



HAL
open science

Minimax adaptive estimation of non-parametric hidden markov models

Yohann de Castro, Elisabeth Gassiat, Claire Lacour

► **To cite this version:**

Yohann de Castro, Elisabeth Gassiat, Claire Lacour. Minimax adaptive estimation of non-parametric hidden markov models. 2015. hal-01105424v1

HAL Id: hal-01105424

<https://hal.science/hal-01105424v1>

Preprint submitted on 20 Jan 2015 (v1), last revised 27 Dec 2015 (v3)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

MINIMAX ADAPTIVE ESTIMATION OF NON-PARAMETRIC HIDDEN MARKOV MODELS

Y. DE CASTRO, É. GASSIAT, AND C. LACOUR

ABSTRACT. In this paper, we consider stationary hidden Markov models with finite state space and non parametric modeling of the emission distributions. We propose a new penalized least-squares estimator for the emission distributions which we prove to be asymptotically rate minimax adaptive up to a logarithmic term when there are two hidden states. This non parametric estimator requires the computation of a preliminary estimator of the transition matrix of the hidden chain for which we propose to use the spectral estimator recently presented in [HKZ12]. We also investigate the asymptotic properties of a spectral estimator of the emission distributions derived from that of [HKZ12]. The spectral estimator can not achieve the asymptotic minimax rate, but it is very useful to avoid initialization problems in our least squares minimization algorithm. Simulations are given that show the improvement obtained when applying the least-squares minimization consecutively to the spectral estimation.

1. INTRODUCTION

1.1. Context and motivations. Finite state space hidden Markov models (HMMs for short) are widely used to model data evolving in time and coming from heterogeneous populations. They seem to be reliable models to depict practical situations in a variety of applications such as economics, genomics, signal processing and image analysis, ecology, environment, speech recognition, to name but a few. From a statistical view point, finite state space HMMs are stochastic processes $(X_j, Y_j)_{j \geq 0}$ where $(X_j)_{j \geq 0}$ is a Markov chain living in a finite state space and conditionally on $(X_j)_{j \geq 0}$ the Y_j 's are independent with a distribution depending only on X_j and living in \mathcal{Y} . The observations are $Y_{1:n} = (Y_1, \dots, Y_n)$ and the associated states $X_{1:n} = (X_1, \dots, X_n)$ are unobserved. The parameters of the model are the initial distribution, the transition matrix of the hidden chain, and the emission distributions of the observations, that is the probability distributions of the Y_j 's conditionally to $X_j = x$ for all possible x 's. In this paper we shall consider stationary ergodic HMMs so that the initial distribution is the stationary distribution of the (ergodic) hidden Markov chain.

Until very recently, asymptotic performances of estimators were proved theoretically only in the parametric frame (that is, with finitely many unknown parameters). Though, non parametric methods for HMMs have been considered in applied papers, see for instance [CC00] for voice activity detection, [LWM03] for climate state identification, [Lef03] for automatic speech recognition, [SC09] for facial expression recognition, [VBMMR13] for methylation comparison of proteins, [YPRH11] for copy number variants identification in DNA analysis. Recent papers that contain theoretical results on different kinds of non parametric HMMs are [GR13], where the emitted distributions are translated of each other, and [DL12] in

Date: January 20, 2015.

Key words and phrases. Non-parametric estimation; Hidden markov models; Spectral Method; Concentration inequality; Model selection; Adaptive estimation.

which the authors consider regression models with hidden regressor variables that can be markovian on a continuous state space.

The preliminary obstacle to obtain theoretical results on general finite state space non parametric HMMs was to understand when such models are indeed identifiable. The papers [AMR09], [HKZ12] and [AHK12] paved the way to obtain identifiability under reasonable assumptions. In [AHK12] the authors point out a structural link between multivariate mixtures with conditionally independent observations and finite state space HMMs. In [HKZ12] the authors propose a spectral method to estimate all parameters for finite state space HMMs (with finitely many observations), under the assumption that the transition matrix of the hidden chain is non singular, and that the (finitely valued) emission distributions are linearly independent. Extension to emission distributions on any space, under the linear independence assumptions (and keeping the assumption of non singularity of the transition matrix), allowed to prove the general identifiability result for finite state space HMMs, see [GCR13], where also model selection likelihood methods and non parametric kernel methods are proposed to get non parametric estimators. Let us notice also [Ver13] that proves theoretical consistency of the posterior in non parametric Bayesian methods for finite state space HMMs with adequate assumptions. Later, [AH14] obtained identifiability when the emission distributions are all distinct (not necessarily linearly independent) and still when the transition matrix of the hidden chain is non singular. In the non parametric multivariate mixture model, [SADX14] prove that any linear functional of the emission distributions may be estimated with parametric rate of convergence in the context of reproducing kernel Hilbert spaces. The latter uses spectral methods, not the same but similar to the ones proposed in [HKZ12] and [AHK12]. Those spectral methods have the extremely interesting characteristic that to compute the estimator the algorithms do not require initialization as is usual in latent variable models estimation when using the EM algorithm. They may be used under the linear independence assumption.

1.2. Contribution. The aim of our paper is to propose a non parametric estimator of the emission distributions that achieves the minimax rate of estimation in an adaptive setting. For this purpose we propose a new penalized least squares estimator in the model selection frame. Our perspective is based on estimating the projections of the emission laws onto nested subspaces of increasing complexity. Our analysis encompasses any family of nested subspaces of Hilbert spaces and works with a large variety of models.

We start from the remarkable works of Anandkumar, Hsu, Kakade and Zhang on spectral methods in the parametric frame. Their papers [HKZ12, AHK12] present an efficient algorithm for learning HMMs. They give theoretical guarantees for observation prediction and the ℓ_2 -error on estimating the parametric emission laws. They present spectral estimators for the stationary distribution and the transition matrix of the hidden Markov chain.

In this paper, we first extend spectral methods to the non-parametric frame. Since projections are linear functionals of the distributions, it is possible to use spectral methods to estimate the projections. Then, to get the best risk for the non parametric estimator of the emission distributions, one has as usual to balance a bias term and a variance term. Our work brings a new quantitative insight on the tradeoff between sampling size and approximation complexity for spectral estimators. Doing so, we get the best possible rate for the non parametric estimation of emission densities using spectral methods. Roughly speaking, when the observations are one dimensional, that is when \mathcal{Y} is a subset of \mathbb{R} , the obtained rate is of order $N^{-s/(2s+3)}$, N being the number of observations and s the smoothness of the emission densities. This would be the right minimax rate if the observations were

living in a 3-dimensional space, that is if \mathcal{Y} was a subset of \mathbb{R}^3 . This rate is easily understood by the fact that the spectral estimators come from empirical estimators of the three dimensional marginal of the process, that is of the distribution of (Y_1, Y_2, Y_3) which is indeed 3-dimensional. But in case \mathcal{Y} is a subset of \mathbb{R} , one could hope to achieve the rate $N^{-s/(2s+1)}$ for the estimators of the emission densities. Indeed, using model selection to estimate the density of (Y_1, Y_2, Y_3) with the HMM modelization, and using sieves for the emission densities on \mathcal{Y} , one can obtain the best rate of $N^{-s/(2s+1)}$ up to a $\log N$ term for L_2 -risk of the estimator of the density of (Y_1, Y_2, Y_3) . The key point is then to be able to go back to the emission densities. This is the cornerstone of our result, see Lemma 2 where we prove that under some appropriate assumptions, the quadratic risk for the density of (Y_1, Y_2, Y_3) is lower bounded by some positive constant multiplied by the quadratic risk of the emission densities.

Thus our approach is the following. First, get a preliminary estimate of the initial distribution and the transition matrix of the hidden chain. We propose one obtained using spectral methods, for which we prove new precise asymptotics. Second, apply penalized least squares estimation on the density of three consecutive observations, using HMM modelization, model selection on the emission densities, and initial distribution and stationary matrix of the hidden chain set at the estimated value. This gives emission density estimators. When the observations form a stationary HMM with two hidden states, this leads to a minimax adaptive estimator, as our main result states, see Theorem 1. Moreover, since the family of sieves we consider is that given by finite dimensional spaces described by an orthonormal basis, we are able to use the spectral estimators of the coefficients of the densities as initial points in the least squares minimization. This is important since here, in the HMM framework, least squares minimization does not have an explicit solution and may lead to several local minima. However, since the spectral estimates are proved to be consistent, we may be confident that their use as initial point is enough. Simulations indeed confirm this point. To conclude we claim that our results support a powerful new approach to estimate non-parametric HMMs with a statistically optimal and practically tractable method.

1.3. Outline of the paper. In Section 2, we set the notations, the model we shall study, and the assumptions we shall consider. We then define the spectral estimators we shall use as preliminary step, and we present our penalized least squares estimation method. In Section 3 we give our main results. We first prove in Section 3.1 that, when the HMM has two hidden states, when the transition matrix is irreducible and aperiodic, when the emission distributions are distinct and the penalty is adequately chosen, then the penalized least squares estimator is asymptotically minimax adaptive up to a $\log N$ term, see Theorem 1 and Corollary 1. The proof is based on three intermediate results. First, we prove an oracle inequality for the least squares estimator of the density of three consecutive observations, see Proposition 2. Then, we prove the key lemma relating the risk of the density of (Y_1, Y_2, Y_3) to that of the emission densities, see Lemma 2. Finally, we need the performances of the spectral estimator of the transition matrix and the stationary distributions which are consequences of the further section. We then give in Section 3.2 precise upper bounds for the quadratic risk of all the spectral estimators, see Theorem 3. Here, the important point is to make explicit the upper bound in the complexity parameter of the sieve, M , together with the number of observations, N . We finally present simulations in Section 4 to illustrate our theoretical results. Those simulations show in particular the improvement obtained when applying the least-squares minimization consecutively to the spectral estimation. Detailed proofs are given in Section 6 and in the Appendices.

2. ESTIMATION METHODS

2.1. Non-parametric Hidden Markov Model. Let K, D be positive integers and let \mathcal{L}^D be the Lebesgue measure on \mathbb{R}^D . Denote by \mathcal{X} the set $\{1, \dots, K\}$ of hidden states, $\mathcal{Y} = [0, 1]^D$ the observation space, and Δ_K the space of probability measures on \mathcal{X} identified to the $(K - 1)$ -dimensional simplex. Let $(X_n)_{n \geq 1}$ be a Markov chain on \mathcal{X} with $K \times K$ transition matrix \mathbf{Q}^* and initial distribution $\pi^* \in \Delta_K$. Let $(Y_n)_{n \geq 1}$ be a sequence of observed random variables on \mathcal{Y} . Assume that, conditional on $(X_n)_{n \geq 1}$, the observations $(Y_n)_{n \geq 1}$ are independent and, for all $n \in \mathbb{N}$, the distribution of Y_n depends only on X_n :

$$\mathcal{L}((Y_n)_{n \geq 1} | (X_n)_{n \geq 1}) = \bigotimes_{n \geq 1} \mathcal{L}(Y_n | X_n).$$

Observe that, for all $n \in \mathbb{N}$ and for all $k \in \mathcal{X}$, conditional on $\{X_n = k\}$, the law of Y_n depends only on the state k . Denote by μ_k^* this conditional law and assume that μ_k^* has density f_k^* with respect to the measure \mathcal{L}^D on \mathcal{Y} :

$$\forall k \in \mathcal{X}, \quad d\mu_k^* = f_k^* d\mathcal{L}^D.$$

Denote by $\mathfrak{F}^* := \{f_1^*, \dots, f_K^*\}$ the set of emission densities with respect to the Lebesgue measure. Then, for any integer n , the distribution of (Y_1, \dots, Y_n) has density with respect to $(\mathcal{L}^D)^{\otimes n}$

$$\sum_{k_1, \dots, k_n=1}^K \pi^*(k_1) \mathbf{Q}^*(k_1, k_2) \dots \mathbf{Q}^*(k_{n-1}, k_n) f_{k_1}^*(y_1) \dots f_{k_n}^*(y_n).$$

We shall denote g^* the density of (Y_1, Y_2, Y_3) .

In this paper we shall address two observations schemes. We shall consider N i.i.d. samples $(Y_1^{(s)}, Y_2^{(s)}, Y_3^{(s)})_{s=1}^N$ of three consecutive observations (**Scenario A**) or consecutive observations of the same chain (**Scenario B**):

$$\forall s \in \{1, \dots, N\}, \quad (Y_1^{(s)}, Y_2^{(s)}, Y_3^{(s)}) := (Y_s, Y_{s+1}, Y_{s+2}).$$

2.2. Projections of the population joint laws. Denote by $(\mathbf{L}^2(\mathcal{Y}, \mathcal{L}^D), \|\cdot\|_2)$ the Banach space of square integrable functions on \mathcal{Y} with respect to the Lebesgue measure \mathcal{L}^D equipped with the usual inner product $\langle \cdot, \cdot \rangle$ on $\mathbf{L}^2(\mathcal{Y}, \mathcal{L}^D)$. Assume $\mathfrak{F}^* \subset \mathbf{L}^2(\mathcal{Y}, \mathcal{L}^D)$.

Let $(M_n)_{n \geq 1}$ be an increasing sequence of integers, and let $(\mathfrak{P}_{M_n})_{n \geq 1}$ be a sequence of nested subspaces such that their union is dense in $\mathbf{L}^2(\mathcal{Y}, \mathcal{L}^D)$. Let $\Phi_{M_n} := \{\varphi_1, \dots, \varphi_{M_n}\}$ be an orthonormal basis of \mathfrak{P}_{M_n} . Note that for all $f \in \mathbf{L}^2(\mathcal{Y}, \mathcal{L}^D)$,

$$(1) \quad \lim_{n \rightarrow \infty} \sum_{m=1}^{M_n} \langle f, \varphi_m \rangle \varphi_m = f,$$

in $\mathbf{L}^2(\mathcal{Y}, \mathcal{L}^D)$. Note that changing M_n may change all functions φ_m , $1 \leq m \leq M_n$ in the basis Φ_{M_n} , which we shall not indicate in the notation for sake of readability.

One can consider the following standard examples:

(Spline) The space of piecewise polynomials of degree bounded by d_n based on the regular partition with p_n^D regular pieces on $\mathcal{Y} = [0, 1]^D$. Consider the Legendre basis on each piece \mathfrak{p} :

$$\varphi_{\mathfrak{p}, r}(x) := p_n^{\frac{D}{2}} \prod_{i=1}^D (2r_i + 1)^{\frac{1}{2}} \mathbf{P}_{r_i}(-1 + 2p_n \rho_{\mathfrak{p}, i}(x))$$

where $\mathfrak{p} \in \{1, \dots, p_n\}^D$, $r \in \{0, \dots, d_n\}^D$, \mathbf{P}_k denotes the k th Legendre polynomial and $\rho_{\mathfrak{p}, i}(x) \in [0, 1/p_n]$ denotes the i th coordinate of x on the piece \mathfrak{p} . It holds that $M_n = (d_n + 1)^D p_n^D$.

(Trig.) The space of real trigonometric polynomials on $\mathcal{Y} = [0, 1]^D$ with degree less than n :

$$\varphi_r(x) := \prod_{i=1}^D \{ \sqrt{2} \cos(2\pi r_i x_i) \mathbb{1}_{\{r_i < 0\}} + \sqrt{2} \sin(2\pi r_i x_i) \mathbb{1}_{\{r_i > 0\}} + \mathbb{1}_{\{r_i = 0\}} \},$$

where $r \in \{-n, \dots, n\}^D$. It holds that $M_n = (2n + 1)^D$.

(Wav.) A wavelet basis Φ_{M_n} of scale n on $\mathcal{Y} = [0, 1]^D$, see [Mey92]. One can consider the collection of functions which are D -tensors of $\mathbb{1}_{[0,1]}$ and $\psi_{j,r}$, $j = 0, \dots, n$, $r = 0, \dots, 2^j - 1$ with for $x \in [0, 1]$

$$\psi_{j,r}(x) = 2^{j/2} \psi(2^j x - r)$$

for some mother function ψ , for instance $\psi = \mathbb{1}_{[0,1/2]} - \mathbb{1}_{[1/2,1]}$. It holds that $M_n = 2^{(n+1)D}$.

For sake of readability, we drop the dependence on n and write M instead of M_n . The following vectors, matrices and tensors will be used:

- Denote by $\mathbf{L}_M \in \mathbb{R}^M$ the projection of the distribution of one observation, for instance Y_1 , on the basis Φ_M :

$$\forall a \in \{1, \dots, M\}, \quad \mathbf{L}_M(a) = \mathbb{E}(\varphi_a(Y_1)).$$

- Denote by $\mathbf{M}_M \in \mathbb{R}^{M \times M \times M}$ the joint distribution of three consecutive observations, for instance (Y_1, Y_2, Y_3) , on the basis Φ_M :

$$\forall (a, b, c) \in \{1, \dots, M\}^3, \quad \mathbf{M}_M(a, b, c) = \mathbb{E}(\varphi_a(Y_1) \varphi_b(Y_2) \varphi_c(Y_3)).$$

- Denote by $\mathbf{N}_M \in \mathbb{R}^{M \times M}$ the joint distribution of two consecutive observations, for instance (Y_1, Y_2) , on the basis Φ_M :

$$\forall (a, b) \in \{1, \dots, M\}^2, \quad \mathbf{N}_M(a, b) = \mathbb{E}(\varphi_a(Y_1) \varphi_b(Y_2)).$$

- Denote by $\mathbf{O}_M \in \mathbb{R}^{M \times \mathcal{X}}$ the conditional distribution of one observation on the basis Φ_M :

$$\forall (m, k) \in \{1, \dots, M\} \times \mathcal{X}, \quad \mathbf{O}_M(m, k) = \mathbb{E}(\varphi_m(Y_1) | X_1 = k) = \langle f_k^*, \varphi_m \rangle.$$

- Define the projection of the emission laws on the subspace \mathfrak{F}_M :

$$\forall k \in \mathcal{X}, \quad f_{M,k}^* := \sum_{m=1}^M \mathbf{O}_M(m, k) \varphi_m,$$

and note $\mathbf{f}_M^* = (f_{M,1}^*, \dots, f_{M,K}^*)$.

- Denote by $\mathbf{P}_M \in \mathbb{R}^{M \times M}$ the joint distribution of (Y_1, Y_3) , on the basis Φ_M :

$$\forall (a, c) \in \{1, \dots, M\}^2, \quad \mathbf{P}_M(a, c) = \mathbb{E}(\varphi_a(Y_1) \varphi_c(Y_3)).$$

2.3. Assumptions and further notations.

2.3.1. *Assumptions on the hidden chain.* We shall use the following assumptions

[H1] The transition matrix \mathbf{Q}^* has full rank,

[H2] The Markov chain $(X_n)_{n \geq 1}$ is irreducible and aperiodic,

[H3] The initial distribution $\pi^* = (\pi_1^*, \dots, \pi_K^*)$ is the stationary distribution

Notice that under **[H1]**, **[H2]** and **[H3]**, one has

$$\forall k \in \mathcal{X}, \quad \pi_k^* \geq \pi_{\min}^* > 0.$$

Those assumptions appear in spectral methods, see for instance [HKZ12, AHK12], and in identifiability issues, see for instance [AMR09, GCR13, AH14].

2.3.2. Norms, singular values and matrix notation. We shall use the following common notation throughout the paper. Let A be a $(p \times q)$ matrix with $p \geq q$. Denote $\sigma_1(A) \geq \sigma_2(A) \geq \dots \geq \sigma_q(A) \geq 0$ its singular values, $\|\cdot\|$ its operator norm and $\|\cdot\|_F$ its Frobenius norm. When A is invertible, denote $\kappa(A) := \sigma_1(A)/\sigma_q(A)$ its condition number. Denote A^\top the transpose matrix of A , $A(k, l)$ its (k, l) th entry, $A(\cdot, l)$ its l th column and $A(k, \cdot)$ its k th line. When A is a $(p \times p)$ diagonalizable matrix, denote $\lambda_1(A) \geq \lambda_2(A) \geq \dots \geq \lambda_p(A)$ its eigenvalues.

For vectors, denote $\|\cdot\|_q$ the usual ℓ_q norm, for $1 \leq q \leq +\infty$. Let v be a vector of size p . We denote by $\mathfrak{Diag}[v]$ the diagonal matrix with diagonal entries v_i and, by abuse of notation, $\mathfrak{Diag}[v] = \mathfrak{Diag}[v^\top]$.

2.3.3. Separation of the emission laws. Assume that the family $\mathfrak{F}^* = \{f_1^*, \dots, f_K^*\}$ is linearly independent. Therefore the $(K \times K)$ Gram matrix denoted by $\mathbf{O}_*^\top \mathbf{O}_*$ and defined by $\mathbf{O}_*^\top \mathbf{O}_* := (\langle f_{k_1}^*, f_{k_2}^* \rangle)_{k_1, k_2 \in \mathcal{X}}$ is invertible. Let:

$$(2) \quad \varepsilon_{\mathfrak{F}^*, M} := \|\mathbf{O}_M^\top \mathbf{O}_M - \mathbf{O}_*^\top \mathbf{O}_*\| = \|(\langle f_{M, k_1}^*, f_{M, k_2}^* \rangle - \langle f_{k_1}^*, f_{k_2}^* \rangle)_{k_1, k_2 \in \mathcal{X}}\|.$$

From (1), one can check that there exists $M_{\mathfrak{F}^*} \geq 1$ such that for all $M \geq M_{\mathfrak{F}^*}$

$$\varepsilon_{\mathfrak{F}^*, M} \leq \frac{3\lambda_K(\mathbf{O}_*^\top \mathbf{O}_*)}{4}.$$

Remark — *One can give an explicit expression of $M_{\mathfrak{F}^*}$ in terms of the regularity of the emission laws. Indeed, standard results in approximation theory [DL93] show that one can upper bound the approximation error $\|f_k^* - f_{M, k}^*\|_2$ by $\mathcal{O}(M^{-s})$ where $s > 0$ denotes a regularity parameter (e.g. \mathfrak{F}^* is included in a well-chosen Besov space). As a matter of fact, under standard hypothesis on the emission laws densities, one can prove that $\varepsilon_{\mathfrak{F}^*, M} \leq C_{\mathfrak{F}^*, s} M^{-s}$ where $C_{\mathfrak{F}^*, s} > 0$ is a constant that may depend on \mathfrak{F}^* and a regularity parameter s . Hence, one can consider:*

$$M_{\mathfrak{F}^*} := \lceil \left(\frac{4C_{\mathfrak{F}^*, s}}{3\lambda_K(\mathbf{O}_*^\top \mathbf{O}_*)} \right)^{\frac{1}{s}} \rceil.$$

Invoke Weyl's inequality (see Theorem 6) to show

$$\sigma_K^2(\mathbf{O}_M) = \lambda_K(\mathbf{O}_M^\top \mathbf{O}_M) \geq \lambda_K(\mathbf{O}_*^\top \mathbf{O}_*)/4.$$

Set $\sigma_K(\mathbf{O}_*) := \lambda_K^{\frac{1}{2}}(\mathbf{O}_*^\top \mathbf{O}_*)$ and notice that for all $M \geq M_{\mathfrak{F}^*}$, $\sigma_K(\mathbf{O}_M) \geq \sigma_K(\mathbf{O}_*)/2$. It shows that one can consider that:

[H4] *There exists a constant $0 < \sigma_{K, \mathfrak{F}^*} \leq 1$ and a positive integer $M_{\mathfrak{F}^*}$ such that:*

$$\forall M \geq M_{\mathfrak{F}^*}, \quad \sigma_K(\mathbf{O}_M) \geq \sigma_{K, \mathfrak{F}^*} > 0.$$

Conversely, if [H4] holds, one can check that $\mathbf{O}_*^\top \mathbf{O}_*$ is invertible. We deduce that [H4] is equivalent to:

[H4b] *The family of emission densities $\mathfrak{F}^* := \{f_1^*, \dots, f_K^*\}$ is linearly independent.* Assumption [H4b] appears in spectral methods, see [HKZ12, AHK12], and in identifiability issues, see for instance [AMR09, GCR13]. Notice that in case $K = 2$, [H4b] reduces to the fact that $f_1^* \neq f_2^*$.

2.3.4. Identifiability issues. For any $\mathbf{f} = (f_1, \dots, f_K) \in (\mathbf{L}^2(\mathcal{Y}, \mathcal{L}^D))^K$ and any irreducible transition matrix \mathbf{Q} , denote $g^{\mathbf{Q}, \mathbf{f}} : \mathcal{Y}^3 \rightarrow \mathbb{R}$ the function given by

$$(3) \quad g^{\mathbf{Q}, \mathbf{f}}(y_1, y_2, y_3) = \sum_{k_1, k_2, k_3=1}^K \pi(k_1) \mathbf{Q}(k_1, k_2) \mathbf{Q}(k_2, k_3) f_{k_1}(y_1) f_{k_2}(y_2) f_{k_3}(y_3),$$

where π is the stationary distribution of \mathbf{Q} . When $\mathbf{Q} = \mathbf{Q}^*$ and $\mathbf{f} = \mathbf{f}^*$, $g^{\mathbf{Q}^*, \mathbf{f}^*} = g^*$. When f_1, \dots, f_K are probability densities on \mathcal{Y} , $g^{\mathbf{Q}, \mathbf{f}}$ is the probability distribution of three consecutive observations of a stationary HMM.

We now state a Lemma that gathers all what we need about identifiability.

Lemma 1 — Assume that \mathbf{Q} is a transition matrix for which **[H1]** and **[H2]** hold. Assume that **[H4]** (or **[H4b]**) holds. Define $T_{\mathbf{Q}}$ the set of permutations τ such that for all i and j , $\mathbf{Q}(\tau(i), \tau(j)) = \mathbf{Q}(i, j)$. Then for any $\mathbf{h} \in (\mathbf{L}^2(\mathcal{Y}, \mathcal{L}^D))^K$,

$$g^{\mathbf{Q}, \mathbf{f}^* + \mathbf{h}} = g^{\mathbf{Q}, \mathbf{f}^*} \iff \exists \tau \in T_{\mathbf{Q}} \text{ such that } h_j = f_{\tau(j)}^* - f_j^*, \quad j = 1, \dots, K.$$

In particular, if $T_{\mathbf{Q}}$ reduces to the identity permutation,

$$g^{\mathbf{Q}, \mathbf{f}^* + \mathbf{h}} = g^{\mathbf{Q}, \mathbf{f}^*} \iff \mathbf{h} = (0, \dots, 0).$$

Proof. In **[HKZ12]** it is proved that when **[H1]**, **[H2]**, **[H3]** hold and $\sigma_K(\mathbf{O}_M) > 0$, the knowledge of \mathbf{M}_M allows to recover \mathbf{O}_M and \mathbf{Q} up to relabelling of the hidden states, using only spectral methods on \mathbf{M}_M . Thus, when **[H1]**, **[H2]**, **[H3]** and **[H4b]** hold, the knowledge of $g^{\mathbf{Q}, \mathbf{f}^*}$ is equivalent to the knowledge of the sequence $(\mathbf{M}_M)_M$ which allows to recover \mathbf{Q} and the sequence $(\mathbf{O}_M)_M$, up to relabelling of the hidden states, which allows to recover $\mathbf{f}^* = (f_1^*, \dots, f_K^*)$ up to relabelling of the hidden states, thanks to **(1)**. See also **[GCR13]**. \square

2.4. Spectral estimation. The following procedure describes a tractable approach to non-parametric emission density estimation and transition matrix estimation. It is based on recent developments in parametric estimation of HMMs. For each fixed M , we estimate the projection of the emission distributions on the basis Φ_M using the spectral method proposed in **[AHK12]**. As the authors of the latter paper explain, this allows further to estimate the transition matrix (we use a modified version of their estimator), and we set the estimator of the stationary distribution as the stationary distribution of the estimator of the transition matrix. The computation of those estimators is particularly simple: it is based on one SVD, some matrix inversions and one diagonalization. One can prove, with overwhelming probability, all matrix inversions and the diagonalization can be done rightfully, see Theorem 4 and Theorem 5.

[Step 1] Consider the following empirical estimators: For any a, b, c in $\{1, \dots, M\}$,

$$\begin{aligned} \hat{\mathbf{L}}_M(a) &:= \frac{1}{N} \sum_{s=1}^N \varphi_a(Y_1^{(s)}), \\ \hat{\mathbf{M}}_M(a, b, c) &:= \frac{1}{N} \sum_{s=1}^N \varphi_a(Y_1^{(s)}) \varphi_b(Y_2^{(s)}) \varphi_c(Y_3^{(s)}), \\ \hat{\mathbf{N}}_M(a, b) &:= \frac{1}{N} \sum_{s=1}^N \varphi_a(Y_1^{(s)}) \varphi_b(Y_2^{(s)}), \\ \text{and } \hat{\mathbf{P}}_M(a, c) &:= \frac{1}{N} \sum_{s=1}^N \varphi_a(Y_1^{(s)}) \varphi_c(Y_3^{(s)}). \end{aligned}$$

[Step 2] Let $\hat{\mathbf{U}}$ be the $M \times K$ matrix of orthonormal right singular vectors of $\hat{\mathbf{P}}_M$ corresponding to its top K singular values.

[Step 3] Form the matrices:

$$\forall b \in \{1, \dots, M\}, \quad \hat{\mathbf{B}}(b) := (\hat{\mathbf{U}}^\top \hat{\mathbf{P}}_M \hat{\mathbf{U}})^{-1} \hat{\mathbf{U}}^\top \hat{\mathbf{M}}_M(\cdot, b, \cdot) \hat{\mathbf{U}}.$$

[Step 4] Set Θ a $(K \times K)$ random unitary matrix uniformly drawn and form the matrices:

$$\forall k \in \{1, \dots, K\}, \quad \hat{\mathbf{C}}(k) := \sum_{b=1}^M (\hat{\mathbf{U}} \Theta)(b, k) \hat{\mathbf{B}}(b).$$

[Step 5] Compute $\hat{\mathbf{R}}$ a $(K \times K)$ unit Euclidean norm columns matrix that diagonalizes the matrix $\hat{\mathbf{C}}(1)$:

$$\hat{\mathbf{R}}^{-1}\hat{\mathbf{C}}(1)\hat{\mathbf{R}} = \mathfrak{D}\text{diag}[(\hat{\Lambda}(1, 1), \dots, \hat{\Lambda}(1, K))].$$

[Step 6] Set:

$$\forall k, k' \in \mathcal{X}, \quad \hat{\Lambda}(k, k') := (\hat{\mathbf{R}}^{-1}\hat{\mathbf{C}}(k)\hat{\mathbf{R}})(k', k'),$$

$$\text{and } \hat{\mathbf{O}}_M := \hat{\mathbf{U}}\hat{\Theta}\hat{\Lambda}.$$

[Step 7] Consider the emission laws estimator $(\hat{f}_{M,k})_{k \in \mathcal{X}}$ defined by:

$$\forall k \in \mathcal{X}, \quad \hat{f}_{M,k} := \sum_{m=1}^M \hat{\mathbf{O}}_M(m, k)\varphi_m,$$

[Step 8] Set

$$\hat{\pi} := (\hat{\mathbf{U}}^\top \hat{\mathbf{O}}_M)^{-1} \hat{\mathbf{U}}^\top \hat{\mathbf{L}}_M.$$

[Step 9] Consider the transition matrix estimator:

$$\hat{\mathbf{Q}} := \Pi_{\text{TM}} \left((\hat{\mathbf{U}}^\top \hat{\mathbf{O}}_M \mathfrak{D}\text{diag}[\hat{\pi}])^{-1} \hat{\mathbf{U}}^\top \hat{\mathbf{N}}_M \hat{\mathbf{U}} (\hat{\mathbf{O}}_M^\top \hat{\mathbf{U}})^{-1} \right),$$

where Π_{TM} denotes the projection (with respect to the scalar product given by the Frobenius norm) onto the convex set of transition matrices, and define $\hat{\pi}$ as the stationary distribution of $\hat{\mathbf{Q}}$.

Remark — *The projection Π_{TM} (and the projection Π_{Δ_K}) can be computed using alternating projections. Indeed, observe the set of transpose transition matrices can be viewed as the product $\Delta_K \times \dots \times \Delta_K$. Note the simplex Δ_K is the intersection between two “simple” convex sets: an affine space and the orthant. Hence, an alternating projection method can be used to compute the projection Π_{TM} . We deduce that with $\mathcal{O}(K^2)$ simple projections (onto affine spaces and orthants) one can compute the projection onto the set of transition matrices.*

2.5. Least squares estimation. In this section we shall estimate the density g^* of (Y_1, Y_2, Y_3) using the so-called penalized least squares method. The idea is the following: starting from the operator $t \mapsto \|t - g^*\|_2^2 - \|g^*\|_2^2 = \|t\|_2^2 - 2 \int t g^*$ which is minimum for the target g^* , we introduce the corresponding empirical contrast γ_N . Namely, for any $t \in \mathbf{L}^2(\mathcal{Y}^3, \mathcal{L}^{D \otimes 3})$, set

$$\gamma_N(t) = \|t\|_2^2 - \frac{2}{N} \sum_{s=1}^N t(Z_s),$$

with $Z_s := (Y_1^s, Y_2^s, Y_3^s)$ (**Scenario A**) or $Z_s := (Y_s, Y_{s+1}, Y_{s+2})$ (**Scenario B**). As N tends to infinity, $\gamma_N(t) - \gamma_N(g^*)$ converges almost surely to $\|t - g^*\|_2^2$, thus the name least squares contrast function. A natural estimator is then a function t such that $\gamma_N(t)$ is minimum over a judicious approximation space S . We thus define a whole collection of estimates \hat{g}_M , each M indexing an approximation subspace $S(M)$ (also called model). It then remains to select the best model, that is to choose M which minimizes $\|\hat{g}_M - g^*\|_2^2 - \|g^*\|_2^2$. This quantity is close to $\gamma_N(\hat{g}_M)$, but we need to take into account the deviations of the process $\gamma - \gamma_N$. Then we rather minimize $\gamma_N(\hat{g}_M) + \text{pen}(N, M)$ where $\text{pen}(N, M)$ is a penalty term to be specified.

