)$

Dulmage-Mendelsohn bound. That is, by regarding  $D = A^{\otimes d}$  and using the Wielandt bound in small cyclicity classes.

### E. Proof of the bounds involving the factor rank

In this subsection, we sketch the proof of Main Theorem 2.

Let  $x_\alpha, y_\alpha \in \mathbb{R}_+^n$ , for  $\alpha = 1, \dots, r$ , be the vectors in factor rank representation (4). Further, Let  $X$  and  $Y$  be the  $n \times r$  matrices whose columns are vectors  $x_\alpha$  and  $y_\alpha$  for  $\alpha = 1, \dots, r$ , and consider the  $(n+r) \times (n+r)$  matrix  $Z$  defined by

$$Z = \begin{pmatrix} 0_{n \times n} & X \\ Y^T & 0_{r \times r} \end{pmatrix}, \quad (7)$$

Then we have

$$Z^{\otimes 2} = \begin{pmatrix} A & 0_{n \times r} \\ 0_{r \times n} & B \end{pmatrix}, \quad (8)$$

where the  $r \times r$  matrix  $B$  is given by

$$b_{\alpha,\beta} = \bigoplus_{i=1}^n y_{\alpha,i} \cdot x_{\beta,i}, \quad \text{for } \alpha, \beta = 1, \dots, r. \quad (9)$$

We will apply the bounds of Main Theorem 1 to the critical nodes of  $B$  and transfer the result to the critical nodes of  $A$ , thanks to the following observation, which can be proved using Lemma 3.2.

*Lemma 3.5:* If  $(k, n+\beta)$  is an edge of  $\mathcal{G}^c(Z)$ , then  $T_k(A) \leq T_\beta(B) + 1$ .

To use this lemma, we need to study the links between  $\mathcal{G}^c(Z)$ ,  $\mathcal{G}^c(A)$  and  $\mathcal{G}^c(B)$ . If  $A$  is irreducible, then so are  $Z$  and  $B$ . Moreover  $\mathcal{G}(B)$  and  $\mathcal{G}(A)$  have the same cyclicity. By construction,  $\mathcal{G}(Z)$  is a bipartite graph, so every walk in  $\mathcal{G}(Z)$  alternates between nodes in  $\{1, \dots, n\}$  and nodes in  $\{n+1, \dots, n+r\}$ . Figure 1 depicts an example of a walk in  $\mathcal{G}(Z)$ .

As all closed walks in  $\mathcal{G}(Z)$  are of even length, the cyclicity of any component of  $\mathcal{G}^c(Z)$  is even, i.e. it is divisible by two. Hence each component  $G$  of  $\mathcal{G}^c(Z)$  splits into two components of  $(\mathcal{G}^c(Z))^2$  such that the (disjoint) union of their node sets is exactly the node set of  $G$ . We call these two components *related*. For a component  $H$  of  $(\mathcal{G}^c(Z))^2$ , the related component will be denoted by  $H'$ .

Each closed walk of  $\mathcal{G}(Z)$  and, therefore, each component of  $\mathcal{G}^c(Z)$  contains nodes both from  $\{1, \dots, n\}$  and from  $\{n+1, \dots, n+r\}$ . Hence, if  $H$  and  $H'$  is a pair of related components of  $(\mathcal{G}^c(Z))^2$  then one of them (say,  $H$ ) contains a node in  $\{1, \dots, n\}$  and the other ( $H'$ ) contains a node in  $\{n+1, \dots, n+r\}$ . Since there are no edges between the two

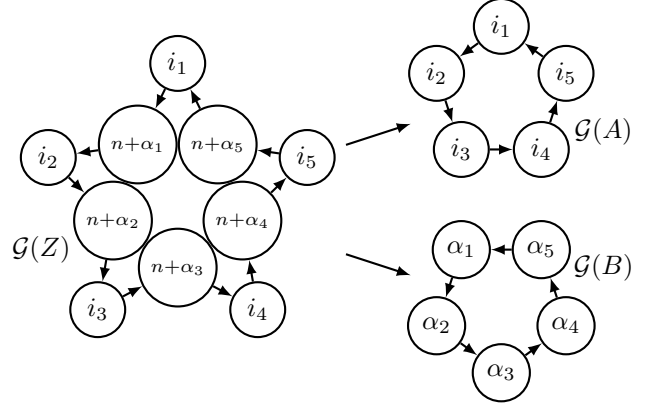


Fig. 2. Correspondence between closed walks of  $\mathcal{G}(Z)$ ,  $\mathcal{G}(A)$  and  $\mathcal{G}(B)$

