REGULARITY RESULTS FOR A CLASS OF HYPERBOLIC EQUATIONS WITH VMO COEFFICIENTS

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Abstract. In this note we show a regularity result for an hyperbolic system with discontinuous coefficients. More precisely, we deal with coefficients in the function space VMO and we prove the existence and uniqueness of a solution $u \in L^{\infty}(0,T;H^2(\Omega))$ with also suitable regularity for $\frac{\partial u}{\partial t}, \frac{\partial^2 u}{\partial t^2}$ and $\frac{\partial^3 u}{\partial t^3}$.

1. Introduction

Let Ω be a bounded open subset of $\mathbb{R}^d$ with $d \geqslant 3$. In the context of photoacoustic tomography process modelling [1], we are led to study the follwing wave equation

$$\begin{cases} \dfrac{\partial^2 p}{\partial t^2}(t,x) - \operatorname{div}(v_s^2 \nabla p)(t,x) = f(t,x) & \text{in } (0,T)\times\Omega \\ p(t,x) = 0 & \text{on } (0,T)\times\partial\Omega \\ p(0,x) = \dfrac{\partial p}{\partial t}(0,x) = 0 & \text{in } \Omega, \end{cases}$$

where $p = p(t,x)$ is an acoustic pressure wave, $v_s = v_s(x)$ is the speed of sound, f is a distibuted source that comes from a lightning process and Ω is the domain where the wave propagates. The coefficient v_s is generally unknown and not smooth. We are interested in establishing new results of regularity of the solution p in the case of discontinuous coefficient v_s.

Hereafter we will assume that $\partial\Omega$ is of class $\mathcal{C}^2$ and we consider the following initial/boundary value problem:

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} + Lu = f & \text{in } (0,T)\times\Omega \\ u = 0 & \text{on } (0,T)\times\partial\Omega \\ u(0,x) = u_0,\ \dfrac{\partial u}{\partial t}(0,x) = u_1 & \text{in } \Omega, \end{cases} \tag{1.1}$$

where $f : (0,T)\times\Omega \to \mathbb{R}$, $u_0, u_1 : \Omega \to \mathbb{R}$ are given and L denotes a second order partial differential operator in the divergence form:

2010 AMS Subject Classification. Primary: 35L20. Secondary: 35B65.

Keywords and phrases. Hyperbolic equations, VMO functions.

Date: January 19, 2015.

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$$Lu = -\sum_{i,j=1}^{d} (a_{ij}(x)u_{x_i})_{x_j} \tag{1.2}$$

where u_{x_i} denotes the partial derivative of u with respect to x_i.

Systems of equations as (1.1) have been extensively studied. Classical results of well-posedness and regularity can be found in [9, §7.2]. In this reference a regularity result similar to our Theorem 3.1 is obtained under coefficient smoothness assumptions, namely $a_{ij} \in C^1(\Omega)$ and $\nabla(a_{ij}) \in [C^1(\Omega)]^d$.

In this work, we consider discontinuous coefficients a_{ij} such that $a_{ij} \in VMO \cap L^\infty(\Omega)$ and $\nabla a_{ij} \in [L^p(\Omega)]^d$ with $p > d$. Assuming the coefficients a_{ij} belong to $L^\infty(\Omega)$, it can proved that System (1.1) admits a unique solution $u \in C^0(0,T; H_0^1(\Omega))$ with $\frac{\partial u}{\partial t} \in C^0(0,T; L^2(\Omega))$ (see first part of the proof of Theorem 3.1).