More precisely, considering (3) we shall introduce a collection of model of functions by projection of possible \mathbf{f} 's on the subspaces $(\mathfrak{P}_M)_M$. We fix a compact subset \mathcal{F} of $\mathbf{L}^2(\mathcal{Y}, \mathcal{L}^D)$ such that for any $f \in \mathcal{F}$, $\int f d\mathcal{L}^D = 1$ and $\|f\|_\infty \leq C_{\mathcal{F}, \infty}$ for some fixed $C_{\mathcal{F}, \infty} > 0$. Recall that \mathcal{F} is a compact subset of $\mathbf{L}^2(\mathcal{Y}, \mathcal{L}^D)$ if and only if

\mathcal{F} is closed, bounded, and for a complete orthonormal basis $(e_j)_{j \geq 1}$ of $\mathbf{L}^2(\mathcal{Y}, \mathcal{L}^D)$, it holds

$$\forall \epsilon > 0, \exists J \text{ such that } \forall f \in \mathcal{F}, \sum_{j > J} \langle f, e_j \rangle^2 < \epsilon.$$

In particular, there exists $C_{\mathcal{F},2} > 0$ such that for all $f \in \mathcal{F}$, $\|f\|_2 \leq C_{\mathcal{F},2}$. Also, when \mathfrak{P}_M is defined as the subspace generated by $(\varphi_m)_{1 \leq m \leq M} = \Phi_M$, for the **(Trig.)** or **(Wav.)** examples, with $\varphi_1 = \mathbf{1}_{\mathcal{Y}}$, one may choose \mathcal{F} by setting a small $\beta > 0$, a large $L > 0$, and let

$$(4) \quad \mathcal{F} = \{f \in \mathbf{L}^2(\mathcal{Y}, \mathcal{L}^D) : \langle f, \varphi_1 \rangle = 1, \|f\|_\infty \leq C_{\mathcal{F},\infty} \text{ and } \sum_{m \geq 2} m^{2\beta} \langle f, \varphi_m \rangle^2 \leq L^2\}.$$

In such a case, when $f \in \mathcal{F}$, then for all M , if f_M is the projection of f onto \mathfrak{P}_M , then $f_M \in \mathcal{F}$. Also, if moreover $\beta > D/2$, embedding theorems of Sobolev spaces (see [AF03] for instance) show that if

$$(5) \quad \mathcal{F} = \{f \in \mathbf{L}^2(\mathcal{Y}, \mathcal{L}^D) : \langle f, \varphi_1 \rangle = 1 \text{ and } \sum_{m \geq 2} m^{2\beta} \langle f, \varphi_m \rangle^2 \leq L^2\}$$

there exists $C_{\mathcal{F},\infty} > 0$ such that $\|f\|_\infty \leq C_{\mathcal{F},\infty}$ for all $f \in \mathcal{F}$.

For any irreducible transition matrix \mathbf{Q} with stationary distribution π , we define $\mathcal{S}(\mathbf{Q}, M)$ as the set of functions $g^{\mathbf{Q}, \mathbf{f}}$ such that, for each $k = 1, \dots, K$, $f_k \in \mathcal{F}$, and there exists $(a_{mk})_{1 \leq m \leq M} \in \mathbb{R}^M$ such that

$$f_k = \sum_{m=1}^M a_{mk} \varphi_m.$$

Let now $\hat{\mathbf{Q}}$ be an estimator of \mathbf{Q}^* . For any M , define \hat{g}_M as a minimizer of $\gamma_N(t)$ for $t \in \mathcal{S}(\hat{\mathbf{Q}}, M)$. Then \hat{g}_M can be written as $\hat{g}_M = g^{\hat{\mathbf{Q}}, \hat{\mathbf{f}}^M}$ with $\hat{\mathbf{f}}^M \in (\mathcal{F})^K$ and

$$\hat{f}_k^M = \sum_{m=1}^M \hat{a}_{mk} \varphi_m, \quad k = 1, \dots, K$$

for some $(\hat{a}_{m,k})_{1 \leq m \leq M} \in \mathbb{R}^M$, $k = 1, \dots, K$. The least squares estimator does not have an explicit form such as in usual nonparametric estimation, so that one has to use numerical minimization algorithms. As initial point of the minimization algorithm, we shall use the spectral estimator, see Section 4 for more details.

Our final estimator will be a penalized least squares estimator. We then set a penalty function $\text{pen}(N, M)$ and choose

$$\hat{M} = \arg \min_{M=1, \dots, N} \{\gamma_N(\hat{g}_M) + \text{pen}(N, M)\}.$$

Notice that, with N observations, we consider N subspaces as candidates for model selection. Then the estimator of g^* is $\hat{g} = \hat{g}_{\hat{M}}$, and the estimator of \mathbf{f}^* is $\hat{\mathbf{f}}$ such that

$$\hat{\mathbf{f}} := \hat{\mathbf{f}}^{\hat{M}}$$

so that $\hat{g} = g^{\hat{\mathbf{Q}}, \hat{\mathbf{f}}}$.

3. MAIN RESULTS

3.1. Adaptive non parametric estimation when $K = 2$. Here we specialize to the situation where $K = 2$. In such a case, $\mathbf{f}^* = (f_1^*, f_2^*)$, and

$$\mathbf{Q}^* = \begin{pmatrix} 1 - p^* & p^* \\ q^* & 1 - q^* \end{pmatrix}$$

for some p^*, q^* in $[0, 1]$. We shall assume

[HB] The coefficients p^* and q^* verify

$$0 < p^* < 1, 0 < q^* < 1, p^* \neq 1 - q^*.$$

[HF] \mathcal{F} is a compact subset of $\mathbf{L}^2(\mathcal{Y}, \mathcal{L}^D)$ such that: for any $f \in \mathcal{F}$, $\int f d\mathcal{L}^D = 1$ and $\|f\|_\infty \leq C_{\mathcal{F},\infty}$ for some fixed $C_{\mathcal{F},\infty} > 0$.

We denote $C_{\mathcal{F},2} = \sup_{f \in \mathcal{F}} \|f\|_2 < \infty$.

Let \hat{f} be the penalized least squares estimator of \mathbf{f}^* when $\hat{\mathbf{Q}} := \hat{\mathbf{Q}}_N$ is chosen as the spectral estimator of \mathbf{Q}^* with $M := M_N$ such that $\eta_3(\Phi_{M_N})$ (defined in Section 3.2) is equal to $\sqrt{\log N}$. The following theorem gives an oracle inequality for the estimators of the emission distributions provided the penalty is adequately chosen.

Theorem 1 (Adaptive estimation) — Assume **[HF]**. Assume that **[HB]** holds for \mathbf{Q}^* . Assume also $\mathbf{f}^* \in \mathcal{F}^2$, $f_1^* \neq f_2^*$, and that for all M , $\mathbf{f}_M^* \in \mathcal{F}^2$. Then, there exists positive constants A^* , B^* (depending on \mathbf{Q}^* , \mathbf{f}^* , $C_{\mathcal{F},2}$ and $C_{\mathcal{F},\infty}$), a positive integer N^* (depending on \mathbf{Q}^* and \mathfrak{F}^*) and a positive constant ρ^* (depending on $C_{\mathcal{F},2}$ and $C_{\mathcal{F},\infty}$ (**Scenario A**) or on \mathbf{Q}^* , $C_{\mathcal{F},2}$ and $C_{\mathcal{F},\infty}$ (**Scenario B**)) such that, if

$$\text{pen}(N, M) \geq \rho^* \frac{M \log M}{N}$$

then for all $x > 0$, for all $N \geq (x \vee x^2) N^* \log N$, there exists a permutation $\tau_N \in \mathcal{S}_2$ such that, with probability larger than $1 - 8e^{-x}$,

$$\begin{aligned} \|f_1^* - \hat{f}_{\tau_N(1)}\|_2^2 + \|f_2^* - \hat{f}_{\tau_N(2)}\|_2^2 \leq \\ A^* \left[\inf_M \{ \|f_1^* - f_{M,1}^*\|_2^2 + \|f_2^* - f_{M,2}^*\|_2^2 + \text{pen}(N, M) \} + \frac{x}{N} \right. \\ \left. + \|\mathbf{Q}^* - \mathbb{P}_{\tau_N} \hat{\mathbf{Q}}_N \mathbb{P}_{\tau_N}^\top\|_F^2 + \|\pi^* - \mathbb{P}_{\tau_N} \hat{\pi}_N\|_2^2 \right] \end{aligned}$$

and

$$\|\mathbf{Q}^* - \mathbb{P}_{\tau_N} \hat{\mathbf{Q}}_N \mathbb{P}_{\tau_N}^\top\| \leq B^* \sqrt{\frac{\log N}{N}} x \quad \text{and} \quad \|\pi^* - \mathbb{P}_{\tau_N} \hat{\pi}_N\|_2 \leq B^* \sqrt{\frac{\log N}{N}} x.$$

Here, \mathbb{P}_{τ_N} is the permutation matrix associated to τ_N .

Remark — An important consequence of the oracle inequality is that a right choice of the penalty leads to a rate minimax adaptive estimator up to a $\log N$ term, see Corollary 1 below.

Remark — Notice that in the situation where \mathcal{F} is given as in (4), $\mathbf{f}^* \in \mathcal{F}^2$ implies that for all M , $\mathbf{f}_M^* \in \mathcal{F}^2$.

Proof. The proof consists in three steps. First, we control the spectral estimator $\hat{\mathbf{Q}}_N$. This is done in Section 3.2, see Corollary 2. Then, we obtain an oracle inequality for the estimation of g^* which is stated below in Proposition 2 and proved in Section 6.1. Notice that this proposition holds for any cardinality K of hidden states (not only $K = 2$) as may be seen from its proof.

Proposition 2 — Assume that **[HB]** holds for \mathbf{Q}^* . Assume also $\mathbf{f}^* \in \mathcal{F}^2$, for all M , $\mathbf{f}_M^* \in \mathcal{F}^2$, and $f_1^* \neq f_2^*$. Then, there exists positive constants ρ^* and A_1^* (depending on $C_{\mathcal{F},2}$ and $C_{\mathcal{F},\infty}$ (**Scenario B**) or on \mathbf{Q}^* , $C_{\mathcal{F},2}$ and $C_{\mathcal{F},\infty}$ (**Scenario B**)) such that, if

$$\text{pen}(N, M) \geq \rho^* \frac{M \log M}{N}$$

for all $x > 0$, for all N , for any permutation $\tau \in \mathcal{S}_2$, one has with probability $1 - (e - 1)^{-1}e^{-x}$

$$\begin{aligned} \|\hat{g} - g^*\|_2^2 &\leq 6 \inf_M \left\{ \|g^* - g^{\mathbf{Q}^*, \mathbf{f}_M^*}\|_2^2 + \text{pen}(N, M) \right\} + A_1^* \frac{x}{N} \\ &\quad + 18C_{\mathcal{F}, 2}^6 (\|\mathbf{Q}^* - \mathbb{P}_\tau \hat{\mathbf{Q}} \mathbb{P}_\tau^\top\|_F^2 + \|\pi^* - \mathbb{P}_\tau \hat{\pi}\|_2^2) \end{aligned}$$

Then, finally we lower bound the risk of \hat{g} by that of $\hat{\mathbf{f}}$ up to relabelling of the states. This uses the following Lemma which is proved in Section 6.2. Recall that $T_{\mathbf{Q}}$ is the set of permutations such that for all i and j , $\mathbf{Q}(i, j) = \mathbf{Q}(\tau(i), \tau(j))$. Thus, if $\mathbf{Q}(1, 2) \neq \mathbf{Q}(2, 1)$, $T_{\mathbf{Q}}$ reduces to the identity, while if $\mathbf{Q}(1, 2) = \mathbf{Q}(2, 1)$, $T_{\mathbf{Q}} = \mathcal{S}_2$. For any $\mathbf{h} = (h_1, h_2) \in (\mathbf{L}^2(\mathcal{Y}, \mathcal{L}^D))^2$ denote $\|\mathbf{h}\|_2 = \|h_1\|_2 + \|h_2\|_2$.

Lemma 2 — *Let \mathcal{K} be a compact subset of $(\mathbf{L}^2(\mathcal{Y}, \mathcal{L}^D))^2$ such that if $\mathbf{h} = (h_1, h_2) \in \mathcal{K}$, then $\int h_i d\mathcal{L}^D = 0$, $i = 1, 2$. Let \mathcal{V} be a compact neighborhood of \mathbf{Q}^* such that, for all $\mathbf{Q} \in \mathcal{V}$, \mathbf{Q} verifies $0 < p < 1$, $0 < q < 1$, $p \neq 1 - q$. Assume that [HB] holds for \mathbf{Q}^* and that $f_1^* \neq f_2^*$. Then there exists a positive constant $c(\mathcal{K}, \mathcal{V}, \mathfrak{F}^*)$ such that*

$$\forall \mathbf{h} = (h_1, h_2) \in \mathcal{K}^2, \forall \mathbf{Q} \in \mathcal{V}, \|g^{\mathbf{Q}, \mathbf{f}^* + \mathbf{h}} - g^{\mathbf{Q}, \mathbf{f}^*}\|_2 \geq c(\mathcal{K}, \mathcal{V}, \mathfrak{F}^*) \|\mathbf{h}\|_{\mathbf{Q}}.$$

Here, $\|\mathbf{h}\|_{\mathbf{Q}} := \min_{\tau \in T_{\mathbf{Q}}} \{\|h_1 + f_1^* - f_{\tau(1)}^*\|_2 + \|h_2 + f_2^* - f_{\tau(2)}^*\|_2\}$.

Let $\mathcal{K} = \{\mathbf{h} = \mathbf{f} - \mathbf{f}^*, \mathbf{f} \in \mathcal{F}^2\}$. Let \mathcal{V} be a compact set such as in Lemma 2 and let $\epsilon > 0$ be such that if \mathbf{Q} is a transition matrix such that $\|\mathbf{Q} - \mathbf{Q}^*\| \leq \epsilon$, then $\mathbf{Q} \in \mathcal{V}$. We shall use Theorem 3 stated in the forthcoming section. Set $B^* = \mathcal{C}(\mathbf{Q}^*, \mathfrak{F}^*)$ and let N^* be an integer larger than $\mathcal{C}(\mathbf{Q}^*, \mathfrak{F}^*)^2/\epsilon^2$ and larger than $\mathbf{N}(\mathbf{Q}^*, \mathfrak{F}^*)$. Observe that for all x and $N \geq (x \vee x^2)N^* \log N$, one has $\epsilon \geq x\mathcal{C}(\mathbf{Q}^*, \mathfrak{F}^*)\sqrt{\log N/N}$. Now using Proposition 2 and Theorem 3 with M_N such that $\eta_3(\Phi_{M_N}) = \sqrt{\log N}$ we get that for all $x > 0$, for all $N \geq (x \vee x^2)N^* \log N$, there exists a permutation τ_N such that with probability $1 - 8e^{-x}$, one has

$$(6) \quad \begin{aligned} \|\hat{g} - g^*\|_2^2 &\leq 6 \inf_M \left\{ \|g^* - g^{\mathbf{Q}^*, \mathbf{f}_M^*}\|_2^2 + \text{pen}(N, M) \right\} + A_1^* \frac{x}{N} \\ &\quad + 18C_{\mathcal{F}, 2}^6 \left(\|\mathbf{Q}^* - \mathbb{P}_{\tau_N} \hat{\mathbf{Q}} \mathbb{P}_{\tau_N}^\top\|_F^2 + \|\pi^* - \mathbb{P}_{\tau_N} \hat{\pi}\|_2^2 \right) \end{aligned}$$

and

$$(7) \quad \|\mathbf{Q}^* - \mathbb{P}_{\tau_N} \hat{\mathbf{Q}} \mathbb{P}_{\tau_N}^\top\| \leq B^* \sqrt{\frac{\log N}{N}} x \text{ and } \|\pi^* - \mathbb{P}_{\tau_N} \hat{\pi}\|_2 \leq B^* \sqrt{\frac{\log N}{N}} x.$$

so that in particular $\mathbb{P}_{\tau_N} \hat{\mathbf{Q}} \mathbb{P}_{\tau_N}^\top \in \mathcal{V}$. Notice that writing

$$\begin{aligned} \hat{g}(y_1, y_2, y_3) &= \sum_{k_1, k_2, k_3=1}^2 (\mathbb{P}_{\tau_N} \hat{\pi}^*)(k_1) (\mathbb{P}_{\tau_N} \hat{\mathbf{Q}} \mathbb{P}_{\tau_N}^\top)(k_1, k_2) (\mathbb{P}_{\tau_N} \hat{\mathbf{Q}} \mathbb{P}_{\tau_N}^\top)(k_2, k_3) \\ &\quad \times \hat{f}_{\tau_N(k_1)}(y_1) \hat{f}_{\tau_N(k_2)}(y_2) \hat{f}_{\tau_N(k_3)}(y_3), \end{aligned}$$

and applying Lemma 2 we get,

$$(8) \quad \|f_1^* - \hat{f}_{\tau_N(1)}\|_2 + \|f_2^* - \hat{f}_{\tau_N(2)}\|_2 \leq \frac{1}{c(\mathcal{K}, \mathcal{V}, \mathfrak{F}^*)} \|\hat{g} - g^{\mathbb{P}_{\tau_N} \hat{\mathbf{Q}} \mathbb{P}_{\tau_N}^\top, \mathbf{f}^*}\|_2.$$

Indeed, in case $\mathbf{Q}^*(1, 2) \neq \mathbf{Q}^*(2, 1)$ one may set \mathcal{V} so that for all $\mathbf{Q} \in \mathcal{V}$, $\mathbf{Q}(1, 2) \neq \mathbf{Q}(2, 1)$ and in case $\mathbf{Q}^*(1, 2) = \mathbf{Q}^*(2, 1)$ one can swap the labels in the estimation of \mathbf{Q}^* so that one may choose τ_N in such a way that (8) holds.

Now by the triangular inequality

$$(9) \quad \|\hat{g} - g^{\mathbb{P}_{\tau_N} \hat{\mathbf{Q}} \mathbb{P}_{\tau_N}^\top, \mathbf{f}^*}\|_2 \leq \|\hat{g} - g^*\|_2 + \|g^{\mathbf{Q}^*, \mathbf{f}^*} - g^{\mathbb{P}_{\tau_N} \hat{\mathbf{Q}} \mathbb{P}_{\tau_N}^\top, \mathbf{f}^*}\|_2$$

We have

$$\begin{aligned} & \left(g^{\mathbf{Q}^*, \mathbf{f}^*} - g^{\mathbb{P}_{\tau_N} \hat{\mathbf{Q}} \mathbb{P}_{\tau_N}^\top, \mathbf{f}^*} \right) (y_1, y_2, y_3) = \\ & \sum_{k_1, k_2, k_3=1}^2 \left(\pi^*(k_1) \mathbf{Q}^*(k_1, k_2) \mathbf{Q}^*(k_2, k_3) - (\mathbb{P}_{\tau_N} \hat{\pi}^*)(k_1) (\mathbb{P}_{\tau_N} \hat{\mathbf{Q}} \mathbb{P}_{\tau_N}^\top)(k_1, k_2) (\mathbb{P}_{\tau_N} \hat{\mathbf{Q}} \mathbb{P}_{\tau_N}^\top)(k_2, k_3) \right) \\ & \quad \times f_{k_1}^*(y_1) f_{k_2}^*(y_2) f_{k_3}^*(y_3), \end{aligned}$$

so that

$$(10) \quad \|g^{\mathbf{Q}^*, \mathbf{f}^*} - g^{\mathbb{P}_{\tau_N} \hat{\mathbf{Q}} \mathbb{P}_{\tau_N}^\top, \mathbf{f}^*}\|_2 \leq \sqrt{3} C_{\mathcal{F}, 2}^3 \left[\|\pi^* - \mathbb{P}_{\tau_N} \hat{\pi}^*\|_2 + 2 \|\mathbf{Q}^* - \mathbb{P}_{\tau_N} \hat{\mathbf{Q}} \mathbb{P}_{\tau_N}^\top\|_F \right].$$

In the same way,

$$\begin{aligned} & \left(g^* - g^{\mathbf{Q}^*, \mathbf{f}_M^*} \right) (y_1, y_2, y_3) = \\ & \sum_{k_1, k_2, k_3=1}^2 \pi^*(k_1) \mathbf{Q}^*(k_1, k_2) \mathbf{Q}^*(k_2, k_3) \left(f_{k_1}^*(y_1) f_{k_2}^*(y_2) f_{k_3}^*(y_3) - f_{M, k_1}^*(y_1) f_{M, k_2}^*(y_2) f_{M, k_3}^*(y_3) \right) \end{aligned}$$

so that

$$\|g^* - g^{\mathbf{Q}^*, \mathbf{f}_M^*}\|_2 \leq 3 C_{\mathcal{F}, 2}^2 \max\{\|f_1^* - f_{M, 1}^*\|; \|f_2^* - f_{M, 2}^*\|\}.$$

Thus collecting (6), (7), (8), (9), (10) and with an appropriate choice of A^* we get Theorem 1. \square

Corollary 1 — *Under the assumptions of Theorem 1, there exists a sequence of permutations $\tau_N \in \mathcal{S}_2$ such that as N tends to infinity,*

$$\begin{aligned} & \mathbb{E} \left[\|f_1^* - \hat{f}_{\tau_N(1)}\|_2^2 + \|f_2^* - \hat{f}_{\tau_N(2)}\|_2^2 \right] = \\ & \quad O \left(\inf_M \left\{ \|f_1^* - f_{M, 1}^*\|^2 + \|f_2^* - f_{M, 2}^*\|^2 + \text{pen}(N, M) \right\} + \frac{\log N}{N} \right), \end{aligned}$$

and

$$\mathbb{E} \left[\|\mathbf{Q}^* - \mathbb{P}_{\tau_N} \hat{\mathbf{Q}} \mathbb{P}_{\tau_N}^\top\| \right] = O \left(\sqrt{\frac{\log N}{N}} \right) \text{ and } \mathbb{E} [\|\pi^* - \mathbb{P}_{\tau_N} \hat{\pi}_N\|_2] = O \left(\sqrt{\frac{\log N}{N}} \right).$$

Thus, choosing $\text{pen}(N, M) = \rho M \log M/N$ for a large ρ leads to the minimax asymptotic rate of convergence up to $\log N$. Indeed, as already said in Section 2.3.3, standard results in approximation theory [DL93] show that one can upper bound the approximation error $\|f_k^* - f_{M, k}^*\|_2$ by $\mathcal{O}(M^{-s})$ where $s > 0$ denotes a regularity parameter. Then the trade-off is obtained for $M \sim (N/\log N)^{1/(2s+D)}$, which leads to the quasi-optimal rate $(N/\log N)^{-s/(2s+D)}$ for the non parametric estimation when the minimal smoothness of the emission densities is s . Notice that the algorithm automatically selects the best M leading to this rate.

To implement the estimator, it remains to choose a value for ρ in the penalty. The calibration of this parameter is a classical issue and could be the subject of a full paper. In practice one can use the slope heuristic [BMM12].

Proof. We shall give the proof concerning the risk of $\hat{\mathbf{Q}}_N$, the proofs for $\hat{\pi}_N$ and for $\hat{\mathbf{f}}$ are similar. First of all, we have

$$\begin{aligned} & \limsup_{N \rightarrow +\infty} \mathbb{E} \left[\sqrt{\frac{N}{\log N}} \|\mathbf{Q}^* - \mathbb{P}_{\tau_N} \hat{\mathbf{Q}}_N \mathbb{P}_{\tau_N}^\top\| \right] \leq \\ & \quad B^* \int_0^{+\infty} \limsup_{N \rightarrow +\infty} \mathbb{P} \left(\frac{\sqrt{N}}{B^* \sqrt{\log N}} \|\mathbf{Q}^* - \mathbb{P}_{\tau_N} \hat{\mathbf{Q}}_N \mathbb{P}_{\tau_N}^\top\| \geq x \right) dx. \end{aligned}$$

Then, Theorem 1 gives that for any $x > \log 8$,

$$\limsup_{N \rightarrow +\infty} \mathbb{P} \left(\frac{\sqrt{N}}{B^* \sqrt{\log N}} \|\mathbf{Q}^* - \mathbb{P}_{\tau_N} \hat{\mathbf{Q}}_N \mathbb{P}_{\tau_N}^T\| \geq x \right) \leq 8e^{-x}$$

so that

$$\limsup_{N \rightarrow +\infty} \mathbb{E} \left[\sqrt{\frac{N}{\log N}} \|\mathbf{Q}^* - \mathbb{P}_{\tau_N} \hat{\mathbf{Q}}_N \mathbb{P}_{\tau_N}^T\| \right] \leq B^* \left[\log 8 + 8 \int_{\log 8}^{+\infty} e^{-x} dx \right].$$

□

3.2. Risk of the spectral estimators. We can now state the Theorem which allows to derive the asymptotic properties of the spectral estimators. As usual in non parametric estimation, the risk is decomposed in a bias term, which comes from the approximating properties of the spaces $(\mathfrak{P}_M)_M$ and decreases when M increases, and in a variance term which comes from the estimator, and increases when M increases. A good choice of M has to balance those two terms. The aim of the following result is to bound the so-called variance term, and what is important is to get a precise behavior of the upper bound with respect to both N and M . The way it depends in M is described by the following quantity. Let us define

$$\eta_3^2(\Phi_M) := \sup_{y, y' \in \mathcal{Y}^3} \sum_{a, b, c=1}^M (\varphi_a(y_1) \varphi_b(y_2) \varphi_c(y_3) - \varphi_a(y'_1) \varphi_b(y'_2) \varphi_c(y'_3))^2.$$

Note that in the examples (**Spline**), (**Trig.**) and (**Wav.**) we have:

$$\eta_3(\Phi_M) \leq C_\eta M^{\frac{3}{2}}$$

where $C_\eta > 0$ is a constant.

Theorem 3 (Spectral estimators) — Assume **[H1]-[H4]**. Then, there exist positive constant numbers $\mathcal{C}(\mathbf{Q}^*, \mathfrak{F}^*)$ and $\mathbf{N}(\mathbf{Q}^*, \mathfrak{F}^*)$ such that the following holds. For any $x > 0$, for any $M \geq M_{\mathfrak{F}^*}$, there exists a permutation $\tau_M \in \mathcal{S}_K$ such that the spectral method estimators $\hat{f}_{M,k}$, $\hat{\pi}$ and $\hat{\mathbf{Q}}$ enjoy: for any $N \geq \mathbf{N}(\mathbf{Q}^*, \mathfrak{F}^*) \eta_3(\Phi_M)^2 x$, with probability greater than $1 - 6e^{-x}$,

$$\|f_{M,k}^* - \hat{f}_{M, \tau_M(k)}\|_2 \leq \mathcal{C}(\mathbf{Q}^*, \mathfrak{F}^*) \frac{\eta_3(\Phi_M)}{\sqrt{N}} x,$$

$$\|\pi^* - \mathbb{P}_{\tau_M} \hat{\pi}\|_2 \leq \mathcal{C}(\mathbf{Q}^*, \mathfrak{F}^*) \frac{\eta_3(\Phi_M)}{\sqrt{N}} x,$$

$$\|\mathbf{Q}^* - \mathbb{P}_{\tau_M} \hat{\mathbf{Q}} \mathbb{P}_{\tau_M}^T\| \leq \mathcal{C}(\mathbf{Q}^*, \mathfrak{F}^*) \frac{\eta_3(\Phi_M)}{\sqrt{N}} x.$$

Let us set the consequences of this theorem.

Corollary 2 — Assume **[H1]-[H4]**. Let M_N be a sequence of integers tending to infinity such that $\eta_3(\Phi_{M_N}) = o(\sqrt{N})$. For each N , define $\hat{\mathbf{f}}_N$, $\hat{\mathbf{Q}}_N$ and $\hat{\pi}_N$ as the estimators obtained by the spectral algorithm with this choice of M_N . Then there exists a sequence of permutations $\tau_N \in \mathcal{S}_K$ such that

$$\mathbb{E}[\|f_{M_N, k}^* - \hat{f}_{N, \tau_N(k)}\|_2] \vee \mathbb{E}[\|\mathbf{Q}^* - \mathbb{P}_{\tau_N} \hat{\mathbf{Q}}_N \mathbb{P}_{\tau_N}^T\|] \vee \mathbb{E}[\|\pi^* - \mathbb{P}_{\tau_N} \hat{\pi}_N\|_2] = O\left(\frac{\eta_3(\Phi_{M_N})}{\sqrt{N}}\right).$$

Here, the expectations are taken on the observations and on the random unitary matrix drawn at **[Step 4]** of the spectral algorithm.

The proof of Corollary 2 follows the same lines as that of Corollary 1. Let us comment on this Corollary. Concerning the parametric part, if we choose M_N such as $\eta_3(\Phi_{M_N}) = (\log N)^\delta$ for some positive δ , as is done to compute the penalized least squares estimator, we get that

$$\mathbb{E} \left[\|\mathbf{Q}^* - \mathbb{P}_{\tau_N} \hat{\mathbf{Q}}_N \mathbb{P}_{\tau_N}^T\| \right] = O \left(\frac{(\log N)^\delta}{\sqrt{N}} \right)$$

and

$$\mathbb{E} [\|\pi^* - \mathbb{P}_{\tau_N} \hat{\pi}_N\|_2] = O \left(\frac{(\log N)^\delta}{\sqrt{N}} \right).$$

Concerning the non parametric part, Corollary 2 gives a control of the so-called variance term, which is of order M^{3D}/N in typical situations such as in the examples **(Spline)**, **(Trig.)** and **(Wav.)**. To get a control on the risk $\|f_k^* - \hat{f}_{N, \tau_N(k)}\|_2$ one has to make a trade-off with the bias term $\|f_k^* - f_{M_N, k}^*\|_2$, which has order $\mathcal{O}(M^{-s})$ where s is the minimal regularity of the emission laws. Choosing $M_N^{3D+2s} \sim N$, this leads to the rate $N^{-s/(2s+3D)}$ for the non parametric estimation. This is similar to the rate of estimation of a density in dimension $3D$ with smoothness s . This loss of rate compared to the minimax rate obtained in Theorem 1 may be understood by the fact that spectral estimators are built from the empirical estimate of the three-dimensional marginal distribution (thus in a $3D$ -dimensional space) of the observations.

4. NUMERICAL EXPERIMENTS

This section is devoted to numerically advocate the performances of the adaptive estimation method studied in this paper. We recall that the experimenter knows nothing about the underlying hidden Markov model but the number of hidden states K . In this set of experiments, we consider the regular histogram basis for estimating $K = 2$ emission laws given by beta laws of parameters $(2, 5)$ and $(4, 3)$ from a single chain of size $N = 30,000$.

Our method is based on the computation of least squares estimators \hat{g}_M defined as minimizers of the empirical contrast γ_N . Then, the values of $\gamma_N(\hat{g}_M)$ (as M varies) are used to heuristically calibrate the penalty $\text{pen}(N, M)$ so as to, eventually, compute the adaptive choice of the size of the model, namely

$$\hat{M} = \arg \min_{M=1, \dots, N} \{ \gamma_N(\hat{g}_M) + \text{pen}(N, M) \}.$$

We understand that a crucial step lies in computing least squares estimators \hat{g}_M . One may struggle to compute \hat{g}_M since the function γ_N is non-convex. It follows that an acceptable procedure must start from a good approximation of \hat{g}_M . This is done by the spectral method. Observe that the key leitmotiv throughout this paper is a two steps estimation procedure that starts by the spectral estimator. This latter has rate of convergence of the order of $N^{-s/(2s+3)}$ and seems to be a good candidate to initialize an iterative scheme that will converge towards \hat{g}_M . Hence we compute \hat{g}_M for each $M = 1, \dots, N$ as follows

- First compute the spectral estimator. This is straightforward using the procedure described by **[Step1-9]**, Section 2.4. In particular, the spectral estimator gives an estimation $\hat{Q}, \hat{\pi}$ of the transition matrix and its stationary distribution which is used to compute the least squares contrast function.
- Use the spectral estimator of the emission densities as a starting point for “Covariance Matrix Adaptation Evolution Strategy” (CMA-ES), see **[Han06]**. This iterative algorithm may ultimately find a local/global minimum of the contrast function.

Our numerical experiments follow this path.

