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REGULARITY RESULTS FOR A CLASS OF HYPERBOLIC EQUATIONS WITH VMO COEFFICIENTS

MAÏTINE BERGOUNIOUX AND ERICA L. SCHWINDT

ABSTRACT. In this note we show a regularity result for an hyperbolic system with discontinuous coefficients. More precisely, we deal with coefficients in the function space VMO and we prove the existence and uniqueness of a solution $u \in L^\infty(0, T; H^2(\Omega))$ with also suitable regularity for $\frac{\partial u}{\partial t}$, $\frac{\partial^2 u}{\partial t^2}$ and $\frac{\partial^3 u}{\partial t^3}$.

1. INTRODUCTION

Let Ω be a bounded open subset of \mathbb{R}^d with $d \geq 3$. In the context of photoacoustic tomography process modelling [1], we are led to study the following wave equation

$$\begin{cases} \frac{\partial^2 p}{\partial t^2}(t, x) - \operatorname{div}(v_s^2 \nabla p)(t, x) = f(t, x) & \text{in } (0, T) \times \Omega \\ p(t, x) = 0 & \text{on } (0, T) \times \partial\Omega \\ p(0, x) = \frac{\partial p}{\partial t}(0, x) = 0 & \text{in } \Omega, \end{cases}$$

where $p = p(t, x)$ is an acoustic pressure wave, $v_s = v_s(x)$ is the speed of sound, f is a distributed source that comes from a lightning process and Ω is the domain where the wave propagates. The coefficient v_s is generally unknown and not smooth. We are interested in establishing new results of regularity of the solution p in the case of discontinuous coefficient v_s .

Hereafter we will assume that $\partial\Omega$ is of class \mathcal{C}^2 and we consider the following initial/boundary value problem:

$$(1.1) \quad \begin{cases} \frac{\partial^2 u}{\partial t^2} + Lu = f & \text{in } (0, T) \times \Omega \\ u = 0 & \text{on } (0, T) \times \partial\Omega \\ u(0, x) = u_0, \quad \frac{\partial u}{\partial t}(0, x) = u_1 & \text{in } \Omega, \end{cases}$$

where $f : (0, T) \times \Omega \rightarrow \mathbb{R}$, $u_0, u_1 : \Omega \rightarrow \mathbb{R}$ are given and L denotes a second order partial differential operator in the divergence form:

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$$(1.2) \quad Lu = - \sum_{i,j=1}^d (a_{ij}(x)u_{x_i})_{x_j}$$

where u_{x_i} denotes the partial derivative of u with respect to x_i .

Systems of equations as (1.1) have been extensively studied. Classical results of well-posedness and regularity can be found in [9, §7.2]. In this reference a regularity result similar to our Theorem 3.1 is obtained under coefficient smoothness assumptions, namely $a_{ij} \in C^1(\Omega)$ and $\nabla(a_{ij}) \in [C^1(\Omega)]^d$.

In this work, we consider discontinuous coefficients a_{ij} such that $a_{ij} \in VMO \cap L^\infty(\Omega)$ and $\nabla a_{ij} \in [L^p(\Omega)]^d$ with $p > d$. Assuming the coefficients a_{ij} belong to $L^\infty(\Omega)$, it can be proved that System (1.1) admits a unique solution $u \in C^0(0, T; H_0^1(\Omega))$ with $\frac{\partial u}{\partial t} \in C^0(0, T; L^2(\Omega))$ (see first part of the proof of Theorem 3.1).

Roughly speaking, the improved regularity, with respect to space, of the solution u is associated with the *elliptic regularity* of the equation for almost every $t \in [0, T]$, that is, with the regularity of $Lu(t) = f(t) - \frac{\partial^2 u}{\partial t^2}(t)$. Several regularity results for elliptic operator L have been obtained with more general elliptic operators of

type $\bar{L}u = - \sum_{i,j=1}^d a_{ij}(t, x)u_{x_i x_j} + \sum_{i=1}^d b_i(t, x)u_{x_i} + c(t, x)u$, and there exists a non-

exhaustive list of papers devoted to results of regularity associated with the operator \bar{L} with different hypothesis on the coefficients a_{ij} , b_i and c (see for example [4, 5, 6, 8, 7, 9, 14, 15] and references therein). Other results for parabolic equations with VMO coefficients can be found in [2, 11].