Roughly speaking, the improved regularity, with respect to space, of the solution u is associated with the *elliptic regularity* of the equation for almost every $t \in [0,T]$, that is, with the regularity of $Lu(t) = f(t) - \dfrac{\partial^2 u}{\partial t^2}(t)$. Several regularity results for elliptic operator L have been obtained with more general elliptic operators of type $\bar{L}u = -\sum_{i,j=1}^{d} a_{ij}(t,x)u_{x_i x_j} + \sum_{i=1}^{d} b_i(t,x)u_{x_i} + c(t,x)u$, and there exists a non-exhaustive list of papers devoted to results of regularity associated with the operator $\bar{L}$ with different hypothesis on the coefficients a_{ij}, b_i and c (see for example [4, 5, 6, 8, 7, 9, 14, 15] and references therein). Other results for parabolic equations with VMO coefficients can be found in [2, 11].

In Section 2 we introduce some definitions and notations and the variational formulation of System (1.1). Section 3 is devoted to the proof of Theorem 3.1 which is based on the regularity results obtained in [14].

2. Preliminaries

In the sequel, $L^p(\Omega)$ is the space of measurable functions u on Ω such that $\int_\Omega |u|^p < +\infty$ for $1 \leqslant p < \infty$, $L^\infty(\Omega)$ is the space of essentially bounded functions on Ω. $\mathcal{C}^k(\Omega)$ is the set of all functions k-times continuously differentiable and its derivates of order $|\alpha|$ are continuous for all multiindex α such that $|\alpha| \leqslant k$, $\mathcal{C}_c^\infty(\Omega)$ denote the subspace of all functions u infinitely differentiable with compact support in Ω. We will denote $H^k(\Omega)$ the usual Sobolev space of all functions u such that $D^\alpha u$ exists in the distributional sense and belongs to $L^2(\Omega)$ for all multiindex α with $|\alpha| \leqslant k$. The subspace $H_0^1(\Omega)$ is the closure of $\mathcal{C}_c^\infty(\Omega)$ in $H^1(\Omega)$ and the subspace $H^{-1}(\Omega)$ denotes the dual subspace to $H_0^1(\Omega)$. Let X be a Banach space: we will denote by $L^p(0,T;X)$ the space of the all measurable functions u such that $u : [0,T] \to X$ defined by $u(t)(x) = u(t,x)$ (by abuse of notation) satisfies

$$\|u\|_{L^p(0,T;X)} = \left(\int_0^T \|u(t)\|_X^p \, dt \right)^{1/p} < +\infty, \quad \text{if } p \in [1, +\infty)$$

and

$$\|u\|_{L^\infty(0,T;X)} = ess \sup_{0 \leqslant t \leqslant T} \|u(t)\|_X < +\infty, \quad \text{if } p = +\infty.$$

The space $W^{1,p}(0,T;X)$ denotes all the functions $u \in L^p(0,T;X)$ such that $\dfrac{\partial u}{\partial t} \in L^p(0,T;X)$. For simplicity, we will use often the notation $W^{1,p}(X)$ instead of $W^{1,p}(0,T;X)$.
Recall that the partial differential operator L is `elliptic` if there exists a constant $\kappa > 0$ such that

$$\sum_{i,j=1}^{d} a_{ij}(x)\xi_i\xi_j \geqslant \kappa|\xi|^2$$

for $a.e.$ $x \in \Omega$ and for all $\xi \in \mathbb{R}^d$. Moreover, we assume

$$a_{ij} = a_{ji} \text{ and } 0 < a_{min} \leqslant a_{ij} \leqslant a^{max}, \text{ for all } i,j \in \{1,2.\dots,d\}. \tag{2.3}$$

so that the operator defined by (1.2) is elliptic.

2.1. **Elliptic regularity results.** Here, we recall the results obtained in [14]. We first introduce useful functional spaces.

Definition 2.1. *A function u is a bounded mean oscillation (BMO) function, if u is a real-valued function whose mean oscillation is bounded (finite). This function space is also called John–Nirenberg space. More precisely, we say that a locally integrable function u is a BMO function if*

$$\sup_B ⨍_B |u(x) - u_B| \, dx =: \|u\|_* < +\infty$$

where B ranges in the class of the balls of $\mathbb{R}^d$ and $u_B = ⨍_B u(x)\, dx = \dfrac{1}{|B|}\displaystyle\int_B u(x)\, dx$.