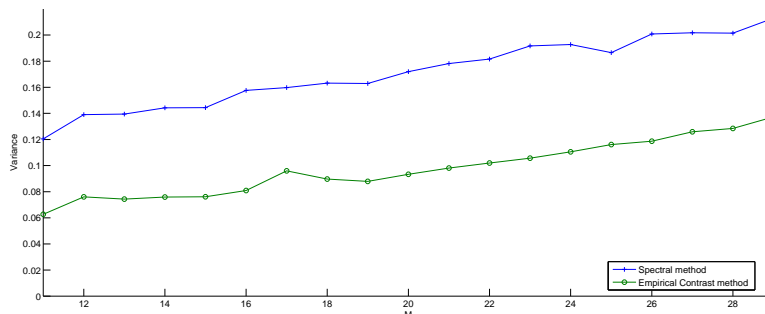


FIGURE 1. Comparison of the variances of the spectral and the least squares estimators.

A first numerical experiment, depicted in Figure 1, compares, for each M , the variances (i.e. the ℓ_2 -distance between the estimator and the orthogonal projection onto the subspace generated by the basis Φ_M) obtained by the spectral method and the empirical least squares method over 100 iterations on chains of length 40,000. It consolidates the idea that the least square method significantly improves the ℓ_2 -distance to the best approximation of the emission laws. Indeed, even for small values of M , one may see that the variance is divided by two in Figure 1.

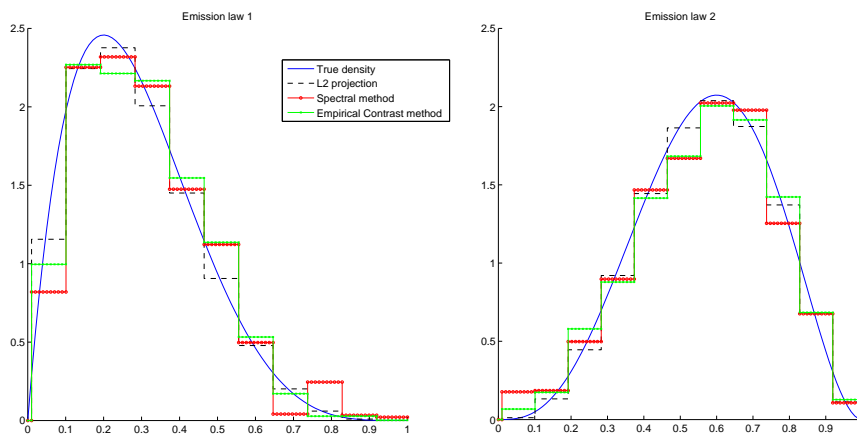


FIGURE 2. Estimators of the emissions densities.

Interestingly, we are able to proceed the slope heuristic procedure in the non-parametric HMM's frame. This compelling data-driven procedure allows us to tune the penalty appearing in our estimator. More precisely, typical behaviors of the function $M \mapsto \gamma(\hat{g}_M)$ and the function $\rho \mapsto \arg \min\{\gamma(\hat{g}_M) + \rho \text{pen}(M)\}$ usher the experimenter to the right tuning parameter, an interested reader may consult [BMM12]. Note that the size M of the projection space for the spectral estimator has been set as the one chosen by the slope heuristic for the least squares estimators. One can see on Figure 2 that our method also qualitatively improve upon the spectral method.

5. DISCUSSION

We have proposed two procedures to estimate the law of a hidden Markov chain. The spectral method has the advantage of being based on simple matrix computations and it provides an estimator of the dynamics of the hidden chain. As for it, the penalized least-squares method allows to obtain quasi-optimal estimates of the emission laws. The result is stated only for $K = 2$ but we can conjecture that it can be generalized to any K . Indeed, the limitation comes only from Lemma 2 and seems purely technical.

Another interesting extension of our work would be to consider emission laws f_k^* with different smoothnesses s_k and to find a procedure which adapts to each behaviour, i.e. which selects a different approximation level M for each hidden state k .

6. PROOFS

6.1. Proof of Proposition 2. The proof is written for any fixed K , not only for $K = 2$. Throughout the proof N is fixed, and we write γ (instead of γ_N) for the contrast function.

6.1.1. *Beginning of the proof: algebraic manipulations.*

Let us fix some M and some permutation τ . Using the definitions of \hat{g}_M and \hat{M} , we can write

$$\gamma(\hat{g}_{\hat{M}}) + \text{pen}(\hat{M}, N) \leq \gamma(\hat{g}_M) + \text{pen}(M, N) \leq \gamma(g^{\hat{\mathbf{Q}}, \mathbf{f}_{M, \tau^{-1}}^*}) + \text{pen}(M, N),$$

where $\mathbf{f}_{M, \tau^{-1}}^* = (f_{M, \tau^{-1}(1)}^*, \dots, f_{M, \tau^{-1}(K)}^*)$ (here we use that $\mathbf{f}_{M, \tau^{-1}}^* \in \mathcal{F}^K$). But we can compute for all functions t_1, t_2 ,

$$\gamma(t_1) - \gamma(t_2) = \|t_1 - g^*\|_2^2 - \|t_2 - g^*\|_2^2 - 2\nu(t_1 - t_2),$$

where ν is the centered empirical process

$$\nu(t) = \frac{1}{N} \sum_{s=1}^N t(Y_1^{(s)}, Y_2^{(s)}, Y_3^{(s)}) - \int t g^*.$$

This gives

$$(11) \quad \|\hat{g}_{\hat{M}} - g^*\|_2^2 \leq \|g^{\hat{\mathbf{Q}}, \mathbf{f}_{M, \tau^{-1}}^*} - g^*\|_2^2 + 2\nu(\hat{g}_{\hat{M}} - g^{\hat{\mathbf{Q}}, \mathbf{f}_{M, \tau^{-1}}^*}) + \text{pen}(M, N) - \text{pen}(\hat{M}, N)$$

Now, we denote $R_{\hat{M}} = \|\hat{g}_{\hat{M}} - g^*\|_2^2$ the squared risk and $B_M = \|g^{\mathbf{Q}^*, \mathbf{f}_M^*} - g^*\|_2^2$ a bias term. We also set $S_M = \cup_{\mathbf{Q}} \mathcal{S}(\mathbf{Q}, M)$ and

$$Z_M = \sup_{t \in S_M} \left[\frac{|\nu(t - g^*)|}{\|t - g^*\|_2^2 + x_M^2} \right]$$

for x_M to be determined later. Notice that $g^{\hat{\mathbf{Q}}, \mathbf{f}_{M, \tau^{-1}}^*} = g^{\mathbb{P}_\tau \hat{\mathbf{Q}} \mathbb{P}_\tau^\top, \mathbf{f}_M^*}$. Then

$$\begin{aligned} \|g^{\hat{\mathbf{Q}}, \mathbf{f}_{M, \tau^{-1}}^*} - g^*\|_2^2 &\leq 2\|g^{\hat{\mathbf{Q}}, \mathbf{f}_{M, \tau^{-1}}^*} - g^{\mathbf{Q}^*, \mathbf{f}_M^*}\|_2^2 + 2\|g^{\mathbf{Q}^*, \mathbf{f}_M^*} - g^*\|_2^2 \\ &\leq 2\|g^{\mathbb{P}_\tau \hat{\mathbf{Q}} \mathbb{P}_\tau^\top, \mathbf{f}_M^*} - g^{\mathbf{Q}^*, \mathbf{f}_M^*}\|_2^2 + 2B_M. \end{aligned}$$

But, using Schwarz inequality, $\|g^{\mathbf{Q}_1, \mathbf{f}_M^*} - g^{\mathbf{Q}_2, \mathbf{f}_M^*}\|_2^2$ can be bounded by

$$\begin{aligned} & \sum_{m_1, m_2, m_3=1}^M \left| \sum_{k_1, k_2, k_3=1}^K (\pi_1(k_1) \mathbf{Q}_1(k_1, k_2) \mathbf{Q}_1(k_2, k_3) - \pi_2(k_1) \mathbf{Q}_2(k_1, k_2) \mathbf{Q}_2(k_2, k_3)) \right. \\ & \qquad \qquad \qquad \left. \langle f_{k_1}^*, \varphi_{m_1} \rangle \langle f_{k_2}^*, \varphi_{m_2} \rangle \langle f_{k_3}^*, \varphi_{m_3} \rangle \right|^2 \\ & \leq \sum_{k_1, k_2, k_3=1}^K (\pi_1(k_1) \mathbf{Q}_1(k_1, k_2) \mathbf{Q}_1(k_2, k_3) - \pi_2(k_1) \mathbf{Q}_2(k_1, k_2) \mathbf{Q}_2(k_2, k_3))^2 C_{\mathcal{F}, 2}^2 C_{\mathcal{F}, 2}^2 C_{\mathcal{F}, 2}^2 \\ & \leq 3C_{\mathcal{F}, 2}^6 (\|\pi_1 - \pi_2\|_2^2 + 2\|\mathbf{Q}_1 - \mathbf{Q}_2\|_F^2) \end{aligned}$$

so that

$$\|g^{\hat{\mathbf{Q}}, \mathbf{f}_{M, \tau^{-1}}^*} - g^*\|_2^2 \leq 6C_{\mathcal{F}, 2}^6 \left(\|\mathbb{P}_\tau \hat{\pi} - \pi^*\|_2^2 + 2\|\mathbb{P}_\tau \hat{\mathbf{Q}} \mathbb{P}_\tau^\top - \mathbf{Q}^*\|_F^2 \right) + 2B_M.$$

Next

$$\begin{aligned} \nu(\hat{g}_{\hat{M}} - g^{\hat{\mathbf{Q}}, \mathbf{f}_{M, \tau^{-1}}^*}) &= \nu(\hat{g}_{\hat{M}} - g^*) + \nu(g^* - g^{\hat{\mathbf{Q}}, \mathbf{f}_{M, \tau^{-1}}^*}) \\ &\leq Z_{\hat{M}} (\|\hat{g}_{\hat{M}} - g^*\|_2^2 + x_{\hat{M}}^2) + Z_M (\|g^{\hat{\mathbf{Q}}, \mathbf{f}_{M, \tau^{-1}}^*} - g^*\|_2^2 + x_M^2) \end{aligned}$$

so that (11) becomes

$$\begin{aligned} R_{\hat{M}} &\leq 6C_{\mathcal{F}, 2}^6 \left(\|\mathbb{P}_\tau \hat{\pi} - \pi^*\|_2^2 + 2\|\mathbb{P}_\tau \hat{\mathbf{Q}} \mathbb{P}_\tau^\top - \mathbf{Q}^*\|_F^2 \right) + 2B_M + 2Z_{\hat{M}} (R_{\hat{M}} + x_{\hat{M}}^2) \\ &\quad + 2Z_M (6C_{\mathcal{F}, 2}^6 \left(\|\mathbb{P}_\tau \hat{\pi} - \pi^*\|_2^2 + 2\|\mathbb{P}_\tau \hat{\mathbf{Q}} \mathbb{P}_\tau^\top - \mathbf{Q}^*\|_F^2 \right) + 2B_M + x_M^2) \\ &\quad + 2\text{pen}(M, N) - \text{pen}(\hat{M}, N) - \text{pen}(M, N), \\ R_{\hat{M}} (1 - 2Z_{\hat{M}}) &\leq (2 + 4Z_M) B_M + 2\text{pen}(M, N) \\ &\quad + (1 + 2Z_M) 6C_{\mathcal{F}, 2}^6 \left(\|\mathbb{P}_\tau \hat{\pi} - \pi^*\|_2^2 + 2\|\mathbb{P}_\tau \hat{\mathbf{Q}} \mathbb{P}_\tau^\top - \mathbf{Q}^*\|_F^2 \right) \\ &\quad + 2 \sup_{M'} (2Z_{M'} x_{M'}^2 - \text{pen}(M', N)). \end{aligned}$$

To conclude it is then sufficient to establish the bounds, with probability greater than $1 - (e - 1)^{-1} e^{-x}$, it holds

$$\sup_{M'} Z_{M'} \leq \frac{1}{4} \quad \text{and} \quad \sup_{M'} (2Z_{M'} x_{M'}^2 - \text{pen}(M', N)) \leq A \frac{x}{N},$$

with A a constant depending only on \mathbf{Q}^* and \mathbf{f}^* and not on N, M, x . Thus we will have, for any M ,

$$\begin{aligned} \frac{1}{2} R_{\hat{M}} &\leq 3B_M + 2\text{pen}(M, N) + 2A \frac{x}{N} \\ &\quad + 9C_{\mathcal{F}, 2}^6 \left(\|\mathbb{P}_\tau \hat{\pi} - \pi^*\|_2^2 + 2\|\mathbb{P}_\tau \hat{\mathbf{Q}} \mathbb{P}_\tau^\top - \mathbf{Q}^*\|_F^2 \right) \end{aligned}$$

which is the announced result.

The heart of the proof is then the study of Z_M . We introduce u_M the projection of g^* on S_M and we split Z_M in two terms: $Z_M \leq 4Z_{M,1} + Z_{M,2}$ with

$$\begin{cases} Z_{M,1} = \sup_{t \in S_M} \left[\frac{|\nu(t - u_M)|}{\|t - u_M\|_2^2 + 4x_M^2} \right] \\ Z_{M,2} = \frac{|\nu(u_M - g^*)|}{\|u_M - g^*\|_2^2 + x_M^2} \end{cases}$$

Indeed u_M verifies: for all $t \in S_M$,

$$\|u_M - g^*\|_2 \leq \|t - g^*\|_2 \quad \text{and} \quad \|u_M - t\|_2 \leq 2\|t - g^*\|_2.$$

6.1.2. Deviation inequality for $Z_{M,2}$.

Bernstein inequality for HMM (31) gives, with probability larger than $1 - e^{-z}$:

$$|\nu(u_M - g^*)| \leq 2\sqrt{2c^*\|u_M - g^*\|_2^2\|g^*\|_\infty} \frac{z}{N} + 2\sqrt{2}c^*\|u_M - g^*\|_\infty \frac{z}{N}.$$

Then, using $a^2 + b^2 \geq 2ab$, with probability larger than $1 - e^{-z}$:

$$\frac{|\nu(u_M - g^*)|}{\|u_M - g^*\|_2^2 + x_M^2} \leq 2\sqrt{2c^*\|g^*\|_\infty} \frac{1}{2x_M} \sqrt{\frac{z}{N}} + 2\sqrt{2}c^* \frac{\|u_M\|_\infty + \|g^*\|_\infty}{x_M^2} \frac{z}{N}.$$

But any function t in S_M can be written

$$t = \sum_{k_1, k_2, k_3=1}^K \pi(k_1) \mathbf{Q}(k_1, k_2) \mathbf{Q}(k_2, k_3) f_{k_1} \otimes f_{k_2} \otimes f_{k_3},$$

with $f_k \in \mathcal{F}$ for $k = 1, \dots, K$, so that $\sup_{t \in S_M} \|t\|_\infty \leq C_{\mathcal{F}, \infty}^3$. Then, with probability larger than $1 - e^{-z_M - z}$

$$Z_{M,2} \leq \sqrt{2c^*\|g^*\|_\infty} \sqrt{\frac{z_M + z}{x_M^2 N}} + 4\sqrt{2}c^* C_{\mathcal{F}, \infty}^3 \frac{z_M + z}{x_M^2 N}.$$

6.1.3. Deviation inequality for $Z_{M,1}$.

We shall first study the term $\sup_{t \in B_\sigma} |\nu(t - u_M)|$ where

$$B_\sigma = \{t \in S_M, \|t - u_M\|_2 \leq \sigma\}.$$

Remark that, for all $t \in \mathcal{S}(\mathbf{Q}, M)$,

$$\|t\|_2^2 \leq \sum_{k_1, k_2, k_3=1}^K \pi^2(k_1) \mathbf{Q}^2(k_1, k_2) \mathbf{Q}^2(k_2, k_3) \sum_{k_1, k_2, k_3=1}^K C_{\mathcal{F}, 2}^2 C_{\mathcal{F}, 2}^2 C_{\mathcal{F}, 2}^2 \leq K^3 C_{\mathcal{F}, 2}^6.$$

Then, if $t \in B_\sigma$, $\|t - u_M\|_2 \leq \sigma \wedge 2K^{3/2} C_{\mathcal{F}, 2}^3$. Notice also that for all $t \in S_M$, $\|t - u_M\|_\infty \leq 2C_{\mathcal{F}, \infty}^3$. Now Proposition 10 (applied to a countable dense set in B_σ) gives

$$\mathbb{E}^A \left(\sup_{t \in B_\sigma} |\nu(t - u_M)| \right) \leq C^* \left[\frac{E}{N} + \sigma \sqrt{\frac{1}{N} \log \left(\frac{1}{\mathbb{P}(A)} \right)} + \frac{2C_{\mathcal{F}, \infty}^3}{N} \log \left(\frac{1}{\mathbb{P}(A)} \right) \right],$$

and

$$E = \sqrt{N} \int_0^\sigma \sqrt{H(u) \wedge N} du + (2C_{\mathcal{F}, \infty}^3 + 2K^{3/2} C_{\mathcal{F}, 2}^3) H(\sigma).$$

We shall compute this term later and find σ_M and φ such that

$$(12) \quad \forall \sigma \geq \sigma_M \quad E \leq (1 + 2C_{\mathcal{F}, \infty}^3 + 2K^{3/2} C_{\mathcal{F}, 2}^3) \varphi(\sigma) \sqrt{N}.$$

(see Section 6.1.4). We then use Lemma 4.23 in [Mas07] to write (for $x_M \geq \sigma_M$)

$$\mathbb{E}^A \left(\sup_{t \in S_M} \left[\frac{|\nu(t - u_M)|}{\|t - u_M\|_2^2 + 4x_M^2} \right] \right) \leq \frac{C^*}{x_M^2} \left[C \frac{\varphi(2x_M)}{\sqrt{N}} + 2x_M \sqrt{\frac{1}{N} \log \left(\frac{1}{\mathbb{P}(A)} \right)} + \frac{2C_{\mathcal{F}, \infty}^3}{N} \log \left(\frac{1}{\mathbb{P}(A)} \right) \right]$$

Finally, Lemma 2.4 in [Mas07] ensures that, with probability $1 - e^{-z_M - z}$:

$$(13) \quad Z_{M,1} = \sup_{t \in S_M} \left[\frac{|\nu(t - u_M)|}{\|t - u_M\|_2^2 + 4x_M^2} \right] \leq C^* \left[C \frac{\varphi(2x_M)}{x_M^2 \sqrt{N}} + 2\sqrt{\frac{z_M + z}{x_M^2 N}} + 2C_{\mathcal{F}, \infty}^3 \frac{z_M + z}{x_M^2 N} \right]$$

6.1.4. *Computation of the entropy and function φ .*

The definition of H given in Proposition 10 shows that $H(\delta)$ is bounded by the classical bracketing entropy for \mathbf{L}^2 distance at point $\delta/C_{\mathcal{F},\infty}^3$ (where $C_{\mathcal{F},\infty}^3$ bounds the sup norm of g^*): $H(\delta) \leq H(\delta/C_{\mathcal{F},\infty}^3, S_M, \mathbf{L}^2)$. We denote by $N(u, S, \mathbf{L}^2) = e^{H(u, S, \mathbf{L}^2)}$ the minimal number of \mathbf{L}^2 -balls of radius u to cover S . Now, observe that $S_M = \cup_{\mathbf{Q}} \mathcal{S}(\mathbf{Q}, M)$ is a set of mixture of parametric functions. Denoting $\mathbf{k} = (k_1, k_2, k_3)$, S_M is included in

$$\left\{ \sum_{\mathbf{k} \in \{1, \dots, K\}^3} \mu(\mathbf{k}) f_{k_1} \otimes f_{k_2} \otimes f_{k_3}, \mu \geq 0, \sum_{\mathbf{k} \in \{1, \dots, K\}^3} \mu(\mathbf{k}) = 1, f_k \in \mathcal{F} \cap \text{Span}(\varphi_1, \dots, \varphi_M) \right\}.$$

Set

$$\begin{aligned} \mathcal{A}_{\mathbf{k}} &= \{f_{k_1} \otimes f_{k_2} \otimes f_{k_3}, f_{k_i} \in \mathcal{F} \cap \text{Span}(\varphi_1, \dots, \varphi_M)\} \\ &= \{t(\cdot, a, \mathbf{k}) := \sum_{m_1, m_2, m_3=1}^M a_{m_1 k_1} a_{m_2 k_2} a_{m_3 k_3} \varphi_{m_1} \otimes \varphi_{m_2} \otimes \varphi_{m_3}, \sum_m a_{m k_i} \varphi_m \in \mathcal{F}\}. \end{aligned}$$

such that S_M is included in the mixture of $\mathcal{A}_{\mathbf{k}}$, $\mathbf{k} \in \{1, \dots, K\}^3$. Following the proof in Appendix A of [BT13], we can prove

$$N(\varepsilon, S_M, \mathbf{L}^2) \leq \left(\frac{C_1}{\varepsilon}\right)^{K^3-1} \prod_{\mathbf{k} \in \{1, \dots, K\}^3} N\left(\frac{\varepsilon}{3}, \mathcal{A}_{\mathbf{k}}, \mathbf{L}^2\right).$$

where C_1 depends on K and $C_{\mathcal{F},2}$. Notice that if $t(\cdot, a, \mathbf{k}) \in \mathcal{A}_{\mathbf{k}}$, the vector $a \in \mathbb{R}^{MK}$ verifies $\sum_{m=1}^M |a_{m k_i}|^2 = \|f_{k_i}\|_2^2 \leq C_{\mathcal{F},2}^2$. Moreover, for $t(\cdot, a, \mathbf{k})$ and $t(\cdot, b, \mathbf{k}) \in \mathcal{A}_{\mathbf{k}}$

$$\begin{aligned} \|t(\cdot, a, \mathbf{k}) - t(\cdot, b, \mathbf{k})\|_2^2 &= \sum_{m_1, m_2, m_3=1}^M |a_{m_1 k_1} a_{m_2 k_2} a_{m_3 k_3} - b_{m_1 k_1} b_{m_2 k_2} b_{m_3 k_3}|^2 \\ &\leq \sum_{m_1, m_2, m_3=1}^M |a_{m_1 k_1} a_{m_2 k_2} (a_{m_3 k_3} - b_{m_3 k_3}) \\ &\quad + a_{m_1 k_1} (a_{m_2 k_2} - b_{m_2 k_2}) a_{m_3 k_3} + (a_{m_1 k_1} - b_{m_1 k_1}) b_{m_2 k_2} b_{m_3 k_3}|^2 \\ &\leq 3C_{\mathcal{F},2}^4 \|a - b\|_{MK}^2 \end{aligned}$$

Thus, $\mathcal{A}_{\mathbf{k}}$ is a parametric family where the \mathbf{L}^2 -distance is controlled by the ℓ_2 distance in \mathbb{R}^{MK} . Then combinatorial computations give

$$N(u, \mathcal{A}_{\mathbf{k}}, \mathbf{L}^2) \leq \left(\frac{2C_{\mathcal{F},2}\sqrt{MK}}{u/(\sqrt{3}C_{\mathcal{F},2}^2)}\right)^{MK}.$$

We deduce

$$N(u, S_M, \mathbf{L}^2) \leq \left(\frac{C_1}{u}\right)^{K^3-1} \prod_{\mathbf{k}} \left(\frac{2C_{\mathcal{F},2}\sqrt{MK}}{u/(3\sqrt{3}C_{\mathcal{F},2}^2)}\right)^{MK},$$

and then

$$H(u, S_M, \mathbf{L}^2) \leq (K^3 - 1) \log(C_1/u) + MK^4 \log\left(\frac{C_2 M^{1/2}}{u}\right),$$

with C_2 depending on K and $C_{\mathcal{F},2}$. To conclude we use that $\int_0^\sigma \sqrt{\log(\frac{1}{x})} dx \leq \sigma(\sqrt{\pi} + \sqrt{\log(\frac{1}{\sigma})})$ (see [MM08]). Finally we can write

$$\int_0^\sigma \sqrt{H(u)} du \leq C_3 \sqrt{M} \sigma \left(1 + \sqrt{\log\left(\frac{M^{1/2}}{\sigma}\right)} \right),$$

where C_3 depends on K , $C_{\mathcal{F},2}$ and $C_{\mathcal{F},\infty}$. Set

$$\varphi(x) = C_3 \sqrt{M} x \left(1 + \sqrt{\log\left(\frac{M^{1/2}}{x}\right)} \right)$$

The function φ is increasing, and $\varphi(x)/x$ is decreasing. Moreover $\varphi(\sigma) \geq \int_0^\sigma \sqrt{H(u)} du$ and $\varphi^2(\sigma) \geq \sigma^2 H(\sigma)$.

6.1.5. End of the proof, choice of parameters.

We define σ_M as the solution of equation $\varphi(x) = \sqrt{N}x^2$. Then, for all $\sigma \geq \sigma_M$,

$$H(\sigma) \leq \frac{\varphi(\sigma)^2}{\sigma^2} \leq \frac{\varphi(\sigma)}{\sigma} \sigma \sqrt{N}.$$

This yields, for all $\sigma \geq \sigma_M$,

$$E \leq (1 + 2C_{\mathcal{F},\infty}^3 + 2K^{3/2}C_{\mathcal{F},2}^3)\varphi(\sigma)\sqrt{N},$$

which was required in (12).

Moreover $\frac{\varphi(2x_M)}{x_M \sqrt{N}} \leq 2\sigma_M$ as soon as $x_M \geq \sigma_M$. Combining (13) and (12), we obtain, with probability $1 - e^{-z_M - z}$:

$$Z_M \leq C^{**} \left[\frac{\sigma_M}{x_M} + \sqrt{\frac{z_M + z}{x_M^2 N}} + \frac{z_M + z}{x_M^2 N} \right],$$

where C^{**} depends on K , $C_{\mathcal{F},2}$, $C_{\mathcal{F},\infty}$, \mathbf{Q}^* . Now let us choose $x_M = \theta^{-1} \sqrt{\sigma_M^2 + \frac{z_M + z}{N}}$ with θ such that $2\theta + \theta^2 \leq (C^{**})^{-1}/4$. This choice entails: $x_M \geq \theta^{-1}\sigma_M$ and $x_M^2 \geq \theta^{-2} \frac{z_M + z}{N}$. Then with probability $1 - e^{-z_M - z}$:

$$Z_M \leq C^{**}(\theta + \theta + \theta^2).$$

We now choose $z_M = M$ which implies $\sum_{M \geq 1} e^{-z_M} = (e-1)^{-1}$. Then, with probability $1 - (e-1)^{-1}e^{-z}$

$$\forall M \quad Z_M \leq C^{**}(2\theta + \theta^2) \leq \frac{1}{4},$$

and for all M ,

$$\begin{aligned} Z_M x_M^2 &\leq C^{**} \left[\sigma_M x_M + x_M \sqrt{\frac{z_M + z}{N}} + \frac{z_M + z}{N} \right] \\ &\leq C^{**} \theta^{-1} \left(\sigma_M + \sqrt{\frac{z_M + z}{N}} \right)^2 + C^{**} \frac{z_M + z}{N}. \end{aligned}$$

Then, with probability $1 - (e-1)^{-1}e^{-z}$, for all M ,

$$Z_M x_M^2 - C^{**} \left(2\theta^{-1} \sigma_M^2 + (2\theta^{-1} + 1) \frac{M}{N} \right) \leq C^{**} (2\theta^{-1} + 1) \frac{z}{N}.$$

Then the result is proved as soon as $\text{pen}(M, N) \geq 2C^{**} (2\theta^{-1}\sigma_M^2 + (2\theta^{-1} + 1)\frac{M}{N})$. Recall that σ_M is defined as the solution of equation $2C_3\sqrt{M}x(1 + \sqrt{\log(\frac{M^{1/2}}{2x})}) = 4\sqrt{N}x^2$. Then we obtain that

$$\sigma_M \geq C_4\sqrt{\frac{M}{N}}(1 + \sqrt{\log(M)}),$$

and $\text{pen}(M, N) \geq \rho^* \frac{M \log(M)}{N}$. Notice that the dependence on \mathbf{Q}^* in ρ^* arises from Propositions 9 and 10, so that it vanishes in **Scenario A**.

6.2. Proof of Lemma 2. Let $\mathbf{h} = (h_1, h_2) \in (\mathbf{L}^2(\mathcal{Y}, \mathcal{L}^D))^2$ and let

$$\mathbf{Q} = \begin{pmatrix} 1-p & p \\ q & 1-q \end{pmatrix}$$

be a 2×2 transition matrix. The fact that $\mathbf{Q} \in \mathcal{V}$ implies that

$$(14) \quad 0 < p < 1, \quad 0 < q < 1, \quad \text{and} \quad p \neq 1 - q.$$

Denote $N(\mathbf{Q}, \mathbf{h}) = \|g^{\mathbf{Q}, \mathbf{f}^* + \mathbf{h}} - g^{\mathbf{Q}, \mathbf{f}^*}\|_2^2$. What we want to prove is that

$$c := c(\mathcal{K}, \mathcal{V}, \mathfrak{F}^*) := \inf_{\mathbf{Q} \in \mathcal{V}, \mathbf{h} \in \mathcal{K}^2, \|\mathbf{h}\|_{\mathbf{Q}} \neq 0} \frac{N(\mathbf{Q}, \mathbf{h})}{\|\mathbf{h}\|_{\mathbf{Q}}^2} > 0.$$

Let $(\mathbf{Q}_n, \mathbf{h}_n)_n$ be a sequence in $\mathcal{V} \times \mathcal{K}$ such that $c = \lim_n \frac{N(\mathbf{Q}_n, \mathbf{h}_n)}{\|\mathbf{h}_n\|_{\mathbf{Q}_n}^2}$. Let $(\tilde{\mathbf{Q}}, \tilde{\mathbf{h}})$ be a limit point of the sequence in the compact set $\mathcal{V} \times \mathcal{K}$. Then one has (considering the cases $\tilde{\mathbf{Q}}(1, 2) \neq \tilde{\mathbf{Q}}(2, 1)$ and $\tilde{\mathbf{Q}}(1, 2) = \tilde{\mathbf{Q}}(2, 1)$) that

$$\lim_{n \rightarrow +\infty} \|\mathbf{h}_n\|_{\mathbf{Q}_n} \geq \|\tilde{\mathbf{h}}\|_{\tilde{\mathbf{Q}}}.$$

Thus if $\|\tilde{\mathbf{h}}\|_{\tilde{\mathbf{Q}}} \neq 0$, then using Lemma 1

$$c = \frac{N(\tilde{\mathbf{Q}}, \tilde{\mathbf{h}})}{\lim_{n \rightarrow +\infty} \|\mathbf{h}_n\|_{\mathbf{Q}_n}^2} > 0.$$

If $\|\tilde{\mathbf{h}}\|_{\tilde{\mathbf{Q}}} = 0$, we shall consider separately the situation where $\tilde{\mathbf{Q}}(1, 2) \neq \tilde{\mathbf{Q}}(2, 1)$ and the situation where $\tilde{\mathbf{Q}}(1, 2) = \tilde{\mathbf{Q}}(2, 1)$.

Let us first consider the situation where $\tilde{\mathbf{Q}}(1, 2) \neq \tilde{\mathbf{Q}}(2, 1)$. In this situation, $\|\mathbf{h}\|_{\tilde{\mathbf{Q}}} = \|\mathbf{h}\|_2$. Direct computation shows that $N(\mathbf{Q}, \mathbf{h})$ is polynomial in the variables $p, q, \langle f_i^*, f_j^* \rangle, \langle h_i, f_j^* \rangle, \langle h_i, h_j \rangle$ without linear part. Let $D(\mathbf{Q}, \mathbf{h})$ denote the quadratic part with respect to the variable \mathbf{h} . One gets

$$N(\mathbf{Q}, \mathbf{h}) = D(\mathbf{Q}, \mathbf{h}) + O(\|\mathbf{h}\|_2^3)$$

where the $O(\cdot)$ depends only on \mathbf{f}^* . Let us first notice that $D(\cdot, \cdot)$ is always non negative. Indeed, since for all $\mathbf{Q} \in \mathcal{V}$ and all $\mathbf{h} \in (\mathbf{L}^2(\mathcal{Y}, \mathcal{L}^D))^2$ one has $N(\mathbf{Q}, \mathbf{h}) \geq 0$, it holds

$$\forall \mathbf{Q} \in \mathcal{V}, \forall \mathbf{h} \in (\mathbf{L}^2(\mathcal{Y}, \mathcal{L}^D))^2, \frac{D(\mathbf{Q}, \mathbf{h})}{\|\mathbf{h}\|_2^2} + O(\|\mathbf{h}\|_2) \geq 0,$$

so that, since for all $\lambda \in \mathbb{R}$, $D(\mathbf{Q}, \lambda \mathbf{h}) = \lambda^2 D(\mathbf{Q}, \mathbf{h})$,

$$(15) \quad \forall \mathbf{Q} \in \mathcal{V}, \forall \mathbf{h} \in (\mathbf{L}^2(\mathcal{Y}, \mathcal{L}^D))^2, D(\mathbf{Q}, \mathbf{h}) \geq 0.$$

Then we obtain in the case (a) where $\|\tilde{\mathbf{h}}\|_{\tilde{\mathbf{Q}}} = 0$ and $\tilde{\mathbf{Q}}(1, 2) \neq \tilde{\mathbf{Q}}(2, 1)$ that

$$c = \lim_{n \rightarrow +\infty} D(\mathbf{Q}_n, \frac{\mathbf{h}_n}{\|\mathbf{h}_n\|_2}).$$

Indeed in this case for large enough n , $\mathbf{Q}_n(1, 2) \neq \mathbf{Q}_n(2, 1)$ so that $\|\mathbf{h}_n\|_{\mathbf{Q}_n} = \|\mathbf{h}_n\|_2$. We shall now study the function $D(\mathbf{Q}, \mathbf{a})$ for $\mathbf{a} = (a_1, a_2)$ of form $\frac{\mathbf{h}}{\|\mathbf{h}\|_2}$ with $\mathbf{h} \in \mathcal{K}^2$.