In Section 2 we introduce some definitions and notations and the variational formulation of System (1.1). Section 3 is devoted to the proof of Theorem 3.1 which is based on the regularity results obtained in [14].

2. PRELIMINARIES

In the sequel, $L^p(\Omega)$ is the space of measurable functions u on Ω such that $\int_\Omega |u|^p < +\infty$ for $1 \leq p < \infty$, $L^\infty(\Omega)$ is the space of essentially bounded functions on Ω . $\mathcal{C}^k(\Omega)$ is the set of all functions k -times continuously differentiable and its derivatives of order $|\alpha|$ are continuous for all multiindex α such that $|\alpha| \leq k$, $\mathcal{C}_c^\infty(\Omega)$ denote the subspace of all functions u infinitely differentiable with compact support in Ω . We will denote $H^k(\Omega)$ the usual Sobolev space of all functions u such that $D^\alpha u$ exists in the distributional sense and belongs to $L^2(\Omega)$ for all multiindex α with $|\alpha| \leq k$. The subspace $H_0^1(\Omega)$ is the closure of $\mathcal{C}_c^\infty(\Omega)$ in $H^1(\Omega)$ and the subspace $H^{-1}(\Omega)$ denotes the dual subspace to $H_0^1(\Omega)$. Let X be a Banach space: we will denote by $L^p(0, T; X)$ the space of the all measurable functions u such that $u : [0, T] \rightarrow X$ defined by $u(t)(x) = u(t, x)$ (by abuse of notation) satisfies

$$\|u\|_{L^p(0, T; X)} = \left(\int_0^T \|u(t)\|_X^p dt \right)^{1/p} < +\infty, \quad \text{if } p \in [1, +\infty)$$

and

$$\|u\|_{L^\infty(0, T; X)} = \text{ess sup}_{0 \leq t \leq T} \|u(t)\|_X < +\infty, \quad \text{if } p = +\infty.$$

The space $W^{1,p}(0, T; X)$ denotes all the functions $u \in L^p(0, T; X)$ such that $\frac{\partial u}{\partial t} \in L^p(0, T; X)$. For simplicity, we will use often the notation $W^{1,p}(X)$ instead of $W^{1,p}(0, T; X)$.

Recall that the partial differential operator L is **elliptic** if there exists a constant $\kappa > 0$ such that

$$\sum_{i,j=1}^d a_{ij}(x)\xi_i\xi_j \geq \kappa|\xi|^2$$

for *a.e.* $x \in \Omega$ and for all $\xi \in \mathbb{R}^d$. Moreover, we assume

$$(2.3) \quad a_{ij} = a_{ji} \text{ and } 0 < a_{\min} \leq a_{ij} \leq a^{\max}, \text{ for all } i, j \in \{1, 2, \dots, d\}.$$

so that the operator defined by (1.2) is elliptic.

2.1. Elliptic regularity results. Here, we recall the results obtained in [14]. We first introduce useful functional spaces.

Definition 2.1. *A function u is a bounded mean oscillation (BMO) function, if u is a real-valued function whose mean oscillation is bounded (finite). This function space is also called John–Nirenberg space. More precisely, we say that a locally integrable function u is a BMO function if*

$$\sup_B \int_B |u(x) - u_B| dx =: \|u\|_* < +\infty$$

where B ranges in the class of the balls of \mathbb{R}^d and $u_B = \int_B u(x) dx = \frac{1}{|B|} \int_B u(x) dx$.

If u a BMO function and $r > 0$ we set

$$\eta(r) = \sup_{\rho \leq r} \int_{B_\rho} |u(x) - u_{B_\rho}| dx$$

where B_ρ ranges in the class of the balls with radius ρ less than or equal to r .

Definition 2.2. *A function u is a vanishing mean oscillation (VMO) function, if u belongs to the subspace of the BMO functions whose BMO norm over a ball vanishes as the radius of the ball tends to zero:*

$$\lim_{r \rightarrow 0} \eta(r) = 0.$$

The space VMO was introduced by D. Sarason in [12]. The characterization of the VMO functions via the norm of the function over balls implies a number of good features of VMO functions not shared by general BMO functions; for example a VMO function can be approximated by smooth functions. The space BMO can be characterized as the dual space to H^1 . Furthermore, if f is a BMO function then for any $q < +\infty$ f is locally in L^q and if f belongs to the Sobolev space $W^{\theta, d/\theta}$ then f is a VMO function, for any $\theta \in (0, 1]$. For more details and properties of BMO and VMO functions we refer [10, 12, 13].