If u a BMO function and $r > 0$ we set

$$\eta(r) = \sup_{\rho \leqslant r} ⨍_{B_\rho} |u(x) - u_{B_\rho}| \, dx$$

where B_ρ ranges in the class of the balls with radius ρ less than or equal to r.

Definition 2.2. *A function u is a vanishing mean oscillation (VMO) function, if u belongs to the subspace of the BMO functions whose BMO norm over a ball vanishes as the radius of the ball tends to zero:*

$$\lim_{r\to 0} \eta(r) = 0.$$

The space VMO was introduced by D. Sarason in [12]. The characterization of the VMO functions via the norm of the function over balls implies a number of good features of VMO functions not shared by general BMO functions; for example a VMO function can be approximated by smooth functions. The space BMO can be characterized as the dual space to H^1. Furthermore, if f is a BMO function then for any $q < +\infty$ f is locally in L^q and if f belongs to the Sobolev space $W^{\theta,d/\theta}$ then f is a VMO function, for any $\theta \in (0,1]$. For more details and properties of BMO and VMO functions we refer [10, 12, 13].

The following theorem have been proved for C. Vitanza in [14]. We consider the elliptic equation in non divergence form

$$\bar{L}u = -\sum_{i,j=1}^{d} \bar{a}_{ij}(x)u_{x_ix_j} + \sum_{i=1}^{d} \bar{b}_i(x)u_{x_i} + \bar{c}(t,x)u = \bar{f}$$

and the associated Dirichlet problem

$$\begin{cases} \bar{L}u = \bar{f} \\ u \in W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega), \ \bar{f} \in L^q(\Omega). \end{cases} \tag{2.4}$$

Theorem 2.1. *Let $\partial\Omega$ be $\mathcal{C}^{1,1}$. Assume $\bar{a}_{ij} = \bar{a}_{ji}$, $\bar{a}_{ij} \in VMO \cap L^\infty(\Omega)$ and that there exists $\lambda > 0$ such that*

$$\forall \xi \in \mathbb{R}^d \quad \lambda^{-1}|\xi|^2 \leqslant \sum_{i,j=1}^{d} \bar{a}_{ij}(x)\xi_i\xi_j \leqslant \lambda|\xi|^2 \text{ a.e. in } \Omega.$$

We also suppose $\bar{b}_i \in L^s(\Omega)$, $s > d$ for $1 < q \leqslant d$, $s = q$ for $q > d$, and $\bar{c} \in L^r(\Omega)$ with $r = \begin{cases} d & \text{if } 1 < q \leqslant d \\ q & \text{if } q > d \end{cases}$ and $\bar{c} \leqslant 0$ a.e. in Ω. Then the Dirichlet problem (2.4) has a unique solution u. Furthermore there exists a positive constant C such that

$$\|u\|_{W^{2,q}(\Omega)\cap W_0^{1,q}(\Omega)} \leqslant C\|\bar{f}\|_{L^q(\Omega)}$$

where the constant C depend on d, $\partial\Omega$, λ, on the VMO modulus of $\bar{a}_{ij}$, on the L^s and L^d norms respectively of $\bar{b}_i$ and $\bar{c}$ and their AC modulus (see [14] for definition of AC modulus).
Here $W^{k,q}(\Omega)$ denotes the space of all functions u such that $D^\alpha u \in L^q(\Omega)$ for all multiindex α with $|\alpha| \leqslant k$ and $1 \leqslant q \leqslant +\infty$.