Let $\mathbf{u} = (u_1, u_2)$ be such that u_i , $i = 1, 2$, is the orthogonal projection of a_i on the subspace of $\mathbf{L}^2(\mathcal{Y}, \mathcal{L}^D)$ orthogonal to f_1^* and f_2^* . Direct computation gives that

$$D(\mathbf{Q}, \mathbf{a}) = T(\mathbf{Q}, \mathbf{u}) + D(\mathbf{Q}, \mathbf{a} - \mathbf{u})$$

where for any \mathbf{Q} and \mathbf{u} ,

$$\begin{aligned} T(\mathbf{Q}, \mathbf{u}) &= \sum_{i,j=1}^2 \{ \langle (\mathbf{Q}^T \mathbf{A} \mathbf{u})_i, (\mathbf{Q}^T \mathbf{A} \mathbf{u})_j \rangle \langle f_i^*, f_j^* \rangle \langle (\mathbf{Q} \mathbf{f}^*)_i, (\mathbf{Q} \mathbf{f}^*)_j \rangle \\ &\quad + \langle (\mathbf{Q}^T \mathbf{A} \mathbf{f}^*)_i, (\mathbf{Q}^T \mathbf{A} \mathbf{f}^*)_j \rangle \langle u_i, u_j \rangle \langle (\mathbf{Q} \mathbf{f}^*)_i, (\mathbf{Q} \mathbf{f}^*)_j \rangle \\ &\quad + \langle (\mathbf{Q}^T \mathbf{A} \mathbf{f}^*)_i, (\mathbf{Q}^T \mathbf{A} \mathbf{f}^*)_j \rangle \langle f_i^*, f_j^* \rangle \langle (\mathbf{Q} \mathbf{u})_i, (\mathbf{Q} \mathbf{u})_j \rangle \}. \end{aligned}$$

Here, $A = \text{Diag}[\pi]$ with π the stationary distribution of \mathbf{Q} . But when \mathbf{Q} and A are full rank, and \mathfrak{F}^* is linearly independent, the matrices

$$\langle \langle f_i^*, f_j^* \rangle \rangle_{i,j}, \quad \langle \langle (\mathbf{Q}^T \mathbf{A} \mathbf{f}^*)_i, (\mathbf{Q}^T \mathbf{A} \mathbf{f}^*)_j \rangle \rangle_{i,j}, \quad \text{and} \quad \langle \langle (\mathbf{Q} \mathbf{f}^*)_i, (\mathbf{Q} \mathbf{f}^*)_j \rangle \rangle_{i,j},$$

are positive definite.

Remark — *We shall now use the fact that if B and C are symmetric positive definite $K \times K$ matrices, then the $K \times K$ matrix E given by $E_{i,j} = B_{i,j} C_{i,j}$, $i, j = 1, \dots, K$ is positive definite. Indeed, let U_1, \dots, U_K be the eigenvectors of B with corresponding eigenvalues $\lambda_1(B) \geq \dots \geq \lambda_K(B) > 0$, so that $B = \sum_{r=1}^K \lambda_r(B) U_r U_r^T$. Let also $\lambda_1(C) \geq \dots \geq \lambda_K(C) > 0$ be the eigenvalues of C . Let now $x \in \mathbb{R}^K$. Then*

$$\begin{aligned} x^T E x &= \sum_{i,j=1}^K x_i B_{i,j} C_{i,j} x_j = \sum_{r=1}^K \lambda_r(B) \sum_{i,j=1}^K x_i (U_r)_i (U_r)_j C_{i,j} x_j \\ &\geq \left(\min_{r=1, \dots, K} \lambda_r(C) \right) \left(\min_{r=1, \dots, K} \lambda_r(B) \right) \sum_{r=1}^K \sum_{i=1}^K (x_i (U_r)_i)^2 \\ &= \left(\min_{r=1, \dots, K} \lambda_r(C) \right) \left(\min_{r=1, \dots, K} \lambda_r(B) \right) \|x\|^2. \end{aligned}$$

Thus, if R, V, W are the matrices given by, for all $i, j = 1, 2$,

$$\begin{aligned} R_{i,j} &= \langle \langle (\mathbf{Q}^T \mathbf{A} \mathbf{f}^*)_i, (\mathbf{Q}^T \mathbf{A} \mathbf{f}^*)_j \rangle \rangle_{i,j} \langle \langle f_i^*, f_j^* \rangle \rangle_{i,j}, \\ V_{i,j} &= \langle \langle f_i^*, f_j^* \rangle \rangle_{i,j} \langle \langle (\mathbf{Q} \mathbf{f}^*)_i, (\mathbf{Q} \mathbf{f}^*)_j \rangle \rangle_{i,j}, \\ W_{i,j} &= \langle \langle (\mathbf{Q}^T \mathbf{A} \mathbf{f}^*)_i, (\mathbf{Q}^T \mathbf{A} \mathbf{f}^*)_j \rangle \rangle_{i,j} \langle \langle (\mathbf{Q} \mathbf{f}^*)_i, (\mathbf{Q} \mathbf{f}^*)_j \rangle \rangle_{i,j}, \end{aligned}$$

and if we denote $\Lambda(\mathbf{Q})$ the minimum of their eigenvalues, then $\Lambda(\mathbf{Q}) > 0$ and we have for any $\mathbf{u} \in (\mathbf{L}^2(\mathcal{Y}, \mathcal{L}^D))^2$,

$$\begin{aligned} T(\mathbf{Q}, \mathbf{u}) &= \int \left((\mathbf{Q}^T \mathbf{A} \mathbf{u})^T(y) V (\mathbf{Q}^T \mathbf{A} \mathbf{u})(y) + \mathbf{u}^T(y) W \mathbf{u}(y) + (\mathbf{Q} \mathbf{u})^T(y) R (\mathbf{Q} \mathbf{u})(y) \right) d(\mathcal{L}^D)^{\otimes 3}(y) \\ &\geq \int \Lambda(\mathbf{Q}) \left(\|\mathbf{Q}^T \mathbf{A} \mathbf{u}(y)\|^2 + \|\mathbf{u}(y)\|^2 + \|\mathbf{Q} \mathbf{u}(y)\|^2 \right) d(\mathcal{L}^D)^{\otimes 3}(y) \\ &= \Lambda(\mathbf{Q}) \left(\|\mathbf{Q}^T \mathbf{A} \mathbf{u}\|_1^2 + \|\mathbf{Q}^T \mathbf{A} \mathbf{u}\|_2^2 + \|u_1\|_2^2 + \|u_2\|_2^2 + \|(\mathbf{Q} \mathbf{u})_1\|_2^2 + \|(\mathbf{Q} \mathbf{u})_2\|_2^2 \right). \end{aligned}$$

Moreover, we have

$$\lim_{n \rightarrow \infty} \Lambda(\mathbf{Q}_n) = \Lambda(\tilde{\mathbf{Q}}) > 0.$$

Let $\mathbf{a}_n = \frac{\mathbf{h}_n}{\|\mathbf{h}_n\|_2}$, \mathbf{u}_n the orthogonal projection (coordinate by coordinate) of \mathbf{a}_n on the subspace of $\mathbf{L}^2(\mathcal{Y}, \mathcal{L}^D)$ orthogonal to f_1^* and f_2^* . We get $c \geq \Lambda(\tilde{\mathbf{Q}}) \liminf_{n \rightarrow \infty} \|\mathbf{u}_n\|_2$ so that in case $\liminf_{n \rightarrow \infty} \|\mathbf{u}_n\|_2 > 0$, we get $c > 0$.

Else, using the subsequence for which $\liminf_{n \rightarrow \infty} \|\mathbf{u}_n\|_2 = 0$ we have

$$c \geq \liminf_{n \rightarrow \infty} D(\mathbf{Q}_n, \mathbf{a}_n - \mathbf{u}_n).$$

But $(\mathbf{a}_n - \mathbf{u}_n)_n$ is a sequence with coordinates in the finite dimensional space spanned by f_1^* and f_2^* , so that it has a limit point $\mathbf{a} = (a_1, a_2)$, and $c \geq D(\tilde{\mathbf{Q}}, \mathbf{a})$. Since on the subsequence $\lim_{n \rightarrow \infty} \|\mathbf{u}_n\|_2 = 0$, we get $\int a_1 d\mathcal{L}^D = 0$ and $\int a_2 d\mathcal{L}^D = 0$, so that there exist real numbers α and β such that $a_1 = \alpha(f_1^* - f_2^*)$ and $a_2 = \beta(f_1^* - f_2^*)$. Now brute force computation gives $D(\tilde{\mathbf{Q}}, \mathbf{a}) = D_{1,1}\alpha^2 + 2D_{1,2}\alpha\beta + D_{2,2}\beta^2$ with, denoting $p = \tilde{\mathbf{Q}}(1, 2)$ and $q = \tilde{\mathbf{Q}}(2, 1)$:

$$\begin{aligned} \frac{D_{1,1}}{q^2(p+q)^2} = & 2(1-p)^2 \|f_1^* - f_2^*\|^2 \|f_1^*\|^2 \|(1-p)f_1^* + pf_2^*\|^2 + \|(1-p)f_1^* + pf_2^*\|^4 \|f_1^* - f_2^*\|^2 \\ & + 4p(1-p) (\langle (1-p)f_1^* + pf_2^*, qf_1^* + (1-q)f_2^* \rangle \langle f_1^*, f_2^* \rangle \|f_1^* - f_2^*\|^2 \\ & + 2p^2 \|f_1^* - f_2^*\|^2 \|f_2^*\|^2 \|qf_1^* + (1-q)f_2^*\|^2 \\ & + 2(1-p)^2 (\langle (1-p)f_1^* + pf_2^*, f_1^* - f_2^* \rangle)^2 \|f_1^*\|^2 \\ & + 2p^2 (\langle qf_1^* + (1-q)f_2^*, f_1^* - f_2^* \rangle)^2 \|f_2^*\|^2 \\ & + 4p(1-p) \langle qf_1^* + (1-q)f_2^*, f_1^* - f_2^* \rangle \langle (1-p)f_1^* + pf_2^*, f_1^* - f_2^* \rangle \langle f_1^*, f_2^* \rangle \\ & + 4(1-p) \langle (1-p)f_1^* + pf_2^*, f_1^* - f_2^* \rangle \langle f_1^*, f_1^* - f_2^* \rangle \|(1-p)f_1^* + pf_2^*\|^2 \\ & + 4p \langle (1-p)f_1^* + pf_2^*, f_1^* - f_2^* \rangle \langle f_2^*, f_1^* - f_2^* \rangle \langle (1-p)f_1^* + pf_2^*, qf_1^* + (1-q)f_2^* \rangle, \end{aligned}$$

$$\begin{aligned} \frac{D_{2,2}}{p^2(p+q)^2} = & 2q^2 \|f_1^* - f_2^*\|^2 \|f_1^*\|^2 \|(1-p)f_1^* + pf_2^*\|^2 + \|qf_1^* + (1-q)f_2^*\|^4 \|f_1^* - f_2^*\|^2 \\ & + 4(1-q)q (\langle (1-p)f_1^* + pf_2^*, qf_1^* + (1-q)f_2^* \rangle \langle f_1^*, f_2^* \rangle \|f_1^* - f_2^*\|^2 \\ & + 2(1-q)^2 \|f_1^* - f_2^*\|^2 \|f_2^*\|^2 \|qf_1^* + (1-q)f_2^*\|^2 \\ & + 2q^2 (\langle (1-p)f_1^* + pf_2^*, f_1^* - f_2^* \rangle)^2 \|f_1^*\|^2 \\ & + 2(1-q)^2 (\langle qf_1^* + (1-q)f_2^*, f_1^* - f_2^* \rangle)^2 \|f_2^*\|^2 \\ & + 4q(1-q) \langle qf_1^* + (1-q)f_2^*, f_1^* - f_2^* \rangle \langle (1-p)f_1^* + pf_2^*, f_1^* - f_2^* \rangle \langle f_1^*, f_2^* \rangle \\ & + 4q \langle qf_1^* + (1-q)f_2^*, f_1^* - f_2^* \rangle \langle f_1^*, f_1^* - f_2^* \rangle \langle (1-p)f_1^* + pf_2^*, qf_1^* + (1-q)f_2^* \rangle \\ & + 4(1-q) \langle qf_1^* + (1-q)f_2^*, f_1^* - f_2^* \rangle \langle f_2^*, f_1^* - f_2^* \rangle \|qf_1^* + (1-q)f_2^*\|^2, \end{aligned}$$

and:

$$\begin{aligned} \frac{D_{1,2}}{pq(p+q)} = & 2(1-p)q \|f_1^* - f_2^*\|^2 \|f_1^*\|^2 \|(1-p)f_1^* + pf_2^*\|^2 \\ & + 2[pq + (1-p)(1-q)] (\langle (1-p)f_1^* + pf_2^*, qf_1^* + (1-q)f_2^* \rangle \langle f_1^*, f_2^* \rangle \|f_1^* - f_2^*\|^2 \\ & + (\langle (1-p)f_1^* + pf_2^*, qf_1^* + (1-q)f_2^* \rangle)^2 \|f_1^* - f_2^*\|^2 \\ & + 2p(1-q) \|f_1^* - f_2^*\|^2 \|f_2^*\|^2 \|qf_1^* + (1-q)f_2^*\|^2 \\ & + 2q(1-p) (\langle (1-p)f_1^* + pf_2^*, f_1^* - f_2^* \rangle)^2 \|f_1^*\|^2 \\ & + 2p(1-q) (\langle qf_1^* + (1-q)f_2^*, f_1^* - f_2^* \rangle)^2 \|f_2^*\|^2 \\ & + 2pq \langle qf_1^* + (1-q)f_2^*, f_1^* - f_2^* \rangle \langle (1-p)f_1^* + pf_2^*, f_1^* - f_2^* \rangle \langle f_1^*, f_2^* \rangle \\ & + 2(1-p)(1-q) \langle qf_1^* + (1-q)f_2^*, f_1^* - f_2^* \rangle \langle (1-p)f_1^* + pf_2^*, f_1^* - f_2^* \rangle \langle f_1^*, f_2^* \rangle \\ & + q \langle (1-p)f_1^* + pf_2^*, f_1^* - f_2^* \rangle \langle f_1^*, f_1^* - f_2^* \rangle \|(1-p)f_1^* + pf_2^*\|^2 \\ & + 2(1-p) \langle qf_1^* + (1-q)f_2^*, f_1^* - f_2^* \rangle \langle f_1^*, f_1^* - f_2^* \rangle \langle (1-p)f_1^* + pf_2^*, qf_1^* + (1-q)f_2^* \rangle \\ & + 2(1-q) \langle (1-p)f_1^* + pf_2^*, f_1^* - f_2^* \rangle \langle f_2^*, f_1^* - f_2^* \rangle \langle (1-p)f_1^* + pf_2^*, qf_1^* + (1-q)f_2^* \rangle \\ & + 2p \langle qf_1^* + (1-q)f_2^*, f_1^* - f_2^* \rangle \langle f_2^*, f_1^* - f_2^* \rangle \|qf_1^* + (1-q)f_2^*\|^2. \end{aligned}$$

Since we already know that the quadratic form is non negative, it only remains to prove that the determinant of the matrix defining the quadratic form is positive,

that is to prove that:

$$\text{DET}(f_1^*, f_2^*, p, q) := \frac{D_{1,1}D_{2,2} - D_{1,2}^2}{p^2q^2(p+q)^2} > 0,$$

whenever f_1^* and f_2^* are distinct probability densities, and (14) holds. We shall now write $\text{DET}(f_1^*, f_2^*, p, q)$ using

$$n_1 = \|f_1^*\|_2, \quad n_2 = \|f_2^*\|_2, \quad a = \frac{\langle f_1^*, f_2^* \rangle}{\|f_1^*\|_2 \|f_2^*\|_2},$$

for which the range is $[1, \infty[^2 \times]0, 1[$. Doing so, we obtain a polynomial P_1 in the variables n_1, n_2, a, p and q .

First observe that, by symmetry,

$$\text{DET}(f_1^*, f_2^*, p, q) = \text{DET}(f_2^*, f_1^*, q, p).$$

so that it is sufficient to prove that the polynomial P_1 is positive on the domain

$$(16) \quad 1 \leq n_2 \leq n_1,$$

and $0 \leq a < 1$ and $0 < p \neq q < 1$.

Furthermore, consider the change of variable

$$q = 1 - p + d$$

then we have a polynomial P_2 in the variables n_1, n_2, a, p and d which factorizes with

$$\frac{p^2(1-a^2)d^2n_1^2n_2^2(1+d-p)^2}{(1+d)^4}.$$

Dividing by this factor, one gets a polynomial P_3 which is homogeneous of degree 8 in n_1 and n_2 , so that one may set $n_1 = 1$ and keep $b = n_2 \in]0, 1]$ (observe that we have used (16) to reduce the problem to the domain $n_2/n_1 \leq 1$) and obtain a polynomial P_4 in the variables b, a, p and d . It remains to prove that P_4 is positive on $\mathcal{D}_4 = \{b \in]0, 1], a \in [0, 1[, p \in]0, 1[, d \in]p-1, 0[\cup]0, p[\}$.

Consider now the following change of variables

$$b = \frac{1}{1+x^2}, \quad a = \frac{y^2}{1+y^2}, \quad p = \frac{z^2}{1+z^2}, \quad \text{and} \quad d = \frac{(tz)^2 - 1}{(1+t^2)(1+z^2)},$$

mapping $(x, y, z, t) \in \mathbb{R}^4$ onto $(b, a, p, d) \in \mathcal{D}_5 = \{b \in]0, 1], a \in [0, 1[, p \in [0, 1[, d \in]p-1, p[\}$ which contains \mathcal{D}_4 . This change of variables maps P_4 onto a rational fraction with positive denominator, namely

$$(1+t^2)^4(1+y^2)^4(1+z^2)^4(1+x^2)^8$$

So it remains to prove that its numerator P_5 , which is polynomial, is positive on \mathbb{R}^4 . An expression of P_5 can be found in Appendix D.

Observe that P_5 is polynomial in x^2, y^2, z^2 and t^2 and there are only three monomials with negative coefficients. These monomials can be expressed as sum of squares using others monomials, namely:

- $-18x^{12}t^2 + 27x^{12} + 1979x^{12}t^4 = 18x^{12} + 9(x^6 - x^6t^2)^2 + 1970x^{12}t^4$,
- $-108x^{10}t^2 + 1970x^{12}t^4 + 495x^8 = 439x^8 + 56(x^4 - x^6t^2)^2 + 1914x^{12}t^4 + 4t^2x^{10}$,
- and $-114x^8t^2 + 972x^4 + 1914x^{12}t^4 = 915x^4 + 57(x^2 - x^6t^2)^2 + 1857x^{12}t^4$.

Thus P_5 is equal to 144 more a sum of squares, hence it is positive. This proves that DET is positive.

There remains to consider the case (b) where $\|\tilde{\mathbf{h}}\|_{\tilde{\mathbf{Q}}} = 0$ and $\tilde{\mathbf{Q}}(1, 2) = \tilde{\mathbf{Q}}(2, 1)$. But from the sequence $(\mathbf{Q}_n, \mathbf{h}_n)$, one may extract a subsequence in which $(\mathbf{Q}_{n_m}, \mathbf{h}_{n_m})$ such that always $\|\mathbf{h}_{n_m}\|_{\mathbf{Q}_{n_m}} = \|\mathbf{h}_{n_m}\|_2$, in which case we may argue as for case (a), or always $\|\mathbf{h}_{n_m}\|_{\mathbf{Q}_{n_m}} = \|(\mathbf{h}_{n_m})_1 + f_1^* - f_2^*\|_2 + \|(\mathbf{h}_{n_m})_2 + f_2^* - f_1^*\|_2$. In this last case, exchanging the states in the transition matrix one is back to case (a).

6.3. Proof of Theorem 3. Let us introduce some notation for latter use. Define the pseudo spectral gap \mathbb{G}_{ps} of the Markov chain $(X_n)_{n \geq 1}$ as:

$$\mathbb{G}_{\text{ps}} := \max_{k \geq 1} \left\{ \frac{\mathbb{G}(\mathfrak{D}\text{diag}[\pi^*]^{-1}(\mathbf{Q}^{*\top})^k \mathfrak{D}\text{diag}[\pi^*] \mathbf{Q}^{*k})}{k} \right\},$$

where \mathbb{G} is the spectral gap of an $\mathbf{L}^2(\pi^*)$ -self-adjoint operator A (with spectral radius 1):

$$\mathbb{G}(A) := \begin{cases} 1 - \max\{\lambda : \lambda \text{ eigenvalue of } A, \lambda \neq 1\} & \text{if eigenvalue 1 has multiplicity 1,} \\ 0 & \text{otherwise.} \end{cases}$$

Remark — *If \mathbf{Q}^* is irreducible then $\mathfrak{D}\text{diag}[\pi^*]^{-1} \mathbf{Q}^{*\top} \mathfrak{D}\text{diag}[\pi^*] \mathbf{Q}^*$ is irreducible and $\mathbb{G}_{\text{ps}} > 0$.*

Remark — *If $(X_n)_{n \geq 1}$ is reversible then \mathbf{Q}^* is an $\mathbf{L}^2(\pi^*)$ -self-adjoint operator (i.e. $\mathfrak{D}\text{diag}[\pi^*] \mathbf{Q}^* = \mathbf{Q}^{*\top} \mathfrak{D}\text{diag}[\pi^*]$) and its pseudo spectral gap enjoys $\mathbb{G}_{\text{ps}} = \mathbb{G}(\mathbf{Q}^*)(2 - \mathbb{G}(\mathbf{Q}^*)) > 0$.*

Define the mixing time \mathbb{T}_{mix} of the Markov chain $(X_n)_{n \geq 1}$ as:

$$\mathbb{T}_{\text{mix}} := \frac{1 + 2 \log 2 - \log \pi_{\min}^*}{\mathbb{G}_{\text{ps}}}.$$

This mixing time has a deeper interpretation in terms of convergence toward the stationary distribution in total variation norm. An interested reader may read [Pau14].

For any $\delta \in (0, 1)$ set:

$$\mathcal{C}_*(\mathbf{Q}^*, \delta) := \begin{cases} 0.71(1 + \sqrt{-\log \delta}) & \text{(Scenario A),} \\ \sqrt{2} + 2\sqrt{-2\mathbb{T}_{\text{mix}} \log \delta} & \text{(Scenario B).} \end{cases}$$

which is a constant that depends only on \mathbf{Q}^* and δ .

Let us also define (recalling the definition of $\eta_3^2(\Phi_M)$)

$$\eta_1^2(\Phi_M) := \sup_{y, y' \in \mathcal{Y}} \sum_{a=1}^M (\varphi_a(y) - \varphi_a(y'))^2,$$

$$\eta_2^2(\Phi_M) := \sup_{y, y' \in \mathcal{Y}^2} \sum_{a, b=1}^M (\varphi_a(y_1) \varphi_b(y_2) - \varphi_a(y'_1) \varphi_b(y'_2))^2,$$

$$\text{and } \eta_3^2(\Phi_M) := \sup_{y, y' \in \mathcal{Y}^3} \sum_{a, b, c=1}^M (\varphi_a(y_1) \varphi_b(y_2) \varphi_c(y_3) - \varphi_a(y'_1) \varphi_b(y'_2) \varphi_c(y'_3))^2.$$

Remark — *Note that in the examples (Spline), (Trig.) and (Wav.) we have:*

$$(17) \quad \forall k \in \{1, 2, 3\} \quad \eta_k \leq C_\eta M^{\frac{k}{2}},$$

where $C_\eta > 0$ is a constant.

Remark — *For any orthonormal family (Φ_M) , it holds:*

$$(18) \quad M\eta_1 \leq \sqrt{M}\eta_2 \leq \eta_3.$$

Indeed, observe that $\|\sum_c \varphi_c^2\|_\infty \geq \sum_c \|\varphi_c\|_2^2 = M$ and for any $\varepsilon > 0$, there exists $y \in \mathcal{Y}$ such that $\sum_c \varphi_c^2(y) \geq M - \varepsilon$. Set $y_3 = y'_3 = y$ to show that $\sqrt{M}\eta_2 \leq \eta_3$.

Theorem 3 follows from the following more precise result.

Theorem 4 — Assume [H1]-[H4]. Let $\delta \in (0, 1)$ then, with probability greater than $1 - 6\delta$, there exists a permutation $\tau \in \mathcal{S}_K$ such that the spectral method estimators $\hat{f}_{M,k}$, $\hat{\pi}$ and $\hat{\mathbf{Q}}$ (see Section 2.4 for a definition) enjoy for any $M \geq M_{\mathfrak{F}^*}$,

$$\forall N \geq \mathbf{N}_1(\mathbf{Q}^*, \mathfrak{F}^*, \Phi_M, \delta), \forall k \in \mathcal{X}, \quad \|f_{M,k}^* - \hat{f}_{M,\tau(k)}\|_2 \leq \mathcal{C}_M(\mathbf{Q}^*, \mathfrak{F}^*, \delta) \mathcal{C}_*(\mathbf{Q}^*, \delta) \frac{\eta_3(\Phi_M)}{\sqrt{N}}, \quad (19)$$

$$\forall N \geq \mathbf{N}_2(\mathbf{Q}^*, \mathfrak{F}^*, \Phi_M, \delta), \quad \|\mathbf{Q}^* - \mathbb{P}_\tau \hat{\mathbf{Q}} \mathbb{P}_\tau^\top\| \leq \mathcal{D}_M(\mathbf{Q}^*, \mathfrak{F}^*, \delta) \mathcal{C}_*(\mathbf{Q}^*, \delta) \frac{\eta_3(\Phi_M)}{\sqrt{N}}, \quad (20)$$

$$\forall N \geq \mathbf{N}_3(\mathbf{Q}^*, \mathfrak{F}^*, \Phi_M, \delta), \quad \|\pi^* - \mathbb{P}_\tau \hat{\pi}\|_2 \leq \mathcal{E}_M(\mathbf{Q}^*, \mathfrak{F}^*, \delta) \mathcal{C}_*(\mathbf{Q}^*, \delta) \frac{\eta_3(\Phi_M)}{\sqrt{N}}, \quad (21)$$

where \mathbb{P}_τ is the permutation matrix associated to τ , and:

$$\begin{aligned} \mathbf{N}_1(\mathbf{Q}^*, \mathfrak{F}^*, \Phi_M, \delta) &:= \frac{4K}{3\sigma_{K,\mathfrak{F}^*}^2} \mathcal{C}_M(\mathbf{Q}^*, \mathfrak{F}^*, \delta)^2 \mathcal{C}_*(\mathbf{Q}^*, \delta)^2 \eta_3(\Phi_M)^2, \\ \mathbf{N}_2(\mathbf{Q}^*, \mathfrak{F}^*, \Phi_M, \delta) &:= \frac{4}{\pi_{\min}^{*2}} \mathcal{D}'_M(\mathbf{Q}^*, \mathfrak{F}^*, \delta)^2 \mathcal{C}_*(\mathbf{Q}^*, \delta)^2 \eta_3(\Phi_M)^2, \\ \mathbf{N}_3(\mathbf{Q}^*, \mathfrak{F}^*, \Phi_M, \delta) &:= \frac{4}{\sigma_K^2(\mathbf{A}_{\mathbf{Q}^*})} \mathcal{D}_M(\mathbf{Q}^*, \mathfrak{F}^*, \delta)^2 \mathcal{C}_*(\mathbf{Q}^*, \delta)^2 \eta_3(\Phi_M)^2, \end{aligned}$$

with notation:

$$\begin{aligned} \mathcal{C}_M(\mathbf{Q}^*, \mathfrak{F}^*, \delta) &:= \left[13 \frac{\kappa^2(\mathbf{Q}^*) K^{\frac{1}{2}}}{\pi_{\min}^* \sigma_K(\mathbf{Q}^{*2})} \frac{\kappa_{\mathfrak{F}^*}^2}{\sigma_{K,\mathfrak{F}^*}^2} + \frac{83}{\delta} \frac{\kappa^6(\mathbf{Q}^*) K^5}{\pi_{\min}^* \sigma_K(\mathbf{Q}^{*2})} \frac{\kappa_{\mathfrak{F}^*}^6 \max_{k \in \mathcal{X}} \|f_k^*\|_2}{\sigma_{K,\mathfrak{F}^*}^3} (1 + (2 \log \frac{K^2}{\delta})^{\frac{1}{2}}) \right] \\ &\quad \times \left[1 + \frac{\|g^*\|_2}{\pi_{\min}^* \sigma_{K,\mathfrak{F}^*}^2 \sigma_K(\mathbf{Q}^{*2})} \frac{1}{\sqrt{M}} \right] + \frac{2}{\sqrt{M}} \frac{\max_{k \in \mathcal{X}} \|f_k^*\|_2}{\sigma_{K,\mathfrak{F}^*}^2 \pi_{\min}^* \sigma_K(\mathbf{Q}^{*2})}, \\ \mathcal{D}'_M(\mathbf{Q}^*, \mathfrak{F}^*, \delta) &:= \frac{2}{3\sigma_{K,\mathfrak{F}^*}^2} \left[4\sqrt{K} \mathcal{C}_M(\mathbf{Q}^*, \mathfrak{F}^*, \delta) \max_{k \in \mathcal{X}} \|f_k^*\|_2 + \frac{3\sqrt{3}\sigma_{K,\mathfrak{F}^*}}{M} \right], \\ \mathcal{D}_M(\mathbf{Q}^*, \mathfrak{F}^*, \delta) &:= \frac{8\|f_{(Y_1, Y_3)}^*\|_2}{3\sigma_{K,\mathfrak{F}^*}^2 \pi_{\min}^{*2}} \\ &\quad \times \left[\mathcal{D}'_M(\mathbf{Q}^*, \mathfrak{F}^*, \delta) + 4\sqrt{3K} \pi_{\min}^* \mathcal{C}_M(\mathbf{Q}^*, \mathfrak{F}^*, \delta) + \frac{5\pi_{\min}^*}{\|f_{(Y_1, Y_3)}^*\|_2 \sqrt{M}} \right], \\ \mathcal{E}_M(\mathbf{Q}^*, \mathfrak{F}^*, \delta) &:= \frac{16\|f_{(Y_1, Y_3)}^*\|_2}{\sigma_K^2(\mathbf{A}_{\mathbf{Q}^*}) \sigma_{K,\mathfrak{F}^*}^2 \pi_{\min}^{*2}} \\ &\quad \times \left[\mathcal{D}'_M(\mathbf{Q}^*, \mathfrak{F}^*, \delta) + 4\sqrt{3K} \pi_{\min}^* \mathcal{C}_M(\mathbf{Q}^*, \mathfrak{F}^*, \delta) + \frac{5\pi_{\min}^*}{\|f_{(Y_1, Y_3)}^*\|_2 \sqrt{M}} \right], \end{aligned}$$

where $\kappa_{\mathfrak{F}^*}$ is given in Lemma 3 and $\sigma_K^2(\mathbf{A}_{\mathbf{Q}^*})$ is the k -th largest singular value of $\begin{pmatrix} \text{Id}_K - (\mathbf{Q}^*)^\top \\ \mathbf{1}_K^\top \end{pmatrix}$ which is positive.

To prove Theorem 4, we shall use the analysis of Anandkumar, Hsu and Kakade in [AHK12] to control the ℓ_2 -error of the estimation using the spectral method described in Section 2.4. To use their result in the non-parametric HMMs frame, it is essential to state explicitly how all constants depend on the dimension M . We thus recast and optimize Anandkumar, Hsu and Kakade's argument. This is exactly what is performed below and proved in Appendix C.