The following theorem have been proved for C. Vitanza in [14]. We consider the elliptic equation in non divergence form

$$\bar{L}u = - \sum_{i,j=1}^d \bar{a}_{ij}(x)u_{x_i x_j} + \sum_{i=1}^d \bar{b}_i(x)u_{x_i} + \bar{c}(t, x)u = \bar{f}$$

and the associated Dirichlet problem

$$(2.4) \quad \begin{cases} \bar{L}u = \bar{f} \\ u \in W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega), \bar{f} \in L^q(\Omega). \end{cases}$$

Theorem 2.1. *Let $\partial\Omega$ be $C^{1,1}$. Assume $\bar{a}_{ij} = \bar{a}_{ji}$, $\bar{a}_{ij} \in VMO \cap L^\infty(\Omega)$ and that there exists $\lambda > 0$ such that*

$$\forall \xi \in \mathbb{R}^d \quad \lambda^{-1}|\xi|^2 \leq \sum_{i,j=1}^d \bar{a}_{ij}(x)\xi_i\xi_j \leq \lambda|\xi|^2 \quad \text{a.e. in } \Omega.$$

We also suppose $\bar{b}_i \in L^s(\Omega)$, $s > d$ for $1 < q \leq d$, $s = q$ for $q > d$, and $\bar{c} \in L^r(\Omega)$ with $r = \begin{cases} d & \text{if } 1 < q \leq d \\ q & \text{if } q > d \end{cases}$ and $\bar{c} \leq 0$ a.e. in Ω . Then the Dirichlet problem (2.4) has a unique solution u . Furthermore there exists a positive constant C such that

$$\|u\|_{W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega)} \leq C \|\bar{f}\|_{L^q(\Omega)}$$

where the constant C depend on d , $\partial\Omega$, λ , on the VMO modulus of \bar{a}_{ij} , on the L^s and L^d norms respectively of \bar{b}_i and \bar{c} and their AC modulus (see [14] for definition of AC modulus).

Here $W^{k,q}(\Omega)$ denotes the space of all functions u such that $D^\alpha u \in L^q(\Omega)$ for all multiindex α with $|\alpha| \leq k$ and $1 \leq q \leq +\infty$.

Remark 2.1. *In this work, we will use Theorem 2.1 with no lower order term ($\bar{c} = 0$).*

2.2. Variational formulation of (1.1). Let $u \in C^2([0, T] \times \Omega)$ be a classical solution of (1.1), (i.e., u satisfies equation (1.1) at any $(t, x) \in (0, T) \times \Omega$). Multiplying the main equation of (1.1) by $\phi \in C_c^\infty(\Omega)$ and integrate by parts, we obtain

$$(2.5) \quad \int_{\Omega} \frac{\partial^2 u}{\partial t^2}(t, x)\phi(x) dx + \int_{\Omega} a_{ij}(x)\nabla u(t, x) \cdot \nabla \phi(x) dx = \int_{\Omega} f(t, x)\phi(x) dx$$

a.e. $t \in (0, T)$. Hence, from the density of $C_c^\infty(\Omega)$ in $H_0^1(\Omega)$, we have (2.5) for all $\phi \in H_0^1(\Omega)$. Now, we recall the definition of a weak solution for (1.1) (see [9])

Definition 2.3. *We say a function*

$$u \in L^2(0, T; H_0^1(\Omega)) \quad \text{with} \quad \frac{\partial u}{\partial t} \in L^2(0, T; L^2(\Omega)) \quad \text{and} \quad \frac{\partial^2 u}{\partial t^2} \in L^2(0, T; H^{-1}(\Omega))$$

is a weak solution of Problem (1.1) provided (2.5) holds true for all $\phi \in H_0^1(\Omega)$ and $0 \leq t \leq T$ a. e., and $u(0, x) = u_0(x)$ and $\frac{\partial u}{\partial t}(0, x) = u_1(x)$.

We remark that the initial conditions $u(0, x) = u_0(x)$ and $\frac{\partial u}{\partial t}(0, x) = u_1(x)$ make sense because of regularity of a weak solution; indeed we have $u \in C(0, T; L^2(\Omega))$ and $\frac{\partial u}{\partial t} \in C(0, T; H^{-1}(\Omega))$.