Remark 2.1. *In this work, we will use Theorem 2.1 with no lower order term ($\bar{c} = 0$).*

2.2. **Variational formulation of (1.1).** Let $u \in \mathcal{C}^2([0,T] \times \Omega)$ be a classical solution of (1.1), (i.e., u satisfies equation (1.1) at any $(t,x) \in (0,T) \times \Omega$). Multiplying the main equation of (1.1) by $\phi \in \mathcal{C}_c^\infty(\Omega)$ and integrate by parts, we obtain

$$\int_\Omega \frac{\partial^2 u}{\partial t^2}(t,x)\phi(x)\, dx + \int_\Omega a_{ij}(x)\nabla u(t,x) \cdot \nabla\phi(x)\, dx = \int_\Omega f(t,x)\phi(x)\, dx \tag{2.5}$$

a.e. $t \in (0,T)$. Hence, from the density of $\mathcal{C}_c^\infty(\Omega)$ in $H_0^1(\Omega)$, we have (2.5) for all $\phi \in H_0^1(\Omega)$. Now, we recall the definition of a weak solution for (1.1) (see [9])

Definition 2.3. *We say a function*

$$u \in L^2(0,T; H_0^1(\Omega)) \text{ with } \frac{\partial u}{\partial t} \in L^2(0,T; L^2(\Omega)) \text{ and } \frac{\partial^2 u}{\partial t^2} \in L^2(0,T; H^{-1}(\Omega))$$

is a weak solution of Problem (1.1) provided (2.5) holds true for all $\phi \in H_0^1(\Omega)$ and $0 \leqslant t \leqslant T$ a. e., and $u(0,x) = u_0(x)$ and $\frac{\partial u}{\partial t}(0,x) = u_1(x)$.

We remark that the initial conditions $u(0,x) = u_0(x)$ and $\frac{\partial u}{\partial t}(0,x) = u_1(x)$ make sense because of regularity of a weak solution; indeed we have $u \in C(0,T; L^2(\Omega))$ and $\frac{\partial u}{\partial t} \in C(0,T; H^{-1}(\Omega))$.

3. The main result

Now, we may give the main result:

Theorem 3.1. *Suppose $a_{ij} \in VMO \cap L^\infty(\Omega)$, $\nabla a_{ij} \in [L^p(\Omega)]^d$ with $p > d$ such that conditions (2.3) are ensured. We also suppose $f \in H^1(L^2(\Omega))$, $u_0 \in H^2(\Omega)$ and $u_1 \in H_0^1(\Omega)$. Then there exists a unique solution u of (1.1) such that*

$$u \in L^\infty(H^2(\Omega)), \quad \frac{\partial u}{\partial t} \in L^\infty(H_0^1(\Omega)), \quad \frac{\partial^2 u}{\partial t^2} \in L^\infty(L^2(\Omega)), \quad \frac{\partial^3 u}{\partial t^3} \in L^2(H^{-1}(\Omega))$$

with the estimate

$$\max_{0 \leqslant t \leqslant T} \left(\|u(t)\|_{H^2(\Omega)} + \left\| \frac{\partial u}{\partial t}(t) \right\|_{H_0^1(\Omega)} + \left\| \frac{\partial^2 u}{\partial t^2}(t) \right\|_{L^2(\Omega)} \right) + \left\| \frac{\partial^3 u}{\partial t^3} \right\|_{L^2(H^{-1}(\Omega))}$$
$$\leqslant C \left(\|f\|_{H^1(L^2(\Omega))} + \|u_0\|_{H^2(\Omega)} + \|u_1\|_{H^1(\Omega)} \right)$$

with the constant C depending on Ω, T and the coefficients a_{ij}.

Proof. We split the proof in several steps.
Step 1: Finite-dimensional approximate solutions.

For sake of simplicity, we denote $u' = \dfrac{\partial u}{\partial t}$, $u'' = \dfrac{\partial^2 u}{\partial t^2}$, $u''' = \dfrac{\partial^3 u}{\partial t^3}$ and $f' = \dfrac{\partial f}{\partial t}$ in the proof. We construct finite-dimensional approximate solutions of (2.5) by the method of Faedo–Galerkin.