Theorem 5 — Let $0 < \delta < 1$. If it holds $3\|\hat{\mathbf{P}}_M - \mathbf{P}_M\| \leq \sigma_K(\mathbf{P}_M)$ and

(22)

$$8.2K^{\frac{5}{2}}(K-1) \frac{\kappa^2(\mathbf{Q}^* \mathbf{O}_M^\top)}{\delta \gamma(\mathbf{O}_M) \sigma_K(\mathbf{P}_M)} \left[\|\hat{\mathbf{M}}_M - \mathbf{M}_M\|_{\infty,2} + \frac{\|\mathbf{M}_M\|_{\infty,2} \|\hat{\mathbf{P}}_M - \mathbf{P}_M\|}{\sigma_K(\mathbf{P}_M)} \right] < 1,$$

(23)

$$43.4K^4(K-1) \frac{\kappa^4(\mathbf{Q}^* \mathbf{O}_M^\top)}{\delta \gamma(\mathbf{O}_M) \sigma_K(\mathbf{P}_M)} \left[\|\hat{\mathbf{M}}_M - \mathbf{M}_M\|_{\infty,2} + \frac{\|\mathbf{M}_M\|_{\infty,2} \|\hat{\mathbf{P}}_M - \mathbf{P}_M\|}{\sigma_K(\mathbf{P}_M)} \right] \leq 1,$$

where $\gamma(\mathbf{O}_M) := \min_{k_1 \neq k_2} \|\mathbf{O}_M(\cdot, k_1) - \mathbf{O}_M(\cdot, k_2)\|_2$ and $\|A\|_{\infty,2} := \max_{\|v\|_2=1} \left\| \sum_{b=1}^M v_b A(\cdot, b, \cdot) \right\|$

for all $A \in \mathbb{R}^{M \times M \times M}$,

then, with probability greater than $1 - 2\delta$, the matrix $\hat{\mathbf{U}}^\top \hat{\mathbf{P}}_M \hat{\mathbf{U}}$ is invertible, the random matrix $\hat{\mathbf{C}}(1)$ is diagonalisable (see Section 2.4), and there exists a permutation $\tau \in \mathcal{S}_K$ such that for all $k \in \mathcal{X}$, $\|\mathbf{O}_M(\cdot, k) - \hat{\mathbf{O}}_M(\cdot, \tau(k))\|_2$ is upper bounded by:

$$\begin{aligned} & \left[13K^{\frac{1}{2}} \frac{\kappa^2(\mathbf{Q}^* \mathbf{O}_M^\top)}{\sigma_K(\mathbf{P}_M)} + 116K^5(1 + (2 \log(K^2/\delta))^{\frac{1}{2}}) \frac{\kappa^6(\mathbf{Q}^* \mathbf{O}_M^\top) \|\mathbf{O}_M\|_{2,\infty}}{\delta \gamma(\mathbf{O}_M) \sigma_K(\mathbf{P}_M)} \right] \\ & \times \left[\|\hat{\mathbf{M}}_M - \mathbf{M}_M\|_{\infty,2} + \frac{\|\mathbf{M}_M\|_{\infty,2} \|\hat{\mathbf{P}}_M - \mathbf{P}_M\|}{\sigma_K(\mathbf{P}_M)} \right] + 2 \frac{\|\hat{\mathbf{P}}_M - \mathbf{P}_M\|}{\sigma_K(\mathbf{P}_M)} \|\mathbf{O}_M\|_{2,\infty}, \end{aligned}$$

where $\|A\|_{2,\infty} := \max_{k \in \mathcal{X}} \|A(\cdot, k)\|_2$ for all $A \in \mathbb{R}^{M \times K}$.

Let us now prove Theorem 4.

6.3.1. Preliminary lemmas.

Lemma 3 — There exists a constant $\kappa_{\mathfrak{F}^*}$ that depends only on \mathfrak{F}^* such that:

$$\forall M \geq M_{\mathfrak{F}^*}, \quad \kappa(\mathbf{O}_M) \leq \kappa_{\mathfrak{F}^*},$$

where $M_{\mathfrak{F}^*}$ is given by [H4]. It holds:

$$\forall M \geq M_{\mathfrak{F}^*}, \quad \kappa(\mathbf{Q}^* \mathbf{O}_M^\top) \leq \kappa_{\mathfrak{F}^*} \kappa(\mathbf{Q}^*).$$

Proof. Note that $\mathbf{O}_*^\top \mathbf{O}_*$ is non-singular. From (1) and (2) we deduce that $\mathbf{O}_M^\top \mathbf{O}_M$ tends to $\mathbf{O}_*^\top \mathbf{O}_*$ as M grows. This proves the first point.

Recall that $\sigma_i(AB) \leq \sigma_1(A)\sigma_i(B)$ for $i = 1, \dots, K$. Take $A = \mathbf{Q}^{*-1}$ and $B = \mathbf{Q}^* \mathbf{O}_M^\top$ to get that $\sigma_K(\mathbf{Q}^*) \sigma_K(\mathbf{O}_M) \leq \sigma_K(\mathbf{Q}^* \mathbf{O}_M^\top)$. It follows that $\kappa(\mathbf{Q}^* \mathbf{O}_M^\top) \leq \kappa(\mathbf{Q}^*) \kappa(\mathbf{O}_M)$. The second claim follows from the first claim. \square

Lemma 4 — It holds:

$$\forall M \geq M_{\mathfrak{F}^*}, \quad \gamma(\mathbf{O}_M) := \min_{k_1 \neq k_2} \|\mathbf{O}_M(\cdot, k_1) - \mathbf{O}_M(\cdot, k_2)\|_2 \geq \sqrt{2} \sigma_{K, \mathfrak{F}^*},$$

and:

$$\|\mathbf{O}_M\|_{2,\infty} := \max_{k \in \mathcal{X}} \|\mathbf{O}_M(\cdot, k)\|_2 \leq \max_{k \in \mathcal{X}} \|f_k^*\|_2.$$

Proof. Observe that $\|\mathbf{O}_M v\|_2 \geq \sigma_K(\mathbf{O}_M) \|v\|_2$. With an appropriate choice of v and [H4] we prove the first point. Since Φ_M is an orthonormal family, it yields $\|\mathbf{O}_M(\cdot, k)\|_2 \leq \|f_k^*\|_2$. This proves the second claim. \square

Lemma 5 — It holds:

$$\|\mathbf{M}_M\|_{\infty,2} := \max_{\|v\|_2=1} \left\| \sum_{b=1}^M v_b \mathbf{M}_M(\cdot, b, \cdot) \right\| \leq \|g^*\|_2.$$

Proof. From [H4b] one can check that $g^* \in \mathbf{L}^2(\mathcal{Y}^3, \mathcal{L}^{D \otimes 3})$. Denote $\langle \cdot, \cdot \rangle_{\mathbf{L}^2(\mathcal{Y}^3, \mathcal{L}^{D \otimes 3})}$ the inner product of $\mathbf{L}^2(\mathcal{Y}^3, \mathcal{L}^{D \otimes 3})$. Check that $\varphi_{a,b,c}(y_1, y_2, y_3) := \varphi_a(y_1)\varphi_b(y_2)\varphi_c(y_3)$ is an orthonormal family of $\mathbf{L}^2(\mathcal{Y}^3, \mathcal{L}^{D \otimes 3})$. Hence:

$$\begin{aligned} \|\mathbf{M}_M\|_{\infty,2} &= \max_{\|v\|_2=1} \left\| \sum_{b=1}^M v_b \mathbf{M}_M(\cdot, b, \cdot) \right\| \leq \max_{\|v\|_2=1} \sum_{b=1}^M |v_b| \|\mathbf{M}_M(\cdot, b, \cdot)\|, \\ &\leq \left(\sum_{b=1}^M \|\mathbf{M}_M(\cdot, b, \cdot)\|^2 \right)^{\frac{1}{2}} \leq \left(\sum_{b=1}^M \|\mathbf{M}_M(\cdot, b, \cdot)\|_F^2 \right)^{\frac{1}{2}}, \\ &= \left(\sum_{a,b,c=1}^M (\mathbb{E} \varphi_a(Y_1) \varphi_b(Y_2) \varphi_c(Y_3))^2 \right)^{\frac{1}{2}} = \left(\sum_{a,b,c=1}^M \langle g^*, \varphi_{a,b,c} \rangle_{\mathbf{L}^2(\mathcal{Y}^3, \mathcal{L}^{D \otimes 3})}^2 \right)^{\frac{1}{2}}, \\ &\leq \|g^*\|_2. \end{aligned}$$

using Cauchy-Schwarz inequality \square

Lemma 6 — *It holds:*

$$\|\hat{\mathbf{M}}_M - \mathbf{M}_M\|_{\infty,2} \leq \|\hat{\mathbf{M}}_M - \mathbf{M}_M\|_F.$$

Proof. One has:

$$\begin{aligned} \|\hat{\mathbf{M}}_M - \mathbf{M}_M\|_{\infty,2} &= \max_{\|v\|_2=1} \left\| \sum_{b=1}^M v_b (\hat{\mathbf{M}}_M - \mathbf{M}_M)(\cdot, b, \cdot) \right\| \leq \max_{\|v\|_2=1} \sum_{b=1}^M |v_b| \|(\hat{\mathbf{M}}_M - \mathbf{M}_M)(\cdot, b, \cdot)\|, \\ &\leq \left(\sum_{b=1}^M \|(\hat{\mathbf{M}}_M - \mathbf{M}_M)(\cdot, b, \cdot)\|^2 \right)^{\frac{1}{2}} \leq \left(\sum_{b=1}^M \|(\hat{\mathbf{M}}_M - \mathbf{M}_M)(\cdot, b, \cdot)\|_F^2 \right)^{\frac{1}{2}}, \\ &= \|\hat{\mathbf{M}}_M - \mathbf{M}_M\|_F. \end{aligned}$$

using Cauchy-Schwarz inequality \square

Lemma 7 — *It holds $\sigma_K(\mathbf{P}_M) \geq \pi_{\min} \sigma_K^2(\mathbf{O}_M) \sigma_K(\mathbf{Q}^2)$. Invoke [H4] to get that:*

$$\forall M \geq M_{\mathfrak{F}^*}, \quad \sigma_K(\mathbf{P}_M) \geq \sigma_{K,\mathfrak{F}^*}^2 \pi_{\min}^* \sigma_K(\mathbf{Q}^{*2}).$$

Proof. From Lemma 14 and [H3], it holds:

$$\begin{aligned} \sigma_K(\mathbf{P}_M) &= \sigma_K(\mathbf{U}^\top \mathbf{P}_M \mathbf{U}) \\ &= \sigma_K((\mathbf{U}^\top \mathbf{O}_M) \mathfrak{Diag}[\pi^*] \mathbf{Q}^{*2} (\mathbf{U}^\top \mathbf{O}_M)^\top) \\ &\geq \sigma_K(\mathbf{U}^\top \mathbf{O}_M) \sigma_K(\mathfrak{Diag}[\pi^*] \mathbf{Q}^{*2} (\mathbf{U}^\top \mathbf{O}_M)^\top) \\ &= \sigma_K(\mathbf{O}_M) \sigma_K(\mathfrak{Diag}[\pi^*] \mathbf{Q}^{*2} (\mathbf{U}^\top \mathbf{O}_M)^\top) \\ &\geq \sigma_K(\mathfrak{Diag}[\pi^*]) \sigma_K(\mathbf{O}_M) \sigma_K((\mathbf{U}^\top \mathbf{O}_M)^\top) \sigma_K(\mathbf{Q}^{*2}) \\ &= \pi_{\min}^* \sigma_K^2(\mathbf{O}_M) \sigma_K(\mathbf{Q}^{*2}), \end{aligned}$$

as claimed. \square

6.3.2. *First step: Estimation of the emission laws using a spectral method.* Appendix B shows that:

$$\begin{aligned} \mathbb{P}\left[\|\hat{\mathbf{L}}_M - \mathbf{L}_M\|_F \geq \mathcal{C}_*(\mathbf{Q}^*, \delta) \frac{\eta_1(\Phi_M)}{\sqrt{N}}\right] &\leq \delta, \\ \mathbb{P}\left[\|\hat{\mathbf{M}}_M - \mathbf{M}_M\|_F \geq \mathcal{C}_*(\mathbf{Q}^*, \delta) \frac{\eta_3(\Phi_M)}{\sqrt{N}}\right] &\leq \delta, \\ \mathbb{P}\left[\|\hat{\mathbf{N}}_M - \mathbf{N}_M\|_F \geq \mathcal{C}_*(\mathbf{Q}^*, \delta) \frac{\eta_2(\Phi_M)}{\sqrt{N}}\right] &\leq \delta, \\ \mathbb{P}\left[\|\hat{\mathbf{P}}_M - \mathbf{P}_M\|_F \geq \mathcal{C}_*(\mathbf{Q}^*, \delta) \frac{\eta_2(\Phi_M)}{\sqrt{N}}\right] &\leq \delta. \end{aligned}$$

Using the preliminary lemmas of Section 6.3.1 and (18), deduce that (22) and (23) along with $3\|\hat{\mathbf{P}}_M - \mathbf{P}_M\| \leq \sigma_K(\mathbf{P}_M)$ are satisfied when $M \geq M_{\mathfrak{F}^*}$ and $N \geq \mathbf{N}_0(\mathbf{Q}^*, \mathfrak{F}^*, \Phi_M, \delta)$ where:

$$\mathbf{N}_0(\mathbf{Q}^*, \mathfrak{F}^*, \Phi_M, \delta) := \frac{942}{\delta^2} \frac{\kappa^8(\mathbf{Q}^*) K^{10}}{\pi_{\min}^* \sigma_K^2(\mathbf{Q}^{*2})} \frac{\kappa_{\mathfrak{F}^*}^8}{\sigma_{K, \mathfrak{F}^*}^6} \left(1 + \frac{\|g^*\|_2}{\pi_{\min}^* \sigma_K^2 \sigma_K(\mathbf{Q}^{*2})} \frac{1}{\sqrt{M}}\right)^2 \mathcal{C}_*(\mathbf{Q}^*, \delta)^2 \eta_3(\Phi_M)^2.$$

Using Theorem 5, one can check that, with probability greater than $1 - 6\delta$, there exists a permutation τ satisfying for any $M \geq M_{\mathfrak{F}^*}$, $N \geq \mathbf{N}_0(\mathbf{Q}^*, \mathfrak{F}^*, \Phi_M, \delta)$ and $k \in \mathcal{X}$,

$$\|\mathbf{O}_M(\cdot, k) - \hat{\mathbf{O}}_M(\cdot, \tau(k))\|_2 \leq \mathcal{C}_M(\mathbf{Q}^*, \mathfrak{F}^*, \delta) \mathcal{C}_*(\mathbf{Q}^*, \delta) \frac{\eta_3(\Phi_M)}{\sqrt{N}}.$$

This proves the first part of Theorem 4.

6.3.3. *Second step: Preliminary estimation of the stationary density using a spectral method.* For sake of readability, assume that τ is the identity permutation. Observe that:

$$\mathbf{N}_1(\mathbf{Q}^*, \mathfrak{F}^*, \Phi_M, \delta) \geq \mathbf{N}_0(\mathbf{Q}^*, \mathfrak{F}^*, \Phi_M, \delta).$$

Recall $\tilde{\pi} := (\hat{\mathbf{U}}^\top \hat{\mathbf{O}}_M)^{-1} \hat{\mathbf{U}}^\top \hat{\mathbf{L}}_M$ and $\pi^* = (\hat{\mathbf{U}}^\top \mathbf{O}_M)^{-1} \hat{\mathbf{U}}^\top \mathbf{L}_M$.

Lemma 8 — *With probability greater than $1 - 6\delta$, if $N > \mathbf{N}_1(\mathbf{Q}^*, \mathfrak{F}^*, \Phi_M, \delta)$ then it holds:*

$$\|\tilde{\pi} - \pi^*\|_2 \leq \frac{2}{\sqrt{3}\sigma_{K, \mathfrak{F}^*}} \left[\mathcal{C}_*(\mathbf{Q}^*, \delta) \frac{\eta_1(\Phi_M)}{\sqrt{N}} + \frac{\sqrt{\mathbf{N}_1(\mathbf{Q}^*, \mathfrak{F}^*, \Phi_M, \delta)}}{\sqrt{N} - \sqrt{\mathbf{N}_1(\mathbf{Q}^*, \mathfrak{F}^*, \Phi_M, \delta)}} (\max_{k \in \mathcal{X}} \|f_k^*\|_2 + \mathcal{C}_*(\mathbf{Q}^*, \delta) \frac{\eta_1(\Phi_M)}{\sqrt{N}}) \right].$$

Proof. Set $A = \hat{\mathbf{U}}^\top \mathbf{O}_M$, $\tilde{A} = \hat{\mathbf{U}}^\top \hat{\mathbf{O}}_M$ and $B = \hat{\mathbf{U}}^\top (\mathbf{O}_M - \hat{\mathbf{O}}_M)$. Compute:

$$\|B\| \leq \|\mathbf{O}_M - \hat{\mathbf{O}}_M\| \leq \|\mathbf{O}_M - \hat{\mathbf{O}}_M\|_F \leq \sqrt{K} \max_k \|\mathbf{O}_M(\cdot, k) - \hat{\mathbf{O}}_M(\cdot, k)\|_2.$$

Hence it holds $\|B\| \leq \sqrt{K} \mathcal{C}_M(\mathbf{Q}^*, \mathfrak{F}^*, \delta) \mathcal{C}_*(\mathbf{Q}^*, \delta) \eta_3(\Phi_M) / \sqrt{N}$. Similarly, invoke Claim (iii) of Lemma 16 to get that:

$$\|A^{-1}B\| \leq \|A^{-1}\| \|B\| \leq \sigma_K^{-1}(A) \|B\| \leq \frac{2\sqrt{K} \max_k \|\mathbf{O}_M(\cdot, k) - \hat{\mathbf{O}}_M(\cdot, k)\|_2}{\sqrt{3}\sigma_K(\mathbf{O}_M)}.$$

So that:

$$\|A^{-1}B\| \leq \frac{2\sqrt{K}}{\sqrt{3}\sigma_{K, \mathfrak{F}^*}} \mathcal{C}_M(\mathbf{Q}^*, \mathfrak{F}^*, \delta) \mathcal{C}_*(\mathbf{Q}^*, \delta) \frac{\eta_3(\Phi_M)}{\sqrt{N}}.$$

Observe that the condition on N and M ensures that $\|A^{-1}B\| < 1$. Apply Theorem 7 to get that:

$$(24) \quad \|(\hat{\mathbf{U}}^\top \mathbf{O}_M)^{-1} - (\hat{\mathbf{U}}^\top \hat{\mathbf{O}}_M)^{-1}\| \leq \frac{2}{\sqrt{3}\sigma_{K, \mathfrak{F}^*}} \frac{\sqrt{\mathbf{N}_1(\mathbf{Q}^*, \mathfrak{F}^*, \Phi_M, \delta)}}{\sqrt{N} - \sqrt{\mathbf{N}_1(\mathbf{Q}^*, \mathfrak{F}^*, \Phi_M, \delta)}}.$$

Furthermore, using (24) it holds:

$$\begin{aligned}
\|\tilde{\pi} - \pi^*\|_2 &= \|(\hat{\mathbf{U}}^\top \hat{\mathbf{O}}_M)^{-1} \hat{\mathbf{U}}^\top \hat{\mathbf{L}}_M - (\hat{\mathbf{U}}^\top \mathbf{O}_M)^{-1} \hat{\mathbf{U}}^\top \mathbf{L}_M\|_2 \\
&= \|(\hat{\mathbf{U}}^\top \hat{\mathbf{O}}_M)^{-1} \hat{\mathbf{U}}^\top \hat{\mathbf{L}}_M - (\hat{\mathbf{U}}^\top \mathbf{O}_M)^{-1} \hat{\mathbf{U}}^\top \hat{\mathbf{L}}_M + (\hat{\mathbf{U}}^\top \mathbf{O}_M)^{-1} \hat{\mathbf{U}}^\top \hat{\mathbf{L}}_M - (\hat{\mathbf{U}}^\top \mathbf{O}_M)^{-1} \hat{\mathbf{U}}^\top \mathbf{L}_M\|_2 \\
&\leq \|(\hat{\mathbf{U}}^\top \mathbf{O}_M)^{-1} - (\hat{\mathbf{U}}^\top \hat{\mathbf{O}}_M)^{-1}\| \|\hat{\mathbf{L}}_M\|_2 + \|A^{-1}\| \|\hat{\mathbf{L}}_M - \mathbf{L}_M\|_2 \\
&\leq \frac{2}{\sqrt{3}\sigma_{K, \mathfrak{F}^*}} \left(\|\hat{\mathbf{L}}_M - \mathbf{L}_M\|_2 + \frac{\sqrt{\mathbf{N}_1(\mathbf{Q}^*, \mathfrak{F}^*, \Phi_M, \delta)}}{\sqrt{N} - \sqrt{\mathbf{N}_1(\mathbf{Q}^*, \mathfrak{F}^*, \Phi_M, \delta)}} (\|\mathbf{L}_M\|_2 + \|\hat{\mathbf{L}}_M - \mathbf{L}_M\|_2) \right).
\end{aligned}$$

Denote $f_{Y_1}^* = \sum_{k_1=1}^K \pi(k_1) f_{k_1}^*(y_1)$ the density of Y_1 . Observe that:

$$\|\mathbf{L}_M\|_2 = \left(\sum_{a=1}^M (\mathbb{E} \varphi_a(Y_1))^2 \right)^{\frac{1}{2}} = \left(\sum_{a=1}^M \langle f_{Y_1}^*, \varphi_a \rangle^2 \right)^{\frac{1}{2}} \leq \|f_{Y_1}^*\|_2 \leq \max_k \|f_k^*\|_2.$$

The result follows. \square

Deduce that

$$(25) \quad \forall N \geq 4\mathbf{N}_1(\mathbf{Q}^*, \mathfrak{F}^*, \Phi_M, \delta), \quad \|\pi^* - \mathbb{P}_\tau \tilde{\pi}\|_2 \leq \mathcal{D}'_M(\mathbf{Q}^*, \mathfrak{F}^*, \delta) \mathcal{C}_*(\mathbf{Q}^*, \delta) \frac{\eta_3(\Phi_M)}{\sqrt{N}},$$

with notation of Theorem 4.

6.3.4. *Third step: Estimation of the transition matrix using a spectral method.* Denote $\tilde{\mathbf{Q}} := (\hat{\mathbf{U}}^\top \hat{\mathbf{O}}_M \mathfrak{D}[\tilde{\pi}])^{-1} \hat{\mathbf{U}}^\top \hat{\mathbf{N}}_M \hat{\mathbf{U}} (\hat{\mathbf{O}}_M^\top \hat{\mathbf{U}})^{-1}$. Observe $\hat{\mathbf{Q}} = \Pi_{T_M}(\tilde{\mathbf{Q}})$ and $\mathbf{Q}^* = \Pi_{T_M}(\mathbf{Q}^*)$ and hence, by non-expansivity of the projection onto convex sets, $\|\hat{\mathbf{Q}} - \mathbf{Q}^*\|_F \leq \|\tilde{\mathbf{Q}} - \mathbf{Q}^*\|_F$. Moreover, notice that:

$$\mathbf{N}_2(\mathbf{Q}^*, \mathfrak{F}^*, \Phi_M, \delta) \geq 4\mathbf{N}_1(\mathbf{Q}^*, \mathfrak{F}^*, \Phi_M, \delta) \geq \mathbf{N}_0(\mathbf{Q}^*, \mathfrak{F}^*, \Phi_M, \delta).$$

Lemma 9 — *With probability greater than $1 - 6\delta$, if $N \geq \mathbf{N}_2(\mathbf{Q}^*, \mathfrak{F}^*, \Phi_M, \delta)$ then it holds:*

$$\|\tilde{\mathbf{Q}} - \mathbf{Q}^*\| \leq \frac{8\|f_{(Y_1, Y_3)}^*\|_2}{3\sigma_{K, \mathfrak{F}^*}^2 \pi_{\min}^*} \|\tilde{\pi} - \pi^*\|_2 + \frac{2}{\pi_{\min}^*} \tilde{\mathcal{E}}_M(\mathbf{Q}^*, \mathfrak{F}^*, \delta) \mathcal{C}_*(\mathbf{Q}^*, \delta) \frac{\eta_3(\Phi_M)}{\sqrt{N}},$$

where:

$$\tilde{\mathcal{E}}_M(\mathbf{Q}^*, \mathfrak{F}^*, \delta) := \frac{16}{\sqrt{3}\sigma_{K, \mathfrak{F}^*}^2} \left[\sqrt{K} \mathcal{C}_M(\mathbf{Q}^*, \mathfrak{F}^*, \delta) \|f_{(Y_1, Y_3)}^*\|_2 + \frac{5}{4\sqrt{3M}} \right].$$

Proof. Observe that (21) shows that $\|\tilde{\pi} - \pi^*\|_2 \leq \pi_{\min}^*/2$. Deduce that for any $k \in \mathcal{X}$:

$$(26) \quad \tilde{\pi}_k \geq \frac{\pi_{\min}^*}{2} > 0.$$

Set $\mathbf{V} = (\hat{\mathbf{U}}^\top \mathbf{O}_M)^{-1} \hat{\mathbf{U}}^\top$ and $\hat{\mathbf{V}} = (\hat{\mathbf{U}}^\top \hat{\mathbf{O}}_M)^{-1} \hat{\mathbf{U}}^\top$. Note $\tilde{\mathbf{Q}} = \mathfrak{D}[\tilde{\pi}]^{-1} \hat{\mathbf{V}} \hat{\mathbf{N}}_M \hat{\mathbf{V}}^\top$ and:

$$\mathbf{Q} = \mathfrak{D}[\pi^*]^{-1} \mathbf{V} \mathbf{N}_M \mathbf{V}^\top.$$

Set $E = \hat{\mathbf{V}} - \mathbf{V}$ and $F = \hat{\mathbf{N}}_M - \mathbf{N}_M$. Using (24) it yields:

$$\|E\| \leq \frac{2}{\sqrt{3}\sigma_{K, \mathfrak{F}^*}} \frac{\sqrt{\mathbf{N}_1(\mathbf{Q}^*, \mathfrak{F}^*, \Phi_M, \delta)}}{\sqrt{N} - \sqrt{\mathbf{N}_1(\mathbf{Q}^*, \mathfrak{F}^*, \Phi_M, \delta)}} \leq \frac{8\sqrt{K}}{3\sigma_{K, \mathfrak{F}^*}^2} \mathcal{C}_M(\mathbf{Q}^*, \mathfrak{F}^*, \delta) \mathcal{C}_*(\mathbf{Q}^*, \delta) \frac{\eta_3(\Phi_M)}{\sqrt{N}}.$$

Invoke Claim (iii) of Lemma 16 to get that:

$$\|\mathbf{V}\| \leq \sigma_K^{-1} (\hat{\mathbf{U}}^\top \mathbf{O}_M) \leq \frac{2}{\sqrt{3}\sigma_{K, \mathfrak{F}^*}}.$$

Furthermore, $\varphi_{a,c}(y_1, y_3) := \varphi_a(y_1)\varphi_c(y_3)$ is an orthonormal family of $\mathbf{L}^2(\mathcal{Y}^2, \mathcal{L}^{\mathcal{D} \otimes 2})$ and it holds:

$$\|\mathbf{N}_M\|_F = \left(\sum_{a,c=1}^M (\mathbb{E}\varphi_a(Y_1)\varphi_c(Y_3))^2 \right)^{\frac{1}{2}} = \left(\sum_{a,c=1}^M \langle f_{(Y_1, Y_3)}^*, \varphi_{a,c} \rangle_{\mathbf{L}^2(\mathcal{Y}^2, \mathcal{L}^{\mathcal{D} \otimes 2})}^2 \right)^{\frac{1}{2}} \leq \|f_{(Y_1, Y_3)}^*\|_2.$$

Compute:

$$\begin{aligned} \|\mathbf{V}\mathbf{N}_M\mathbf{V}^\top - \hat{\mathbf{V}}\hat{\mathbf{N}}_M\hat{\mathbf{V}}^\top\| &= \|\mathbf{V}\mathbf{N}_M\mathbf{V}^\top - (\mathbf{V} + E)(\mathbf{N}_M + F)(\mathbf{V} + E)^\top\|, \\ &= \|\mathbf{V}\mathbf{N}_ME^\top + \mathbf{V}F\mathbf{V}^\top + \mathbf{V}FE^\top + E\mathbf{N}_M\mathbf{V}^\top + E\mathbf{N}_ME^\top + EF\mathbf{V}^\top + EFE^\top\|, \\ &\leq 2\|E\|\|\mathbf{V}\|\|\mathbf{N}_M\| + 2\|E\|\|\mathbf{V}\|\|F\| + \|E\|^2\|\mathbf{N}_M\| + \|\mathbf{V}\|^2\|F\| + \|E\|^2\|F\|. \end{aligned}$$

It yields:

$$\begin{aligned} \|\mathbf{V}\mathbf{N}_M\mathbf{V}^\top - \hat{\mathbf{V}}\hat{\mathbf{N}}_M\hat{\mathbf{V}}^\top\| &\leq \frac{32\sqrt{K}\mathcal{C}_M(\mathbf{Q}^*, \mathfrak{F}^*, \delta)\mathcal{C}_*(\mathbf{Q}^*, \delta)\|f_{(Y_1, Y_3)}^*\|_2}{3\sqrt{3}\sigma_{K, \mathfrak{F}^*}^3} \left[1 + \frac{\mathcal{C}_*(\mathbf{Q}^*, \delta)}{\|f_{(Y_1, Y_3)}^*\|_2} \frac{\eta_3(\Phi_M)}{\sqrt{NM}} \right. \\ &\quad + \frac{2\sqrt{K}\mathcal{C}_M(\mathbf{Q}^*, \mathfrak{F}^*, \delta)\mathcal{C}_*(\mathbf{Q}^*, \delta)}{\sqrt{3}\sigma_{K, \mathfrak{F}^*}} \frac{\eta_3(\Phi_M)}{\sqrt{N}} \\ &\quad + \frac{\sqrt{3}\sigma_{K, \mathfrak{F}^*}}{4\mathcal{C}_M(\mathbf{Q}^*, \mathfrak{F}^*, \delta)\|f_{(Y_1, Y_3)}^*\|_2\sqrt{K}} \frac{1}{\sqrt{M}} \\ &\quad \left. + \frac{2\sqrt{K}\mathcal{C}_M(\mathbf{Q}^*, \mathfrak{F}^*, \delta)\mathcal{C}_*(\mathbf{Q}^*, \delta)^2\eta_3^2(\Phi_M)}{\sqrt{3}\sigma_{K, \mathfrak{F}^*}\|f_{(Y_1, Y_3)}^*\|_2 N\sqrt{M}} \right] \frac{\eta_3(\Phi_M)}{\sqrt{N}} \end{aligned}$$

Note $N \geq \mathbf{N}_2(\mathbf{Q}^*, \mathfrak{F}^*, \Phi_M, \delta) \geq 4\mathbf{N}_1(\mathbf{Q}^*, \mathfrak{F}^*, \Phi_M, \delta) = \frac{16K}{3\sigma_{K, \mathfrak{F}^*}^2}\mathcal{C}_M(\mathbf{Q}^*, \mathfrak{F}^*, \delta)^2\mathcal{C}_*(\mathbf{Q}^*, \delta)^2\eta_3(\Phi_M)^2$.

It yields:

$$(27) \quad \|\mathbf{V}\mathbf{N}_M\mathbf{V}^\top - \hat{\mathbf{V}}\hat{\mathbf{N}}_M\hat{\mathbf{V}}^\top\| \leq \tilde{\mathcal{E}}_M(\mathbf{Q}^*, \mathfrak{F}^*, \delta)\mathcal{C}_*(\mathbf{Q}^*, \delta)\frac{\eta_3(\Phi_M)}{\sqrt{N}}$$

Observe that:

$$\begin{aligned} \|\mathbf{Q}^* - \tilde{\mathbf{Q}}\| &= \|(\mathfrak{D}\text{diag}[\pi^*]^{-1} - \mathfrak{D}\text{diag}[\hat{\pi}]^{-1})\mathbf{V}\mathbf{N}_M\mathbf{V}^\top + \mathfrak{D}\text{diag}[\hat{\pi}]^{-1}(\mathbf{V}\mathbf{N}_M\mathbf{V}^\top - \hat{\mathbf{V}}\hat{\mathbf{N}}_M\hat{\mathbf{V}}^\top)\| \\ &\leq \|\mathfrak{D}\text{diag}[\pi^*]^{-1} - \mathfrak{D}\text{diag}[\hat{\pi}]^{-1}\|\|\mathbf{V}\|^2\|\mathbf{N}_M\| + \|\mathfrak{D}\text{diag}[\hat{\pi}]^{-1}\|\|\mathbf{V}\mathbf{N}_M\mathbf{V}^\top - \hat{\mathbf{V}}\hat{\mathbf{N}}_M\hat{\mathbf{V}}^\top\| \\ &\leq \frac{4\|f_{(Y_1, Y_3)}^*\|_2}{3\sigma_{K, \mathfrak{F}^*}^2} \max_{k \in \mathcal{X}}(\pi_k^{*-1} - \hat{\pi}_k^{-1}) + \max_{k \in \mathcal{X}}\hat{\pi}_k^{-1}\tilde{\mathcal{E}}_M(\mathbf{Q}^*, \mathfrak{F}^*, \delta)\mathcal{C}_*(\mathbf{Q}^*, \delta)\frac{\eta_3(\Phi_M)}{\sqrt{N}} \\ &\leq \frac{8\|f_{(Y_1, Y_3)}^*\|_2}{3\sigma_{K, \mathfrak{F}^*}^2\pi_{\min}^*} \|\tilde{\pi} - \pi^*\|_2 + \frac{2}{\pi_{\min}^*}\tilde{\mathcal{E}}_M(\mathbf{Q}^*, \mathfrak{F}^*, \delta)\mathcal{C}_*(\mathbf{Q}^*, \delta)\frac{\eta_3(\Phi_M)}{\sqrt{N}}, \end{aligned}$$

using (26) and (27). \square

Combining (25) and Lemma 9 we prove the second point of Theorem 4.