3. THE MAIN RESULT

Now, we may give the main result:

Theorem 3.1. *Suppose $a_{ij} \in VMO \cap L^\infty(\Omega)$, $\nabla a_{ij} \in [L^p(\Omega)]^d$ with $p > d$ such that conditions (2.3) are ensured. We also suppose $f \in H^1(L^2(\Omega))$, $u_0 \in H^2(\Omega)$ and $u_1 \in H_0^1(\Omega)$. Then there exists a unique solution u of (1.1) such that*

$$u \in L^\infty(H^2(\Omega)), \quad \frac{\partial u}{\partial t} \in L^\infty(H_0^1(\Omega)), \quad \frac{\partial^2 u}{\partial t^2} \in L^\infty(L^2(\Omega)), \quad \frac{\partial^3 u}{\partial t^3} \in L^2(H^{-1}(\Omega))$$

with the estimate

$$\begin{aligned} & \max_{0 \leq t \leq T} \left(\|u(t)\|_{H^2(\Omega)} + \left\| \frac{\partial u}{\partial t}(t) \right\|_{H_0^1(\Omega)} + \left\| \frac{\partial^2 u}{\partial t^2}(t) \right\|_{L^2(\Omega)} \right) + \left\| \frac{\partial^3 u}{\partial t^3} \right\|_{L^2(H^{-1}(\Omega))} \\ & \leq C \left(\|f\|_{H^1(L^2(\Omega))} + \|u_0\|_{H^2(\Omega)} + \|u_1\|_{H^1(\Omega)} \right) \end{aligned}$$

with the constant C depending on Ω , T and the coefficients a_{ij} .

Proof. We split the proof in several steps.

Step 1: Finite-dimensional approximate solutions.

For sake of simplicity, we denote $u' = \frac{\partial u}{\partial t}$, $u'' = \frac{\partial^2 u}{\partial t^2}$, $u''' = \frac{\partial^3 u}{\partial t^3}$ and $f' = \frac{\partial f}{\partial t}$ in the proof. We construct finite-dimensional approximate solutions of (2.5) by the method of Faedo–Galerkin.

As $H_0^1(\Omega)$ is a separable Hilbert space, there exist a family of functions $\{w_m\}_{m \geq 1}$ in $H_0^1(\Omega)$ such that

$$\{w_m\}_{m \geq 1} \text{ is an orthogonal basis of } H_0^1(\Omega)$$

and

$$\{w_m\}_{m \geq 1} \text{ is an orthonormal basis of } L^2(\Omega).$$

Fix now a positive m , we look for approximate solutions of (2.5) $u_m : [0, T] \rightarrow H_0^1(\Omega)$, as

$$(3.6) \quad u_m(t) = \sum_{i=1}^m g_{im}(t) w_i$$

with $\mathbf{g}_m := (g_{1m}, g_{2m}, \dots, g_{mm})$ satisfying

$$(3.7) \quad \begin{cases} \left(u_m''(t), w_j \right) + (a_{ij} \nabla u_m(t), \nabla w_j) = (f(t), w_j) \\ g_{im}(0) = (u_0, w_i), \quad g'_{im}(0) = (u_1, w_i) \end{cases} \quad (i = 1, 2, \dots, m)$$

where (\cdot, \cdot) denotes the scalar product in $L^2(\Omega)$. The initial conditions in system (3.7) mean that $u_m(0)$ and $u'_m(0)$ are the respective projections of u_0 and u_1 onto the subspace spanned by $\{w_1, w_2, \dots, w_m\}$; thus we have $\lim_{m \rightarrow +\infty} u_m(0) = u_0$ and $\lim_{m \rightarrow +\infty} u'_m(0) = u_1$ (see, for example [3, Chapter 5]). From the classical theory of ordinary differential equations and assumptions of w_i , system (3.7) admits a unique local solution \mathbf{g}_m such that $g_{jm} \in C^2(0, T_m)$ for $j = 1, 2, \dots, m$. Then, for each fixed m , u_m defined by (3.6) is solution of (3.7).

Step 2: a priori estimates.