components of  $\mathcal{G}(Z^{\otimes 2})$ ,  $H$  is a subgraph of  $\mathcal{G}(A)$  and  $H'$  is a subgraph of  $\mathcal{G}(B)$ . Further as  $(\mathcal{G}^c(Z))^2 = \mathcal{G}^c(Z^{\otimes 2})$ ,  $H$  and  $H'$  are components of  $\mathcal{G}^c(Z^{\otimes 2})$ . As  $\mathcal{G}^c(Z^{\otimes 2})$  consists of only such components and the cycles not belonging to such components have a strictly smaller geometric mean, it follows that  $H$  is a component of  $\mathcal{G}^c(A)$ ,  $H'$  is a component of  $\mathcal{G}^c(B)$  and, moreover,  $\mathcal{G}^c(A)$  and  $\mathcal{G}^c(B)$  do not have components that are not formed this way. Take a closed walk  $C$  in  $H$ . Each edge of  $C$  results from a path of  $\mathcal{G}^c(Z)$  of length 2, and inserting these path in  $C$  we obtain a closed walk of  $\mathcal{G}^c(Z)$  (see Figure 2, left). This walk contains nodes from both  $H$  and  $H'$ . In  $Z^{\otimes 2}$  it splits in two closed walks of  $\mathcal{G}^c(Z^{\otimes 2})$  of the same length (see Figure 2, right). One of these closed walks is  $C$  and the other is a closed walk  $\tilde{C}$  of  $H'$  (since  $H$  and  $H'$  are isolated in  $\mathcal{G}^c(Z^{\otimes 2})$ ).

This allows us to transfer the bounds of Main Theorem 1 used for  $B$  to bounds on  $A$  via Lemma 3.5, adding only an additive constant of 1.

## IV. CONCLUSION

We proved that the six bounds on the transient of non-weighted directed graphs continue to hold for critical nodes in the weighted case. To the authors' knowledge, these are the first genuine extensions of unweighted graph transients to the weighted case. This is the main idea of the present contribution. Because they also hold in the nonweighted case, they are independent of the specific weights. More specifically, they only depend on the underlying digraph and the critical digraph. Contrasting this with the fact that the global transient of weighted digraphs can be unbounded even with fixed digraph and critical digraph, our results show that the difference between transients of critical and noncritical nodes can be arbitrarily large. This insight can give guidelines during system design.

Precision of these bounds is one of the possible directions for further research. Namely, it would be interesting to describe, in the most precise and concise way, which weighted digraphs (or associated max-algebraic matrices) attain these bounds. Some preliminary results in this direction are given in [15], Section 8. For instance, Figure 3 displays an example from the work of Schwarz [19] and its weighted

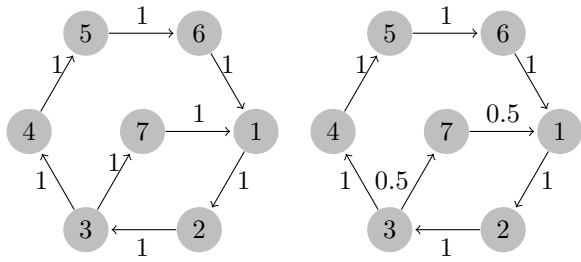


Fig. 3. Schwarz's example (left) and its max-algebraic version (right)

(max-algebraic) version. In this example we have  $T_4(A) = T_4(B) = 11$  for the associated matrices  $A$  (left) and  $B$  (right).

Comparison between the bounds is discussed in Remark 2.5. This comparison is very similar to the well-known unweighted (Boolean) case, and rather straightforward. For future research, it would be desirable to run numerical experiments to get some statistics of how the transients of critical rows and columns typically behave.

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