As $H_0^1(\Omega)$ is a separable Hilbert space, there exist a family of functions $\{w_m\}_{m \geqslant 1}$ in $H_0^1(\Omega)$ such that

$$\{w_m\}_{m \geqslant 1} \text{ is an orthogonal basis of } H_0^1(\Omega)$$

and

$$\{w_m\}_{m \geqslant 1} \text{ is an orthonormal basis of } L^2(\Omega).$$

Fix now a positive m, we look for approximate solutions of (2.5) $u_m : [0, T] \to H_0^1(\Omega)$, as

$$u_m(t) = \sum_{i=1}^m g_{im}(t) w_i \tag{3.6}$$

with $\boldsymbol{g}_m := (g_{1m}, g_{2m}, \ldots, g_{mm})$ satisfying

$$\begin{cases} \left(u_m''(t), w_j\right) + (a_{ij} \nabla u_m(t), \nabla w_j) = (f(t), w_j) \\ g_{im}(0) = (u_0, w_i), \quad g_{im}'(0) = (u_1, w_i) \qquad\qquad (i = 1, 2, \ldots, m) \end{cases} \tag{3.7}$$

where $(\cdot, \cdot)$ denotes the scalar product in $L^2(\Omega)$. The initial conditions in system (3.7) mean that $u_m(0)$ and $u_m'(0)$ are the respective projections of u_0 and u_1 onto the subspace spanned by $\{w_1, w_2, \ldots, w_m\}$; thus we have $\lim_{m \to +\infty} u_m(0) = u_0$ and $\lim_{m \to +\infty} u_m'(0) = u_1$ (see, for example [3, Chapter 5]). From the classical theory of ordinary differential equations and assumptions of w_i, system (3.7) admits a unique local solution $\boldsymbol{g}_m$ such that $g_{jm} \in C^2(0, T_m)$ for $j = 1, 2, \ldots, m$. Then, for each fixed m, u_m defined by (3.6) is solution of (3.7).

Step 2: a priori estimates.
Multiplying (3.7) by g_{jm}', summing for $j = 1, \ldots, m$ and taking relation (3.6) into account, we get

$$\left(u_m''(t), u_m'(t)\right) + \left(a_{ij} \nabla u_m(t), \nabla u_m'(t)\right) = \left(f(t), u_m'(t)\right), \quad \text{a.e.} t \in [0, T] \tag{3.8}$$

or equivalently

$$(3.9)\qquad \underbrace{\frac{\partial}{\partial t}\left\|u_m'(t)\right\|^2_{L^2(\Omega)}}_{\geqslant 0} + \overbrace{\frac{\partial}{\partial t}\|a_{ij}\nabla u_m(t)\|^2_{L^2(\Omega)}}^{\geqslant 0} \leqslant \left(\|f(t)\|^2_{L^2(\Omega)} + \left\|u_m'(t)\right\|^2_{L^2(\Omega)}\right).$$

Integrating on $(0,s)$, we deduce

$$\left\|u_m'(s)\right\|^2_{L^2(\Omega)} \leqslant \|f\|^2_{L^2(L^2(\Omega))} + \int_0^s \left\|u_m'(t)\right\|^2_{L^2(\Omega)}\, dt + \left\|u_m'(0)\right\|^2_{L^2(\Omega)}$$

and with Gronwall's inequality

$$\|u_m'(t)\|^2_{L^2(\Omega)} \leqslant exp(T)\left(\|f\|^2_{L^2(L^2(\Omega))} + \|u_m'(0)\|^2_{L^2(\Omega)}\right).$$

Therefore,

$$(3.10)\qquad \max_{0\leqslant t\leqslant T}\|u_m'(t)\|^2_{L^2(\Omega)} \leqslant C\left(\|f\|^2_{L^2(L^2(\Omega))} + \|u_1\|^2_{L^2(\Omega)}\right)$$

with C depending on T and Ω. Here, we have used that $u_m'(0)$ is the projection of u_1 onto the subspace spanned by $\{w_1,\ldots,w_m\}$.