Multiplying (3.7) by g'_{jm} , summing for $j = 1, \dots, m$ and taking relation (3.6) into account, we get

$$(3.8) \quad \left(u_m''(t), u'_m(t) \right) + \left(a_{ij} \nabla u_m(t), \nabla u'_m(t) \right) = \left(f(t), u'_m(t) \right), \quad \text{a.e. } t \in [0, T]$$

or equivalently

$$(3.9) \quad \underbrace{\frac{\partial}{\partial t} \|u'_m(t)\|_{L^2(\Omega)}^2}_{\geq 0} + \overbrace{\frac{\partial}{\partial t} \|a_{ij} \nabla u_m(t)\|_{L^2(\Omega)}^2}^{\geq 0} \leq \left(\|f(t)\|_{L^2(\Omega)}^2 + \|u'_m(t)\|_{L^2(\Omega)}^2 \right).$$

Integrating on $(0, s)$, we deduce

$$\|u'_m(s)\|_{L^2(\Omega)}^2 \leq \|f\|_{L^2(L^2(\Omega))}^2 + \int_0^s \|u'_m(t)\|_{L^2(\Omega)}^2 dt + \|u'_m(0)\|_{L^2(\Omega)}^2$$

and with Gronwall's inequality

$$\|u'_m(t)\|_{L^2(\Omega)}^2 \leq \exp(T) \left(\|f\|_{L^2(L^2(\Omega))}^2 + \|u'_m(0)\|_{L^2(\Omega)}^2 \right).$$

Therefore,

$$(3.10) \quad \max_{0 \leq t \leq T} \|u'_m(t)\|_{L^2(\Omega)}^2 \leq C \left(\|f\|_{L^2(L^2(\Omega))}^2 + \|u_1\|_{L^2(\Omega)}^2 \right)$$

with C depending on T and Ω . Here, we have used that $u'_m(0)$ is the projection of u_1 onto the subspace spanned by $\{w_1, \dots, w_m\}$.

Using (3.9) again, integrating on $(0, s)$ and using (3.10), we obtain

$$\begin{aligned} \|a_{ij} \nabla u_m(t)\|_{L^2(\Omega)}^2 &\leq \|f\|_{L^2(L^2(\Omega))}^2 + \|a_{ij} \nabla u_m(0)\|_{L^2(\Omega)}^2 + \int_0^t \|u'_m(s)\|_{L^2(\Omega)}^2 ds \\ &\leq \|f\|_{L^2(L^2(\Omega))}^2 + (a^{max})^2 \|\nabla u_0\|_{L^2(\Omega)}^2 + TC \left(\|f\|_{L^2(L^2(\Omega))}^2 + \|h\|_{L^2(\Omega)}^2 \right) \\ &\leq C \left(\|f\|_{L^2(L^2(\Omega))}^2 + \|\nabla u_0\|_{L^2(\Omega)}^2 + \|u_1\|_{L^2(\Omega)}^2 \right) \end{aligned}$$

with C depending on T , Ω and a^{max} . From hypothesis on a_{ij} and Poincaré inequality we get

$$\|a_{ij} \nabla u_m(t)\|_{L^2(\Omega)}^2 \geq Ca_{min}^2 \|u_m(t)\|_{H_0^1(\Omega)}^2.$$

So,

$$(3.11) \quad \max_{0 \leq t \leq T} \|u_m(t)\|_{H_0^1(\Omega)}^2 \leq C \left(\|f\|_{L^2(L^2(\Omega))}^2 + \|u_0\|_{H^1(\Omega)}^2 + \|u_1\|_{L^2(\Omega)}^2 \right)$$

with C depending on Ω , T , a_{min} and a^{max} .

Now, we estimate $\|u_m''\|_{L^2(H^{-1}(\Omega))}$:

$$\begin{aligned} \|u_m''(t)\|_{H^{-1}(\Omega)} &= \sup_{\substack{\phi \in H_0^1(\Omega) \\ \|\phi\|_{H_0^1(\Omega)}=1}} \langle u_m''(t), \phi \rangle_{H^{-1}, H_0^1} \\ &= \sup_{\substack{\phi \in H_0^1(\Omega) \\ \|\phi\|_{H_0^1(\Omega)}=1}} [(f(t), \phi) - (a_{ij} \nabla u_m(t), \nabla \phi)] \\ &\leq \|f(t)\|_{L^2(\Omega)} + \|a_{ij} \nabla u_m(t)\|_{L^2(\Omega)}. \end{aligned}$$

Thus

$$(3.12) \quad \|u_m''\|_{L^2(H^{-1}(\Omega))} \leq C \left(\|f\|_{L^2(L^2(\Omega))}^2 + \|u_0\|_{H^1(\Omega)}^2 + \|u_1\|_{L^2(\Omega)}^2 \right)$$

where C depends on Ω , T , a_{min} and a^{max} .