6.3.5. Last step: Final estimation of the stationary distribution. From our model hypothesis **[H2]**, we know that the transition matrix \mathbf{Q}^* is irreducible and aperiodic. Perron-Frobenius theorem shows that \mathbf{Q}^* has a unique stationary distribution π^* . More precisely, it holds

- $\mathbb{R} \cdot \pi^* = \ker(\text{Id}_K - (\mathbf{Q}^*)^\top)$ so that $(\mathbb{R} \cdot \pi^*)^\perp = \text{range}(\text{Id}_K - \mathbf{Q}^*)$,
- and $\langle \pi^*, \mathbf{1}_K \rangle = 1$,

where $\mathbf{1}_K = (1, \dots, 1) \in \mathbb{R}^K$. We deduce $\mathbf{1}_K \notin \text{range}(\text{Id}_K - \mathbf{Q}^*)$ and

$$\text{Rank} \begin{pmatrix} \text{Id}_K - (\mathbf{Q}^*)^\top \\ \mathbf{1}_K^\top \end{pmatrix} = K.$$

Set

$$A = \begin{pmatrix} \text{Id}_K - \mathbf{Q}^\top \\ \mathbb{1}_K^\top \end{pmatrix} \quad \text{and} \quad A^* = \begin{pmatrix} \text{Id}_K - (\mathbf{Q}^*)^\top \\ \mathbb{1}_K^\top \end{pmatrix}.$$

We first derive an upper bound on $\|A^+ - (A^*)^+\|$ where A^+ denotes the Moore-Penrose pseudo-inverse of A . One has

$$(28) \quad A^+ - (A^*)^+ = (A^*)^+(A^* - A)A^+ - (A^*)^+(\text{Id}_{K+1} - AA^+).$$

The last term can be written as

$$\begin{aligned} (A^*)^+(\text{Id}_{K+1} - AA^+) &= (A^*)^+(A^*(A^*)^+)(\text{Id}_{K+1} - AA^+), \\ &= (A^*)^+P_{\text{range}(A^*)}P_{\text{range}(A)^{\perp}}, \end{aligned}$$

where $P_{\text{range}(A^*)} = A^*(A^*)^+$ denotes the orthogonal projection onto $\text{range}(A^*)$ and $P_{\text{range}(A)^{\perp}} = \text{Id}_{K+1} - AA^+$ denotes the orthogonal projection onto the orthogonal of $\text{range}(A)$. Set $s(\mathbf{Q}^*) = \sigma_K(A^*)$, we have the following perturbation lemma.

Lemma 10 — *If $\|\mathbf{Q} - \mathbf{Q}^*\| \leq s(\mathbf{Q}^*)/2$ then $\text{Rank}(A) = \text{Rank}(A^*) = K$ and*

$$\|P_{\text{range}(A^*)}P_{\text{range}(A)^{\perp}}\| \leq \frac{2\|\mathbf{Q} - \mathbf{Q}^*\|}{s(\mathbf{Q}^*)}.$$

Proof. The first point follows from Weyl's inequality, see Theorem 6. Invoking [Wed72], one has

$$\|P_{\text{range}(A^*)^{\perp}}P_{\text{range}(A)}\| = \|P_{\text{range}(A)^{\perp}}P_{\text{range}(A^*)}\|.$$

Moreover, since projections P are orthogonal

$$(P_{\text{range}(A)^{\perp}}P_{\text{range}(A^*)})^\top = P_{\text{range}(A^*)}P_{\text{range}(A)^{\perp}}.$$

Using notation of [Wed72], one may notice that

$$\|\sin \theta(\text{range}(A), \text{range}(A^*))\| = \|P_{\text{range}(A^*)^{\perp}}P_{\text{range}(A)}\|.$$

Denote $s(\mathbf{Q}^*) = \sigma_K(A^*)$ and invoke Wedin's theorem [Wed72] to get the following result: If

$$\sigma_K(A) \geq \frac{s(\mathbf{Q}^*)}{2}$$

then

$$\|\sin \theta(\text{range}(A), \text{range}(A^*))\| \leq \frac{2\|A - A^*\|}{\sigma_K(A^*)}$$

We conclude using Weyl's inequality, see Theorem 6. \square

Triangular inequality in (28) gives

$$\begin{aligned} \|A^+ - (A^*)^+\| &\leq \|(A^*)^+\| \|\mathbf{Q} - \mathbf{Q}^*\| \left(\|A^+\| + \frac{2}{\sigma_K(A^*)} \right), \\ &\leq \frac{\|\mathbf{Q} - \mathbf{Q}^*\|}{\sigma_K(A^*)} \left(\|A^+ - (A^*)^+\| + \frac{3}{\sigma_K(A^*)} \right), \end{aligned}$$

using that $\|(A^*)^+\| = 1/\sigma_K(A^*)$. Deduce that if $\|\mathbf{Q} - \mathbf{Q}^*\| \leq \sigma_K(A^*)/2$ then

$$\|A^+ - (A^*)^+\| \leq \frac{6\|\mathbf{Q} - \mathbf{Q}^*\|}{\sigma_K^2(A^*)}.$$

From Weyl's inequality, if $\|\mathbf{Q} - \mathbf{Q}^*\| \leq \sigma_K(A^*)/2$ then $\sigma_K(A) \geq \sigma_K(A^*)/2$. As a matter of fact, matrix $\text{Id}_K - \mathbf{Q}^\top$ has rank $K - 1$ and the eigenspace $\ker(\text{Id}_K - \mathbf{Q}^\top)$ has dimension 1. Thus, \mathbf{Q} is an irreducible and aperiodic transition matrix, and π is the unique solution to

$$\begin{pmatrix} \text{Id}_K - \mathbf{Q}^\top \\ \mathbb{1}_K^\top \end{pmatrix} \pi = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Now

$$\|\pi - \pi^*\|_2 \leq \|A^+ - (A^*)^+\|,$$

and the last part of Theorem 4 is proved.

APPENDIX A. MATRIX PERTURBATION

We gather in this section some useful results in matrix perturbation theory. An interested reader can find the proofs of the following theorem in [SS90] for instance.

Theorem 6 (Weyl's inequality) — *Let A, B be $(p \times q)$ matrices with $p \geq q$ then it holds:*

$$\forall i = 1, \dots, q, \quad |\sigma_i(A + B) - \sigma_i(A)| \leq \sigma_1(B).$$

Theorem 7 — *Let A, B be $(p \times p)$ matrices. If A is invertible and $\|A^{-1}B\| < 1$ then $\tilde{A} := A + B$ is invertible and it holds:*

$$\|\tilde{A}^{-1} - A^{-1}\| \leq \frac{\|B\| \|A^{-1}\|^2}{1 - \|A^{-1}B\|}.$$

Theorem 8 (Bauer-Fike) — *Let A, B be $(p \times p)$ matrices and $\tilde{A} := A + B$. Suppose that A is diagonalizable, i.e. $X^{-1}AX = \Lambda$, where $\Lambda = \mathbf{Diag}[(\lambda_1, \dots, \lambda_p)]$. Then:*

$$(29) \quad \text{sv}_A(\tilde{A}) \leq \kappa(X) \|B\|,$$

where $\text{sv}_A(\tilde{A}) := \max_j \min_i |\tilde{\lambda}_j - \lambda_i|$ and $\tilde{\lambda}_j$ denotes the eigenvalues of \tilde{A} .

Remark — *Moreover, if the disks:*

$$\mathcal{D}_i := \{\xi : |\xi - \lambda_i| \leq \kappa(X) \|B\|\}$$

are isolated from the others, then (29) holds with the matching distance:

$$\text{md}(A, \tilde{A}) \leq \kappa(X) \|B\|,$$

where $\text{md}(A, \tilde{A}) := \min_{\tau \in \mathcal{S}_p} \max_i |\hat{\lambda}_{\tau(i)} - \lambda_i|$. Eventually, if Λ, \tilde{A} are real valued matrices then \tilde{A} has p distinct real eigenvalues.

APPENDIX B. CONCENTRATION INEQUALITIES

We first recall results that hold both for **(Scenario A)** (where we consider N i.i.d. samples $(Y_1^{(s)}, Y_2^{(s)}, Y_3^{(s)})_{s=1}^N$ of three consecutive observations) and for **(Scenario B)** (where we consider consecutive observations of the same chain).

The following proposition is the classical Bernstein's inequality for **(Scenario A)** and is proved in [Pau13], Theorem 2.4, for **(Scenario B)**.

Proposition 9 — *Let t be a real valued and measurable bounded function on \mathcal{Y}^3 . Let $V = \mathbb{E}[t^2(Z_1)]$. There exists a positive constant c^* depending only on \mathbf{Q}^* such that for all $0 \leq \lambda \leq 1/(2\sqrt{2}c^*\|t\|_\infty)$:*

$$(30) \quad \log \mathbb{E} \exp \left[\lambda \sum_{s=1}^N (t(Z_s) - \mathbb{E}t(Z_s)) \right] \leq \frac{2Nc^*V\lambda^2}{1 - 2\sqrt{2}c^*\|t\|_\infty\lambda}$$

so that for all $x \geq 0$,

$$(31) \quad \mathbb{P} \left(\sum_{s=1}^N (t(Z_s) - \mathbb{E}t(Z_s)) \geq 2\sqrt{2Nc^*Vx} + 2\sqrt{2}c^*\|t\|_\infty x \right) \leq e^{-x}.$$

We now state a deviation inequality, which comes from [Mas07] Theorem 6.8 and Corollary 6.9 for **(Scenario A)**. For **(Scenario B)** the proof of the following proposition follows mutatis mutandis from the proof of Theorem 6.8 (and then Corollary 6.9) in [Mas07] the early first step being equation (30). Recall that when t_1 and t_2 are real valued functions, the bracket $[t_1, t_2]$ is the set of real valued functions t such that $t_1(\cdot) \leq t(\cdot) \leq t_2(\cdot)$. For any measurable set A such that $\mathbb{P}(A) > 0$, and any integrable random variable Z , denote $E^A[Z] = E[Z\mathbf{1}_A]/\mathbb{P}(A)$.

Proposition 10 — *Let \mathcal{T} be some countable class of real valued and measurable functions on \mathcal{Y}^3 . Assume that there exists some positive numbers σ and b such that for all $t \in \mathcal{T}$, $\|t\|_\infty \leq b$ and $\mathbb{E}[t^2(Z_1)] \leq \sigma^2$.*

Assume furthermore that for any positive number δ , there exists some finite set B_δ of brackets covering \mathcal{F} such that for any bracket $[t_1, t_2] \in B_\delta$, $\|t_1 - t_2\|_\infty \leq b$ and $\mathbb{E}[(t_1 - t_2)^2(Z_1)] \leq \delta^2$. Let $e^{H(\delta)}$ denote the minimal cardinality of such a covering. Then, there exists a positive constant C^ depending only on \mathbf{Q}^* such that: for any measurable set A ,*

$$\mathbb{E}^A \left(\sup_{t \in \mathcal{T}} \sum_{s=1}^N (t(Z_s) - \mathbb{E}t(Z_s)) \right) \leq C^* \left[E + \sigma \sqrt{N \log \left(\frac{1}{\mathbb{P}(A)} \right)} + b \log \left(\frac{1}{\mathbb{P}(A)} \right) \right]$$

and for all positive number x

$$\mathbb{P} \left(\sup_{t \in \mathcal{T}} \sum_{s=1}^N (t(Z_s) - \mathbb{E}t(Z_s)) \geq C^* [E + \sigma \sqrt{Nx} + bx] \right) \leq \exp(-x)$$

where

$$E = \sqrt{N} \int_0^\sigma \sqrt{H(u) \wedge N} du + (b + \sigma)H(\sigma)$$

We now separate statements for the two different scenarios.

B.1. Independent samples. In this subsection we assume independent samples. Consider N i.i.d. samples $Z_s := (Y_1^{(s)}, Y_2^{(s)}, Y_3^{(s)})$ of three consecutive observations.

Lemma 11 — *It holds, for any positive x , any M and any N :*

$$\begin{aligned} \mathbb{P} \left[\|\hat{\mathbf{L}}_M - \mathbf{L}_M\|_F \geq \frac{\eta_1(\Phi_M)}{\sqrt{2N}}(1+x) \right] &\leq \exp(-x^2), \\ \mathbb{P} \left[\|\hat{\mathbf{M}}_M - \mathbf{M}_M\|_F \geq \frac{\eta_3(\Phi_M)}{\sqrt{2N}}(1+x) \right] &\leq \exp(-x^2), \\ \mathbb{P} \left[\|\hat{\mathbf{N}}_M - \mathbf{N}_M\|_F \geq \frac{\eta_2(\Phi_M)}{\sqrt{2N}}(1+x) \right] &\leq \exp(-x^2), \\ \mathbb{P} \left[\|\hat{\mathbf{P}}_M - \mathbf{P}_M\|_F \geq \frac{\eta_2(\Phi_M)}{\sqrt{2N}}(1+x) \right] &\leq \exp(-x^2). \end{aligned}$$

Proof. Set:

$$\begin{aligned} \zeta_{\mathbf{L}_M}(Z_1, Z_2, \dots, Z_N) &:= \|\hat{\mathbf{L}}_M(Z_1, Z_2, \dots, Z_N) - \mathbf{L}_M\|_2, \\ \zeta_{\mathbf{M}_M}(Z_1, Z_2, \dots, Z_N) &:= \|\hat{\mathbf{M}}_M(Z_1, Z_2, \dots, Z_N) - \mathbf{M}_M\|_F, \\ \zeta_{\mathbf{N}_M}(Z_1, Z_2, \dots, Z_N) &:= \|\hat{\mathbf{N}}_M(Z_1, Z_2, \dots, Z_N) - \mathbf{N}_M\|_F, \\ \zeta_{\mathbf{P}_M}(Z_1, Z_2, \dots, Z_N) &:= \|\hat{\mathbf{P}}_M(Z_1, Z_2, \dots, Z_N) - \mathbf{P}_M\|_F, \end{aligned}$$

where, for instance $\hat{\mathbf{L}}_M(Z_1, Z_2, \dots, Z_N)$ denotes the dependence of $\hat{\mathbf{L}}_M$ in Z_1, Z_2, \dots, Z_N . We begin with $\zeta_{\mathbf{M}_M}$, other cases are similar. Form the difference with respect to the coordinate i :

$$c_i := \sup_{z_j \in \mathcal{Y}^3, z'_i \in \mathcal{Y}^3} |\zeta_{\mathbf{M}_M}(z_1, \dots, z_{i-1}, z_i, z_{i+1}, \dots, z_N) - \zeta_{\mathbf{M}_M}(z_1, \dots, z_{i-1}, z'_i, z_{i+1}, \dots, z_N)|.$$

Invoke the triangular inequality to get that:

$$c_i \leq \sup_{z_j \in \mathcal{Y}^3, z'_i \in \mathcal{Y}^3} \|\hat{\mathbf{M}}_M(z_1, \dots, z_{i-1}, z_i, z_{i+1}, \dots, z_N) - \hat{\mathbf{M}}_M(z_1, \dots, z_{i-1}, z'_i, z_{i+1}, \dots, z_N)\|_F$$

So that:

$$c_i \leq \frac{1}{N} \sup_{z_i \in \mathcal{Y}^3, z'_i \in \mathcal{Y}^3} \left(\sum_{a,b,c} (\varphi_a(y_1^{(i)})\varphi_b(y_2^{(i)})\varphi_c(y_3^{(i)}) - \varphi_a(y_1'^{(i)})\varphi_b(y_2'^{(i)})\varphi_c(y_3'^{(i)})) \right)^{\frac{1}{2}}.$$

Eventually, we get that $c_i \leq \eta_3(\Phi_M)/N$. By McDiarmid's inequality, one has:

$$\forall x > 0, \quad \mathbb{P}(\|\hat{\mathbf{M}}_M - \mathbf{M}_M\|_F \geq \mathbb{E}\|\hat{\mathbf{M}}_M - \mathbf{M}_M\|_F + x) \leq \exp\left(-\frac{2Nx^2}{\eta_3^2(\Phi_M)}\right).$$

Futhermore, it holds:

$$\begin{aligned} \mathbb{E}\|\hat{\mathbf{M}}_M - \mathbf{M}_M\|_F &\leq \left[\mathbb{E}\|\hat{\mathbf{M}}_M - \mathbf{M}_M\|_F^2 \right]^{\frac{1}{2}}, \\ &\leq \left[\mathbb{E} \sum_{a,b,c} \left(\frac{1}{N} \sum_{s=1}^N \varphi_a(Y_1^{(s)})\varphi_b(Y_2^{(s)})\varphi_c(Y_3^{(s)}) - \mathbb{E}\varphi_a(Y_1)\varphi_b(Y_2)\varphi_c(Y_3) \right)^2 \right]^{\frac{1}{2}}, \\ &\leq \left[\sum_{a,b,c} \mathbb{E} \left(\sum_{s=1}^N \frac{1}{N} [\varphi_a(Y_1^{(s)})\varphi_b(Y_2^{(s)})\varphi_c(Y_3^{(s)}) - \mathbb{E}\varphi_a(Y_1)\varphi_b(Y_2)\varphi_c(Y_3)] \right)^2 \right]^{\frac{1}{2}}, \\ &\leq \left[\sum_{a,b,c} \sum_{s=1}^N \mathbb{E} \left(\frac{1}{N} [\varphi_a(Y_1^{(s)})\varphi_b(Y_2^{(s)})\varphi_c(Y_3^{(s)}) - \mathbb{E}\varphi_a(Y_1)\varphi_b(Y_2)\varphi_c(Y_3)] \right)^2 \right]^{\frac{1}{2}}, \\ &\leq \frac{1}{\sqrt{N}} \left[\sum_{a,b,c} \mathbb{E} (\varphi_a(Y_1)\varphi_b(Y_2)\varphi_c(Y_3) - \mathbb{E}\varphi_a(Y_1)\varphi_b(Y_2)\varphi_c(Y_3))^2 \right]^{\frac{1}{2}}, \\ &\leq \frac{1}{\sqrt{2N}} \left[\mathbb{E} \sum_{a,b,c} (\varphi_a(Y_1)\varphi_b(Y_2)\varphi_c(Y_3) - \varphi_a(Y'_1)\varphi_b(Y'_2)\varphi_c(Y'_3))^2 \right]^{\frac{1}{2}}, \\ &\leq \frac{\eta_3(\Phi_M)}{\sqrt{2N}}, \end{aligned}$$

using Jensen's inequality and $2\mathbb{E}(U - \mathbb{E}U)^2 \leq \mathbb{E}(U - U')^2$ where U is any real valued random variable with finite second moment and U' an independent copy of U . The proof is similar for \mathbf{L}_M , \mathbf{N}_M and \mathbf{P}_M . \square

B.2. Same chain sampling. Consider consecutive observations of the same hidden Markov chain $Z_s := (Y_s, Y_{s+1}, Y_{s+2})$ for $1 \leq s \leq N$,

Lemma 12 — *It holds, for any positive x , any M and any N :*

$$\begin{aligned} \mathbb{P} \left[\|\hat{\mathbf{L}}_M - \mathbf{L}_M\|_F \geq \frac{\sqrt{2}\eta_1(\Phi_M)}{\sqrt{N\mathbb{G}_{\text{ps}}}} (1 + 2x\sqrt{1 + \log(8/\pi^*_{\min})}) \right] &\leq \exp(-x^2), \\ \mathbb{P} \left[\|\hat{\mathbf{M}}_M - \mathbf{M}_M\|_F \geq \frac{\sqrt{2}\eta_3(\Phi_M)}{\sqrt{N\mathbb{G}_{\text{ps}}}} (1 + 2x\sqrt{1 + \log(8/\pi^*_{\min})}) \right] &\leq \exp(-x^2), \\ \mathbb{P} \left[\|\hat{\mathbf{N}}_M - \mathbf{N}_M\|_F \geq \frac{\sqrt{2}\eta_2(\Phi_M)}{\sqrt{N\mathbb{G}_{\text{ps}}}} (1 + 2x\sqrt{1 + \log(8/\pi^*_{\min})}) \right] &\leq \exp(-x^2), \\ \mathbb{P} \left[\|\hat{\mathbf{P}}_M - \mathbf{P}_M\|_F \geq \frac{\sqrt{2}\eta_2(\Phi_M)}{\sqrt{N\mathbb{G}_{\text{ps}}}} (1 + 2x\sqrt{1 + \log(8/\pi^*_{\min})}) \right] &\leq \exp(-x^2). \end{aligned}$$

Proof. With the same notation as previous lemma, we get that:

$$\sup_{z_j \in \mathcal{Y}^3, z'_i \in \mathcal{Y}^3} |\zeta_{\mathbf{M}_M}(z_1, \dots, z_{i-1}, z_i, z_{i+1}, \dots, z_N) - \zeta_{\mathbf{M}_M}(z_1, \dots, z_{i-1}, z'_i, z_{i+1}, \dots, z_N)| \leq \frac{\eta_3(\Phi_M)}{N}.$$

By McDiarmid's inequality [Pau14], one has:

$$\forall x > 0, \quad \mathbb{P}(\|\hat{\mathbf{M}}_M - \mathbf{M}_M\|_F \geq \mathbb{E}\|\hat{\mathbf{M}}_M - \mathbf{M}_M\|_F + x) \leq \exp\left(-\frac{Nx^2}{8\mathbb{T}_{\text{mix}}\eta_3^2(\Phi_M)}\right).$$

We need the following lemma that can be deduced from [Pau14].

Lemma 13 — *For any $a, b, c \in \{1, \dots, M\}$,*

$$\begin{aligned} & \mathbb{E}\left(\sum_{s=1}^N \frac{1}{N} [\varphi_a(Y_s)\varphi_b(Y_{s+1})\varphi_c(Y_{s+2}) - \mathbb{E}\varphi_a(Y_1)\varphi_b(Y_2)\varphi_c(Y_3)]\right)^2 \\ & \leq \frac{4}{N\mathbb{G}_{\text{ps}}}\mathbb{E}([\varphi_a(Y_1)\varphi_b(Y_2)\varphi_c(Y_3) - \mathbb{E}\varphi_a(Y_1)\varphi_b(Y_2)\varphi_c(Y_3)])^2. \end{aligned}$$

Proof. Notice that $(X_1, Y_1), (X_2, Y_2), \dots$ is homogenous, irreducible, aperiodic and stationary Markov chain on $\mathcal{X} \times \mathcal{Y}$, whose stationary distribution is $\tilde{\pi}(x, dy) := \pi_x \mu_x(dy)$. Observe that its transition kernel $\tilde{\mathbf{Q}}$ enjoys:

$$\forall x, x' \in \mathcal{X}, y, y' \in \mathcal{Y}, \quad \tilde{\mathbf{Q}}(x, y; x', dy') = \mathbf{Q}^*(x, x')\mu_{x'}(dy').$$

The transition kernel $\tilde{\mathbf{Q}}$ can be viewed as an operator \mathbb{Q} on the Hilbert space $\mathbf{L}^2(\tilde{\pi})$ defined by:

$$\forall f \in \mathbf{L}^2(\tilde{\pi}), \quad (\mathbb{Q}f)(x, y) := \mathbb{E}_{\tilde{\mathbf{Q}}(x, y; \dots)}(f) = \sum_{x' \in \mathcal{X}} \mathbf{Q}^*(x, x') \int_{\mathcal{Y}} f(x', y') \mu_{x'}(dy').$$

Notice $\mathbb{Q}f(x, y)$ does not depend on y . Set $E := \{f(x, y) \in \mathbf{L}^2(\tilde{\pi}) : f \text{ does not depend on } y\}$ and observe the $\mathbf{L}^2(\tilde{\pi})$ -self-adjoint operator:

$$\forall f \in \mathbf{L}^2(\tilde{\pi}), \quad (\Pi_E f)(x, y) := \int_{\mathcal{Y}} f(x, y') \mu_x(dy'),$$

is the orthogonal projection onto E . Since $\Pi_E \mathbb{Q} \Pi_E = \mathbb{Q}$ then the set of non-zero eigenvalues of \mathbb{Q} is exactly the set of non-zero eigenvalues of the K dimensional linear operator $\Pi_E \mathbb{Q} \Pi_E$. Eventually, note that the matrix of \mathbb{Q} in the basis $((x, y) \mapsto \mathbb{1}_{k=x})_{k \in \mathcal{X}}$ is the matrix \mathbf{Q}^* . We deduce that the pseudo spectral gap of \mathbb{Q} is equal to \mathbb{G}_{ps} (the pseudo spectral gap of \mathbf{Q}^*).

Furthermore, note the same analysis can be lead for $(X_1, X_2, X_3, Z_1), (X_2, X_3, X_4, Z_2), \dots$ and one can check that its pseudo spectral gap is the pseudo spectral gap of the Markov chain $(X_1, X_2, X_3), (X_2, X_3, X_4), \dots$ which is \mathbb{G}_{ps} . Indeed, the set of non-zero eigenvalues of the Markov chain $(X_1, X_2, X_3), (X_2, X_3, X_4), \dots$ is equal to the set of non-zero eigenvalues of the Markov chain X_1, X_2, \dots

Eventually, set $g(X_s, X_{s+1}, X_{s+2}, Z_s) := (1/N)\varphi_a(Y_s)\varphi_b(Y_{s+1})\varphi_c(Y_{s+2})$ and invoke Theorem 3.7 in [Pau14] to get the result. \square

Hence it holds:

$$\begin{aligned}
\mathbb{E}\|\hat{\mathbf{M}}_M - \mathbf{M}_M\|_F &\leq \left[\mathbb{E}\|\hat{\mathbf{M}}_M - \mathbf{M}_M\|_F^2 \right]^{\frac{1}{2}}, \\
&\leq \left[\mathbb{E} \sum_{a,b,c} \left(\frac{1}{N} \sum_{s=1}^N \varphi_a(Y_s) \varphi_b(Y_{s+1}) \varphi_c(Y_{s+2}) - \mathbb{E} \varphi_a(Y_1) \varphi_b(Y_2) \varphi_c(Y_3) \right)^2 \right]^{\frac{1}{2}}, \\
&\leq \left[\sum_{a,b,c} \mathbb{E} \left(\sum_{s=1}^N \frac{1}{N} [\varphi_a(Y_s) \varphi_b(Y_{s+1}) \varphi_c(Y_{s+2}) - \mathbb{E} \varphi_a(Y_1) \varphi_b(Y_2) \varphi_c(Y_3)] \right)^2 \right]^{\frac{1}{2}}, \\
&\leq \frac{2}{\sqrt{N G_{\text{ps}}}} \left[\sum_{a,b,c} \mathbb{E} (\varphi_a(Y_1) \varphi_b(Y_2) \varphi_c(Y_3) - \mathbb{E} \varphi_a(Y_1) \varphi_b(Y_2) \varphi_c(Y_3))^2 \right]^{\frac{1}{2}}, \\
&\leq \left(\frac{2}{N G_{\text{ps}}} \right)^{\frac{1}{2}} \left[\mathbb{E} \sum_{a,b,c} (\varphi_a(Y_1) \varphi_b(Y_2) \varphi_c(Y_3) - \varphi_a(Y'_1) \varphi_b(Y'_2) \varphi_c(Y'_3))^2 \right]^{\frac{1}{2}}, \\
&\leq \left(\frac{2 \eta_3^2(\Phi_M)}{N G_{\text{ps}}} \right)^{\frac{1}{2}},
\end{aligned}$$

using Jensen's inequality, Lemma 13 and then $2\mathbb{E}(U - \mathbb{E}U)^2 \leq \mathbb{E}(U - U')^2$ where U is any real valued random variable with finite second moment and U' an independent copy of U . The proof is similar for \mathbf{L}_M , \mathbf{N}_M and \mathbf{P}_M . \square

APPENDIX C. PROOF OF THEOREM 5

C.1. Preliminaries lemmas.

Lemma 14 — *It holds:*

$$\forall b \in \{1, \dots, M\}, \quad \mathbf{M}_M(\cdot, b, \cdot) = \mathbf{O}_M \mathfrak{D} \text{diag}[\pi^*] \mathbf{Q}^* \mathfrak{D} \text{diag}[\mathbf{O}_M(b, \cdot)] \mathbf{Q}^* \mathbf{O}_M^\top.$$

Similarly, we get that $\mathbf{P}_M = \mathbf{O}_M \mathfrak{D} \text{diag}[\pi^*] \mathbf{Q}^{*2} \mathbf{O}_M^\top$.

Proof. Let $a, c \in \{1, \dots, M\}^2$ and observe that:

$$\begin{aligned}
&(\mathbf{O}_M \mathfrak{D} \text{diag}[\pi^*] \mathbf{Q}^* \mathfrak{D} \text{diag}[\mathbf{O}_M(b, \cdot)] \mathbf{Q}^* \mathbf{O}_M^\top)(a, c) \\
&= \sum_{(i,j,k) \in \mathcal{X}^3} \mathbf{O}_M(a, i) \pi(i) \mathbf{Q}^*(i, j) \mathbf{O}_M(b, j) \mathbf{Q}^*(j, k) \mathbf{O}_M(c, k), \\
&= \sum_{(i,j,k) \in \mathcal{X}^3} \mathbb{E}(\varphi_a(Y_1) | X_1 = i) \mathbb{P}(X_1 = i) \mathbb{P}(X_2 = j | X_1 = i) \\
&\quad \times \mathbb{E}(\varphi_b(Y_2) | X_2 = j) \mathbb{P}(X_3 = k | X_2 = j) \mathbb{E}(\varphi_c(Y_3) | X_3 = k), \\
&= \mathbb{E}(\varphi_a(Y_1) \varphi_b(Y_2) \varphi_c(Y_3)),
\end{aligned}$$

as claimed. Similarly, it holds:

$$\begin{aligned}
&(\mathbf{O}_M \mathfrak{D} \text{diag}[\pi^*] \mathbf{Q}^{*2} \mathbf{O}_M^\top)(a, c) = \sum_{(i,j,k) \in \mathcal{X}^3} \mathbf{O}_M(a, i) \pi(i) \mathbf{Q}^*(i, j) \mathbf{Q}^*(j, k) \mathbf{O}_M(c, k), \\
&= \sum_{(i,j,k) \in \mathcal{X}^3} \mathbb{E}(\varphi_a(Y_1) | X_1 = i) \mathbb{P}(X_1 = i) \mathbb{P}(X_2 = j | X_1 = i) \\
&\quad \times \mathbb{P}(X_3 = k | X_2 = j) \mathbb{E}(\varphi_c(Y_3) | X_3 = k), \\
&= \mathbb{E}(\varphi_a(Y_1) \varphi_c(Y_3)),
\end{aligned}$$

as announced. \square

Lemma 15 — *Let \mathbf{U} be any $(M \times K)$ matrix such that $\mathbf{P}_M \mathbf{U}$ has rank K .*

- It holds:

$$\forall b \in \{1, \dots, M\}, \quad \mathbf{B}(b) := (\mathbf{P}_M \mathbf{U})^\dagger \mathbf{M}_M(\cdot, b, \cdot) \mathbf{U} = \mathbf{R} \mathfrak{D}\text{diag}[\mathbf{O}_M(b, \cdot)] \mathbf{R}^{-1},$$

where $\mathbf{R}^{-1} := \mathbf{Q}^* \mathbf{O}_M^\top \mathbf{U}$ and $(\mathbf{P}_M \mathbf{U})^\dagger := (\mathbf{U}^\top \mathbf{P}_M^\top \mathbf{P}_M \mathbf{U})^{-1} \mathbf{U}^\top \mathbf{P}_M^\top$ denotes the Moore-Penrose pseudoinverse of the matrix $\mathbf{P}_M \mathbf{U}$.

- Furthermore, observe that $\mathbf{U}^\top \mathbf{P}_M \mathbf{U}$ is invertible and it holds:

$$\forall b \in \{1, \dots, M\}, \quad \mathbf{B}(b) = (\mathbf{U}^\top \mathbf{P}_M \mathbf{U})^{-1} \mathbf{U}^\top \mathbf{M}_M(\cdot, b, \cdot) \mathbf{U} = \mathbf{R} \mathfrak{D}\text{diag}[\mathbf{O}_M(b, \cdot)] \mathbf{R}^{-1}.$$

Proof. Observe that:

$$\begin{aligned} \mathbf{M}_M(\cdot, b, \cdot) \mathbf{U} &= \mathbf{O}_M \mathfrak{D}\text{diag}[\pi^*] \mathbf{Q}^* \mathfrak{D}\text{diag}[\mathbf{O}_M(b, \cdot)] \mathbf{R}^{-1}, \\ &= \mathbf{P}_M \mathbf{U} \mathbf{R} \mathfrak{D}\text{diag}[\mathbf{O}_M(b, \cdot)] \mathbf{R}^{-1}, \end{aligned}$$

as claimed \square

Lemma 16 — Assume that $2\|\hat{\mathbf{P}}_M - \mathbf{P}_M\| < \sigma_K(\mathbf{P}_M)$, then the following is true.