Using (3.9) again, integrating on $(0,s)$ and using (3.10), we obtain

$$\begin{aligned}\|a_{ij}\nabla u_m(t)\|^2_{L^2(\Omega)} &\leqslant \|f\|^2_{L^2(L^2(\Omega))} + \|a_{ij}\nabla u_m(0)\|^2_{L^2(\Omega)} + \int_0^t \|u_m'(s)\|^2_{L^2(\Omega)}\, ds\\ &\leqslant \|f\|^2_{L^2(L^2(\Omega))} + (a^{max})^2\|\nabla u_0\|^2_{L^2(\Omega)} + TC\left(\|f\|^2_{L^2(L^2(\Omega))} + \|h\|^2_{L^2(\Omega)}\right)\\ &\leqslant C\left(\|f\|^2_{L^2(L^2(\Omega))} + \|\nabla u_0\|^2_{L^2(\Omega)} + \|u_1\|^2_{L^2(\Omega)}\right)\end{aligned}$$

with C depending on T, Ω and a^{max}. From hypothesis on a_{ij} and Poincaré inequality we get

$$\|a_{ij}\nabla u_m(t)\|^2_{L^2(\Omega)} \geqslant C a^2_{min}\|u_m(t)\|^2_{H^1_0(\Omega)}.$$

So,

$$(3.11)\qquad \max_{0\leqslant t\leqslant T}\|u_m(t)\|^2_{H^1_0(\Omega)} \leqslant C\left(\|f\|^2_{L^2(L^2(\Omega))} + \|u_0\|^2_{H^1(\Omega)} + \|u_1\|^2_{L^2(\Omega)}\right)$$

with C depending on Ω, T, a_{min} and a^{max}.

Now, we estimate $\|u_m''\|_{L^2(H^{-1}(\Omega))}$:

$$\begin{aligned}\|u_m''(t)\|_{H^{-1}(\Omega)} &= \sup_{\substack{\phi\in H^1_0(\Omega)\\ \|\phi\|_{H^1_0(\Omega)=1}}} \left\langle u_m''(t),\phi\right\rangle_{H^{-1},H^1_0}\\ &= \sup_{\substack{\phi\in H^1_0(\Omega)\\ \|\phi\|_{H^1_0(\Omega)=1}}} \left[(f(t),\phi) - (a_{ij}\nabla u_m(t),\nabla\phi)\right]\\ &\leqslant\leqslant \|f(t)\|_{L^2(\Omega)} + \|a_{ij}\nabla u_m(t)\|_{L^2(\Omega)}.\end{aligned}$$

Thus

$$(3.12)\qquad \|u_m''\|_{L^2(H^{-1}(\Omega))} \leqslant C\left(\|f\|^2_{L^2(L^2(\Omega))} + \|u_0\|^2_{H^1(\Omega)} + \|u_1\|^2_{L^2(\Omega)}\right)$$

where C depends on Ω, T, a_{min} and a^{max}.

From these estimates we can conclude that $T_m = T$, that is $\boldsymbol{g}_m = (g_{1m}, g_{2m},\ldots,g_{mm})$ is a global solution of system (3.7) and consequently a global solution u_m.

Step 3: passage to the limit.
The estimates of step 2 allow us to conclude there exists a subsequence u_{mk} still denoted u_m and a function u such that

$$\begin{aligned} u_m &\rightharpoonup u \quad L^2(0,T;H_0^1(\Omega)) \\ u_m' &\rightharpoonup u' \quad L^2(0,T;L^2(\Omega)) \\ u_m'' &\rightharpoonup u'' \quad L^2(0,T;H^{-1}(\Omega)) \end{aligned}$$

where $\rightharpoonup$ stands for the weak convergence. This yields

$$\int_\Omega \frac{\partial^2 u_m}{\partial t^2}(t) w_j \, dx \to \int_\Omega \frac{\partial^2 u}{\partial t^2}(t) w_j \, dx \quad \text{as } m \to +\infty$$