From these estimates we can conclude that $T_m = T$, that is $\mathbf{g}_m = (g_{1m}, g_{2m}, \dots, g_{mm})$ is a global solution of system (3.7) and consequently a global solution u_m .

Step 3: passage to the limit.

The estimates of step 2 allow us to conclude there exists a subsequence u_{mk} still denoted u_m and a function u such that

$$\begin{aligned} u_m &\rightharpoonup u && L^2(0, T; H_0^1(\Omega)) \\ u'_m &\rightharpoonup u' && L^2(0, T; L^2(\Omega)) \\ u''_m &\rightharpoonup u'' && L^2(0, T; H^{-1}(\Omega)) \end{aligned}$$

where \rightharpoonup stands for the weak convergence. This yields

$$\begin{aligned} \int_{\Omega} \frac{\partial^2 u_m}{\partial t^2}(t) w_j \, dx &\rightarrow \int_{\Omega} \frac{\partial^2 u}{\partial t^2}(t) w_j \, dx \quad \text{as } m \rightarrow +\infty \\ \int_{\Omega} a_{ij}(x) \nabla u_m(t) \cdot \nabla w_j \, dx &\rightarrow \int_{\Omega} a_{ij}(x) \nabla u(t) \cdot \nabla w_j \, dx \quad \text{as } m \rightarrow +\infty \end{aligned}$$

for every w_j , by a density argument, for every H_0^1 function so equation (2.5) is satisfied. Furthermore, by standard arguments it is possible to show that $u(0) = u_0$ and $u'(0) = u_1$. This proves that u is a weak solution of (1.1). Moreover from (3.10)-(3.12) we have $u \in L^\infty(0, T; H_0^1(\Omega))$, $u' \in L^\infty(0, T; L^2(\Omega))$ and $u'' \in L^2(0, T; H^{-1}(\Omega))$.

Step 4: The uniqueness solution of (1.1) follows similarly to the classical results for hyperbolic equations (for example [9, §7.2]) and from the conditions (2.3) for a_{ij} .

Step 5: Regularity improvement.

Let us differentiate the main equation of (3.7) with respect to t and multiply by g_{jm}''

$$\left(u_m'''(t), u_m''(t) \right) + \left(a_{ij} \nabla u_m'(t), \nabla u_m''(t) \right) = \left(f'(t), u_m''(t) \right),$$

that is,

$$(3.13) \quad \underbrace{\frac{\partial}{\partial t} \|u_m''(t)\|_{L^2(\Omega)}^2}_{\geq 0} + \overbrace{\frac{\partial}{\partial t} \|a_{ij} \nabla u_m'(t)\|_{L^2(\Omega)}^2}^{\geq 0} \leq \left(\|f'(t)\|_{L^2(\Omega)}^2 + \|u_m''(t)\|_{L^2(\Omega)}^2 \right).$$

Integrating on $(0, s)$ gives

$$\|u_m''(s)\|_{L^2(\Omega)}^2 \leq \|f'\|_{L^2(L^2(\Omega))}^2 + \int_0^s \|u_m''(t)\|_{L^2(\Omega)}^2 \, dt + \|u_m''(0)\|_{L^2(\Omega)}^2$$

and with (3.8) we deduce

$$\|u_m''(0)\|_{L^2(\Omega)}^2 \leq C \left(\|f\|_{H^1(L^2(\Omega))}^2 + \|u_m(0)\|_{H^2(\Omega)}^2 \right) \leq C \left(\|f\|_{H^1(L^2(\Omega))}^2 + \|u_0\|_{H^2(\Omega)}^2 \right).$$

Then Gronwall's inequality gives

$$\max_{0 \leq t \leq T} \|u_m''(t)\|_{L^2(\Omega)}^2 \leq C \left(\|f\|_{H^1(L^2(\Omega))}^2 + \|u_0\|_{H^2(\Omega)}^2 \right)$$

where C depends on T , a_{min} and Ω .