- (i) It holds:

$$\varepsilon_{\mathbf{P}_M} := \frac{\|\hat{\mathbf{P}}_M - \mathbf{P}_M\|}{\sigma_K(\mathbf{P}_M) - \|\hat{\mathbf{P}}_M - \mathbf{P}_M\|} < 1,$$

- (ii)

$$\sigma_K(\hat{\mathbf{P}}_M) \geq \left[\frac{\sigma_K(\mathbf{P}_M) - \|\hat{\mathbf{P}}_M - \mathbf{P}_M\|}{\sigma_K(\mathbf{P}_M)} \right] \sigma_K(\mathbf{P}_M) > \frac{\sigma_K(\mathbf{P}_M)}{2},$$

$$(iii) \quad \sigma_K(\hat{\mathbf{U}}^\top \mathbf{U}) \geq (1 - \varepsilon_{\mathbf{P}_M}^2)^{\frac{1}{2}},$$

$$(iv) \quad \sigma_K(\hat{\mathbf{U}}^\top \mathbf{P}_M \hat{\mathbf{U}}) \geq (1 - \varepsilon_{\mathbf{P}_M}^2) \sigma_K(\mathbf{P}_M),$$

$$(v) \quad \text{for all } \alpha \in \mathbb{R}^K \text{ and for all } v \in \text{Range}(\mathbf{P}_M), \|\hat{\mathbf{U}}\alpha - v\|_2^2 \leq \|\alpha - \hat{\mathbf{U}}^\top v\|_2^2 + \varepsilon_{\mathbf{P}_M}^2 \|v\|_2^2.$$

- (vi) Furthermore, if $3\|\hat{\mathbf{P}}_M - \mathbf{P}_M\| \leq \sigma_K(\mathbf{P}_M)$ then:

$$\sigma_K(\hat{\mathbf{U}}^\top \hat{\mathbf{P}}_M \hat{\mathbf{U}}) \geq \frac{\sigma_K(\mathbf{P}_M)}{3},$$

- (vii) and:

$$\begin{aligned} \|(\hat{\mathbf{U}}^\top \hat{\mathbf{P}}_M \hat{\mathbf{U}})^{-1} - (\hat{\mathbf{U}}^\top \mathbf{P}_M \hat{\mathbf{U}})^{-1}\| &\leq \frac{\|\hat{\mathbf{P}}_M - \mathbf{P}_M\|}{\sigma_K(\mathbf{P}_M)(1 - \varepsilon_{\mathbf{P}_M}^2)((1 - \varepsilon_{\mathbf{P}_M}^2)\sigma_K(\mathbf{P}_M) - \|\hat{\mathbf{P}}_M - \mathbf{P}_M\|)}, \\ &\leq 3.2 \frac{\|\hat{\mathbf{P}}_M - \mathbf{P}_M\|}{\sigma_K^2(\mathbf{P}_M)}. \end{aligned}$$

Proof. See Lemma C.1 in [AHK12] for the first five claims. The sixth claim follows from the fourth point and Theorem 6. The seventh point follows from the fourth claim and Theorem 7. \square

C.2. Control of the observable operator. Claim (iv) in Lemma 16 and Lemma 15 ensure that:

$$\forall b \in \{1, \dots, M\}, \quad \tilde{\mathbf{B}}(b) := (\hat{\mathbf{U}}^\top \mathbf{P}_M \hat{\mathbf{U}})^{-1} \hat{\mathbf{U}}^\top \mathbf{M}_M(\cdot, b, \cdot) \hat{\mathbf{U}} = \tilde{\mathbf{R}} \mathfrak{D}\text{diag}[\mathbf{O}_M(b, \cdot)] \tilde{\mathbf{R}}^{-1},$$

where one can pick:

$$\tilde{\mathbf{R}}^{-1} := \mathfrak{D}\text{diag}[(\|(\mathbf{Q}^* \mathbf{O}_M^\top \hat{\mathbf{U}})^{-1}(\cdot, 1)\|_2, \dots, \|(\mathbf{Q}^* \mathbf{O}_M^\top \hat{\mathbf{U}})^{-1}(\cdot, K)\|_2)] \mathbf{Q}^* \mathbf{O}_M^\top \hat{\mathbf{U}}.$$

Set $\Lambda := \Theta^\top \hat{\mathbf{U}}^\top \mathbf{O}_M$ and:

$$\forall k \in \{1, \dots, K\}, \quad \tilde{\mathbf{C}}(k) := \sum_{b=1}^M (\hat{\mathbf{U}}\Theta)(b, k) \tilde{\mathbf{B}}(b) = \tilde{\mathbf{R}} \mathfrak{D}\text{diag}[\Lambda(k, \cdot)] \tilde{\mathbf{R}}^{-1}.$$

As a matter of fact, observe $\tilde{\mathbf{R}}$ has unit Euclidean norm columns:

$$\tilde{\mathbf{R}} = (\mathbf{Q}^* \mathbf{O}_M^\top \hat{\mathbf{U}})^{-1} \mathfrak{D}\text{diag}[(\|(\mathbf{Q}^* \mathbf{O}_M^\top \hat{\mathbf{U}})^{-1}(\cdot, 1)\|_2, \dots, \|(\mathbf{Q}^* \mathbf{O}_M^\top \hat{\mathbf{U}})^{-1}(\cdot, K)\|_2)]^{-1},$$

corresponding to unit Euclidean norm eigenvectors of $\tilde{\mathbf{C}}(k)$.

Lemma 17 — Assume that $3\|\hat{\mathbf{P}}_M - \mathbf{P}_M\| \leq \sigma_K(\mathbf{P}_M)$, then it holds for all $b \in \{1, \dots, M\}$,

$$\|\hat{\mathbf{B}}(b) - \tilde{\mathbf{B}}(b)\| \leq 3.2 \frac{\|\mathbf{M}_M(\cdot, b, \cdot)\|}{\sigma_K(\mathbf{P}_M)} \left[\frac{\|\hat{\mathbf{M}}_M(\cdot, b, \cdot) - \mathbf{M}_M(\cdot, b, \cdot)\|}{\|\mathbf{M}_M(\cdot, b, \cdot)\|} + \frac{\|\hat{\mathbf{P}}_M - \mathbf{P}_M\|}{\sigma_K(\mathbf{P}_M)} \right],$$

and:

$$\forall k \in \{1, \dots, K\}, \quad \|\hat{\mathbf{C}}(k) - \tilde{\mathbf{C}}(k)\| \leq 3.2 \frac{\|\mathbf{M}_M\|_{\infty,2}}{\sigma_K(\mathbf{P}_M)} \left[\frac{\|\hat{\mathbf{M}}_M - \mathbf{M}_M\|_{\infty,2}}{\|\mathbf{M}_M\|_{\infty,2}} + \frac{\|\hat{\mathbf{P}}_M - \mathbf{P}_M\|}{\sigma_K(\mathbf{P}_M)} \right].$$

Proof. Observe that:

$$\begin{aligned} \|\hat{\mathbf{B}}(b) - \tilde{\mathbf{B}}(b)\| &\leq \|(\hat{\mathbf{U}}^\top \hat{\mathbf{P}}_M \hat{\mathbf{U}})^{-1} \hat{\mathbf{U}}^\top \hat{\mathbf{M}}_M(\cdot, b, \cdot) \hat{\mathbf{U}} - (\hat{\mathbf{U}}^\top \hat{\mathbf{P}}_M \hat{\mathbf{U}})^{-1} \hat{\mathbf{U}}^\top \mathbf{M}_M(\cdot, b, \cdot) \hat{\mathbf{U}}\| \\ &\quad + \|(\hat{\mathbf{U}}^\top \mathbf{P}_M \hat{\mathbf{U}})^{-1} \hat{\mathbf{U}}^\top \mathbf{M}_M(\cdot, b, \cdot) \hat{\mathbf{U}} - (\hat{\mathbf{U}}^\top \hat{\mathbf{P}}_M \hat{\mathbf{U}})^{-1} \hat{\mathbf{U}}^\top \mathbf{M}_M(\cdot, b, \cdot) \hat{\mathbf{U}}\|, \\ &\leq \|\hat{\mathbf{U}}^\top (\hat{\mathbf{M}}_M(\cdot, b, \cdot) - \mathbf{M}_M(\cdot, b, \cdot)) \hat{\mathbf{U}}\| \|(\hat{\mathbf{U}}^\top \hat{\mathbf{P}}_M \hat{\mathbf{U}})^{-1}\| \\ &\quad + \|(\hat{\mathbf{U}}^\top \mathbf{P}_M \hat{\mathbf{U}})^{-1} - (\hat{\mathbf{U}}^\top \hat{\mathbf{P}}_M \hat{\mathbf{U}})^{-1}\| \|\hat{\mathbf{U}}^\top \mathbf{M}_M(\cdot, b, \cdot) \hat{\mathbf{U}}\|, \\ &\leq \|\hat{\mathbf{M}}_M(\cdot, b, \cdot) - \mathbf{M}_M(\cdot, b, \cdot)\| \sigma_K^{-1}(\hat{\mathbf{U}}^\top \hat{\mathbf{P}}_M \hat{\mathbf{U}}) \\ &\quad + \|\mathbf{M}_M(\cdot, b, \cdot)\| \|(\hat{\mathbf{U}}^\top \mathbf{P}_M \hat{\mathbf{U}})^{-1} - (\hat{\mathbf{U}}^\top \hat{\mathbf{P}}_M \hat{\mathbf{U}})^{-1}\|. \end{aligned}$$

Invoke claims (vi) and (vii) of Lemma 16 to get that:

$$3\sigma_K(\hat{\mathbf{U}}^\top \hat{\mathbf{P}}_M \hat{\mathbf{U}}) \geq \sigma_K(\mathbf{P}_M) \quad \text{and} \quad \|(\hat{\mathbf{U}}^\top \hat{\mathbf{P}}_M \hat{\mathbf{U}})^{-1} - (\hat{\mathbf{U}}^\top \mathbf{P}_M \hat{\mathbf{U}})^{-1}\| \leq 3.2 \frac{\|\hat{\mathbf{P}}_M - \mathbf{P}_M\|}{\sigma_K^2(\mathbf{P}_M)},$$

as claimed. Replacing $\mathbf{M}_M(\cdot, b, \cdot)$ by $\sum_{b=1}^M (\hat{\mathbf{U}}\theta)(b, k) \mathbf{M}_M(\cdot, b, \cdot)$, the same result holds for $\|\hat{\mathbf{C}}(k) - \tilde{\mathbf{C}}(k)\|$. \square

Lemma 18 — Assume that $2\|\hat{\mathbf{P}}_M - \mathbf{P}_M\| < \sigma_K(\mathbf{P}_M)$, then the following is true.

(i) It holds:

$$\kappa(\tilde{\mathbf{R}}) := \|\tilde{\mathbf{R}}\| \|\tilde{\mathbf{R}}^{-1}\| \leq \kappa^2(\mathbf{Q}^* \mathbf{O}_M^\top \hat{\mathbf{U}}) \leq \frac{\kappa^2(\mathbf{Q}^* \mathbf{O}_M^\top)}{1 - \varepsilon_{\mathbf{P}_M}^2},$$

(ii) and:

$$\text{sv}_{\mathbf{C}(1)}(\hat{\mathbf{C}}(1)) \leq \kappa(\tilde{\mathbf{R}}) \|\hat{\mathbf{C}}(1) - \tilde{\mathbf{C}}(1)\| \leq \frac{\kappa^2(\mathbf{Q}^* \mathbf{O}_M^\top)}{1 - \varepsilon_{\mathbf{P}_M}^2} \|\hat{\mathbf{C}}(1) - \tilde{\mathbf{C}}(1)\|,$$

where $\text{sv}_{\mathbf{C}(1)}(\hat{\mathbf{C}}(1)) := \max_{k_1 \in \mathcal{X}} \min_{k_2 \in \mathcal{X}} |\hat{\lambda}(1, k_1) - \lambda(1, k_2)|$.

(iii) If

$$\frac{\kappa^2(\mathbf{Q}^* \mathbf{O}_M^\top)}{1 - \varepsilon_{\mathbf{P}_M}^2} \|\hat{\mathbf{C}}(1) - \tilde{\mathbf{C}}(1)\| < \frac{1}{2} \min_{k, k' \in \mathcal{X}} |\Lambda(1, k) - \Lambda(1, k')|$$

then $\hat{\mathbf{C}}(1)$ has K distinct real eigenvalues and:

$$\text{md}(\mathbf{C}(1), \hat{\mathbf{C}}(1)) \leq \frac{\kappa^2(\mathbf{Q}^* \mathbf{O}_M^\top)}{1 - \varepsilon_{\mathbf{P}_M}^2} \|\hat{\mathbf{C}}(1) - \tilde{\mathbf{C}}(1)\|,$$

where $\text{md}(\mathbf{C}(1), \hat{\mathbf{C}}(1)) := \min_{\tau \in \mathcal{S}_K} \max_{k \in \mathcal{X}} |\hat{\Lambda}(1, \tau(k)) - \Lambda(1, k)|$.

Proof. Observe that \mathbf{U} is an orthonormal basis of range of \mathbf{O}_M . The first point follows from claim (iii) of Lemma 16. The second point is derived from Theorem 8 and the first point. The remark following Theorem 8 proves the last point. \square

C.3. Control of its spectrum.

Lemma 19 — *It holds:*

$$\mathbb{P}\left[\forall k, k_1 \neq k_2, |\Lambda(k, k_1) - \Lambda(k, k_2)| \geq \frac{2\delta(1 - \varepsilon_{\mathbf{P}_M}^2)^{\frac{1}{2}}}{\sqrt{e}K^{\frac{5}{2}}(K-1)}\gamma(\mathbf{O}_M)\right] \geq 1 - \delta.$$

Furthermore:

$$\mathbb{P}\left[\|\Lambda\|_\infty \geq \frac{1 + \sqrt{2\log(K^2/\delta)}}{\sqrt{K}}\|\mathbf{O}_M\|_{2,\infty}\right] \leq \delta.$$

Proof. Observe that:

$$\begin{aligned} \Lambda(k, k_1) - \Lambda(k, k_2) &= \langle \Theta(\cdot, k), (\hat{\mathbf{U}}^\top \mathbf{O}_M)(\cdot, k_1) - (\hat{\mathbf{U}}^\top \mathbf{O}_M)(\cdot, k_2) \rangle \\ &= \langle \Theta(\cdot, k), \hat{\mathbf{U}}^\top (\mathbf{O}_M(\cdot, k_1) - \mathbf{O}_M(\cdot, k_2)) \rangle. \end{aligned}$$

Furthermore, from (iii) in Lemma 16, we get that:

$$\|\hat{\mathbf{U}}^\top (\mathbf{O}_M(\cdot, k_1) - \mathbf{O}_M(\cdot, k_2))\|_2 \geq (1 - \varepsilon_{\mathbf{P}_M}^2)^{\frac{1}{2}} \|\mathbf{O}_M(\cdot, k_1) - \mathbf{O}_M(\cdot, k_2)\|_2 \geq (1 - \varepsilon_{\mathbf{P}_M}^2)^{\frac{1}{2}} \gamma(\mathbf{O}_M).$$

Similarly, note that:

$$\|\Lambda\|_\infty = \max_{k, k'} |\langle \Theta(\cdot, k), \hat{\mathbf{U}}^\top \mathbf{O}_M(\cdot, k') \rangle|,$$

and $\|\hat{\mathbf{U}}^\top \mathbf{O}_M(\cdot, k')\|_2 \leq \|\mathbf{O}_M(\cdot, k')\|_2 \leq \|\mathbf{O}_M\|_{2,\infty}$. For sake of readability, we borrow the result of Lemma F.2 and the argument of Lemma C.6 in [AHK12] to conclude. \square

C.4. Perturbation of simultaneously diagonalizable matrices.

Lemma 20 — *If $3\|\hat{\mathbf{P}}_M - \mathbf{P}_M\| \leq \sigma_K(\mathbf{P}_M)$ and:*

(32)

$$8.2K^{\frac{5}{2}}(K-1) \frac{\kappa^2(\mathbf{Q}\mathbf{O}_M^\top)}{\delta\gamma(\mathbf{O}_M)\sigma_K(\mathbf{P}_M)} \left[\|\hat{\mathbf{M}}_M - \mathbf{M}_M\|_{\infty,2} + \frac{\|\mathbf{M}_M\|_{\infty,2}\|\hat{\mathbf{P}}_M - \mathbf{P}_M\|}{\sigma_K(\mathbf{P}_M)} \right] < 1,$$

(33)

$$43.4K^4(K-1) \frac{\kappa^4(\mathbf{Q}\mathbf{O}_M^\top)}{\delta\gamma(\mathbf{O}_M)\sigma_K(\mathbf{P}_M)} \left[\|\hat{\mathbf{M}}_M - \mathbf{M}_M\|_{\infty,2} + \frac{\|\mathbf{M}_M\|_{\infty,2}\|\hat{\mathbf{P}}_M - \mathbf{P}_M\|}{\sigma_K(\mathbf{P}_M)} \right] \leq 1,$$

and:

$$\forall k, k_1 \neq k_2, |\Lambda(k, k_1) - \Lambda(k, k_2)| \geq \frac{\sqrt{3}\delta}{\sqrt{e}K^{\frac{5}{2}}(K-1)}\gamma(\mathbf{O}_M),$$

and:

$$\|\Lambda\|_\infty \leq \frac{1 + \sqrt{2\log(K^2/\delta)}}{\sqrt{K}}\|\mathbf{O}_M\|_{2,\infty},$$

then there exists $\tau \in \mathcal{S}_K$ such that $\forall k \in \mathcal{X}$:

$$\begin{aligned} \|\Lambda(\cdot, k) - \hat{\Lambda}(\cdot, \tau(k))\|_\infty &\leq \left[13 \frac{\kappa^2(\mathbf{Q}\mathbf{O}_M^\top)}{\sigma_K(\mathbf{P}_M)} + 116K^{\frac{7}{2}}(K-1)(1 + (2\log(K^2/\delta))^{\frac{1}{2}}) \right. \\ &\quad \left. \times \frac{\kappa^6(\mathbf{Q}\mathbf{O}_M^\top)\|\mathbf{O}_M\|_{2,\infty}}{\delta\gamma(\mathbf{O}_M)\sigma_K(\mathbf{P}_M)} \right] \times \left[\|\hat{\mathbf{M}}_M - \mathbf{M}_M\|_{\infty,2} + \frac{\|\mathbf{M}_M\|_{\infty,2}\|\hat{\mathbf{P}}_M - \mathbf{P}_M\|}{\sigma_K(\mathbf{P}_M)} \right]. \end{aligned}$$

Proof. Note $\varepsilon_{\mathbf{P}_M} \leq 1/2$. Invoke the last part of Claim 4 of Lemma C.4 in [AHK12]

with:

- $\gamma_A \leftarrow \frac{\sqrt{3}\delta}{\sqrt{e}K^{\frac{5}{2}}(K-1)}\gamma(\mathbf{O}_M)$
- $\kappa(R) \leftarrow \frac{4\kappa^2(\mathbf{Q}\mathbf{O}_M^\top)}{3}$
- $\|\tilde{R}\|_2^2 \leftarrow \frac{4\kappa^2(\mathbf{Q}\mathbf{O}_M^\top)}{3}$

- $\epsilon_A \leftarrow 3.2 \frac{\|\mathbf{M}_M\|_{\infty,2}}{\sigma_K(\mathbf{P}_M)} \left[\frac{\|\hat{\mathbf{M}}_M - \mathbf{M}_M\|_{\infty,2}}{\|\mathbf{M}_M\|_{\infty,2}} + \frac{\|\hat{\mathbf{P}}_M - \mathbf{P}_M\|}{\sigma_K(\mathbf{P}_M)} \right]$.
- $\lambda_{\max} \leftarrow \frac{1 + \sqrt{2 \log(K^2/\delta)}}{\sqrt{K}} \|\mathbf{O}_M\|_{2,\infty}$

and observe that (32) agrees with $\epsilon_3 < 1/2$ and (33) agrees with $\epsilon_4 \leq 1/2$. \square

Since Θ^\top is an isometry, observe that:

$$\|\hat{\mathbf{U}}^\top \mathbf{O}_M(\cdot, k) - \Theta \hat{\Lambda}(\cdot, \tau(k))\|_2 = \|\Lambda(\cdot, k) - \hat{\Lambda}(\cdot, \tau(k))\|_2 \leq \sqrt{K} \|\Lambda(\cdot, k) - \hat{\Lambda}(\cdot, \tau(k))\|_\infty.$$

Claim (v) in Lemma 16 (with $\alpha = \Theta \hat{\Lambda}(\cdot, \tau(k))$ and $v = \mathbf{O}_M(\cdot, k)$) gives:

$$\begin{aligned} \|\mathbf{O}_M(\cdot, k) - \hat{\mathbf{O}}_M(\cdot, \tau(k))\|_2 &\leq \|\hat{\mathbf{U}}^\top \mathbf{O}_M(\cdot, k) - \Theta \hat{\Lambda}(\cdot, \tau(k))\|_2 + \frac{3\|\hat{\mathbf{P}}_M - \mathbf{P}_M\|}{2\sigma_K(\mathbf{P}_M)} \|\mathbf{O}_M(\cdot, k)\|_2 \\ &\leq \sqrt{K} \|\Lambda(\cdot, k) - \hat{\Lambda}(\cdot, \tau(k))\|_\infty + \frac{3\|\hat{\mathbf{P}}_M - \mathbf{P}_M\|}{2\sigma_K(\mathbf{P}_M)} \|\mathbf{O}_M(\cdot, k)\|_2. \end{aligned}$$

Using Lemma 20, Theorem 5 follows.

APPENDIX D. EXPRESSION OF POLYNOMIAL P_5

Computer assisted computations (available on Yohann's web page) give that:

$P_5 =$

$$\begin{aligned} &144 - 114 t^2 x^8 - 108 t^2 x^{10} - 18 t^2 x^{12} + \\ &192 t^2 + 128 t^4 + 256 t^6 + 176 t^8 + 576 x^2 + 624 t^2 x^2 + \\ &672 t^4 x^2 + 1776 t^6 x^2 + 1152 t^8 x^2 + 972 x^4 + 720 t^2 x^4 + \\ &1884 t^4 x^4 + 5496 t^6 x^4 + 3360 t^8 x^4 + 900 x^6 + 264 t^2 x^6 + \\ &3556 t^4 x^6 + 9920 t^6 x^6 + 5728 t^8 x^6 + 495 x^8 + \\ &4551 t^4 x^8 + 11424 t^6 x^8 + 6264 t^8 x^8 + 162 x^{10} + \\ &3810 t^4 x^{10} + 8592 t^6 x^{10} + 4512 t^8 x^{10} + \\ &27 x^{12} + 1979 t^4 x^{12} + 4120 t^6 x^{12} + \\ &2096 t^8 x^{12} + 576 t^4 x^{14} + 1152 t^6 x^{14} + 576 t^8 x^{14} + \\ &72 t^4 x^{16} + 144 t^6 x^{16} + 72 t^8 x^{16} + 144 y^2 + 480 t^2 y^2 + \\ &784 t^4 y^2 + 704 t^6 y^2 + 256 t^8 y^2 + 576 x^2 y^2 + \\ &2064 t^2 x^2 y^2 + 4192 t^4 x^2 y^2 + 4496 t^6 x^2 y^2 + \\ &1792 t^8 x^2 y^2 + 1080 x^4 y^2 + 4104 t^2 x^4 y^2 + \\ &10760 t^4 x^4 y^2 + 13528 t^6 x^4 y^2 + 5792 t^8 x^4 y^2 + \\ &1224 x^6 y^2 + 5016 t^2 x^6 y^2 + 17592 t^4 x^6 y^2 + \\ &25032 t^6 x^6 y^2 + 11232 t^8 x^6 y^2 + 900 x^8 y^2 + \\ &4224 t^2 x^8 y^2 + 19924 t^4 x^8 y^2 + 30776 t^6 x^8 y^2 + \\ &14176 t^8 x^8 y^2 + 432 x^{10} y^2 + 2520 t^2 x^{10} y^2 + \\ &15584 t^4 x^{10} y^2 + 25336 t^6 x^{10} y^2 + 11840 t^8 x^{10} y^2 + \\ &108 x^{12} y^2 + 936 t^2 x^{12} y^2 + 7916 t^4 x^{12} y^2 + \\ &13456 t^6 x^{12} y^2 + 6368 t^8 x^{12} y^2 + 144 t^2 x^{14} y^2 + \\ &2304 t^4 x^{14} y^2 + 4176 t^6 x^{14} y^2 + 2016 t^8 x^{14} y^2 + \\ &288 t^4 x^{16} y^2 + 576 t^6 x^{16} y^2 + 288 t^8 x^{16} y^2 + 144 y^4 + \\ &480 t^2 y^4 + 624 t^4 y^4 + 384 t^6 y^4 + 96 t^8 y^4 + 576 x^2 y^4 + \\ &2208 t^2 x^2 y^4 + 3392 t^4 x^2 y^4 + 2464 t^6 x^2 y^4 + \\ &704 t^8 x^2 y^4 + 1188 x^4 y^4 + 5256 t^2 x^4 y^4 + \\ &9636 t^4 x^4 y^4 + 8256 t^6 x^4 y^4 + 2688 t^8 x^4 y^4 + \\ &1548 x^6 y^4 + 8112 t^2 x^6 y^4 + 18076 t^4 x^6 y^4 + \\ &18008 t^6 x^6 y^4 + 6496 t^8 x^6 y^4 + 1359 x^8 y^4 + \\ &8598 t^2 x^8 y^4 + 23375 t^4 x^8 y^4 + 26392 t^6 x^8 y^4 + \\ &10256 t^8 x^8 y^4 + 810 x^{10} y^4 + 6156 t^2 x^{10} y^4 + \\ &20442 t^4 x^{10} y^4 + 25656 t^6 x^{10} y^4 + 10560 t^8 x^{10} y^4 + \\ &243 x^{12} y^4 + 2574 t^2 x^{12} y^4 + 11299 t^4 x^{12} y^4 + \\ &15848 t^6 x^{12} y^4 + 6880 t^8 x^{12} y^4 + 432 t^2 x^{14} y^4 + \\ &3456 t^4 x^{14} y^4 + 5616 t^6 x^{14} y^4 + 2592 t^8 x^{14} y^4 + \end{aligned}$$

$$\begin{aligned}
& 432 t^4 x^{16} y^4 + 864 t^6 x^{16} y^4 + 432 t^8 x^{16} y^4 + \\
& 216 x^4 y^6 + 720 t^2 x^4 y^6 + 952 t^4 x^4 y^6 + 608 t^6 x^4 y^6 + \\
& 160 t^8 x^4 y^6 + 648 x^6 y^6 + 2592 t^2 x^6 y^6 + \\
& 4168 t^4 x^6 y^6 + 3152 t^6 x^6 y^6 + 928 t^8 x^6 y^6 + \\
& 918 x^8 y^6 + 4428 t^2 x^8 y^6 + 8502 t^4 x^8 y^6 + \\
& 7392 t^6 x^8 y^6 + 2400 t^8 x^8 y^6 + 756 x^{10} y^6 + \\
& 4392 t^2 x^{10} y^6 + 10036 t^4 x^{10} y^6 + 9920 t^6 x^{10} y^6 + \\
& 3520 t^8 x^{10} y^6 + 270 x^{12} y^6 + 2268 t^2 x^{12} y^6 + \\
& 6766 t^4 x^{12} y^6 + 7808 t^6 x^{12} y^6 + 3040 t^8 x^{12} y^6 + \\
& 432 t^2 x^{14} y^6 + 2304 t^4 x^{14} y^6 + 3312 t^6 x^{14} y^6 + \\
& 1440 t^8 x^{14} y^6 + 288 t^4 x^{16} y^6 + 576 t^6 x^{16} y^6 + \\
& 288 t^8 x^{16} y^6 + 108 x^8 y^8 + 360 t^2 x^8 y^8 + 468 t^4 x^8 y^8 + \\
& 288 t^6 x^8 y^8 + 72 t^8 x^8 y^8 + 216 x^{10} y^8 + 864 t^2 x^{10} y^8 + \\
& 1368 t^4 x^{10} y^8 + 1008 t^6 x^{10} y^8 + 288 t^8 x^{10} y^8 + \\
& 108 x^{12} y^8 + 648 t^2 x^{12} y^8 + 1404 t^4 x^{12} y^8 + \\
& 1296 t^6 x^{12} y^8 + 432 t^8 x^{12} y^8 + 144 t^2 x^{14} y^8 + \\
& 576 t^4 x^{14} y^8 + 720 t^6 x^{14} y^8 + 288 t^8 x^{14} y^8 + \\
& 72 t^4 x^{16} y^8 + 144 t^6 x^{16} y^8 + 72 t^8 x^{16} y^8 + 192 z^2 + \\
& 416 t^2 z^2 + 288 t^4 z^2 + 320 t^6 z^2 + 256 t^8 z^2 + \\
& 912 x^2 z^2 + 1664 t^2 x^2 z^2 + 1248 t^4 x^2 z^2 + \\
& 2304 t^6 x^2 z^2 + 1808 t^8 x^2 z^2 + 1728 x^4 z^2 + \\
& 2520 t^2 x^4 z^2 + 2776 t^4 x^4 z^2 + 7624 t^6 x^4 z^2 + \\
& 5640 t^8 x^4 z^2 + 1704 x^6 z^2 + 1736 t^2 x^6 z^2 + \\
& 4664 t^4 x^6 z^2 + 14808 t^6 x^6 z^2 + 10176 t^8 x^6 z^2 + \\
& 966 x^8 z^2 + 494 t^2 x^8 z^2 + 6098 t^4 x^8 z^2 + \\
& 18218 t^6 x^8 z^2 + 11648 t^8 x^8 z^2 + 324 x^{10} z^2 + \\
& 36 t^2 x^{10} z^2 + 5468 t^4 x^{10} z^2 + 14444 t^6 x^{10} z^2 + \\
& 8688 t^8 x^{10} z^2 + 54 x^{12} z^2 + 6 t^2 x^{12} z^2 + \\
& 3002 t^4 x^{12} z^2 + 7186 t^6 x^{12} z^2 + 4136 t^8 x^{12} z^2 + \\
& 896 t^4 x^{14} z^2 + 2048 t^6 x^{14} z^2 + 1152 t^8 x^{14} z^2 + \\
& 112 t^4 x^{16} z^2 + 256 t^6 x^{16} z^2 + 144 t^8 x^{16} z^2 + \\
& 480 y^2 z^2 + 1312 t^2 y^2 z^2 + 1888 t^4 y^2 z^2 + \\
& 1760 t^6 y^2 z^2 + 704 t^8 y^2 z^2 + 1776 x^2 y^2 z^2 + \\
& 5248 t^2 x^2 y^2 z^2 + 9504 t^4 x^2 y^2 z^2 + \\
& 10624 t^6 x^2 y^2 z^2 + 4592 t^8 x^2 y^2 z^2 + 3096 x^4 y^2 z^2 + \\
& 9904 t^2 x^4 y^2 z^2 + 23104 t^4 x^4 y^2 z^2 + \\
& 30288 t^6 x^4 y^2 z^2 + 13992 t^8 x^4 y^2 z^2 + 3144 x^6 y^2 z^2 + \\
& 11344 t^2 x^6 y^2 z^2 + 35712 t^4 x^6 y^2 z^2 + \\
& 53424 t^6 x^6 y^2 z^2 + 25912 t^8 x^6 y^2 z^2 + 2064 x^8 y^2 z^2 + \\
& 9016 t^2 x^8 y^2 z^2 + 38552 t^4 x^8 y^2 z^2 + \\
& 63192 t^6 x^8 y^2 z^2 + 31592 t^8 x^8 y^2 z^2 + 936 x^{10} y^2 z^2 + \\
& 5248 t^2 x^{10} y^2 z^2 + 29072 t^4 x^{10} y^2 z^2 + \\
& 50464 t^6 x^{10} y^2 z^2 + 25704 t^8 x^{10} y^2 z^2 + 216 x^{12} y^2 z^2 + \\
& 1872 t^2 x^{12} y^2 z^2 + 14192 t^4 x^{12} y^2 z^2 + \\
& 26056 t^6 x^{12} y^2 z^2 + 13520 t^8 x^{12} y^2 z^2 + \\
& 264 t^2 x^{14} y^2 z^2 + 3896 t^4 x^{14} y^2 z^2 + \\
& 7808 t^6 x^{14} y^2 z^2 + 4176 t^8 x^{14} y^2 z^2 + \\
& 448 t^4 x^{16} y^2 z^2 + 1024 t^6 x^{16} y^2 z^2 + \\
& 576 t^8 x^{16} y^2 z^2 + 480 y^4 z^2 + 1632 t^2 y^4 z^2 + \\
& 2208 t^4 y^4 z^2 + 1440 t^6 y^4 z^2 + 384 t^8 y^4 z^2 + \\
& 1632 x^2 y^4 z^2 + 6528 t^2 x^2 y^4 z^2 + 10688 t^4 x^2 y^4 z^2 + \\
& 8320 t^6 x^2 y^4 z^2 + 2528 t^8 x^2 y^4 z^2 + 3240 x^4 y^4 z^2 + \\
& 14280 t^2 x^4 y^4 z^2 + 27448 t^4 x^4 y^4 z^2 + \\
& 25048 t^6 x^4 y^4 z^2 + 8640 t^8 x^4 y^4 z^2 + 3936 x^6 y^4 z^2 + \\
& 19992 t^2 x^6 y^4 z^2 + 46552 t^4 x^6 y^4 z^2 + \\
& 49352 t^6 x^6 y^4 z^2 + 18856 t^8 x^6 y^4 z^2 + 3198 x^8 y^4 z^2 + \\
& 19518 t^2 x^8 y^4 z^2 + 55218 t^4 x^8 y^4 z^2 +
\end{aligned}$$