$$\int_\Omega a_{ij}(x) \nabla u_m(t) \cdot \nabla w_j \, dx \to \int_\Omega a_{ij}(x) \nabla u(t) \cdot \nabla w_j \, dx \quad \text{as } m \to +\infty$$

for every w_j, by a density argument, for every H_0^1 function so equation (2.5) is satisfied. Furthermore, by standard arguments is possible to show that $u(0) = u_0$ and $u'(0) = u_1$. This proves that u is a weak solution of (1.1). Moreover from (3.10)-(3.12) we have $u \in L^\infty(0,T;H_0^1(\Omega))$, $u' \in L^\infty(0,T;L^2(\Omega))$ and $u'' \in L^2(0,T;H^{-1}(\Omega))$.

Step 4: The uniqueness solution of (1.1) follows similarly to the classical results for hyperbolic equations (for example [9, §7.2]) and from the conditions (2.3) for a_{ij}.

Step 5: Regularity improvment.
Let us differentiate the mais equation of (3.7) with respect to t and multiply by g_{jm}''

$$\left(u_m'''(t), u_m''(t)\right) + \left(a_{ij} \nabla u_m'(t), \nabla u_m''(t)\right) = \left(f'(t), u_m''(t)\right),$$

that is,

$$\underbrace{\frac{\partial}{\partial t} \left\| u_m''(t) \right\|_{L^2(\Omega)}^2}_{\geqslant 0} + \overbrace{\frac{\partial}{\partial t} \left\| a_{ij} \nabla u_m'(t) \right\|_{L^2(\Omega)}^2}^{\geqslant 0} \leqslant \left(\left\| f'(t) \right\|_{L^2(\Omega)}^2 + \left\| u_m''(t) \right\|_{L^2(\Omega)}^2 \right). \tag{3.13}$$

Integrating on $(0,s)$ gives

$$\left\| u_m''(t) \right\|_{L^2(\Omega)}^2 \leqslant \left\| f' \right\|_{L^2(L^2(\Omega))}^2 + \int_0^s \left\| u_m''(t) \right\|_{L^2(\Omega)}^2 \, dt + \left\| u_m''(0) \right\|_{L^2(\Omega)}^2$$

and with (3.8) we deduce

$$\left\| u_m''(0) \right\|_{L^2(\Omega)}^2 \leqslant C \left(\|f\|_{H^1(L^2(\Omega))}^2 + \|u_m(0)\|_{H^2(\Omega)}^2 \right) \leqslant C \left(\|f\|_{H^1(L^2(\Omega))}^2 + \|u_0\|_{H^2(\Omega)}^2 \right).$$

Then Gronwall's inequality gives

$$\max_{0 \leqslant t \leqslant T} \left\| u_m''(t) \right\|_{L^2(\Omega)}^2 \leqslant C \left(\|f\|_{H^1(L^2(\Omega))}^2 + \|u_0\|_{H^2(\Omega)}^2 \right)$$

where C depends on T, a_{min} and Ω.

On the other hand, by integrating on $(0,s)$ in (3.13) and using the last inequality, we obtain

$$\begin{aligned}\left\|a_{ij}\nabla u_m'(t)\right\|_{L^2(\Omega)} &\leqslant \left\|f'\right\|^2_{L^2(L^2(\Omega))} + \left\|u_m''\right\|^2_{L^2(L^2(\Omega))} + \left\|a_{ij}\nabla u_m'(0)\right\|_{L^2(\Omega)} \\ &\leqslant C\left(\|f\|^2_{H^1(L^2(\Omega))} + \|u_0\|^2_{H^2(\Omega)} + \|u_m'(0)\|^2_{H^1(\Omega)}\right) \\ &\leqslant C\left(\|f\|^2_{H^1(L^2(\Omega))} + \|u_0\|^2_{H^2(\Omega)} + \|u_1\|^2_{H^1(\Omega)}\right)\end{aligned}$$