On the other hand, by integrating on $(0, s)$ in (3.13) and using the last inequality, we obtain

$$\begin{aligned} \left\| a_{ij} \nabla u'_m(t) \right\|_{L^2(\Omega)} &\leq \left\| f' \right\|_{L^2(L^2(\Omega))}^2 + \left\| u''_m \right\|_{L^2(L^2(\Omega))}^2 + \left\| a_{ij} \nabla u'_m(0) \right\|_{L^2(\Omega)} \\ &\leq C \left(\|f\|_{H^1(L^2(\Omega))}^2 + \|u_0\|_{H^2(\Omega)}^2 + \|u'_m(0)\|_{H^1(\Omega)}^2 \right) \\ &\leq C \left(\|f\|_{H^1(L^2(\Omega))}^2 + \|u_0\|_{H^2(\Omega)}^2 + \|u_1\|_{H^1(\Omega)}^2 \right) \end{aligned}$$

with C depending on T , a_{min} , a^{max} and Ω . Therefore,

$$(3.14) \quad \max_{0 \leq t \leq T} \left(\left\| u''_m(t) \right\|_{L^2(\Omega)}^2 + \left\| u'_m(t) \right\|_{H_0^1(\Omega)} \right) \leq C \left(\|f\|_{H^1(L^2(\Omega))}^2 + \|u_0\|_{H^2(\Omega)}^2 + \|u_1\|_{H^1(\Omega)}^2 \right)$$

where C depends on T , a_{min} , a^{max} and Ω .

In order to establish the higher regularity for u , we remark that, from (3.8), for a.e $t \in [0, T]$ we have

$$(-\operatorname{div}(a_{ij} \nabla u_m(t)), \phi) = (f(t) - u''_m(t), \phi)$$

for every $\phi \in H_0^1(\Omega)$. We taking $q = 2$, $\bar{a}_{ij} = a_{ij}$, $\bar{b}_i = \frac{\partial a_{i1}}{\partial x_1} + \dots + \frac{\partial a_{id}}{\partial x_d}$ and $\bar{c} = 0$ in Theorem 2.1 and from hypothesis for a_{ij} and ∇a_{ij} , we get $u_m(t) \in H^2(\Omega)$ and

$$\|u_m(t)\|_{H^2(\Omega)} \leq C \|f(t) - u''_m(t)\|_{L^2(\Omega)}$$

where C depends on Ω and the coefficients a_{ij} (via $\|\nabla a_{ij}\|_{L^p}$ and the VMO modulus of a_{ij}).

Hence, by using (3.14) we deduce

$$\begin{aligned} \max_{0 \leq t \leq T} \|u_m(t)\|_{H^2(\Omega)} &\leq C \max_{0 \leq t \leq T} \|f(t) - u''_m(t)\|_{L^2(\Omega)} \\ &\leq C \left(\|f\|_{H^1(L^2(\Omega))} + \|u_0\|_{H^1(\Omega)} + \|u_1\|_{L^2(\Omega)}^2 \right). \end{aligned}$$

with C depending on Ω , T and the coefficients a_{ij} .

Last, we estimate $\|u'''_m\|_{L^2(H^{-1}(\Omega))}$

$$\begin{aligned} \|u'''_m(t)\|_{H^{-1}(\Omega)} &= \sup_{\substack{\phi \in H_0^1(\Omega) \\ \phi \neq 0}} \left\langle u'''_m(t), \phi \right\rangle_{H^{-1}, H_0^1} \frac{1}{\|\phi\|_{H_0^1(\Omega)}} \\ &= \sup_{\substack{\phi \in H_0^1(\Omega) \\ \phi \neq 0}} \left[(f'(t), \phi) - (a_{ij} \nabla u'_m(t), \nabla \phi) \right] \frac{1}{\|\phi\|_{H_0^1(\Omega)}} \\ &\leq \|f'(t)\|_{L^2(\Omega)} + \|a_{ij} \nabla u'_m(t)\|_{L^2(\Omega)}. \end{aligned}$$

Thus, from (3.14)

$$\|u'''_m\|_{L^2(H^{-1}(\Omega))} \leq C \left(\|f\|_{H^1(L^2(\Omega))}^2 + \|u_0\|_{H^2(\Omega)}^2 + \|u_1\|_{H^1(\Omega)}^2 \right)$$

where C depends on Ω , T and the coefficients a_{ij} . Passing to limit as $m \rightarrow +\infty$, we obtain the same regularity and bounds for u . This concludes the proof of theorem. \square

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