$$\begin{aligned}
& 66170 t^6 x^8 y^4 z^2 + 27272 t^8 x^8 y^4 z^2 + 1836 x^{10} y^4 z^2 + \\
& 13332 t^2 x^{10} y^4 z^2 + 44988 t^4 x^{10} y^4 z^2 + \\
& 59580 t^6 x^{10} y^4 z^2 + 26088 t^8 x^{10} y^4 z^2 + 486 x^{12} y^4 z^2 + \\
& 5214 t^2 x^{12} y^4 z^2 + 22994 t^4 x^{12} y^4 z^2 + \\
& 34194 t^6 x^{12} y^4 z^2 + 15928 t^8 x^{12} y^4 z^2 + \\
& 792 t^2 x^{14} y^4 z^2 + 6312 t^4 x^{14} y^4 z^2 + \\
& 11136 t^6 x^{14} y^4 z^2 + 5616 t^8 x^{14} y^4 z^2 + \\
& 672 t^4 x^{16} y^4 z^2 + 1536 t^6 x^{16} y^4 z^2 + \\
& 864 t^8 x^{16} y^4 z^2 + 720 x^4 y^6 z^2 + 2480 t^2 x^4 y^6 z^2 + \\
& 3472 t^4 x^4 y^6 z^2 + 2384 t^6 x^4 y^6 z^2 + 672 t^8 x^4 y^6 z^2 + \\
& 1728 x^6 y^6 z^2 + 7440 t^2 x^6 y^6 z^2 + 13072 t^4 x^6 y^6 z^2 + \\
& 10736 t^6 x^6 y^6 z^2 + 3376 t^8 x^6 y^6 z^2 + 2268 x^8 y^6 z^2 + \\
& 11484 t^2 x^8 y^6 z^2 + 23812 t^4 x^8 y^6 z^2 + \\
& 22276 t^6 x^8 y^6 z^2 + 7680 t^8 x^8 y^6 z^2 + 1800 x^{10} y^6 z^2 + \\
& 10568 t^2 x^{10} y^6 z^2 + 25560 t^4 x^{10} y^6 z^2 + \\
& 26872 t^6 x^{10} y^6 z^2 + 10080 t^8 x^{10} y^6 z^2 + 540 x^{12} y^6 z^2 + \\
& 4836 t^2 x^{12} y^6 z^2 + 15420 t^4 x^{12} y^6 z^2 + \\
& 18964 t^6 x^{12} y^6 z^2 + 7840 t^8 x^{12} y^6 z^2 + \\
& 792 t^2 x^{14} y^6 z^2 + 4520 t^4 x^{14} y^6 z^2 + \\
& 7040 t^6 x^{14} y^6 z^2 + 3312 t^8 x^{14} y^6 z^2 + \\
& 448 t^4 x^{16} y^6 z^2 + 1024 t^6 x^{16} y^6 z^2 + \\
& 576 t^8 x^{16} y^6 z^2 + 360 x^8 y^8 z^2 + 1224 t^2 x^8 y^8 z^2 + \\
& 1656 t^4 x^8 y^8 z^2 + 1080 t^6 x^8 y^8 z^2 + 288 t^8 x^8 y^8 z^2 + \\
& 576 x^{10} y^8 z^2 + 2448 t^2 x^{10} y^8 z^2 + 4176 t^4 x^{10} y^8 z^2 + \\
& 3312 t^6 x^{10} y^8 z^2 + 1008 t^8 x^{10} y^8 z^2 + 216 x^{12} y^8 z^2 + \\
& 1488 t^2 x^{12} y^8 z^2 + 3616 t^4 x^{12} y^8 z^2 + \\
& 3640 t^6 x^{12} y^8 z^2 + 1296 t^8 x^{12} y^8 z^2 + \\
& 264 t^2 x^{14} y^8 z^2 + 1208 t^4 x^{14} y^8 z^2 + \\
& 1664 t^6 x^{14} y^8 z^2 + 720 t^8 x^{14} y^8 z^2 + \\
& 112 t^4 x^{16} y^8 z^2 + 256 t^6 x^{16} y^8 z^2 + 144 t^8 x^{16} y^8 z^2 + \\
& 128 z^4 + 288 t^2 z^4 + 352 t^4 z^4 + 384 t^6 z^4 + 256 t^8 z^4 + \\
& 352 x^2 z^4 + 1056 t^2 x^2 z^4 + 1408 t^4 x^2 z^4 + \\
& 1952 t^6 x^2 z^4 + 1504 t^8 x^2 z^4 + 764 x^4 z^4 + \\
& 2104 t^2 x^4 z^4 + 2616 t^4 x^4 z^4 + 5016 t^6 x^4 z^4 + \\
& 4252 t^8 x^4 z^4 + 804 x^6 z^4 + 1912 t^2 x^6 z^4 + \\
& 2920 t^4 x^6 z^4 + 8536 t^6 x^6 z^4 + 7364 t^8 x^6 z^4 + \\
& 471 x^8 z^4 + 898 t^2 x^8 z^4 + 2694 t^4 x^8 z^4 + \\
& 10058 t^6 x^8 z^4 + 8335 t^8 x^8 z^4 + 162 x^{10} z^4 + \\
& 252 t^2 x^{10} z^4 + 2164 t^4 x^{10} z^4 + 7980 t^6 x^{10} z^4 + \\
& 6226 t^8 x^{10} z^4 + 27 x^{12} z^4 + 42 t^2 x^{12} z^4 + \\
& 1182 t^4 x^{12} z^4 + 4018 t^6 x^{12} z^4 + 2979 t^8 x^{12} z^4 + \\
& 352 t^4 x^{14} z^4 + 1152 t^6 x^{14} z^4 + 832 t^8 x^{14} z^4 + \\
& 44 t^4 x^{16} z^4 + 144 t^6 x^{16} z^4 + 104 t^8 x^{16} z^4 + \\
& 784 y^2 z^4 + 1888 t^2 y^2 z^4 + 2208 t^4 y^2 z^4 + \\
& 1888 t^6 y^2 z^4 + 784 t^8 y^2 z^4 + 2080 x^2 y^2 z^4 + \\
& 5600 t^2 x^2 y^2 z^4 + 8832 t^4 x^2 y^2 z^4 + 9952 t^6 x^2 y^2 z^4 + \\
& 4640 t^8 x^2 y^2 z^4 + 3368 x^4 y^2 z^4 + 9440 t^2 x^4 y^2 z^4 + \\
& 18928 t^4 x^4 y^2 z^4 + 25952 t^6 x^4 y^2 z^4 + \\
& 13224 t^8 x^4 y^2 z^4 + 2840 x^6 y^2 z^4 + 9056 t^2 x^6 y^2 z^4 + \\
& 25872 t^4 x^6 y^2 z^4 + 42464 t^6 x^6 y^2 z^4 + \\
& 23192 t^8 x^6 y^2 z^4 + 1524 x^8 y^2 z^4 + 6072 t^2 x^8 y^2 z^4 + \\
& 25016 t^4 x^8 y^2 z^4 + 46792 t^6 x^8 y^2 z^4 + \\
& 26900 t^8 x^8 y^2 z^4 + 576 x^{10} y^2 z^4 + 3184 t^2 x^{10} y^2 z^4 + \\
& 17216 t^4 x^{10} y^2 z^4 + 35024 t^6 x^{10} y^2 z^4 + \\
& 20928 t^8 x^{10} y^2 z^4 + 108 x^{12} y^2 z^4 + 1008 t^2 x^{12} y^2 z^4 + \\
& 7584 t^4 x^{12} y^2 z^4 + 16968 t^6 x^{12} y^2 z^4 + \\
& 10572 t^8 x^{12} y^2 z^4 + 120 t^2 x^{14} y^2 z^4 +
\end{aligned}$$

$$\begin{aligned}
& 1816 t^4 x^{14} y^2 z^4 + 4736 t^6 x^{14} y^2 z^4 + \\
& 3136 t^8 x^{14} y^2 z^4 + 176 t^4 x^{16} y^2 z^4 + \\
& 576 t^6 x^{16} y^2 z^4 + 416 t^8 x^{16} y^2 z^4 + 624 y^4 z^4 + \\
& 2208 t^2 y^4 z^4 + 3168 t^4 y^4 z^4 + 2208 t^6 y^4 z^4 + \\
& 624 t^8 y^4 z^4 + 1600 x^2 y^4 z^4 + 6976 t^2 x^2 y^4 z^4 + \\
& 12672 t^4 x^2 y^4 z^4 + 10816 t^6 x^2 y^4 z^4 + \\
& 3520 t^8 x^2 y^4 z^4 + 3364 x^4 y^4 z^4 + 14456 t^2 x^4 y^4 z^4 + \\
& 29416 t^4 x^4 y^4 z^4 + 29016 t^6 x^4 y^4 z^4 + \\
& 10692 t^8 x^4 y^4 z^4 + 3452 x^6 y^4 z^4 + 17336 t^2 x^6 y^4 z^4 + \\
& 43896 t^4 x^6 y^4 z^4 + 51032 t^6 x^6 y^4 z^4 + \\
& 21020 t^8 x^6 y^4 z^4 + 2495 x^8 y^4 z^4 + 14658 t^2 x^8 y^4 z^4 + \\
& 45814 t^4 x^8 y^4 z^4 + 61162 t^6 x^8 y^4 z^4 + \\
& 27607 t^8 x^8 y^4 z^4 + 1242 x^{10} y^4 z^4 + 8892 t^2 x^{10} y^4 z^4 + \\
& 33252 t^4 x^{10} y^4 z^4 + 49644 t^6 x^{10} y^4 z^4 + \\
& 24234 t^8 x^{10} y^4 z^4 + 243 x^{12} y^4 z^4 + 2914 t^2 x^{12} y^4 z^4 + \\
& 14758 t^4 x^{12} y^4 z^4 + 25538 t^6 x^{12} y^4 z^4 + \\
& 13643 t^8 x^{12} y^4 z^4 + 360 t^2 x^{14} y^4 z^4 + \\
& 3336 t^4 x^{14} y^4 z^4 + 7296 t^6 x^{14} y^4 z^4 + \\
& 4416 t^8 x^{14} y^4 z^4 + 264 t^4 x^{16} y^4 z^4 + \\
& 864 t^6 x^{16} y^4 z^4 + 624 t^8 x^{16} y^4 z^4 + 952 x^4 y^6 z^4 + \\
& 3472 t^2 x^4 y^6 z^4 + 5232 t^4 x^4 y^6 z^4 + 3856 t^6 x^4 y^6 z^4 + \\
& 1144 t^8 x^4 y^6 z^4 + 1544 x^6 y^6 z^4 + 7760 t^2 x^6 y^6 z^4 + \\
& 15696 t^4 x^6 y^6 z^4 + 14288 t^6 x^6 y^6 z^4 + \\
& 4808 t^8 x^6 y^6 z^4 + 1942 x^8 y^6 z^4 + 10532 t^2 x^8 y^6 z^4 + \\
& 24556 t^4 x^8 y^6 z^4 + 25380 t^6 x^8 y^6 z^4 + \\
& 9414 t^8 x^8 y^6 z^4 + 1332 x^{10} y^6 z^4 + 8408 t^2 x^{10} y^6 z^4 + \\
& 22952 t^4 x^{10} y^6 z^4 + 26776 t^6 x^{10} y^6 z^4 + \\
& 10900 t^8 x^{10} y^6 z^4 + 270 x^{12} y^6 z^4 + 2972 t^2 x^{12} y^6 z^4 + \\
& 11492 t^4 x^{12} y^6 z^4 + 16244 t^6 x^{12} y^6 z^4 + \\
& 7486 t^8 x^{12} y^6 z^4 + 360 t^2 x^{14} y^6 z^4 + \\
& 2632 t^4 x^{14} y^6 z^4 + 4992 t^6 x^{14} y^6 z^4 + \\
& 2752 t^8 x^{14} y^6 z^4 + 176 t^4 x^{16} y^6 z^4 + \\
& 576 t^6 x^{16} y^6 z^4 + 416 t^8 x^{16} y^6 z^4 + 468 x^8 y^8 z^4 + \\
& 1656 t^2 x^8 y^8 z^4 + 2376 t^4 x^8 y^8 z^4 + 1656 t^6 x^8 y^8 z^4 + \\
& 468 t^8 x^8 y^8 z^4 + 504 x^{10} y^8 z^4 + 2448 t^2 x^{10} y^8 z^4 + \\
& 4752 t^4 x^{10} y^8 z^4 + 4176 t^6 x^{10} y^8 z^4 + \\
& 1368 t^8 x^{10} y^8 z^4 + 108 x^{12} y^8 z^4 + 1024 t^2 x^{12} y^8 z^4 + \\
& 3136 t^4 x^{12} y^8 z^4 + 3656 t^6 x^{12} y^8 z^4 + \\
& 1436 t^8 x^{12} y^8 z^4 + 120 t^2 x^{14} y^8 z^4 + \\
& 760 t^4 x^{14} y^8 z^4 + 1280 t^6 x^{14} y^8 z^4 + \\
& 640 t^8 x^{14} y^8 z^4 + 44 t^4 x^{16} y^8 z^4 + 144 t^6 x^{16} y^8 z^4 + \\
& 104 t^8 x^{16} y^8 z^4 + 256 z^6 + 320 t^2 z^6 + 384 t^4 z^6 + \\
& 352 t^6 z^6 + 160 t^8 z^6 + 272 x^2 z^6 + 256 t^2 x^2 z^6 + \\
& 1120 t^4 x^2 z^6 + 1408 t^6 x^2 z^6 + 784 t^8 x^2 z^6 + \\
& 232 x^4 z^6 + 456 t^2 x^4 z^6 + 2104 t^4 x^4 z^6 + \\
& 2712 t^6 x^4 z^6 + 1856 t^8 x^4 z^6 + 96 x^6 z^6 + 472 t^2 x^6 z^6 + \\
& 2072 t^4 x^6 z^6 + 3208 t^6 x^6 z^6 + 2792 t^8 x^6 z^6 + \\
& 24 x^8 z^6 + 298 t^2 x^8 z^6 + 1178 t^4 x^8 z^6 + 2686 t^6 x^8 z^6 + \\
& 2870 t^8 x^8 z^6 + 108 t^2 x^{10} z^6 + 396 t^4 x^{10} z^6 + \\
& 1668 t^6 x^{10} z^6 + 2020 t^8 x^{10} z^6 + 18 t^2 x^{12} z^6 + \\
& 66 t^4 x^{12} z^6 + 726 t^6 x^{12} z^6 + 934 t^8 x^{12} z^6 + \\
& 192 t^2 x^{14} z^6 + 256 t^4 x^{14} z^6 + 24 t^6 x^{14} z^6 + \\
& 32 t^8 x^{14} z^6 + 704 y^2 z^6 + 1760 t^2 y^2 z^6 + \\
& 1888 t^4 y^2 z^6 + 1312 t^6 y^2 z^6 + 480 t^8 y^2 z^6 + \\
& 1136 x^2 y^2 z^6 + 3456 t^2 x^2 y^2 z^6 + 5152 t^4 x^2 y^2 z^6 + \\
& 5248 t^6 x^2 y^2 z^6 + 2416 t^8 x^2 y^2 z^6 + 1768 x^4 y^2 z^6 + \\
& 5200 t^2 x^4 y^2 z^6 + 9152 t^4 x^4 y^2 z^6 +
\end{aligned}$$

$$\begin{aligned}
& 11696 t^6 x^4 y^2 z^6 + 6232 t^8 x^4 y^2 z^6 + 1144 x^6 y^2 z^6 + \\
& 3760 t^2 x^6 y^2 z^6 + 9984 t^4 x^6 y^2 z^6 + \\
& 16720 t^6 x^6 y^2 z^6 + 10120 t^8 x^6 y^2 z^6 + 456 x^8 y^2 z^6 + \\
& 1752 t^2 x^8 y^2 z^6 + 7592 t^4 x^8 y^2 z^6 + \\
& 16024 t^6 x^8 y^2 z^6 + 10880 t^8 x^8 y^2 z^6 + 72 x^{10} y^2 z^6 + \\
& 544 t^2 x^{10} y^2 z^6 + 3952 t^4 x^{10} y^2 z^6 + \\
& 10304 t^6 x^{10} y^2 z^6 + 7848 t^8 x^{10} y^2 z^6 + \\
& 72 t^2 x^{12} y^2 z^6 + 1160 t^4 x^{12} y^2 z^6 + \\
& 4192 t^6 x^{12} y^2 z^6 + 3680 t^8 x^{12} y^2 z^6 + \\
& 128 t^4 x^{14} y^2 z^6 + 952 t^6 x^{14} y^2 z^6 + \\
& 1016 t^8 x^{14} y^2 z^6 + 96 t^6 x^{16} y^2 z^6 + 128 t^8 x^{16} y^2 z^6 + \\
& 384 y^4 z^6 + 1440 t^2 y^4 z^6 + 2208 t^4 y^4 z^6 + \\
& 1632 t^6 y^4 z^6 + 480 t^8 y^4 z^6 + 608 x^2 y^4 z^6 + \\
& 3200 t^2 x^2 y^4 z^6 + 6848 t^4 x^2 y^4 z^6 + 6528 t^6 x^2 y^4 z^6 + \\
& 2272 t^8 x^2 y^4 z^6 + 1760 x^4 y^4 z^6 + 7128 t^2 x^4 y^4 z^6 + \\
& 15128 t^4 x^4 y^4 z^6 + 16008 t^6 x^4 y^4 z^6 + \\
& 6248 t^8 x^4 y^4 z^6 + 1288 x^6 y^4 z^6 + 6856 t^2 x^6 y^4 z^6 + \\
& 19576 t^4 x^6 y^4 z^6 + 25176 t^6 x^6 y^4 z^6 + \\
& 11168 t^8 x^6 y^4 z^6 + 832 x^8 y^4 z^6 + 4730 t^2 x^8 y^4 z^6 + \\
& 17242 t^4 x^8 y^4 z^6 + 26382 t^6 x^8 y^4 z^6 + \\
& 13230 t^8 x^8 y^4 z^6 + 216 x^{10} y^4 z^6 + 1980 t^2 x^{10} y^4 z^6 + \\
& 10092 t^4 x^{10} y^4 z^6 + 18420 t^6 x^{10} y^4 z^6 + \\
& 10476 t^8 x^{10} y^4 z^6 + 274 t^2 x^{12} y^4 z^6 + \\
& 3186 t^4 x^{12} y^4 z^6 + 7806 t^6 x^{12} y^4 z^6 + \\
& 5278 t^8 x^{12} y^4 z^6 + 384 t^4 x^{14} y^4 z^6 + \\
& 1704 t^6 x^{14} y^4 z^6 + 1512 t^8 x^{14} y^4 z^6 + \\
& 144 t^6 x^{16} y^4 z^6 + 192 t^8 x^{16} y^4 z^6 + 608 x^4 y^6 z^6 + \\
& 2384 t^2 x^4 y^6 z^6 + 3856 t^4 x^4 y^6 z^6 + 2992 t^6 x^4 y^6 z^6 + \\
& 912 t^8 x^4 y^6 z^6 + 496 x^6 y^6 z^6 + 3568 t^2 x^6 y^6 z^6 + \\
& 8848 t^4 x^6 y^6 z^6 + 8976 t^6 x^6 y^6 z^6 + 3200 t^8 x^6 y^6 z^6 + \\
& 752 x^8 y^6 z^6 + 4356 t^2 x^8 y^6 z^6 + 11780 t^4 x^8 y^6 z^6 + \\
& 13596 t^6 x^8 y^6 z^6 + 5420 t^8 x^8 y^6 z^6 + 288 x^{10} y^6 z^6 + \\
& 2552 t^2 x^{10} y^6 z^6 + 8984 t^4 x^{10} y^6 z^6 + \\
& 12232 t^6 x^{10} y^6 z^6 + 5512 t^8 x^{10} y^6 z^6 + \\
& 404 t^2 x^{12} y^6 z^6 + 3156 t^4 x^{12} y^6 z^6 + \\
& 5940 t^6 x^{12} y^6 z^6 + 3252 t^8 x^{12} y^6 z^6 + \\
& 384 t^4 x^{14} y^6 z^6 + 1320 t^6 x^{14} y^6 z^6 + \\
& 1000 t^8 x^{14} y^6 z^6 + 96 t^6 x^{16} y^6 z^6 + 128 t^8 x^{16} y^6 z^6 + \\
& 288 x^8 y^8 z^6 + 1080 t^2 x^8 y^8 z^6 + 1656 t^4 x^8 y^8 z^6 + \\
& 1224 t^6 x^8 y^8 z^6 + 360 t^8 x^8 y^8 z^6 + 144 x^{10} y^8 z^6 + \\
& 1008 t^2 x^{10} y^8 z^6 + 2448 t^4 x^{10} y^8 z^6 + \\
& 2448 t^6 x^{10} y^8 z^6 + 864 t^8 x^{10} y^8 z^6 + \\
& 184 t^2 x^{12} y^8 z^6 + 1064 t^4 x^{12} y^8 z^6 + \\
& 1600 t^6 x^{12} y^8 z^6 + 720 t^8 x^{12} y^8 z^6 + \\
& 128 t^4 x^{14} y^8 z^6 + 376 t^6 x^{14} y^8 z^6 + 248 t^8 x^{14} y^8 z^6 + \\
& 24 t^6 x^{16} y^8 z^6 + 32 t^8 x^{16} y^8 z^6 + 176 z^8 + 256 t^2 z^8 + \\
& 256 t^4 z^8 + 160 t^6 z^8 + 48 t^8 z^8 + 256 x^2 z^8 + \\
& 240 t^2 x^2 z^8 + 544 t^4 x^2 z^8 + 496 t^6 x^2 z^8 + \\
& 192 t^8 x^2 z^8 + 224 x^4 z^8 + 152 t^2 x^4 z^8 + 892 t^4 x^4 z^8 + \\
& 848 t^6 x^4 z^8 + 396 t^8 x^4 z^8 + 96 x^6 z^8 + 32 t^2 x^6 z^8 + \\
& 900 t^4 x^6 z^8 + 840 t^6 x^6 z^8 + 516 t^8 x^6 z^8 + 24 x^8 z^8 + \\
& 8 t^2 x^8 z^8 + 575 t^4 x^8 z^8 + 510 t^6 x^8 z^8 + \\
& 463 t^8 x^8 z^8 + 210 t^4 x^{10} z^8 + 180 t^6 x^{10} z^8 + \\
& 290 t^8 x^{10} z^8 + 35 t^4 x^{12} z^8 + 30 t^6 x^{12} z^8 + \\
& 123 t^8 x^{12} z^8 + 32 t^8 x^{14} z^8 + 4 t^8 x^{16} z^8 + 256 y^2 z^8 + \\
& 704 t^2 y^2 z^8 + 784 t^4 y^2 z^8 + 480 t^6 y^2 z^8 + \\
& 144 t^8 y^2 z^8 + 256 x^2 y^2 z^8 + 1040 t^2 x^2 y^2 z^8 +
\end{aligned}$$

$$\begin{aligned}
& 1632 t^4 x^2 y^2 z^8 + 1424 t^6 x^2 y^2 z^8 + 576 t^8 x^2 y^2 z^8 + \\
& 416 x^4 y^2 z^8 + 1560 t^2 x^4 y^2 z^8 + 2696 t^4 x^4 y^2 z^8 + \\
& 2760 t^6 x^4 y^2 z^8 + 1336 t^8 x^4 y^2 z^8 + 224 x^6 y^2 z^8 + \\
& 1032 t^2 x^6 y^2 z^8 + 2616 t^4 x^6 y^2 z^8 + 3416 t^6 x^6 y^2 z^8 + \\
& 1992 t^8 x^6 y^2 z^8 + 96 x^8 y^2 z^8 + 472 t^2 x^8 y^2 z^8 + \\
& 1780 t^4 x^8 y^2 z^8 + 2800 t^6 x^8 y^2 z^8 + 1972 t^8 x^8 y^2 z^8 + \\
& 88 t^2 x^{10} y^2 z^8 + 736 t^4 x^{10} y^2 z^8 + 1432 t^6 x^{10} y^2 z^8 + \\
& 1296 t^8 x^{10} y^2 z^8 + 140 t^4 x^{12} y^2 z^8 + \\
& 400 t^6 x^{12} y^2 z^8 + 548 t^8 x^{12} y^2 z^8 + 40 t^6 x^{14} y^2 z^8 + \\
& 136 t^8 x^{14} y^2 z^8 + 16 t^8 x^{16} y^2 z^8 + 96 y^4 z^8 + \\
& 384 t^2 y^4 z^8 + 624 t^4 y^4 z^8 + 480 t^6 y^4 z^8 + \\
& 144 t^8 y^4 z^8 + 64 x^2 y^4 z^8 + 544 t^2 x^2 y^4 z^8 + \\
& 1472 t^4 x^2 y^4 z^8 + 1568 t^6 x^2 y^4 z^8 + 576 t^8 x^2 y^4 z^8 + \\
& 448 x^4 y^4 z^8 + 1696 t^2 x^4 y^4 z^8 + 3524 t^4 x^4 y^4 z^8 + \\
& 3784 t^6 x^4 y^4 z^8 + 1508 t^8 x^4 y^4 z^8 + 224 x^6 y^4 z^8 + \\
& 1400 t^2 x^6 y^4 z^8 + 4156 t^4 x^6 y^4 z^8 + 5488 t^6 x^6 y^4 z^8 + \\
& 2508 t^8 x^6 y^4 z^8 + 176 x^8 y^4 z^8 + 992 t^2 x^8 y^4 z^8 + \\
& 3367 t^4 x^8 y^4 z^8 + 5190 t^6 x^8 y^4 z^8 + 2735 t^8 x^8 y^4 z^8 + \\
& 264 t^2 x^{10} y^4 z^8 + 1578 t^4 x^{10} y^4 z^8 + \\
& 3084 t^6 x^{10} y^4 z^8 + 1962 t^8 x^{10} y^4 z^8 + \\
& 315 t^4 x^{12} y^4 z^8 + 998 t^6 x^{12} y^4 z^8 + 875 t^8 x^{12} y^4 z^8 + \\
& 120 t^6 x^{14} y^4 z^8 + 216 t^8 x^{14} y^4 z^8 + 24 t^8 x^{16} y^4 z^8 + \\
& 160 x^4 y^6 z^8 + 672 t^2 x^4 y^6 z^8 + 1144 t^4 x^4 y^6 z^8 + \\
& 912 t^6 x^4 y^6 z^8 + 280 t^8 x^4 y^6 z^8 + 32 x^6 y^6 z^8 + \\
& 656 t^2 x^6 y^6 z^8 + 2056 t^4 x^6 y^6 z^8 + 2272 t^6 x^6 y^6 z^8 + \\
& 840 t^8 x^6 y^6 z^8 + 160 x^8 y^6 z^8 + 880 t^2 x^8 y^6 z^8 + \\
& 2534 t^4 x^8 y^6 z^8 + 3100 t^6 x^8 y^6 z^8 + 1286 t^8 x^8 y^6 z^8 + \\
& 320 t^2 x^{10} y^6 z^8 + 1556 t^4 x^{10} y^6 z^8 + \\
& 2408 t^6 x^{10} y^6 z^8 + 1172 t^8 x^{10} y^6 z^8 + \\
& 350 t^4 x^{12} y^6 z^8 + 916 t^6 x^{12} y^6 z^8 + 598 t^8 x^{12} y^6 z^8 + \\
& 120 t^6 x^{14} y^6 z^8 + 152 t^8 x^{14} y^6 z^8 + 16 t^8 x^{16} y^6 z^8 + \\
& 72 x^8 y^8 z^8 + 288 t^2 x^8 y^8 z^8 + 468 t^4 x^8 y^8 z^8 + \\
& 360 t^6 x^8 y^8 z^8 + 108 t^8 x^8 y^8 z^8 + 144 t^2 x^{10} y^8 z^8 + \\
& 504 t^4 x^{10} y^8 z^8 + 576 t^6 x^{10} y^8 z^8 + 216 t^8 x^{10} y^8 z^8 + \\
& 140 t^4 x^{12} y^8 z^8 + 288 t^6 x^{12} y^8 z^8 + 148 t^8 x^{12} y^8 z^8 + \\
& 40 t^6 x^{14} y^8 z^8 + 40 t^8 x^{14} y^8 z^8 + 4 t^8 x^{16} y^8 z^8
\end{aligned}$$

ACKNOWLEDGEMENTS

We would like to thank N. Curien and D. Henrion for their help in the computation of P_5 . We are deeply grateful to V. Magron for pointing us that P_5 can be explicitly expressed as a Sum-Of-Squares.

REFERENCES

- [AF03] R. A. Adams and J. F. Fournier. *Sobolev spaces*, volume 140 of *Pure and Applied Mathematics (Amsterdam)*. Elsevier/Academic Press, Amsterdam, second edition, 2003.
- [AH14] G. Alexandrovich and H. Holzmann. Nonparametric identification of hidden Markov models. *arXiv preprint arXiv:1404.4210*, 2014.
- [AHK12] A. Anandkumar, D. Hsu, and S. M. Kakade. A method of moments for mixture models and hidden Markov models. *arXiv preprint arXiv:1203.0683*, 2012.
- [AMR09] E. S. Allman, C. Matias, and J. A. Rhodes. Identifiability of parameters in latent structure models with many observed variables. *Ann. Statist.*, 37(6A):3099–3132, 12 2009.
- [BMM12] J.-P. Baudry, C. Maugis, and B. Michel. Slope heuristics: overview and implementation. *Stat. Comput.*, 22(2):455–470, 2012.
- [BT13] D. Bontemps and W. Toussile. Clustering and variable selection for categorical multivariate data. *Electron. J. Stat.*, 7:2344–2371, 2013.

- [CC00] L. Couvreur and C. Couvreur. Wavelet based non-parametric HMMs: theory and methods. In *ICASSP '00 Proceedings*, pages 604–607, 2000.
- [DL93] R. A. DeVore and G. G. Lorentz. *Constructive approximation*, volume 303. Springer, 1993.
- [DL12] T. Dumont and S. Le Corff. Nonparametric regression on hidden phi-mixing variables: identifiability and consistency of a pseudo-likelihood based estimation procedure. *arXiv preprint arXiv:1209.0633*, 2012.
- [GCR13] É. Gassiat, A. Cleynen, and S. Robin. Finite state space non parametric hidden Markov models are in general identifiable. *Stat. Comput.*, 2013. To appear.
- [GR13] É. Gassiat and J. Rousseau. Non parametric finite translation hidden Markov models and extensions. *Bernoulli*, 2013. To appear.
- [Han06] N. Hansen. The CMA evolution strategy: a comparing review. In *Towards a new evolutionary computation*, pages 75–102. Springer, 2006.
- [HKZ12] D. Hsu, S. M. Kakade, and T. Zhang. A spectral algorithm for learning hidden Markov models. *J. Comput. System Sci.*, 78(5):1460–1480, 2012.
- [Lef03] F. Lefèvre. Non-parametric probability estimation for HMM-based automatic speech recognition. *Computer Speech and Language*, 17:113–136, 2003.
- [LWM03] M. F. Lambert, J. P. Whiting, and A. V. Metcalfe. A non-parametric hidden Markov model for climate state identification. *Hydrology and Earth System Sciences*, 7(5):652–667, 2003.
- [Mas07] P. Massart. *Concentration inequalities and model selection*, volume 1896 of *Lecture Notes in Mathematics*. Springer, Berlin, 2007. Lectures from the 33rd Summer School on Probability Theory held in Saint-Flour, July 6–23, 2003, With a foreword by Jean Picard.
- [Mey92] Y. Meyer. *Wavelets and operators*, volume 37 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1992. Translated from the 1990 French original by D. H. Salinger.
- [MM08] C. Maugis and B. Michel. Slope heuristics for variable selection and clustering via gaussian mixtures. <https://hal.inria.fr/inria-00284613v2>, 2008.
- [Pau13] D. Paulin. Concentration inequalities for Markov chains by Marton couplings. *arXiv preprint arXiv:1212.2015v2*, 2013.
- [Pau14] D. Paulin. Concentration inequalities for Markov chains by Marton couplings. *arXiv preprint arXiv:1212.2015v3*, 2014.
- [SADX14] L. Song, A. Anandkumar, B. Dai, and B. Xie. Nonparametric estimation of multi-view latent variable models. In *ICML*, 2014.
- [SC09] L. Shang and K.P. Chan. Nonparametric discriminant HMM and application to facial expression recognition. In *2009 IEEE Computer Society Conference on Computer Vision and Pattern Recognition Workshops*, pages 2090–2096, 2009.
- [SS90] G. W. Stewart and J.-G. Sun. *Matrix perturbation theory*. Academic press, 1990.
- [VBMMR13] S. Volant, C. Bérard, M.-L. Martin-Magniette, and S. Robin. Hidden markov models with mixtures as emission distributions. *Statistics and Computing*, pages 1–12, 2013.
- [Ver13] E. Vernet. Posterior consistency for nonparametric Hidden Markov Models with finite state space. *arXiv preprint arXiv:1311.3092*, 2013.
- [Wed72] P.-Å. Wedin. Perturbation bounds in connection with singular value decomposition. *BIT Numerical Mathematics*, 12(1):99–111, 1972.
- [YPRH11] C. Yau, O. Papaspiliopoulos, G. O. Roberts, and C. Holmes. Bayesian non-parametric hidden Markov models with applications in genomics. *J. R. Stat. Soc. Ser. B Stat. Methodol.*, 73(1):37–57, 2011.

(LM-Orsay) LABORATOIRE DE MATHÉMATIQUES D’ORSAY (CNRS UMR 8628), UNIVERSITÉ PARIS-SUD, FACULTÉ DES SCIENCES D’ORSAY, 91405 ORSAY, FRANCE.

E-mail address: `yohann.decastro,elisabeth.gassiat,claire.lacour@math.u-psud.fr`