with C depending on T, a_{min}, a^{max} and Ω. Therefore,

$$\max_{0\leqslant t\leqslant T}\left(\left\|u_m''(t)\right\|^2_{L^2(\Omega)} + \left\|u_m'(t)\right\|_{H^1_0(\Omega)}\right) \leqslant C\left(\|f\|^2_{H^1(L^2(\Omega))} + \|u_0\|^2_{H^2(\Omega)} + \|u_1\|^2_{H^1(\Omega)}\right) \tag{3.14}$$

where C depends on T, a_{min}, a^{max} and Ω.

In order to establish the higher regularity for u, we remark that, from (3.8), for $a.e\ t \in [0,T]$ we have

$$(-\operatorname{div}(a_{ij}\nabla u_m(t)),\phi) = (f(t) - u_m''(t)),\phi)$$

for every $\phi \in H^1_0(\Omega)$. We taking $q = 2$, $\bar{a}_{ij} = a_{ij}$, $\bar{b}_i = \dfrac{\partial a_{i1}}{\partial x_1} + \ldots + \dfrac{\partial a_{id}}{\partial x_d}$ and $\bar{c} = 0$ in Theorem 2.1 and from hypothesis for a_{ij} and ∇a_{ij}, we get $u_m(t) \in H^2(\Omega)$ and

$$\|u_m(t)\|_{H^2(\Omega)} \leqslant C\|f(t) - u_m''(t)\|_{L^2(\Omega)}$$

where C depends on Ω and the coefficients a_{ij} (via $\|\nabla a_{ij}\|_{L^p}$ and the VMO modulus of a_{ij}).

Hence, by using (3.14) we deduce

$$\begin{aligned}\max_{0\leqslant t\leqslant T}\|u_m(t)\|_{H^2(\Omega)} &\leqslant C\max_{0\leqslant t\leqslant T}\|f(t) - u_m''(t)\|_{L^2(\Omega)} \\ &\leqslant C\left(\|f\|_{H^1(L^2(\Omega))} + \|u_0\|_{H^1(\Omega)} + \|u_1\|^2_{L^2(\Omega)}\right).\end{aligned}$$

with C depending on Ω, T and the coefficients a_{ij}.

Last, we estimate $\|u_m'''\|_{L^2(H^{-1}(\Omega))}$

$$\begin{aligned}\|u_m'''(t)\|_{H^{-1}(\Omega)} = &\sup_{\substack{\phi\in H^1_0(\Omega)\\ \phi\neq 0}}\left\langle u_m'''(t),\phi\right\rangle_{H^{-1},H^1_0}\frac{1}{\|\phi\|_{H^1_0(\Omega)}} \\ &\sup_{\substack{\phi\in H^1_0(\Omega)\\ \phi\neq 0}}\left[(f'(t),\phi) - (a_{ij}\nabla u_m'(t),\nabla\phi)\right]\frac{1}{\|\phi\|_{H^1_0(\Omega)}} \\ &\leqslant \|f'(t)\|_{L^2(\Omega)} + \|a_{ij}\nabla u_m'(t)\|_{L^2(\Omega)}.\end{aligned}$$

Thus, from (3.14)

$$\|u_m'''\|_{L^2(H^{-1}(\Omega))} \leqslant C\left(\|f\|^2_{H^1(L^2(\Omega))} + \|u_0\|^2_{H^2(\Omega)} + \|u_1\|^2_{H^1(\Omega)}\right)$$

where C depends on Ω, T and the coefficients a_{ij}. Passing to limit as $m \to +\infty$, we obtain the same regularity and bounds for u. This concludes the proof of theorem. □

The work of MB was partially supported by AVENTURES - ANR-12-BLAN-BS01-0001-01. The work of ELS was partially supported by AVENTURES - ANR-12-BLAN-BS01-0001-01 and Ecos-Conicyt Grant C13E05.

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